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On the global existence of solutions of systems of linear differential equations with constant coefficients

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Let $P(D)$ be a matrix of linear differential operators with constant coefficients and f be a vector of functions defined on a domain Ω in $\boldsymbol{R}^n$. We consider the problem whether a solution u of the equation $P(D)u=f$ exists or not on the domain Ω assuming that f satisfies the compatibility condition. For example, let $P(D)$ be the Cauchy-Riemann system (resp. the de Rham system). Then the solution always exists if the domain Ω is pseudo-convex (resp. simply connected). If Ω is convex, this problem has been affirmatively solved for general $P(D)$ by Ehrenpreis [**3**], Malgrange [**9**], Hörmander [**6**], Palamodov [**11**] and Komatsu [**7**] in many function spaces.

In this note we discuss two special cases of the problem in § 2 and § 3 respectively. The first one is the case where the domain has *a compact hole* and the second one is the case where $P(D)$ is *a partial de Rham system*. First we introduce some notations in § 1.

§ **1.** We denote by $\mathcal{P}$ the ring of linear partial differential operators with constant coefficients in $\boldsymbol{R}^n$, by $\mathcal{A}$, $\mathcal{B}$, $\mathcal{D}'$, $\mathcal{E}$ the sheaves of real analytic functions, hyperfunctions, distributions and infinitely differentiable functions over $\boldsymbol{R}^n$ respectively and generally by $\mathcal{F}$ one of these sheaves. Let M be *a finitely generated $\mathcal{P}$-module.* Then M defines an equation $P(D)u=f$ in the following way:

M has *a free resolution*

$$0 \longleftarrow M \longleftarrow \mathcal{P}^{r_0} \xleftarrow{{}^tP(D)} \mathcal{P}^{r_1} \xleftarrow{{}^tP_1(D)} \mathcal{P}^{r_2} \xleftarrow{{}^tP_2(D)} \cdots, \tag{1.1}$$

where ${}^tP(D)$ is the transposed matrix of the $r_1 \times r_0$ matrix $P(D)$. We regard $\mathcal{P}$ and M as constant sheaves over $\boldsymbol{R}^n$. Then M and $\mathcal{F}$ are sheaves of $\mathcal{P}$-Modules in the natural way. Applying the functor $\mathcal{H}om_{\mathcal{P}}(\cdot, \mathcal{F})$ to (1.1), we have a cochain complex of sheaves of $\mathcal{P}$-Modules:

$$0 \longrightarrow \mathcal{F}^M \longrightarrow \mathcal{F}^{r_0} \xrightarrow{P(D)} \mathcal{F}^{r_1} \xrightarrow{P_1(D)} \mathcal{F}^{r_2} \xrightarrow{P_2(D)} \cdots, \tag{1.2}$$

where we denote by $\mathcal{F}^M$ the *solution sheaf* $\mathcal{H}om_{\mathcal{P}}(M, \mathcal{F})$.

We denote by $\mathcal{F}(\Omega)$ the space of the sections $\Gamma(\Omega, \mathcal{F})$ and apply the functor $\mathrm{Hom}_{\mathcal{P}}(\cdot, \mathcal{F}(\Omega))$ to (1.1) or the functor $\Gamma(\Omega, \cdot)$ to (1.2), then we have a cochain complex of $\mathcal{P}$-modules:

$$(1.3) \qquad \cdots \longrightarrow \mathcal{F}(\Omega)^{r_{i-1}} \xrightarrow{P_{i-1}(D)} \mathcal{F}(\Omega)^{r_i} \xrightarrow{P_i(D)} \mathcal{F}(\Omega)^{r_{i+1}} \longrightarrow \cdots.$$

Since the i-th cohomology group of (1.3) is $\mathrm{Ext}^i_{\mathcal{P}}(M, \mathcal{F}(\Omega))$ by definition, a vector of functions $u \in \mathcal{F}(\Omega)^{r_0}$ satisfies $P(D)u=0$ if and only if $u \in \mathrm{Hom}_{\mathcal{P}}(M, \mathcal{F}(\Omega))$, and assuming that $\mathrm{Ext}^1_{\mathcal{P}}(M, \mathcal{F}(\Omega))=0$, the equation $P(D)u=f$ on Ω has a solution u if and only if the *compatibility condition* $P_1(D)f=0$ holds.

In Ehrenpreis [**3**], Malgrange [**9**], Hörmander [**6**], Palamodov [**11**] and Komatsu [**7**] the following theorem is proved by the method of Fourier analysis.

THEOREM 1.1. *Let $\mathcal{F}$ be one of $\mathcal{B}$, $\mathcal{D}'$ and $\mathcal{E}$ (or $\mathcal{A}$) and W be a convex open (resp. convex compact) set in $\boldsymbol{R}^n$. Then $\mathcal{F}(W)$ is an injective $\mathcal{P}$-module, i.e., $\mathrm{Ext}^1_{\mathcal{P}}(M, \mathcal{F}(W))=0$ for any $\mathcal{P}$-module M.*

Since the above sets W form a fundamental system of neighbourhoods at any point of $\boldsymbol{R}^n$, the sequence (1.2) is exact. Namely (1.2) is a resolution of $\mathcal{F}^M$. Moreover, $\mathcal{B}$, $\mathcal{D}'$ and $\mathcal{E}$ are *soft* sheaves and $H^i(\Omega, \mathcal{A})=0$ for $i \geqq 1$ by a theorem of Malgrange. Hence the i-th cohomology group of (1.3) is equal to $H^i(\Omega, \mathcal{F}^M)$. Thus we have

$$(1.4) \qquad H^i(\Omega, \mathcal{F}^M)=\mathrm{Ext}^i_{\mathcal{P}}(M, \mathcal{F}(\Omega)) \qquad \text{for } i \geqq 0.$$

