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Mixed-type boundary conditions for second order elliptic differential equations

By Yoshio KATO

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§ 0. Introduction.

Let Ω be a bounded domain in R^m with C^∞ boundary Γ of dimension $m-1$ and let there be given two differential operators A on $\bar{\Omega}=\Omega\cup\Gamma$,

$$A=\sum_{|\mu|,\,|\nu|\leqq 1} D^\mu a_{\nu\mu}(x)D^\nu\,, \tag{0.1}$$

and B_1 on Γ,

$$B_1=a_0(x)\frac{\partial}{\partial n}+b(x,D)\,, \tag{0.2}$$

where n denotes the exterior normal of Γ and $b(x, D)$ is a tangential differential operator of first order on Γ. The notations are the usual ones: $\nu=(\nu_1,\cdots,\nu_m)$ with non-negative integers ν_j, $|\nu|=\nu_1+\cdots+\nu_m$, $D=(D_1,\cdots,D_m)$ with $D_j=-i\partial/\partial x_j$ and $D^\nu=D_1^{\nu_1}\cdots D_m^{\nu_m}$. All coefficients are assumed to be complex-valued and C^∞ in their sets of definition indicated. It is also assumed that A is elliptic and moreover that $a_0(x)\neq 0$ for all $x\in\Gamma$.

Many authors had studied the mixed problem: To find u such that

$$\begin{cases} Au=f & \text{in } \Omega \\ B_1u=0 & \text{on } \Gamma_1 \\ u=0 & \text{on } \Gamma_2 \end{cases} \tag{0.3}$$

when f is given, where Γ_1 and Γ_2 are two open sets of Γ such that $\bar{\Gamma}_1\cup\bar{\Gamma}_2=\Gamma$ and $\Gamma_1\cup\Gamma_2=\emptyset$. But it seems to be more natural to replace and generalize (0.3) with

$$\begin{cases} Au=f & \text{in } \Omega \\ \alpha B_1u+\beta u=0 & \text{on } \Gamma\,, \end{cases} \tag{0.4}$$

where α and β are two functions on Γ such that $\alpha\geqq 0$, $\beta\geqq 0$ and $\alpha+\beta=1$ on Γ. The first reason is that for the problem (0.3) no regularity can hold, i.e. even if f is smooth, u may not be smooth (this fact is pointed out in [**1**, **5**]), while the regularity holds for the problem (0.4), provided α is smooth. The second is that all solutions of (0.3) are approximated by solutions u_n of

(0.4) with $\alpha=\alpha_n$ such that $\alpha_n=1$ on Γ_1 and $\alpha_n\to 0$ on Γ_2 as $n\to\infty$ (see Theorem 4). To establish this regularity is the main purpose of this paper. This kind of mixed-type boundary conditions, however, has been thoroughly studied by Itô [3]. Here he uses fundamental solutions of parabolic differential equations. Recently Hayashida [2] treated again this problem from the viewpoint of functional analysis, but he had to assume that β is sufficiently large, in order to assure the regularity of solutions. In this paper we shall also use the tool of functional analysis and our main results are stated in Theorems 1~6.

The plan of the present paper is the following. In Section 1 we shall show the existence of the Green kernels G and $\tilde{G}$ for the systems (A_λ, B) and (A_λ^*, B^*), respectively, in a weak sense. Here it is only assumed that α is measurable. The definition of weak solutions is introduced in Section 2, where Theorem 4 is stated. We shall discuss in Section 3 the regularity of G and $\tilde{G}$ when $\alpha\in\mathscr{B}^2(\Gamma)$, i.e. if $f\in H^0(\Omega)$, then Gf and $\tilde{G}f$ are in $H^2(\Omega)$. Section 4 is continued from the preceding section and is devoted to the further regularity of G and $\tilde{G}$, i.e. $f\in H^{k-2}(\Omega;p)$ implies that Gf and $\tilde{G}f$ are in $H_B^k(\Omega;p)$, if $\alpha\in\mathscr{B}^k(\Gamma)$, $k\geqq 3$, and if Condition (H) (see § 4) is satisfied. It should be noticed that Condition (H) is verified in [2, 3]. If Condition (H) is not satisfied, we must further assume, in order to establish the above regularity, that $\sqrt{\alpha}\in\mathscr{B}^k(\Gamma)$. These regularities will be obtained in both cases by using what is called elliptic regularization technique. To do so we introduce, in Appendix, the extensions q and p of α and $\sqrt{\alpha}$, respectively, and their mollifiers q_ε and p_ε.

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§ 1. Existence of the Green kernels.

Let $B[u, v]$ be a differential bilinear form associated with the differential operator A of (0.1) presented in Introduction,

$$B[u, v]=\int_\Omega \sum_{|\nu|,|\mu|\leqq 1} a_{\nu\mu}(x)D^\nu u\cdot\overline{D^\mu v}\,dx$$

and B_1 is the differential operator (0.2). Throughout the present paper we assume that there exists a positive constant c_0 such that

$$\mathscr{R}e \sum_{|\nu|=|\mu|=1} a_{\nu\mu}(x)\xi^{\nu+\mu}\geqq c_0|\xi|^2, \qquad \xi\in R^m$$

for all $x\in\bar{\Omega}$. Then it follows that there exist a nonvanishing function $a(x)$ in $C^\infty(\bar{\Omega})$ and a first order tangential differential operator $\gamma(x, D)$ on Γ satisfying for all u, $v\in C^\infty(\bar{\Omega})$

(1.1) $$B[u, v]+\int_{\Gamma}\gamma(x, D)u\cdot\bar{v}d\sigma=(Au, v)+\int_{\Gamma}a(x)B_1u\cdot\bar{v}d\sigma ,$$

where (u, v) is the usual inner product in $L_2(\Omega)$. In practice we have

$$a(x)=\sum_{|\nu|=|\mu|=1}a_{\nu\mu}(x)n^{\nu+\mu}/a_0(x) .$$

Moreover we can easily find another differential operator B_1' on Γ similar to (0.2) such that for all $u, v\in C^\infty(\bar{\Omega})$

(1.1*) $$B[u, v]+\int_{\Gamma}\gamma(x, D)u\cdot\bar{v}d\sigma=(u, A^*v)+\int_{\Gamma}a(x)u\overline{B_1'v}d\sigma$$

where A^* is the formal adjoint of A. As a matter of fact we have only to take as

$$B_1'=\overline{a_0(x)}\frac{\partial}{\partial n}+b'(x, D) ,$$

$b'(x, D)$ being an appropriate first order tangential differential operator on Γ.

Now let α and β be two functions on Γ satisfying

(1.2) $$\alpha\geqq 0,\quad \beta\geqq 0 \quad\text{and}\quad \alpha+\beta=1 \qquad\text{on}\quad \Gamma .$$

Throughout this section we assume α to be measurable on Γ. Setting

$$\begin{cases} Bu=\alpha B_1u+\beta u \\ B^*u=\alpha B_1'u+\beta u \end{cases}$$

we introduce two function spaces,

$$\begin{cases} \mathcal{D}_B=\{u\in H^2(\Omega);\ Bu=0 \quad\text{on}\quad \Gamma\} \\ \mathcal{D}_{B^*}=\{u\in H^2(\Omega);\ B^*u=0 \quad\text{on}\quad \Gamma\} , \end{cases}$$

where and from now on by $H^s(\Omega)$, s real, we mean the set of all $u\in\mathcal{D}'(\Omega)$ such that there exists a distribution $U\in H^s(R^m)$ with $U=u$ in Ω. The norm of u is defined by

$$\|u\|_{s,\Omega}=\inf\|U\|_s ,$$

the infimum being taken over all such U. Similarly we can define the space $H^s(\Gamma)$ and the norm $\|\cdot\|_{s,\Gamma}$. It is well-known by the trace operator theorems that if $s>0$, $u\in H^{s+1/2}(\Omega)$ implies $u\in H^s(\Gamma)$ and there really exists a positive constant C_s such that

(1.3) $$\|u\|_{s,\Gamma}\leqq C_s\|u\|_{s+1/2,\Omega} ,\qquad u\in H^{s+1/2}(\Omega) .$$