Let Z be a closed set in $\boldsymbol{R}^n$. Then the i-th cohomology group of

$$(1.5) \qquad \cdots \longrightarrow \mathcal{B}_Z(\boldsymbol{R}^n)^{r_{i-1}} \xrightarrow{P_{i-1}(D)} \mathcal{B}_Z(\boldsymbol{R}^n)^{r_i} \xrightarrow{P_i(D)} \mathcal{B}_Z(\boldsymbol{R}^n)^{r_{i+1}} \longrightarrow \cdots$$

is equal to $H^i_Z(\boldsymbol{R}, \mathcal{B}^M)$ since $\mathcal{B}$ is a flabby sheaf, where we denote by $\mathcal{B}_Z(\boldsymbol{R}^n)$ the space of the global sections of $\mathcal{B}$ whose supports are contained in Z. We get also (1.5) by applying the functor $\mathrm{Hom}_{\mathcal{P}}(\cdot, \mathcal{B}_Z(\boldsymbol{R}^n))$ to (1.1), so that we have

$$(1.6) \qquad H^i_Z(\boldsymbol{R}^n, \mathcal{B}^M)=\mathrm{Ext}^i_{\mathcal{P}}(M, \mathcal{B}_Z(\boldsymbol{R}^n)) \qquad \text{for } i \geqq 0.$$

§**2.** In this section we discuss the first case in the space of hyperfunctions.

THEOREM 2.1. *Let Ω be a domain in $\boldsymbol{R}^n$ with a compact hole, i.e., there exist a domain V in $\boldsymbol{R}^n$ and a compact subset $K \neq \emptyset$ of V such that $\Omega = V-K$ and M be a finitely generated $\mathcal{P}$-module. Then the following condition* (1) *implies* (2) *for an arbitrary positive integer i.*

(1) $$\mathrm{Ext}^i_{\mathcal{P}}(M, \mathcal{B}(\Omega))=0.$$

(2) $$\mathrm{Ext}^{i+1}_{\mathcal{P}}(M, \mathcal{P})=0.$$

PROOF. According to Komatsu [8] we have the following exact sequence:

(2.1) $$0 \longrightarrow H^i(V, \mathcal{B}^M) \longrightarrow H^i(\Omega, \mathcal{B}^M) \longrightarrow H^{i+1}_K(V, \mathcal{B}^M) \longrightarrow 0.$$

Therefore (1) and (1.4) imply $H^{i+1}_K(V, \mathcal{B}^M)=0$. Since $H^{i+1}_K(V, \mathcal{B}^M)$ is equal to $H^{i+1}_K(\boldsymbol{R}^n, \mathcal{B}^M)$, the $i+1$-th part of (1.5) and its Fourier-Laplace transform

(2.2) $$\widetilde{\mathcal{B}_K(\boldsymbol{R}^n)^{r_i}} \xrightarrow{P_i(\zeta)} \widetilde{\mathcal{B}_K(\boldsymbol{R}^n)^{r_{i+1}}} \xrightarrow{P_{i+1}(\zeta)} \widetilde{\mathcal{B}_K(\boldsymbol{R}^n)^{r_{i+2}}}$$

are exact, where $P_i(\zeta)$ and $P_{i+1}(\zeta)$ are the matrices with polynomial elements which we get replacing $-\sqrt{-1}\partial/\partial x_j$ by ζ_j. We can assume $\{0\}\in K$ without loss of generality, then the space of polynomials with n variables, which we denote by A, is contained in $\widetilde{\mathcal{B}_K(\boldsymbol{R}^n)}$ by the Paley-Wiener theorem and if a vector $F(\zeta)\in A^{r_{i+1}}$ satisfies $P_{i+1}(\zeta)F(\zeta)=0$, there exists $U(\zeta)\in\widetilde{\mathcal{B}_K(\boldsymbol{R}^n)^{r_i}}$ such that $P_i(\zeta)U(\zeta)=F(\zeta)$. Applying Hörmander [6] Theorem 7.6.11, we can prove that there exists $U'(\zeta)\in A^{r_i}$ such that $P_i(\zeta)U'(\zeta)=F(\zeta)$. (See the proof of Komatsu [8] Theorem 4.4.)

Since the ring $\mathcal{P}$ is isomorphic to the ring A by the above correspondence, we have proved the following sequence is exact:

(2.3) $$\mathcal{P}^{r_i} \xrightarrow{P_i(D)} \mathcal{P}^{r_{i+1}} \xrightarrow{P_{i+1}(D)} \mathcal{P}^{r_{i+2}}.$$

We also get the sequence (2.3) by applying the function $\mathrm{Hom}_{\mathcal{P}}(\cdot, \mathcal{P})$ to (1.1). This implies (2). q. e. d.

REMARK. This theorem does not hold in the space $\mathcal{A}$ nor $\mathcal{E}$. (See Example 3.2 iii).)

Conversely we have the following theorem because of the flabbiness of $\mathcal{B}$.

THEOREM 2.2. *Let k be a positive integer and assume that a finitely generated $\mathcal{P}$-module M satisfies the condition* (2) *for any $i\geqq k$. Then the condition* (1) *holds for any domain Ω in $\boldsymbol{R}^n$ and any $i\geqq k$.*

PROOF. Any finitely generated $\mathcal{P}$-module N has a free resolution:

(2.4) $$0 \longleftarrow N \longleftarrow \mathcal{P}^{s_0} \xleftarrow{Q(D)} \mathcal{P}^{s_1} \xleftarrow{Q_1(D)} \cdots.$$

From the short exact sequence

$$0 \longrightarrow \mathrm{Im}\, Q_j(D) \longrightarrow \mathcal{P}^{s_j} \longrightarrow \mathrm{Im}\, Q_{j-1}(D) \longrightarrow 0,$$

we obtain the long exact sequence:

$$\cdots \longrightarrow \mathrm{Ext}^i_{\mathcal{P}}(M, \mathrm{Im}\, Q_j(D)) \longrightarrow \mathrm{Ext}^i_{\mathcal{P}}(M, \mathcal{P}^{s_j}) \longrightarrow \mathrm{Ext}^i_{\mathcal{P}}(M, \mathrm{Im}\, Q_{j-1}(D)) \longrightarrow \cdots.$$

Hence the assumption implies $\mathrm{Ext}^i_{\mathcal{P}}(M, \mathrm{Im}\, Q_{j-1}(D))=\mathrm{Ext}^{i+1}_{\mathcal{P}}(M, \mathrm{Im}\, Q_j(D))$ for

$i\geqq k+1$. Thus we have $\mathrm{Ext}_{\mathscr{P}}^{k+1}(M, N)=\mathrm{Ext}_{\mathscr{P}}^{k+2}(M, \mathrm{Im}\, Q(D))=\cdots=\mathrm{Ext}_{\mathscr{P}}^{n+1}(M, \mathrm{Im}\, Q_{n-k-1}(D))$, which vanishes because the *global dimension* of $\mathscr{P}$ is equal to n (see [2]). This implies that the *projective dimension* of M is not larger than k.

Set $Z=\boldsymbol{R}^n-\Omega$. Then we have the long exact sequence of the *relative cohomology* of the sheaf $\mathscr{B}^M$ with respect to the pair $Z\subset\boldsymbol{R}^n$:

$$\cdots\longrightarrow H^i(\boldsymbol{R}^n, \mathscr{B}^M)\longrightarrow H^i(\Omega, \mathscr{B}^M)\longrightarrow H_Z^{i+1}(\boldsymbol{R}^n, \mathscr{B}^M)\longrightarrow\cdots.$$

Theorem 1.1 and (1.4) show that $H^i(\boldsymbol{R}^n, \mathscr{B}^M)=0$ for $i\geqq 1$. The projective dimension of M and (1.6) show that $H_Z^i(\boldsymbol{R}^n, \mathscr{B}^M)=0$ for $i\geqq k+1$. Combining these facts, we see that $\mathrm{Ext}_{\mathscr{P}}^i(M, \mathscr{B}(\Omega))=H^i(\Omega, \mathscr{B}^M)=0$ for $i\geqq k$. q. e. d.

REMARK. The assumption of Theorem 2.2 is equivalent to the following condition (3). (See the above proof.)

$$\text{(3)} \qquad \mathrm{proj\,dim}_{\mathscr{P}} M\leqq k\,.$$

Since the global dimension of $\mathscr{P}$ equals n, we have $\mathrm{Ext}_{\mathscr{P}}^i(M, \mathscr{B}(\Omega))=0$ unconditionally for $i\geqq n$.

THEOREM 2.3. *Assume that the space dimension n equals 2 and M is a finitely generated $\mathscr{P}$-module. In the case where $\mathrm{Ext}_{\mathscr{P}}^2(M, \mathscr{P})=0$, we have $\mathrm{Ext}_{\mathscr{P}}^1(M, \mathscr{B}(\Omega))=0$ for any domain Ω in $\boldsymbol{R}^2$. In the case where $\mathrm{Ext}_{\mathscr{P}}^2(M, \mathscr{P})\neq 0$, we have $\mathrm{Ext}_{\mathscr{P}}^1(M, \mathscr{B}(\Omega))=0$ if and only if $H^1(\Omega, \boldsymbol{C})=0$.*

PROOF. Considering Theorem 2.1, Theorem 2.2 and the above remark, we have only to prove that $\mathscr{B}(\Omega)$ is an injective $\mathscr{P}$-module if $H^1(\Omega, \boldsymbol{C})=0$.

Let $\mathscr{J}$ be an ideal of $\mathscr{P}$ and its generators be $P_1(D), \cdots, P_m(D)$. We find a solution u of the equations $P_i(D)u=f_i$ $(1\leqq i\leqq m)$ on Ω as follows if f_i satisfy the compatibility condition. We set $P_i(D)=Q_i(D)R(D)$ where $Q_i(D)$ have no non-trivial common factor for $1\leqq i\leqq m$, and define the equations $Q_i(D)v=f_i$ $(1\leqq i\leqq m)$ satisfying the compatibility condition. Considering the space dimension, we see that the equations form *a maximally overdetermined system*. We can find a solution v by the assumption $H^1(\Omega, \boldsymbol{C})=0$ because the solution sheaf of such system is a constant sheaf (cf. Matsuura [10]). And then we can solve the *single equation* $R(D)u=v$ by Theorem 2.2 because the $\mathscr{P}$-module $\mathscr{P}/\mathscr{P}R(D)$ satisfies (3) for $k=1$. (This solvability was proved first by Harvey [5].)

Thus we have $\mathrm{Ext}_{\mathscr{P}}^1(\mathscr{P}/\mathscr{J}, \mathscr{B}(\Omega))=0$ for any ideal $\mathscr{J}$ of $\mathscr{P}$. This implies that $\mathscr{B}(\Omega)$ is an injective $\mathscr{P}$-module. (See [2].) q. e. d.

§3. Throughout this section we assume the $\mathscr{P}$-module M equals $\mathscr{P}/\mathscr{J}$ where $\mathscr{J}$ is the ideal of $\mathscr{P}$ generated by $\partial/\partial x_1, \cdots, \partial/\partial x_k$ $(1\leqq k\leqq n)$. We denote by y and z the coordinates $x_1, \cdots, x_k$ and $x_{k+1}, \cdots, x_n$ respectively and by π

the projection from a domain Ω in $\boldsymbol{R}^n$ to $\boldsymbol{R}^{n-k}$ defined by $(y, z) \overset{\pi}{\longmapsto} z$. Moreover we denote by $\mathcal{F}$ one of the sheaves $\mathcal{A}$, $\mathcal{B}$, $\mathcal{D}'$, $\mathcal{E}$ over $\boldsymbol{R}^n$ as in § 1 and by $\mathcal{F}_{n-k}$ the corresponding sheaf over $\boldsymbol{R}^{n-k}$. Then the solution sheaf $\mathcal{F}^M$ is isomorphic to $\pi^*\mathcal{F}_{n-k}$ because $\mathcal{F}^M$ is constant along the fibre of π. Here we denote by $\pi^*\mathcal{G}$ the inverse image of the sheaf $\mathcal{G}$ over $\boldsymbol{R}^{n-k}$ under the map π.