Put for $u\in H^2(\Omega)$

$$\varphi=u-B_1u \qquad\text{on}\quad \Gamma .$$

Then it follows from (1.3) that $\varphi \in H^{1/2}(\Gamma)$ and that

$$(1.4) \qquad \begin{cases} u = Bu + \alpha\varphi \\ B_1 u = Bu - \beta\varphi \end{cases}$$

on Γ. Similarly putting for $v \in H^2(\Omega)$

$$\psi = v - B_1' v \qquad \text{on} \quad \Gamma\,,$$

we have $\psi \in H^{1/2}(\Gamma)$, and on Γ

$$(1.4^*) \qquad \begin{cases} v = B^* v + \alpha\psi \\ B_1' v = B^* v - \beta\psi\,. \end{cases}$$

PROPOSITION 1.1. *Let $Q[u, v]$ be an integro-differential bilinear form defined by*

$$(1.5) \qquad Q[u, v] = B[u, v] + \int_\Gamma \gamma(x, D) u \cdot \bar{v}\, d\sigma + \int_{\Gamma_1} \frac{\beta}{\alpha} au \cdot \bar{v}\, d\sigma\,,$$

where

$$\Gamma_1 = \{x \in \Gamma\,;\ \alpha(x) \neq 0\}\,.$$

Then we have for all u, $v \in H^2(\Omega)$

$$Q[u, v] = \begin{cases} (Au, v) + \displaystyle\int_{\Gamma_1} \frac{1}{\alpha} aBu \cdot \bar{v}\, d\sigma\,, & \text{if} \quad v \in \mathcal{D}_B \cup \mathcal{D}_{B^*} \\ (u, A^* v) + \displaystyle\int_{\Gamma_1} \frac{1}{\alpha} au \cdot \overline{B^* v}\, d\sigma\,, & \text{if} \quad u \in \mathcal{D}_B \cup \mathcal{D}_{B^*}\,. \end{cases}$$

PROOF. Let $u \in H^2(\Omega)$ and $v \in \mathcal{D}_B \cup \mathcal{D}_{B^*}$. From (1.4), (1.4*) we have

$$\begin{aligned} \int_\Gamma aB_1 u \cdot \bar{v}\, d\sigma &= \int_{\Gamma_1} \frac{1}{\alpha} a(Bu - \beta u)\bar{v}\, d\sigma \\ &= \int_{\Gamma_1} \frac{1}{\alpha} aBu \cdot \bar{v}\, d\sigma - \int_{\Gamma_1} \frac{\beta}{\alpha} au \cdot \bar{v}\, d\sigma\,, \end{aligned}$$

and hence by (1.1) and (1.5)

$$Q[u, v] = (Au, v) + \int_{\Gamma_1} \frac{1}{\alpha} aBu \cdot \bar{v}\, d\sigma\,.$$

The same argument as above and the use of (1.1*) complete the proof.

Q. E. D.

CONDITION 1. The integro-differential bilinear form defined by

$$(1.6) \qquad Q^0[u, v] = B[u, v] + \int_\Gamma \gamma(x, D) u \cdot \bar{v}\, d\sigma$$

is coercive over $\mathcal{D}_B$, i. e. there exist two positive constants c_1 and C such that

$$\mathcal{R}_e Q^0[u, u] \geqq c_1\|u\|_{1,\Omega}^2 - C\|u\|_{0,\Omega}^2, \qquad u \in \mathcal{D}_B .$$

CONDITION 1*. The form $Q[u, v]$ is coercive over $\mathcal{D}_{B^*}$.

This kind of the coerciveness is characterized in [4] when α is a characteristic function of an open set of Γ.

CONDITION 2. There exists a positive constant c_2 such that

$$\mathcal{R}_e\, a(x) \geqq c_2, \qquad x \in \Gamma .$$

PROPOSITION 1.2. *Suppose that Conditions* 1 *and* 2 (*resp.* 1* *and* 2) *hold. Then we can find two positive constants* c, λ *such that*

$$\mathcal{R}_e\, Q[u, u] \geqq c\Big(\|u\|_{1,\Omega}^2 + \int_{\Gamma_1} \frac{1}{\alpha}|u|^2 d\sigma\Big) - \lambda\|u\|_{0,\Omega}^2, \qquad u \in \mathcal{D}_B \ (resp.\ u \in \mathcal{D}_{B^*}) .$$

PROOF. From (1.2) and (1.5) we have for any $u \in \mathcal{D}_B$ (or $u \in \mathcal{D}_{B^*}$)

$$\begin{aligned}\mathcal{R}_e\, Q[u, u] &= \mathcal{R}_e\, Q^0[u, u] + \int_{\Gamma_1} \frac{\beta}{\alpha} \mathcal{R}_e\, a|u|^2 d\sigma \\ &= \mathcal{R}_e\, Q^0[u, u] + \int_{\Gamma_1} \frac{1}{\alpha} \mathcal{R}_e\, a|u|^2 - \int_{\Gamma_1} \mathcal{R}_e\, a|u|^2 d\sigma \\ &\geqq c_1\|u\|_{1,\Omega}^2 - C\|u\|_{0,\Omega}^2 + c_2\int_{\Gamma_1} \frac{1}{\alpha}|u|^2 d\sigma - \max_{\Gamma}|\mathcal{R}_e\, a| \int_{\Gamma}|u|^2 d\sigma .\end{aligned}$$

Using an inequality well known in the theory of trace operators; for any $\delta > 0$ there exists a constant $C_\delta > 0$ such that

$$\int_{\Gamma} |f|^2 d\sigma \leqq \delta\|f\|_{1,\Omega}^2 + C_\delta\|f\|_{0,\Omega}^2, \qquad f \in H^1(\Omega), \tag{1.7}$$

we can assert the proposition. Q.E.D.

PROPOSITION 1.3. *If we denote by* V *the Banach space obtained by the completion of* $\mathcal{D}_B$ *with respect to the norm* $|||\cdot|||$ *defined by*

$$|||u|||^2 = \|u\|_{1,\Omega}^2 + \int_{\Gamma_1} \frac{1}{\alpha}|u|^2 d\sigma ,$$

it then follows that $\mathcal{D}_{B^*}$ *is a dense subset of* V.

PROOF. For any $u \in \mathcal{D}_{B^*}$, we can choose a sequence $\{v_n\}$ in $H^2(\Omega)$ so that

$$\begin{cases} Bv_n = Bu & \text{on } \Gamma \\ |||v_n||| \longrightarrow 0 & (n \to \infty) . \end{cases} \tag{1.8}$$

In fact, from $B^*u = 0$ on Γ we have

$$Bu = Bu - B^*u = \alpha(B_1 u - B_1' u) \qquad \text{on } \Gamma .$$

It then follows from (1.3) that

$$\varphi=(B_1u-B_1'u)|_\Gamma\in H^{1/2}(\Gamma).$$

Hence we can find $v\in H^2(\Omega)$ so that

$$\begin{cases} B_1v=\varphi \\ v=0 \end{cases}$$

on Γ. Letting

$$\Omega_n=\left\{x\in\Omega\,;\ \mathrm{dis}\,(x,\Gamma)>\frac{1}{n}\right\},\qquad n=1,2,\cdots,$$

we introduce function $\zeta_n(x)\in C^\infty(R^m)$ such that $0\leqq\zeta_n(x)\leqq 1$ on R^m, $\zeta_n(x)=0$ in Ω_n and $=1$ in $R^m-\Omega$. Thus

$$v_n=\zeta_n v$$

satisfies

$$Bv_n=Bv=\alpha\varphi=Bu\qquad\text{on}\quad\Gamma$$

and

$$|||v_n|||=\|v_n\|_{1,\Omega}=\|\zeta_n v\|_{1,\Omega}\longrightarrow 0\qquad(n\to\infty),$$

since $v_n=0$ on Γ and a Poincaré-type inequality

$$\|v\|_{0,\Omega-\Omega_n}\leqq\frac{M}{n}\|v\|_{1,\Omega-\Omega_n}$$

holds with a constant M independent of n and v. In a way quite similar to the above we can show that for any $u\in\mathcal{D}_B$ there exists a sequence $\{v_n\}$ in $H^2(\Omega)$ such that

$$\text{(1.8*)}\qquad\begin{cases} B^*v_n=B^*u & \text{on}\quad\Gamma \\ |||v_n|||\longrightarrow 0 & (n\to\infty). \end{cases}$$

Now let $u\in\mathcal{D}_{B^*}$ and $\{v_n\}$ be a sequence satisfying (1.8). Set $u_n=u-v_n$. It then easily follows that $u_n\in\mathcal{D}_B$ and $|||u-u_n|||\to 0$ $(n\to\infty)$. This asserts $u\in V$ and hence $\mathcal{D}_{B^*}\subset V$.