We cite some examples. Let k equal 2 in the examples, which means that we think the system of the equations $\frac{\partial u}{\partial x_i} = f_i$ $(i=1, 2)$. Then the compatibility condition is the equation $\frac{\partial f_2}{\partial x_1} = \frac{\partial f_1}{\partial x_2}$. Assume that $\Omega = \boldsymbol{R}^n - A$, $n \geqq 3$ and A is as follows:

EXAMPLE 3.1.

$$A = \{(x_1, x_2, z) \in \boldsymbol{R}^n ;\ x_1 = z = 0\} .$$

Set $D = \boldsymbol{R}^{n-2} - \{0\}$. Then we have

$$\mathrm{Ext}^1_{\mathcal{P}}(M, \mathcal{F}(\Omega)) \cong \Gamma(D, \mathcal{F}_{n-2})/\Gamma(\boldsymbol{R}^{n-2}, \mathcal{F}_{n-2}) . \tag{3.1}$$

This vanishes if and only if $\mathcal{F}$ is $\mathcal{B}$.

For instance, the following system on Ω has no solution in $\mathcal{E}$.

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial x_1} = \left\{ \begin{array}{ll} (a(x_1/\|z\|^2)/\|z\|^2) \cdot b(z) , & \text{if } \|z\| \neq 0 , \\ 0 , \quad \text{if } \|z\| = 0 , & \end{array} \right. \\ \frac{\partial u}{\partial x_2} = 0 , \end{array} \right. \tag{3.2}$$

where $a(t) \in \mathcal{D}(\boldsymbol{R}^1)$ (which denotes the space of the infinitely differentiable functions with compact supports on $\boldsymbol{R}^1$), $\int a(t)dt \neq 0$, $\|z\| = (x_3^2 + \cdots + x_n^2)^{1/2}$, $b(z) \in \Gamma(D, \mathcal{E})$ and $b(z) \notin \Gamma(\boldsymbol{R}^{n-2}, \mathcal{E})$.

To prove (3.1) we define

$$U_1 = \{(x_1, x_2, z) \in \boldsymbol{R}^n ;\ z \neq 0\}$$

and

$$U_2 = \{(x_1, x_2, z) \in \boldsymbol{R}^n ;\ x_1 \neq 0\} .$$

Then by Leray's theorem on cohomology groups of the covering $\Omega = U_1 \cup U_2$, $H^1(\Omega, \mathcal{F}^M)$ is isomorphic to the cokernel of the map

$$\begin{array}{ccc} \Gamma(U_1, \mathcal{F}^M) \oplus \Gamma(U_2, \mathcal{F}^M) & \longrightarrow & \Gamma(U_1 \cap U_2, \mathcal{F}^M) \\ \Downarrow \qquad\qquad \Downarrow & & \Downarrow \\ (\varphi_1, \qquad\qquad \varphi_2) & \longmapsto & \varphi_1 - \varphi_2 . \end{array}$$

Therefore we have (3.1) by the following isomorphisms:

$$\Gamma(U_1, \mathcal{F}^M) \cong \Gamma(D, \mathcal{F}_{n-2}) ,$$

$$\Gamma(U_2, \mathcal{F}^M) \cong \Gamma(\boldsymbol{R}^{n-2}, \mathcal{F}_{n-2}) \oplus \Gamma(\boldsymbol{R}^{n-2}, \mathcal{F}_{n-2})$$

and

$$\Gamma(U_1 \cap U_2, \mathcal{F}^M) \cong \Gamma(D, \mathcal{F}_{n-2}) \oplus \Gamma(D, \mathcal{F}_{n-2}).$$

EXAMPLE 3.2. i) $A = \{(x_1, x_2, z) \in \boldsymbol{R}^n ;\ x_1 = x_2 = 0\}$,

ii) $A = \{(x_1, x_2, z) \in \boldsymbol{R}^n ;\ x_1 = x_2 = 0,\ x_3 \geqq 0\}$,

iii) $A = \{(x_1, x_2, z) \in \boldsymbol{R}^n ;\ x_1 = x_2 = z = 0\}$.

In i), ii) and iii) we have

$$\operatorname{Ext}^1_{\mathcal{P}}(M, \mathcal{F}(\Omega)) \cong \Gamma_{\pi(A)}(\boldsymbol{R}^{n-2}, \mathcal{F}_{n-2}), \tag{3.3}$$

which vanishes in and only in the following cases respectively:

i) It never vanishes,

ii) $\mathcal{F} = \mathcal{A}$,

iii) $\mathcal{F} = \mathcal{A},\ \mathcal{E}$.

The system on Ω

$$\tag{3.4} \begin{cases} \dfrac{\partial u}{\partial x_1} = \dfrac{1}{x_1 + \sqrt{-1}x_2} \cdot b(z), \\[2ex] \dfrac{\partial u}{\partial x_2} = \dfrac{\sqrt{-1}}{x_1 + \sqrt{-1}x_2} \cdot b(z) \end{cases}$$

has no solution for non-zero function $b(z) \in \Gamma_{\pi(A)}(\boldsymbol{R}^{n-k}, \mathcal{F}_{n-k})$. We can prove (3.3) by the same method as in Example 3.1.

We have the following theorems as expected by these examples.

THEOREM 3.3. *Assume that M is the $\mathcal{P}$-module $\mathcal{P}/\mathcal{J}$ where $\mathcal{J}$ is the ideal of $\mathcal{P}$ generated by $\partial/\partial x_1, \cdots, \partial/\partial x_k$. Then the two conditions*

$$\operatorname{Ext}^1_{\mathcal{P}}(M, \mathcal{B}(\Omega)) = 0, \tag{4}$$

$$H^1(\pi^{-1}(z), \boldsymbol{C}) = 0 \qquad \textit{for any} \quad z \in \boldsymbol{R}^{n-k} \tag{5}$$

are equivalent for a domain Ω in $\boldsymbol{R}^n$.

To prove the theorem we employ a method similar to Suzuki [**13**], which argues the problem in the holomorphic category in the case $k=1$. First we give some definitions. Given a point $x \in \Omega$, let L_x be the connected component of the set $\pi^{-1}\cdot\pi(x)$ containing x. We denote by X the quotient space of Ω with the quotient topology by the equivalence relation "$L_x = L_{x'}$" for $x, x' \in \Omega$. We write the natural projections $\pi_1 : \Omega \to X$ and $\pi_2 : X \to \boldsymbol{R}^{n-k}$. Then the following is clear:

$$\tag{3.5} \begin{cases} \pi = \pi_2 \cdot \pi_1 \text{ and } \mathcal{F}^M \cong \pi^* \mathcal{F}_{n-k} = \pi_1^* \cdot \pi_2^* \mathcal{F}_{n-k}; \\ \pi_1 \text{ is } \textit{an open map with connected fibres}; \\ \pi_2 \text{ is } \textit{a local homeomorphism}. \end{cases}$$

We prepare two lemmas:

LEMMA 3.4. *The sheaf $\pi_2^* \mathcal{B}_{n-k}$ is flabby.*

PROOF. Let P be an arbitrary point of X. Since the map π_2 is a local homeomorphism, there exists a neighbourhood U of P such that $\pi_2^*\mathcal{B}_{n-k}|U$ is a flabby sheaf. Therefore we can prove easily by Zorn's lemma that $\pi_2^*\mathcal{B}_{n-k}$ is flabby (cf. [4], Chapter II, § 3.1). q. e. d.

LEMMA 3.5. *Using the above notations and assuming that* $H^1(\pi_1^{-1}(P), \boldsymbol{C})=0$ *at every point* P *in* X, *we have* $\mathcal{H}^1_{\pi_1}(\pi_1^*\mathcal{G})=0$ *for any sheaf* $\mathcal{G}$ *of* $\boldsymbol{C}$*-Module over* X. *Here we denote by* $\mathcal{H}^q_{\pi_1}(\mathcal{G}')$ *the q-th direct image of a sheaf* $\mathcal{G}'$ *over* Ω *under the projection* π_1. (*For the definition see* [1], *Chapter* IV, 4. $\mathcal{H}^q_{\pi_1}(\mathcal{G}')$ *is called there the Leray sheaf in degree q.*)

PROOF. Consider the stalk of $\mathcal{H}^1_{\pi_1}(\pi_1^*\mathcal{G})$ at every point $P\in X$. Then we have by definition

$$\mathcal{H}^1_{\pi_1}(\pi_1^*\mathcal{G})_P=\varinjlim H^1(\pi_1^{-1}(U), \pi_1^*\mathcal{G})\,, \tag{3.6}$$

where U ranges over the open sets in X containing P. To calculate (3.6) we write the canonical flabby resolution of $\pi_1^*\mathcal{G}$:

$$0\longrightarrow \pi_1^*\mathcal{G}\longrightarrow \mathcal{G}_0\xrightarrow{p_0}\mathcal{G}_1\xrightarrow{p_1}\mathcal{G}_2\longrightarrow\cdots.$$

Let u be a section of $\mathcal{G}_1$ over $\pi_1^{-1}(U)$ satisfying $p_1u=0$. Then there exist a convex open set $V_x\subset\pi_1^{-1}(U)$ and a section $v_x\in\mathcal{G}_0(V_x)$ for every $x\in\pi_1^{-1}(U)$ such that $V_x\ni x$ and $p_0v_x=u|V_x$. We choose a point $x^0\in\pi_1^{-1}(P)$ and denote by U' the open set $\pi_1(V_{x^0})$ containing P. Then we can find $v\in\mathcal{G}_0(\pi_1^{-1}(U'))$ satisfying $p_0v=u|\pi_1^{-1}(U')$ as follows:

For a point $x\in\pi_1^{-1}(U')$ there exist finite points $x^1,\cdots,x^r$ contained in $\pi_1^{-1}\cdot\pi_1(x)$ such that $V_{xi}\cap V_{xi+1}\neq\emptyset$ for $0\leqq i\leqq r$ where we denote by x^{r+1} the point x. Set $U''=U'\cap\bigcap_{0\leqq i\leqq r}\pi_1(V_{xi}\cap V_{xi+1})$. Since $v_{xi}-v_{xi+1}$ is an element of $\Gamma(V_{xi}\cap V_{xi+1}, \pi_1^*\mathcal{G})$ and $\pi_1|V_{xi}\cap V_{xi+1}$ is an open map with connected fibres, we can find the unique section $w_i\in\mathcal{G}(U'')$ such that $\pi_1^*w_i=v_{xi}-v_{xi+1}$ on $V_{xi}\cap V_{xi+1}\cap\pi_1^{-1}(U'')$. Then we define v by the equality

$$v\mid V_x\cap\pi_1^{-1}(U'')=v_x+\pi_1^*\sum_{i=0}^{r}w_i\,.$$

The well-definedness of v is due to the assumption meaning that $H^1(\pi_1^{-1}\cdot\pi_1(x), \mathcal{G}_{\pi_1(x)})=0$.

This shows the right side of (3.6) equals 0 by definition. So we have $\mathcal{H}^1_{\pi_1}(\pi_1^*\mathcal{G})_P=0$ for $P\in X$, thus $\mathcal{H}^1_{\pi_1}(\pi_1^*\mathcal{G})=0$. q. e. d.

PROOF OF THEOREM 3.3. (4) ⇒ (5). It suffices to prove that the equations on $\pi^{-1}(z^0)$

$$\frac{\partial v}{\partial x_i}=f_i(x_1,\cdots,x_k)\qquad(1\leqq i\leqq k) \tag{3.7}$$

satisfying the compatibility condition have a solution v for $z^0\in\boldsymbol{R}^{n-k}$.