Next we shall prove that $\mathcal{D}_{B^*}$ is dense in V. Let $u\in V$ and $\{u_m\}$ be a sequence in $\mathcal{D}_B$ converging to u in V. For each u_m there exists a sequence $\{v_{m,n}\}$ in $H^2(\Omega)$ satisfying (1.8*). Set $u_{m,n}=u_m-v_{m,n}$. Then $u_{m,n}\in\mathcal{D}_{B^*}$ and $|||u_m-u_{m,n}|||\to 0$ $(n\to\infty)$. Hence for each m there exists an integer m' such that

$$|||u_m-u_{m,m'}|||<\frac{1}{m}.$$

Therefore

$$|||u-u_{m,m'}|||\leqq|||u-u_m|||+|||u_m-u_{m,m'}|||\longrightarrow 0\qquad(m\to\infty).$$

This completes the proof. Q. E. D.

Denote by V' the dual space of V with norm

$$|||f|||' = \sup_{0 \neq v \in V} \frac{|(f, v)|}{|||v|||}.$$

It is easily seen that V and $H^0(\Omega)$ are contained continuously in $H^1(\Omega)$ and V', respectively, i.e.

$$(1.9) \qquad \begin{cases} \|u\|_{1,\Omega} \leqq |||u|||, & u \in V \\ |||f|||' \leqq \|f\|_0, & f \in H^0(\Omega). \end{cases}$$

THEOREM 1. *Suppose that Conditions* 1 *and* 2 *hold. Then there exists an isomorphism G of V' onto V, which is called a Green kernel for the system* (A_λ, B), *such that for every $f \in V'$*

$$Q_\lambda[Gf, v] = (f, v), \qquad v \in V,$$

where λ is the constant which appeared in Proposition 1.2 *and we set $A_\lambda = A + \lambda$ and*

$$Q_\lambda[u, v] = Q[u, v] + \lambda(u, v).$$

PROOF. From Propositions 1.2 and 1.3 we have

$$(1.10) \qquad \mathcal{R}e\, Q_\lambda[u, u] \geqq c|||u|||^2, \qquad u \in V$$

and

$$(1.11) \qquad |Q_\lambda[u, v]| \leqq \text{const.}\, |||u||| \cdot |||v|||, \qquad u, v \in V.$$

Using the theorem of Lax-Milgram, we can assert that for every $f \in V'$ there exists the unique $u \in V$ such that

$$Q_\lambda[u, v] = (f, v), \qquad v \in V.$$

We now define G by $u = Gf$. Applying this to $v = Gf$, we get by (1.10)

$$c|||Gf||| \leqq |||f|||'.$$

If $Gf = 0$, it follows that $(f, v) = 0$ for every $v \in V$ and hence $f = 0$. Finally we shall show that G is onto and its inverse is continuous. Let $u \in V$ and $\{u_n\}$ be a sequence in $\mathcal{D}_B$ such that $u_n \to u$ in V as $n \to \infty$. By proposition 1.1,

$$(1.12) \qquad (A_\lambda u_n, v) = Q_\lambda[u_n, v], \qquad v \in \mathcal{D}_B$$

and hence by (1.11)

$$|||A_\lambda u_n - A_\lambda u_m|||' \leqq \text{const.}\, |||u_n - u_m|||.$$

Therefore $A_\lambda u_n$ converges to some $f \in V'$. Thus, as $n \to \infty$, (1.12) becomes

$$(f, v) = Q_\lambda[u, v], \qquad v \in \mathcal{D}_B.$$

Hence

$$|||f|||' \leqq \text{const.}\, |||u|||.$$

These show that $u=Gf$ and G^{-1} is continuous. Q.E.D.

THEOREM 1*. *Suppose that Conditions* 1* *and* 2 *hold. Then there exists an isomorphism* $\tilde{G}$ *of* V' *onto* V, *the Green kernel for the system* (A_λ^*, B^*), *such that for every* $g\in V'$

$$Q_\lambda[u, \tilde{G}g]=(u, g), \qquad u\in V.$$

The proof can be done in quite parallel with one of Theorem 1.

THEOREM 2. *Suppose that Conditions* 1, 1* *and* 2 *hold, and let* G *and* G *be the isomorphisms which appeared in Theorems* 1 *and* 1*, *respectively. If we consider* G, $\tilde{G}$ *as operators of* $H^0(\Omega)$ *to itself, then they are completely continuous and* $\tilde{G}$ *is equal to* G^* *which is defined by*

$$(Gf, g)=(f, G^*g) \qquad \textit{for} \quad f, g\in H^0(\Omega).$$

PROOF. From (1.9) and Theorems 1, 1* we have

$$\|Gf\|_{1,\Omega}\leqq \text{const.}\, \|f\|_{0,\Omega}, \qquad f\in H^0(\Omega),$$

$$\|\tilde{G}g\|_{1,\Omega}\leqq \text{const.}\, \|g\|_{0,\Omega}, \qquad g\in H^0(\Omega).$$

Hence G and $\tilde{G}$ are completely continuous on $H^0(\Omega)$, for the injection: $H^1(\Omega)\rightarrow H^0(\Omega)$ is completely continuous.

We shall now prove $\tilde{G}=G^*$. It follows from Theorems 1 and 1* that for every f, $g\in H^0(\Omega)$

$$(Gf, g)=Q_\lambda[Gf, \tilde{G}g]=(f, \tilde{G}g).$$

This shows $G^*g=\tilde{G}g$ for every $g\in H^0(\Omega)$. Q.E.D.

REMARK 1. Proposition 1.3 guarantees that Conditions 1 and 1* are equivalent.

REMARK 2. The form $Q[u, v]$ defined by (1.5) is hermitian if $\overline{a_{\nu\mu}(x)}=a_{\mu\nu}(x)$ for all ν, μ such that $|\nu|, |\mu|\leqq 1$, $\gamma(x, D)^*=\gamma(x, D)$ and $\overline{a_0(x)}=a_0(x)$. Then it immediately follows that $A^*=A$ and $B_1'=B_1$ and hence $B^*=B$. Thus $\tilde{G}=G$. Hence it follows from Theorem 2 that G is self-adjoint as an operator on $H^0(\Omega)$.

§2. Weak solutions and mixed problems.

As in the preceding section we assume α and β to be measurable and satisfy (1.2), and the notations are all the same as there. Moreover we assume that Conditions 1, 1* and 2 hold.

Now we shall say u to be a *weak solution* of Problem

$$[A, f] \qquad \begin{cases} Au=f & \text{in } \Omega \\ Bu=0 & \text{on } \Gamma, \end{cases}$$

if u is a square integrable function on Ω and if

(2.1) $$(u, A^*v)=(f, v), \qquad v\in \mathcal{D}_{B^*}$$

is valid. We shall also say v to be a *weak solution* of Problem

$[A^*, g]$ $$\begin{cases} A^*v=g & \text{in } \Omega \\ B^*v=0 & \text{on } \Gamma, \end{cases}$$

if v is a square integrable function on Ω and if

$$(Au, v)=(u, g), \qquad u\in \mathcal{D}_B$$

is valid. We denote by D_B the set of u such that there exists a sequence $u_j\in \mathcal{D}_B$ satisfying

$$\begin{cases} \|u_j-u\|_0 \longrightarrow 0 \\ \|Au_j-Au_k\|_0 \longrightarrow 0 \end{cases}$$

as $j, k\rightarrow\infty$. Then we set

$$\bar{A}u=\lim_{j\to\infty} Au_j .$$

PROPOSITION 2.1. *Let $u\in V$ and $f\in V'$. Then the u is a weak solution of Problem $[A, f]$ if and only if*

(2.2) $$Q[u, v]=(f, v), \qquad v\in V$$

holds.