Let $\delta(z)$ be the Dirac δ-function on $\boldsymbol{R}^{n-k}$. Then by the assumption (4) the equations on Ω

$$\frac{\partial u}{\partial x_i}=f_i(x_1, \cdots, x_k)\cdot\delta(z-z^0) \qquad (1\leqq i\leqq k) \tag{3.8}$$

have a solution u. The functions in the right side of the equation (3.8) are 0 on $\Omega-\pi^{-1}(z^0)$, which implies that $u|\Omega-\pi^{-1}(z^0)\in\mathcal{B}^M(\Omega-\pi^{-1}(z^0))$. Since π_1 is an open map with connected fibres and $\mathcal{B}^M\cong\pi_1{}^*\cdot\pi_2{}^*\mathcal{B}_{n-k}$, we see that $\mathcal{B}^M(\Omega-\pi^{-1}(z^0))$ and $\mathcal{B}^M(\Omega)$ are isomorphic to $\Gamma(X-\pi_2^{-1}(z^0), \pi_2{}^*\mathcal{B}_{n-k})$ and $\Gamma(X, \pi_2{}^*\mathcal{B}_{n-k})$ respectively. Therefore we can find a section $\tilde{u}\in\mathcal{B}^M(\Omega)$ by Lemma 3.4 such that

$$(u-\tilde{u}) \mid \Omega-\pi^{-1}(z^0)=0\,. \tag{3.9}$$

Since $u-\tilde{u}$ is a solution of (3.8), we see that the section $\int(u-\tilde{u})dx_{k+1}\cdots dx_n$ over $\pi^{-1}(z^0)$ is a solution of (3.7). In fact, the well-definedness of the integral follows from (3.9).

(5) $\Rightarrow$ (4). Consider the following *Leray spectral sequence* of the map π_1 (cf. [1], Chapter IV, 6):

$$E_2^{p,q}=H^p(X, \mathcal{H}_{\pi_1}^q(\mathcal{B}^M))\Rightarrow H^{p+q}(\Omega, \mathcal{B}^M)\,. \tag{3.10}$$

Since π_1 is an open map with connected fibres, we have

$$\mathcal{H}_{\pi_1}^0(\mathcal{B}^M)=\pi_{1*}\mathcal{B}^M\cong\pi_{1*}\pi_1{}^*\pi_2{}^*\mathcal{B}_{n-k}\cong\pi_2{}^*\mathcal{B}_{n-k}\,.$$

And by Lemma 3.5 we have

$$\mathcal{H}_{\pi_1}^1(\mathcal{B}^M)\cong\mathcal{H}_{\pi_1}^1(\pi_1{}^*\pi_2{}^*\mathcal{B}_{n-k})=0\,.$$

Now in the exact sequence of the edge homomorphisms (cf. [4], Chapter I, Theorem 4.5.1)

$$0\longrightarrow E_2^{1,0}\longrightarrow H^1\longrightarrow E_2^{0,1}\longrightarrow E_2^{2,0}\longrightarrow H^2\,, \tag{3.11}$$

we have proved that $E_2^{0,1}=0$. Combining the above facts with Lemma 3.4 and (1.4), we have

$$\begin{aligned}\mathrm{Ext}^1_{\mathcal{P}}(M, \mathcal{B}(\Omega))&=H^1(\Omega, \mathcal{B}^M)\\&\cong E_2^{1,0}=H^1(X, \mathcal{H}_{\pi_1}^0(\mathcal{B}^M))\\&\cong H^1(X, \pi_2{}^*\mathcal{B}_{n-k})=0\,.\end{aligned}$$

This completes the proof of the theorem.

In the space of distributions we have the following theorem.

THEOREM 3.6. *Assume that M is the same as in Theorem 3.3. Then the followings are equivalent conditions for a domain Ω in $\boldsymbol{R}^n$.*

$$\mathrm{Ext}^1_{\mathcal{P}}(M, \mathcal{D}'(\Omega))=0\,, \tag{6}$$

(7) $$H^1(\pi^{-1}(z), \boldsymbol{C})=0 \qquad \text{for any} \quad z\in \boldsymbol{R}^{n-k}$$

and the topology of X is Hausdorff.

In our proof of Theorem 3.6 we need the following lemmas.

LEMMA 3.7. *Let u be an element of $\Gamma(\Omega-\pi^{-1}(z^0), \mathcal{D}'^M)$ where $z^0\in \boldsymbol{R}^{n-k}$. Assume u has an extension $u'\in\Gamma(\Omega, \mathcal{D}')$. Then we can also extend u over Ω as a section of $\mathcal{D}'^M$.*

PROOF. The set $\pi^{-1}(z^0)$ has the following decomposition into the connected components:

$$\pi^{-1}(z^0)=\bigcup_{\lambda\in\Lambda} L_{x\lambda}.$$

For every $\lambda\in\Lambda$ we can find convex open sets $V\subset\boldsymbol{R}^k$ and $W\subset\boldsymbol{R}^{n-k}$ such that $x^\lambda\in V\times W\subset\Omega$. Choose a function $\varphi(y)\in\mathcal{D}(V)$ satisfying $\int\varphi(y)dy\neq 0$. We define a distribution $w_\lambda\in\mathcal{D}'(W)$ by the equality

$$w_\lambda=\int\varphi(y)u'dy\Big/\int\varphi(y)dy.$$

That is, $\langle w_\lambda, \rho(z)\rangle=\langle u', \varphi(y)\rho(z)\big/\int\varphi(y)dy\rangle$ for any $\rho(z)\in\mathcal{D}(W)$. Let $U_\lambda=\pi_1^{-1}\cdot\pi_1(V\times W)$. Since the distribution $u'|U_\lambda-\pi^{-1}(z^0)$ is constant along the fibre of π and $\pi|U_\lambda$ has connected fibres, it is clear that the distribution $\pi^*w_\lambda\in\mathcal{D}'^M(U_\lambda)$ equals u' and also u on $U_\lambda-\pi^{-1}(z^0)$. Hence there exists $\tilde{u}\in\mathcal{D}'^M(\Omega)$ such that

$$\tilde{u}\mid\Omega-\pi^{-1}(z^0)=u,$$

$$\tilde{u}\mid U_\lambda=\pi^*w_\lambda \qquad \text{for} \quad \lambda\in\Lambda.$$ q. e. d.