PROOF. Let $u_n\in \mathcal{D}_B$ such that $u_n\rightarrow u$ in V as $n\rightarrow\infty$. By Proposition 1.1

$$Q[u_n, v]=(u_n, A^*v), \qquad v\in \mathcal{D}_{B^*},$$

and hence

$$Q[u, v]=(u, A^*v), \qquad v\in \mathcal{D}_{B^*}.$$

This shows that (2.1) and (2.2) are equivalent, for $\mathcal{D}_{B^*}$ is dense in V.

Q. E. D.

PROPOSITION 2.2. *If $u\in D_B$, the u is a weak solution of Problem $[A, \bar{A}u]$ and moreover is in V.*

PROOF. Setting for $u, v\in H^2(\Omega)$

$$\varphi=u-B_1u, \qquad \psi=v-B_1'v \qquad \text{on } \Gamma,$$

we have $\varphi, \psi\in H^{1/2}(\Gamma)$ and moreover (1.4) and (1.4*) are valid. From (1.1) and (1.1*)

$$(Au, v)=(u, A^*v)=\int_\Gamma a(x)(u\overline{B_1'v}-B_1u\cdot\bar{v})d\sigma .$$

Thus

$$u\overline{B_1'v}-B_1u\cdot\bar{v}=(Bu+\alpha\varphi)(\overline{B^*v-\beta\psi})-(Bu-\beta\varphi)(\overline{B^*v+\alpha\psi})$$
$$=\varphi\overline{B^*v}-Bu\cdot\bar{\psi} .$$

Hence

(2.3) $$(Au, v)-(u, A^*v)=\int_{\Gamma} a(x)(\varphi\overline{B^*v}-Bu\cdot\bar{\psi})d\sigma\,.$$

Now let $u\in D_B$ and let $u_j\in \mathcal{D}_B$ be such that

$$\begin{cases} \|u_j-u\|_0 \longrightarrow 0 \\ \|Au_j-\bar{A}u\|_0 \longrightarrow 0 \end{cases}$$

as $j\to\infty$. Applying (2.3), we have

$$(Au_j, v)-(u_j, A^*v)=0\,,\qquad v\in \mathcal{D}_{B^*}\,.$$

Letting j tend to infinity, we can see that u is a weak solution of Problem $[A, \bar{A}u]$.

It easily follows from (1.10) and (1.11) that

$$c|||u|||\leqq |||A_\lambda u|||'\leqq \text{const.}\, |||u|||\,,\qquad u\in\mathcal{D}_B\,.$$

In particular

(2.4) $$c|||u|||\leqq \|A_\lambda u\|_0\,,\qquad u\in\mathcal{D}_B\,,$$

from which we can assert the proposition. Q. E. D.

PROPOSITION 2.3. *Every weak solution of Problem $[A, f]$ belonging to $H^2(\Omega)$ is in $\mathcal{D}_B$ and $Au=f$ in Ω, provided $\alpha\in H^{1+1/2}(\Gamma)$.*

PROOF. Let u be a weak solution of $[A, f]$ such that $u\in H^2(\Omega)$. Then we have by (2.3)

$$(Au, v)-(f, v)=0\,,\qquad v\in C_0^\infty(\Omega)\,.$$

This means $Au=f$ in Ω, which implies

(2.5) $$\int_{\Gamma} a(x)Bu\overline{(v-B_1'v)}d\sigma=0\,,\qquad v\in\mathcal{D}_{B^*}\,.$$

Let χ be arbitrarily given in $C^\infty(\Gamma)$. Then we can easily find a function $v\in H^2(\Omega)$ such that

$$\begin{cases} B_1'v=-\beta\chi \\ v=\alpha\chi \end{cases}$$

on Γ, since $\alpha\chi$ and $\beta\chi$ are belonging to $H^{1+1/2}(\Gamma)$. It is clear that $v\in\mathcal{D}_{B^*}$ and $v-B_1'v=\chi$ on Γ. This together with (2.5) yields $Bu=0$ on Γ, i.e. $u\in\mathcal{D}_B$.

THEOREM 3. *Denoting by N_{B^*} the set of weak solutions of Problem $[A^*, 0]$, we have*

$$H^0(\Omega)=\bar{A}_\lambda(D_B)\oplus N_{B^*}\,.$$

PROOF. We have only to prove that $\bar{A}_\lambda(D_B)$ is the closure of $A_\lambda(\mathcal{D}_B)$ in $H^0(\Omega)$. This is easily shown by (2.4). Q. E. D.

Now we consider the mixed problem (0.3). Here we assume that $\bar{\Gamma}_1\cap\bar{\Gamma}_2$ is C^∞ of dimension $m-2$. Then we can prove

THEOREM 4. *Let* $\alpha=1$ *on* Γ_1, $=0$ *on* Γ_2 *and* $\beta=1-\alpha$. *For each* $n=1, 2, \cdots$, *we introduce two measurable functions* α_n, β_n *satisfying* (1.2) *and that* $\alpha_n=1$ *on* Γ_1 *and* $=0$ *for every* $x\in\Gamma_2$ *such that* $\mathrm{dis}(x, \Gamma_1)\geqq 1/n$. *Then for every* $f\in H^0(\Omega)$, G_nf *converges to* Gf *weakly in* $H^1(\Omega)$, *where* G_n *is the Green kernel for the system* $(A_\lambda, B_n=\alpha_nB_1+\beta_n)$.

PROOF. Let V_n be the completion of $\mathscr{D}_{B_n}$ with respect to the norm

$$|||u|||_n^2=\|u\|_{1,\Omega}^2+\int_{\alpha_n\neq 0}\frac{1}{\alpha_n}|u|^2d\sigma$$

and

$$Q_\lambda^n[u, v]=Q_\lambda^0[u, v]+\int_{\alpha_n\neq 0}\frac{\beta_n}{\alpha_n}u\bar{v}d\sigma .$$

For $f\in H^0(\Omega)$, we put $u_n=G_nf$. Then

$$Q_\lambda^n[u_n, v]=(f, v), \qquad v\in V_n .$$

By (1.9) and (1.10) we have

$$c|||u_n|||_n\leqq \|f\|_0 .$$

Therefore there exists a subsequence $u_{n'}$ such that

$$u_{n'}\longrightarrow u \quad \text{in} \quad H^1(\Omega) . \tag{2.6}$$

Now using the same argument as in Proof of Proposition 1.3, we can assert $\mathscr{D}_B\subset V_n$. Hence

$$Q_\lambda^0[u_n, v]=(f, v), \qquad v\in\mathscr{D}_B ,$$

since $\beta_n=0$ on Γ_1 and $v=0$ on Γ_2. Thus it follows from (2.6) that $u\in H^1(\Omega)$, $u=0$ on Γ_2 and

$$Q_\lambda[u, v]=(f, v), \qquad v\in V . \tag{2.7}$$

The use of Lemma 5.2 in [4] guarantees $u\in V$. This together with (2.7) shows $u=Gf$. Q. E. D.

§ 3. Regularity of the Green kernels, I.

By $\mathscr{B}^s(\Gamma)$, $s=1, 2, \cdots$, we denote the set of s-times differentiable functions on Γ whose derivatives of order s are all bounded on Γ. Let α, β be functions on Γ satisfying (1.2) and assume that $\alpha\in\mathscr{B}^2(\Gamma)$. By ε we denote, throughout the present paper, positive numbers not greater than 1. All other notations are the same as in § 1. Now let $q_\varepsilon(x)$ be a C^∞-function on R^m defined in Lemma A.2 of Appendix, and denote by α_ε the restriction of q_ε onto Γ, and set

$$\beta_\varepsilon=1+\varepsilon-\alpha_\varepsilon .$$

Then if follows that α_ε and β_ε are in $C^\infty(\Gamma)$ and uniformly convergent to α

and β on Γ as $\varepsilon\to 0$, and that their derivatives are all uniformly bounded on Γ with respect to ε, and moreover that $\varepsilon\leqq\alpha_\varepsilon\leqq 1+\varepsilon$ and $\alpha_\varepsilon\geqq\alpha$ on Γ for every ε. Accordingly $0\leqq\beta_\varepsilon\leqq 1$ on Γ.

PROPOSITION 3.1. *Let $Q^\varepsilon[u, v]$ be an integro-differential bilinear form defined by*

$$Q^\varepsilon[u, v]=Q^0[u, v]+\int_\Gamma \frac{\beta_\varepsilon}{\alpha_\varepsilon} a(x)u\bar{v}\, d\sigma ,$$

and set

$$\begin{cases} B_\varepsilon u=\alpha_\varepsilon B_1 u+\beta_\varepsilon u \\ B_\varepsilon^* u=\alpha_\varepsilon B_1' u+\beta_\varepsilon u . \end{cases}$$

Then we have for every $u, v\in H^2(\Omega)$

$$\begin{aligned} Q^\varepsilon[u, v]&=(Au, v)+\int_\Gamma \frac{1}{\alpha_\varepsilon} aB_\varepsilon u\cdot\bar{v}\, d\sigma \\ &=(u, A^*v)+\int_\Gamma \frac{1}{\alpha_\varepsilon} au\cdot\overline{B_\varepsilon^* v}\, d\sigma . \end{aligned}$$

The proof may be done by the same argument as in Proposition 1.1. Throughout this section we must assume Condition 2 as well as the following one, stronger than Conditions 1 and 1*.

CONDITION 3. The form $Q^0[u, v]$ is coercive over $H^1(\Omega)$, i. e. there exist positive constants c_1 and C such that

$$\mathcal{R}e\, Q^0[u, u]\geqq c_1\|u\|_{1,\Omega}^2-C\|u\|_{0,\Omega}^2 , \qquad u\in H^1(\Omega) .$$

PROPOSITION 3.2. *Suppose that Conditions 2 and 3 hold. Then we can find two positive constants c, λ independent of ε such that*

$$\mathcal{R}e\, Q^\varepsilon[u, u]\geqq c\Big(\|u\|_{1,\Omega}^2+\int_\Gamma \frac{1}{\alpha_\varepsilon}|u|^2 d\sigma\Big)-\lambda\|u\|_{0,\Omega}^2 , \qquad u\in H^1(\Omega) .$$

The proof is immediately obtained in parallel with one of Proposition 1.2. For each ε, the completion of $C^\infty(\bar{\Omega})$ with respect to the norm

$$|||u|||_\varepsilon^2=\|u\|_{1,\Omega}^2+\int_\Gamma \frac{1}{\alpha_\varepsilon}|u|^2 d\sigma$$

is obviously equal to $H^1(\Omega)$.

PROPOSITION 3.3. *Suppose that Conditions 2 and 3 hold. Then it follows that for every $f\in C^\infty(\bar{\Omega})$ there exists the unique $u_\varepsilon\in H^1(\Omega)$ such that*

$$Q_\lambda^\varepsilon[u_\varepsilon, v]=(f, v) , \qquad v\in H^1(\Omega) , \tag{3.1}$$

and moreover that u_ε is in $C^\infty(\bar{\Omega})$ and satisfies

$$\begin{cases} A_\lambda u_\varepsilon=f & \text{in } \Omega \\ B_\varepsilon u_\varepsilon=0 & \text{on } \Gamma \end{cases} \tag{3.2}$$

and

(3.3) $$c|||u_\varepsilon|||_\varepsilon \leqq \|f\|_{0,\Omega} ,$$

where λ and c are the constants which appeared in Proposition 3.2.

PROOF. We have from Proposition 3.2

(3.4) $$\mathcal{R}e\, Q_\lambda^\varepsilon[u, u] \geqq c|||u|||_\varepsilon^2 , \qquad u \in H^1(\Omega) ,$$

and immediately

$$|Q_\lambda^\varepsilon[u, v]| \leqq C|||u|||_\varepsilon |||v|||_\varepsilon , \qquad u, v \in H^1(\Omega)$$

with a constant $C>0$ not depending on ε. These inequalities assert the unique existence of $u_\varepsilon \in H^1(\Omega)$ such that (3.1) is valid. Obviously $f \in C^\infty(\bar{\Omega})$ implies $u_\varepsilon \in C^\infty(\bar{\Omega})$ and Proposition 3.1 assures (3.2). From (3.1) and (3.4)

$$c|||u_\varepsilon|||_\varepsilon^2 \leqq |(f, u_\varepsilon)| \leqq \|f\|_{0,\Omega}\|u_\varepsilon\|_{0,\Omega}$$

and hence (3.3) follows from $\|u_\varepsilon\|_{0,\Omega} \leqq |||u_\varepsilon|||_\varepsilon$. Q. E. D.

Let Ω_0 be an open subset of Ω and assume that there exists a C^∞-coordinate transformation $y=\kappa(x)$ such that Ω_0 is mapped in a one-to-one way onto an open portion Σ of a half space $y_m>0$ and $\Gamma_0=\bar{\Omega}_0 \cap \Gamma$ is transformed onto an open portion σ of $y_m=0$. For functions u on Ω and φ on Γ, we write

$$\begin{cases} \tilde{u}(y)=u(\kappa^{-1}(y)) & \text{for } y \in \Sigma \\ \tilde{\varphi}(y')=\varphi(\kappa^{-1}(y')) & \text{for } y'=(y_1, \cdots, y_{m-1}) \in \sigma \end{cases}$$

and assume that the form $Q_\lambda^\varepsilon[u, v]$ is altered by the transformation κ to

$$P^\varepsilon[\tilde{u}, \tilde{v}]=P_1[\tilde{u}, \tilde{v}]+P_2^\varepsilon[\tilde{u}, \tilde{v}] ,$$

provided $u, v \in H^1(\Omega)$ and $\operatorname{supp}[v] \subset \Omega_0 \cup \Gamma_0$. Here

$$P_1[\tilde{u}, \tilde{v}]=\int_\Sigma \sum_{|\nu|,|\mu|\leqq 1} b_{\nu\mu}(y)D^\nu\tilde{u}\overline{D^\mu\tilde{v}}dy+\int_\sigma \delta(y', D')\tilde{u}\cdot\bar{\tilde{v}}dy' ,$$

$$P_2^\varepsilon[\tilde{u}, \tilde{v}]=\int_\sigma \frac{\tilde{\beta}_\varepsilon}{\tilde{\alpha}_\varepsilon} b(y')\tilde{u}\bar{\tilde{v}}d\sigma ,$$

all coefficients being infinitely differentiable and $\delta(y', D')$ of first order in $D'=(D_1, \cdots, D_{m-1})$. In the following propositions we always assume Conditions 2 and 3. Then it follows from Proposition 3.2 that there exists a positive constant c' such that

(3.5) $$\mathcal{R}e\, P^\varepsilon[\tilde{u}, \tilde{u}] \geqq c'\Big(\|\tilde{u}\|_{1,\Sigma}^2+\int_\sigma \frac{1}{\tilde{\alpha}_\varepsilon}|\tilde{u}|^2dy'\Big)$$

for every $u \in H^1(\Omega)$ satisfying $\operatorname{supp}[u] \subset \Omega_0 \cup \Gamma_0$.