LEMMA 3.8. *Assume $\mathcal{F}$ is $\mathcal{A}$, $\mathcal{D}'$ or $\mathcal{E}$. Then the followings are equivalent conditions for X:*

(8) *The topology of X is Hausdorff.*

(9) $$H^1(X, \pi_2{}^*\mathcal{F}_{n-k})=0.$$

PROOF. (8) ⇒ (9). This is clear, because (8) implies that X is an $(n-k)$-dimensional real analytic manifold and that $\pi_2{}^*\mathcal{F}_{n-k}$ is a sheaf of real analytic functions, distributions or infinitely differentiable functions over the manifold.

(9) ⇒ (8). Suppose that the topology of X is not Hausdorff. Let P and P' be distinct points in X which cannot be separated by open sets. Let U and U' be open neighbourhoods of P and P' respectively such that $\pi_2|U$ and $\pi_2|U'$ are into-homeomorphisms. We denote shortly by V the open set $X-\{P\}$ and by $\mathcal{G}$ the sheaf $\pi_2{}^*\mathcal{F}_{n-k}$. Consider the following commutative diagram:

$$\begin{array}{ccccccc} \longrightarrow \Gamma(U, \mathcal{G})\oplus\Gamma(V, \mathcal{G}) & \xrightarrow{p_1} & \Gamma(U\cap V, \mathcal{G}) & \longrightarrow & H^1(U\cup V, \mathcal{G}) \longrightarrow \\ r_1\downarrow & & r_2\downarrow & & \\ \Gamma(U, \mathcal{G})\oplus\Gamma(U', \mathcal{G}) & \xrightarrow{p_2} & \Gamma(U\cap U', \mathcal{G}). & & \end{array}$$

The first row is the exact *Mayer-Vietoris sequence* (cf. [1], Chapter II, 13), the map p_1 (or p_2) is defined by $(u_1, u_2)\mapsto u_1-u_2$, and r_1 and r_2 are restrictions. Note that $U\cup V=X$, $U\cap V=U-\{P\}$. Condition (9) implies that p_1 is surjective, therefore $\operatorname{Im} r_2\subset \operatorname{Im} p_2$. Considering $\Gamma(U, \mathcal{G})\cong\Gamma(\pi_2(U), \mathcal{F}_{n-k})$ etc., we define the maps

$$r_2' : \Gamma(\pi_2(U)-\pi_2(P), \mathcal{F}_{n-k}) \longrightarrow \Gamma(\pi_2(U\cap U'), \mathcal{F}_{n-k})$$

and

$$p_2' : \Gamma(\pi_2(U), \mathcal{F}_{n-k})\oplus\Gamma(\pi_2(U'), \mathcal{F}_{n-k}) \longrightarrow \Gamma(\pi_2(U\cap U'), \mathcal{F}_{n-k}),$$

then $\operatorname{Im} r_2'\subset \operatorname{Im} p_2'$. We can find $f\in \operatorname{Im} r_2'-\operatorname{Im} p_2'$ in the undermentioned way, which contradicts this fact and completes the proof:

There exists a sequence $\{P_i\}$ of points in $U\cap U'$ which converges to P and P'. Then the sequence $\{\pi_2(P_i)\}$ converges to the point $\pi_2(P)=\pi_2(P')$. In the case where $\mathcal{F}$ is $\mathcal{A}$ or $\mathcal{E}$, we set $f=1/\|z-\pi_2(P)\|^2$ and in the case where $\mathcal{F}$ is $\mathcal{D}'$, we set $f=\sum_{i=1}^{\infty}\delta^{(i)}(z-\pi_2(P_i))$ where we denote by $\delta^{(i)}(z)$ the i-th derivative of the Dirac δ-function on $\boldsymbol{R}^{n-k}$. Since the both open sets $\pi_2(U)$ and $\pi_2(U')$ contain $\pi_2(P)$, it is clear that $f\notin \operatorname{Im} p_2'$. q.e.d.

PROOF OF THEOREM 3.6. Refer to the proof of Theorem 3.3.

(6) $\Rightarrow$ (7). Using Lemma 3.7 in place of Lemma 3.4, we can prove that (6) implies (5) in the same way as in Theorem 3.3. On the other hand, since the map

$$i : H^1(X, \pi_2{}^*\mathcal{D}'_{n-k}) \longrightarrow H^1(\Omega, \mathcal{D}'^M)$$

is injective (cf. (3.11)), it follows immediately from (1.4) and Lemma 3.8 that (6) implies (8).

(7) $\Rightarrow$ (6) follows from Lemma 3.5 and Lemma 3.8. See the proof of "(5) $\Rightarrow$ (4)". q.e.d.

We define the conditions for a domain Ω in $\boldsymbol{R}^n$

(10) $$\operatorname{Ext}^1_{\mathcal{P}}(M, \mathcal{A}(\Omega))=0,$$

(11) $$\operatorname{Ext}^1_{\mathcal{P}}(M, \mathcal{E}(\Omega))=0,$$

then the following theorem also follows from the same proof as above.

THEOREM 3.9. *On the same assumption as in Theorem* 3.6,

i) (7) *implies* (10) *and* (11),

ii) (10) *implies* (8), (11) *implies* (8).

Note that neither (10) nor (11) implies (7) except in the cases where $k=1$

and where $k=n$ (cf. Example 3.2).