Now let $\zeta \in C_0^\infty(\Omega_0 \cup \Gamma_0)$ and assume $\zeta \geqq 0$ there, and put

$$T=D_j\zeta \quad (D_j=-i\partial/\partial y_j) , \qquad j \neq m .$$

PROPOSITION 3.4. *There exist a constant* $C_1>0$ *such that for every* $u\in C^{\infty}(\bar{\Omega})$

$$|P_1[T\tilde{u}, T\tilde{u}]-P_1[\tilde{u}, T^*T\tilde{u}]|\leqq C_1\|\tilde{u}\|_{1,\Sigma}\|T\tilde{u}\|_{1,\Sigma}.$$

PROOF. For the sake of simplicity, we shall write $\tilde{u}$ as u and set

$$\begin{cases} R=b_{\nu\mu}(y)D^{\nu} \\ S=D^{\mu} \qquad (|\nu|,\ |\mu|\leqq 1). \end{cases}$$

Thus

$$\begin{aligned}(RTu, STu)&=(TRu, STu)+([R, T]u, STu)\\ &=(Ru, T^*STu)+([R, T]u, STu)\\ &=(Ru, ST^*Tu)+(Ru, [T^*, S]Tu)+([R, T]u, STu),\end{aligned}$$

where $[A, B]$ denotes a commutator $AB-BA$ and (u, v) does the usual inner product in $L_2(\Sigma)$. Hence

$$|(RTu, STu)-(Ru, ST^*Tu)|\leqq K_1\|u\|_{1,\Sigma}\|Tu\|_{1,\Sigma}. \tag{3.6}$$

In this section we always denote by K_j a positive constant not depending on u or ε.

Now,

$$\begin{aligned}(\delta Tu, Tu)_{\sigma}&=(T\delta u, Tu)_{\sigma}+([\delta, T]u, Tu)_{\sigma}\\ &=(\delta u, T^*Tu)_{\sigma}+([\delta, T]u, Tu)_{\sigma},\end{aligned}$$

and hence by (1.3)

$$\begin{aligned}|(\delta Tu, Tu)_{\sigma}-(\delta u, T^*Tu)_{\sigma}|&\leqq K_2\|[\delta, T]u\|_{-1/2,\sigma}\|Tu\|_{1/2,\sigma}\\ &\leqq K_3\|u\|_{1,\Sigma}\|Tu\|_{1,\Sigma},\end{aligned} \tag{3.7}$$

where and in the following we put

$$(u, v)_{\sigma}=\int_{\sigma}u\bar{v}dy'.$$

The proof is completed by (3.6) and (3.7). Q. E. D.

PROPOSITION 3.5. *There exists a constant* $C_2>0$ *such that for every* $f\in C^{\infty}(\bar{\Omega})$ *and every* ε

$$\begin{aligned}&|P_2^{\varepsilon}[T\tilde{u}_{\varepsilon}, T\tilde{u}_{\varepsilon}]-P_2^{\varepsilon}[\tilde{u}_{\varepsilon}, T^*T\tilde{u}_{\varepsilon}]|\\ &\leqq C_2\Big(\int_{\sigma}\frac{1}{\tilde{\alpha}_{\varepsilon}}|\tilde{u}_{\varepsilon}|^2dy'+\int_{\sigma}|\widetilde{B_1u_{\varepsilon}}|^2dy'\Big)^{1/2}\Big(\int_{\sigma}\frac{1}{\tilde{\alpha}_{\varepsilon}}|T\tilde{u}_{\varepsilon}|^2dy'\Big)^{1/2},\end{aligned}$$

u_{ε} *being a* C^{∞}*-function on* $\bar{\Omega}$ *settled in Proposition* 3.3.

PROOF. In the proof we omit the wave sign $\tilde{}$. Thus

$$P_2^\varepsilon[Tu_\varepsilon, Tu_\varepsilon]-P_2^\varepsilon[u_\varepsilon, T^*Tu_\varepsilon]=\int_\sigma\Big[\frac{\beta_\varepsilon}{\alpha_\varepsilon}b, T\Big]u_\varepsilon\cdot\overline{Tu_\varepsilon}dy'$$

$$=\int_\sigma\frac{1}{\alpha_\varepsilon}[\beta_\varepsilon b, T]u_\varepsilon\overline{Tu_\varepsilon}dy'+\int_\sigma\Big[\frac{1}{\alpha_\varepsilon}, T\Big](\beta_\varepsilon bu_\varepsilon)\overline{Tu_\varepsilon}dy' .$$

By (3.2)

$$\beta_\varepsilon bu_\varepsilon=-b\alpha_\varepsilon B_1u_\varepsilon \quad \text{on} \quad \sigma .$$

Hence

$$\Big[\frac{1}{\alpha_\varepsilon}, T\Big](\beta_\varepsilon bu_\varepsilon)=\Big[T, \frac{1}{\alpha_\varepsilon}\Big](b\alpha_\varepsilon B_1u_\varepsilon)$$

$$=\Big[D_j, \frac{1}{\alpha_\varepsilon}\Big](\zeta b\alpha_\varepsilon B_1u_\varepsilon)$$

$$=-\frac{D_j\alpha_\varepsilon}{\alpha_\varepsilon^2}\zeta b\alpha_\varepsilon B_1u_\varepsilon=-\frac{D_j\alpha_\varepsilon}{\alpha_\varepsilon}\zeta bB_1u_\varepsilon .$$

Therefore by the Schwarz inequality and by Remark 1 in Appendix we can conclude the proposition. Q. E. D.

PROPOSITION 3.6. *Under the same situation as in Proposition* 3.5, *we can find a constant* $C_3>0$ *such that for every* $f\in C^\infty(\bar{\Omega})$ *and every* ε

$$\|T\tilde{u}_\varepsilon\|_{1,\Sigma}^2+\int_\sigma\frac{1}{\tilde{\alpha}_\varepsilon}|T\tilde{u}_\varepsilon|^2dy'\leqq C_3\Big(\|f\|_{0,\Omega}^2+\int_\Gamma|B_1u_\varepsilon|^2d\sigma\Big) . \tag{3.8}$$

PROOF. Applying (3.5) to $\tilde{u}=T\tilde{u}_\varepsilon$, we obtain

$$c'\Big(\|T\tilde{u}_\varepsilon\|_{1,\Sigma}^2+\int_\sigma\frac{1}{\tilde{\alpha}_\varepsilon}|T\tilde{u}_\varepsilon|^2dy'\Big)\leqq \mathcal{R}_e\, P^\varepsilon[T\tilde{u}_\varepsilon, T\tilde{u}_\varepsilon]$$

$$\leqq|P^\varepsilon[T\tilde{u}_\varepsilon, T\tilde{u}_\varepsilon]-P^\varepsilon[\tilde{u}_\varepsilon, T^*T\tilde{u}_\varepsilon]|+|P^\varepsilon[\tilde{u}_\varepsilon, T^*T\tilde{u}_\varepsilon]| .$$

Let $v_\varepsilon\in C_0^\infty(\Omega_0\cup\Gamma_0)$ be such that $\tilde{v}_\varepsilon=T^*T\tilde{u}_\varepsilon$ on Σ. Then from Proposition 3.3

$$P^\varepsilon[\tilde{u}_\varepsilon, T^*T\tilde{u}_\varepsilon]=Q_\lambda^\varepsilon[u_\varepsilon, v_\varepsilon]=(f, v_\varepsilon) .$$

Thus by Propositions 3.4, 3.5 and the Cauchy inequality we obtain that the left hand side of (3.8)

$$\leqq K_4\Big(\|\tilde{u}_\varepsilon\|_{1,\Sigma}^2+\int_\sigma\frac{1}{\tilde{\alpha}_\varepsilon}|\tilde{u}_\varepsilon|^2dy'+\int_\sigma|\widetilde{B_1u_\varepsilon}|^2dy'+\|\tilde{f}\|_{0,\Sigma}^2\Big) .$$

This together with (3.3) completes the proof. Q. E. D.

PROPOSITION 3.7. *Under the same situation as in Proposition* 3.5, *we can find a constant* $C_4>0$ *such that for every* $f\in C^\infty(\bar{\Omega})$ *and every* ε

$$\|u_\varepsilon\|_{2,\Omega}^2+\int_\Gamma\frac{1}{\alpha_\varepsilon}|D_\tau u_\varepsilon|^2d\sigma\leqq C_4\|f\|_{0,\Omega}^2 ,$$

where D_τ *denotes the tangential derivative of first order.*

PROOF. Rewriting as $T_j=D_j\zeta, j=1, \cdots, m-1$, we get for every $u\in C^\infty(\bar{\Omega})$

$$\|\widetilde{\zeta u}\|_{2,\Sigma}^2 \leqq K_5(\|\widetilde{\zeta u}\|_{1,\Sigma}^2 + \sum_{j=1}^{m-1} \|T_j\tilde{u}\|_{1,\Sigma}^2 + \|D_m^2(\widetilde{\zeta u})\|_{0,\Sigma}^2)$$

and from (3.2) and the ellipticity of A

$$\|D_m^2(\widetilde{\zeta u_\varepsilon})\|_{0,\Sigma}^2 \leqq K_6(\|f\|_{0,\Omega}^2 + \|u_\varepsilon\|_{1,\Omega}^2 + \sum_{j=1}^{m-1} \|T_j\tilde{u}_\varepsilon\|_{1,\Sigma}^2) .$$

These together with Proposition 3.6 and (3.3) give

$$\|\widetilde{\zeta u_\varepsilon}\|_{2,\Sigma}^2 \leqq K_7\Big(\|f\|_{0,\Omega}^2 + \int_\Gamma |B_1 u_\varepsilon|^2 d\sigma\Big) . \tag{3.9}$$

Now again from Proposition 3.6 and (3.3)

$$\int_\sigma \frac{1}{\tilde{\alpha}_\varepsilon} \zeta^2 |D_j \tilde{u}_\varepsilon|^2 dy' \leqq 2\int_\sigma \frac{1}{\tilde{\alpha}_\varepsilon} |T_j\tilde{u}_\varepsilon|^2 dy' + 2\int_\sigma \frac{1}{\tilde{\alpha}_\varepsilon} |D_j\zeta|^2 |\tilde{u}_\varepsilon|^2 dy' \tag{3.10}$$

$$\leqq K_8\Big(\|f\|_{0,\Omega}^2 + \int_\Gamma |B_1 u_\varepsilon|^2 d\sigma\Big) .$$

Using the partition of unity of $\bar{\Omega}$, we obtain by (3.9) and (3.10)

$$\|u_\varepsilon\|_{2,\Omega}^2 + \int_\Gamma \frac{1}{\alpha_\varepsilon} |D_\tau u_\varepsilon|^2 d\sigma \leqq K_9\Big(\|f\|_{0,\Omega}^2 + \int_\Gamma |B_1 u_\varepsilon|^2 d\sigma\Big).$$

Applying (1.7) to $f = B_1 u_\varepsilon$, we can assert the proposition. Q. E. D.

PROPOSITION 3.8. *If $f \in C^\infty(\bar{\Omega})$, then Gf is contained in $\mathcal{D}_B$ and $A_\lambda Gf = f$ in Ω. Moreover it follows that there exists a constant $C_5 > 0$ such that for every $f \in C^\infty(\bar{\Omega})$*

$$\|Gf\|_{2,\Omega} \leqq C_5 \|f\|_{0,\Omega} .$$

PROOF. From Proposition 3.7

$$\|u_\varepsilon\|_{2,\Omega}^2 \leqq C_4 \|f\|_{0,\Omega}^2 .$$

It then follows from the theorem of Banach-Sacks that there exists a decreasing sequence $\varepsilon_1, \varepsilon_2, \cdots$ converging to zero such that

$$v_n = \frac{u_{\varepsilon_1} + \cdots + u_{\varepsilon_n}}{u}$$

converges to some u in $H^2(\Omega)$ by the norm $\|\cdot\|_{2,\Omega}$. Thus, from Proposition 3.3,

$$A_\lambda v_n = f \quad \text{in } \Omega$$

and

$$Bv_n = \frac{1}{n} \sum_{j=1}^{n} \{(\alpha - \alpha_{\varepsilon_j}) B_1 u_{\varepsilon_j} + (\beta - \beta_{\varepsilon_j}) u_{\varepsilon_j}\} .$$

Hence, letting n tend to infinity, we obtain

$$\begin{cases} A_\lambda u=f & \text{in } \Omega \\ Bu=0 & \text{on } \Gamma, \end{cases}$$

which shows that $u=Gf$ and hence $A_\lambda Gf=f$ in Ω. The latter half of the proposition is obvious. Q.E.D.

THEOREM 5. *Suppose that $\alpha \in \mathcal{B}^2(\Gamma)$ and that Conditions 2 and 3 hold. Then the Green kernel G stated in Theorem 1 is also an isomorphism of $H^0(\Omega)$ onto $\mathcal{D}_B$ with $G^{-1}=A_\lambda$, provided $\mathcal{D}_B$ is equipped with the norm $\|\cdot\|_{2,\Omega}$.*

PROOF. Let f be arbitrarily given in $H^0(\Omega)$ and let $f_j \in C^\infty(\bar{\Omega})$, $j=1, 2, \cdots$, such that $f_j \to f$ in $H^0(\Omega)$. Proposition 3.8 guarantees that $Gf_j \in \mathcal{D}_B$, $A_\lambda Gf_j = f_j$ in Ω and

$$\|G(f_j-f_i)\|_{2,\Omega} \leqq C_5\|f_j-f_i\|_{0,\Omega}.$$

Hence $Gf \in \mathcal{D}_B$, $A_\lambda Gf=f$ in Ω and

$$\|Gf\|_{2,\Omega} \leqq C_5\|f\|_{0,\Omega}.$$

Now for any $u \in \mathcal{D}_B$, we set

$$U=GA_\lambda u.$$

Clearly $U \in \mathcal{D}_B$, for $A_\lambda u \in H^0(\Omega)$. Moreover

$$A_\lambda U=A_\lambda u \quad \text{in } \Omega$$

and $U-u \in \mathcal{D}_B$. Thus $u=GA_\lambda u$ and

$$\|A_\lambda u\|_{0,\Omega} \leqq K_{10}\|u\|_{2,\Omega}.$$

Q.E.D.

THEOREM 5*. *Under the same supposition as in Theorem 5, it follows that the Green kernel $\tilde{G}$ stated in Theorem 1* is also an isomorphism of $H^0(\Omega)$ onto $\mathcal{D}_{B^*}$ with $G^{*-1}=A_\lambda^*$, provided $\mathcal{D}_{B^*}$ is equipped with the norm $\|\cdot\|_{2,\Omega}$.*

The proof is just the same as one of Theorem 5.

REMARK 1. It follows from Theorems 5, 5* that there exists a constant $C>0$ such that

$$|||u||| \leqq C\|u\|_{2,\Omega}, \qquad u \in \mathcal{D}_B \cup \mathcal{D}_{B^*}.$$

But this fact immediately follows from (1.4) and (1.4*) too.

REMARK 2. In Theorems 5 and 5*, we can replace $\alpha \in \mathcal{B}^2(\Gamma)$ with $\sqrt{\alpha} \in \mathcal{B}^1(\Gamma)$. Then we must use the $q_\varepsilon(x)$ defined in Remark 2 of Appendix, instead of one defined in Lemma A.2 there.

COROLLARY. *Every weak solution of Problem $[A, f]$ (resp. $[A^*, g]$) which belongs to V is in $\mathcal{D}_B$ and $Au=f$ in Ω, if $f \in H^0(\Omega)$ (resp. $g \in H^0(\Omega)$).*

PROOF. Let $u \in V$ be a weak solution of $[A, f]$ with $f \in H^0(\Omega)$, i.e. from Proposition 2.1, we assume that the u satisfies

$$Q[u, v]=(f, v), \qquad v \in V.$$

Accordingly

$$Q_\lambda[u, v]=(f+\lambda u, v), \qquad v \in V.$$

Thus $u=G(f+\lambda u)$. Hence by Theorem 5 we have $u \in \mathcal{D}_B$ and $A_\lambda u=f+\lambda u$.

§4. Regularity of the Green kernels, II.

This section is continued from the preceding section, that is, we shall discuss the further regularity of the Green kernels G and $\tilde{G}$. Thus let k be a fixed integer such that $k \geqq 3$. We always assume that $\alpha \in \mathcal{B}^k(\Gamma)$ when the bilinear form $Q^0[u, v]$ defined by (1.6) satisfies the following condition:

CONDITION (H). For all $x \in \bar{\Omega}$ and all ν, μ such that $|\nu|=|\mu|=1$

$$a_{\nu\mu}(x)=\overline{a_{\mu\nu}(x)}$$

is valid, and $\gamma(x, D)-\gamma(x, D)^*$ is of order zero.