PROPOSITION 3.10. *Assume that $M=\mathcal{P}/\mathcal{J}$ where $\mathcal{J}$ is an ideal of $\mathcal{P}$ generated by $\partial/\partial x_1$ and $\partial/\partial x_2$ (i.e. $k=2$) and that a domain Ω in $\boldsymbol{R}^n$ satisfies the conditions* (11) *and*

$$\Omega=\operatorname{int}\bar{\Omega}\ (=\text{the interior of the closure of } \Omega)\,. \tag{12}$$

Then (7) *holds.*

PROOF. It suffices to prove (5). Suppose that there exists a point $z^0\in\boldsymbol{R}^{n-2}$ such that $H^1(\pi^{-1}(z^0), \boldsymbol{C})\neq 0$. Then there exist a Jordan curve C in $\boldsymbol{R}^2$ and a point y^0 contained in the domain which is surrounded by C such that

$$C\times z^0\subset\Omega\,,\qquad (y^0, z^0)\notin\Omega\,.$$

Assume that $(y^0, z^0)\in\partial\Omega$. Then we can find convex open sets V in $\boldsymbol{R}^2$ and W in $\boldsymbol{R}^{n-2}$ such that

$$V\ni y^0\,,\quad W\ni z^0\,,\quad C\cap V=\emptyset\,,\quad C\times W\subset\Omega\,.$$

Since (12) implies that $\partial\Omega=\partial(\boldsymbol{R}^n-\bar{\Omega})$, the set $V\times W\cap(\boldsymbol{R}^n-\bar{\Omega})$ is non-void. Choose a point (y^1, z^1) in the set. Thus we can assume that $(y^0, z^0)\notin\bar{\Omega}$ from the beginning replacing (y^0, z^0) by (y^1, z^1) if necessary. So there exists an open neighbourhood W' of z^0 such that $y^0\times W'\cap\Omega=\emptyset$. Then the following system has no solution:

$$\begin{cases} \dfrac{\partial u}{\partial x_1}=\dfrac{1}{(x_1-x_1^0)+\sqrt{-1}(x_2-x_2^0)}\cdot b(z)\,, \\[2ex] \dfrac{\partial u}{\partial x_2}=\dfrac{\sqrt{-1}}{(x_1-x_1^0)+\sqrt{-1}(x_2-x_2^0)}\cdot b(z)\,, \end{cases}$$

where $y^0=(x_1^0, x_2^0)$, $b(z)\in\mathcal{D}(W')$ and $b(z^0)\neq 0$. This is a contradiction.

q. e. d.

REMARK. These theorems hold for a $\mathcal{P}$-module M' in place of M if the solution sheaf $\mathcal{F}^{M'}$ is isomorphic to $\mathcal{F}^M$. We owe the following to Sato, Kawai and Kashiwara [**12**]:

Let M' be $\mathcal{P}/\mathcal{J}'$ where $\mathcal{J}'$ is an ideal of $\mathcal{P}$ generated by $P_1(D), \cdots, P_k(D)$ and $\mathcal{J}$ be the radical of the ideal of $\mathcal{P}$ generated by the principal symbols of $P_1(D), \cdots, P_k(D)$. Assume that $\mathcal{J}$ is as before (i.e. $\mathcal{J}$ is generated by $\partial/\partial x_1, \cdots, \partial/\partial x_k$ and $M=\mathcal{P}/\mathcal{J}$). Then one of the two modules M and M' (precisely $\bar{\mathcal{P}}\otimes_{\mathcal{P}} M$ and $\bar{\mathcal{P}}\otimes_{\mathcal{P}} M'$) is isomorphic to a direct summand of a direct sum of finite copies of the other in the ring $\bar{\mathcal{P}}$ of linear differential operators *of infinite order* with constant coefficients, which operates $\mathcal{A}$ and $\mathcal{B}$. And $\bar{\mathcal{P}}$ is *faithfully flat* over $\mathcal{P}$.

Therefore one of the solution sheaves $\mathcal{A}^M$ and $\mathcal{A}^{M'}$ (or $\mathcal{B}^M$ and $\mathcal{B}^{M'}$) is isomorphic to a direct summand of a direct sum of finite copies of the other as sheaves of $\mathcal{P}$-Modules. Hence Theorem 3.3 and "(7) ⇒ (10), (10) ⇒ (8)" in Theorem 3.9 hold even if we replace M by M'. For example, we can apply the theorems to the following system:

$$(3.12)\qquad \begin{cases} P_1(D)u \equiv \dfrac{\partial^2 u}{\partial x_1^2} - \dfrac{\partial u}{\partial x_2} = f_1 , \\ P_2(D)u \equiv \dfrac{\partial^2 u}{\partial x_2^2} - \dfrac{\partial u}{\partial x_3} = f_2 . \end{cases}$$

References

[1] G. Bredon, Sheaf theory, McGraw-Hill, New York, 1967.
[2] H. Cartan and S. Eilenberg, Homological algebra, Princeton, 1956.
[3] L. Ehrenpreis, Fourier analysis in several complex variables, Wiley-Interscience, New York, 1970.
[4] R. Godement, Topologie algébrique et théorie des faisceaux, Hermann, Paris, 1958.
[5] R. Harvey, Hyperfunctions and partial differential equations, Thesis, Stanford University, 1966, a part of it is published in Proc. Nat. Acad. Sci. U.S.A., **55** (1966), 1042–1046.
[6] L. Hörmander, An introduction to complex analysis in several variables, Van Nostrand, Princeton, 1966.
[7] H. Komatsu, Resolution by hyperfunctions of sheaves of solutions of differential equations with constant coefficients, Math. Ann., **176** (1968), 77–86.
[8] H. Komatsu, Relative cohomology of sheaves of solutions of differential equations, Séminaire Lions-Schwartz, 1966–67, Hyperfunctions and pseudo-differential equations, Part I, Lecture Notes in Math., No. 287, Springer, Berlin, 1973, 192–261.
[9] B. Malgrange, Systèmes différentiels à coefficients constants, Séminaire Bourbaki, Paris, 1962/63, No. 246.
[10] S. Matsuura, Finite type system of partial differential operators and decomposition of solutions of partial differential equations, Proceedings of the Symposium on Partial Differential Equations RIMS, 1966, Sûrikaiseki-kenkyûsho Kôkyûroku, **22** (1967), 10–17.
[11] V.P. Palamodov, Linear differential operators with constant coefficients, Nauka, Moskva, 1967 (in Russian).
[12] M. Sato, T. Kawai and M. Kashiwara, Microfunctions and pseudo-differential equations, Hyperfunctions and pseudo-differential equations, Part II, Lecture Notes in Math., No. 287, Springer, Berlin, 1973, 265–529.
[13] H. Suzuki, On the global existence of holomorphic solutions of the equation $\partial u/\partial x_1 = f$, Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A., **11** (1972), 253–258.

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