Then q and q_ε are functions defined in Lemma A.2, and p and p_ε are defined by (A.4) in Remark 1 of Appendix. When Condition (H) is not satisfied, we must further assume $\sqrt{\alpha} \in \mathcal{B}^k(\Gamma)$. Then p and p_ε are functions defined in Remark 2 of Appendix, and q and q_ε are defined by (A.5). In either case we denote by α_ε the restriction of q_ε onto Γ and set $\beta_\varepsilon=1+\varepsilon-\alpha_\varepsilon$. All other notations are the same as in §3 and we always assume Conditions 2 and 3. Put for any integer $r \geqq 1$

$$\begin{cases} M_r=\sum_{|\nu|\leqq r} \sup|D^\nu p_\varepsilon(x)| \\ N_r=\sum_{|\nu|\leqq r} \sup|D^\nu q_\varepsilon(x)|, \end{cases}$$

the supremum being taken over all $x \in \Omega$ and all ε such that $0<\varepsilon \leqq 1$. It then follows from Appendix that M_1 and N_k are finite when $\alpha \in \mathcal{B}^k(\Gamma)$ and that M_k, N_k are finite when $\sqrt{\alpha} \in \mathcal{B}^k(\Gamma)$. In the following we denote by C_j and L_j positive constants not depending on u or ε. Their dependencies on M_r, N_s are denoted by $C_j(M_r, N_s)$ and $L_j(M_r, N_s)$.

PROPOSITION 4.1. *There exists a constant* $C_1=C_1(M_1)>0$ *such that for every* $u \in C^\infty(\bar{\Omega})$ *and every* ε

$$\begin{aligned} &|Q_\lambda^\varepsilon[p_\varepsilon u, p_\varepsilon u]-Q_\lambda^\varepsilon[u, q_\varepsilon u]| \\ &\qquad \leqq C_1(\|u\|_{0,\Omega}\|p_\varepsilon u\|_{1,\Omega}+\|u\|_{0,\Omega}^2+\|u\|_{-1/2,\Gamma}\|p_\varepsilon u\|_{1/2,\Gamma}). \end{aligned}$$

PROOF. Setting

$$\begin{cases} R=a_{\nu\mu}(x)D^\nu \\ S=D^\mu \qquad (|\nu|, |\mu| \leqq 1), \end{cases}$$

we have

$$(Rp_\varepsilon u, Sp_\varepsilon u)=(p_\varepsilon Ru, Sp_\varepsilon u)+([R, p_\varepsilon]u, Sp_\varepsilon u)$$
$$=(Ru, Sp_\varepsilon^2 u)+(Ru, [p_\varepsilon, S]p_\varepsilon u)+([R, p_\varepsilon]u, Sp_\varepsilon u)$$
$$=(Ru, Sp_\varepsilon^2 u)+(Rp_\varepsilon u, [p_\varepsilon, S]u)+([p_\varepsilon, R]u, [p_\varepsilon, S]u)$$
$$+([R, p_\varepsilon]u, Sp_\varepsilon u).$$

Thus

$$|(Rp_\varepsilon u, Sp_\varepsilon u)-(Ru, Sp_\varepsilon^2 u)|\leqq L_1(\|u\|_{0,\Omega}\|p_\varepsilon u\|_{1,\Omega}+\|u\|_{0,\Omega}^2).$$

Now

$$(\gamma p_\varepsilon u, p_\varepsilon u)_\Gamma=(p_\varepsilon \gamma u, p_\varepsilon u)_\Gamma+([\gamma, p_\varepsilon]u, p_\varepsilon u)_\Gamma$$
$$=(\gamma u, p_\varepsilon^2 u)_\Gamma+([\gamma, p_\varepsilon]u, p_\varepsilon u)_\Gamma,$$

and hence

$$|(\gamma p_\varepsilon u, p_\varepsilon u)_\Gamma-(\gamma u, p_\varepsilon^2 u)_\Gamma|\leqq L_2\|u\|_{-1/2,\Gamma}\|p_\varepsilon u\|_{1/2,\Gamma}.$$

Q. E. D.

Let $\zeta\in C_0^\infty(\Omega_0\cup\Gamma_0)$ and assume $\zeta\geqq 0$ there. Put

$$T=D^\rho\zeta, \qquad |\rho|=r,$$

where $\rho_m=0$ and $2\leqq r\leqq k-1$.

PROPOSITION 4.2. *There exists a constant* $C_2=C_2(M_r, N_{r+1})>0$ *such that for every* $u\in C^\infty(\bar{\Omega})$ *and every* ε

$$\mathcal{R}e\,\{P^\varepsilon[T\tilde{u}, \tilde{q}_\varepsilon T\tilde{u}]-P^\varepsilon[\tilde{u}, \tilde{q}_\varepsilon T^*T\tilde{u}]\}$$
$$\leqq C_2(\|\tilde{u}\|_{r,\Sigma}\|p_\varepsilon T\tilde{u}\|_{1,\Sigma}+\|\tilde{u}\|_{r,\Sigma}^2).$$

In particular $C_2=C_2(M_1, N_{r+1})$ when Condition (H) is satisfied.

PROOF. For the sake of simplicity we omit the wave sign ~ and set

$$\begin{cases} R=b_{\nu\mu}(y)D^\nu \\ S=D^\mu \qquad (|\nu|, |\mu|\leqq 1). \end{cases}$$

Then

$$(RTu, Sq_\varepsilon Tu)=(TRu, Sq_\varepsilon Tu)+([R, T]u, Sq_\varepsilon Tu)$$
$$=(Ru, T^*Sq_\varepsilon Tu)+([R, T]u, Sq_\varepsilon Tu)$$
$$=(Ru, Sq_\varepsilon T^*Tu)+(Ru, [T^*, Sq_\varepsilon]Tu)+([R, T]u, Sq_\varepsilon Tu)$$
$$=(Ru, Sq_\varepsilon T^*Tu)+(Ru, [T^*, S]q_\varepsilon Tu)+([R, T]u, Sq_\varepsilon Tu)$$
$$+(Ru, S[T^*, q_\varepsilon]Tu).$$

Hence

$$\mathcal{R}e\,\{(RTu, Sq_\varepsilon Tu)-(Ru, Sq_\varepsilon T^*Tu)\} \tag{4.1}$$
$$\leqq L_3(M_1)\|u\|_{r,\Sigma}\|p_\varepsilon Tu\|_{1,\Sigma}+\mathcal{R}e\,(Ru, S[T^*, q_\varepsilon]Tu).$$

Now

$$(\delta Tu,\ q_\varepsilon Tu)_\sigma=(T\delta u,\ q_\varepsilon Tu)_\sigma+([\delta,\ T]u,\ q_\varepsilon Tu)_\sigma$$
$$=(\delta u,\ T^*q_\varepsilon Tu)_\sigma+([\delta,\ T]u,\ q_\varepsilon Tu)_\sigma$$
$$=(\delta u,\ q_\varepsilon T^*Tu)_\sigma+(\delta u,\ [T^*,\ q_\varepsilon]Tu)_\sigma+([\delta,\ T]u,\ q_\varepsilon Tu)_\sigma .$$

Hence

$$\mathcal{R}e\ \{(\delta Tu,\ q_\varepsilon Tu)_\sigma-(\delta u,\ q_\varepsilon T^*Tu)_\sigma\}$$
$$\leqq L_4(M_1)\|u\|_{r,\Sigma}\,\|p_\varepsilon Tu\|_{1,\Sigma}+\mathcal{R}e\ (\delta u,\ [T^*,\ q_\varepsilon]Tu)_\sigma .$$

This together with (4.1) gives

$$\mathcal{R}e\ \{P_1[Tu,\ q_\varepsilon Tu]-P_1[u,\ q_\varepsilon T^*Tu]\} \tag{4.2}$$
$$\leqq L_5(M_1)\|u\|_{r,\Sigma}\,\|p_\varepsilon Tu\|_{1,\Sigma}+\mathcal{R}e\ P_1[u,\ [T^*,\ q_\varepsilon]Tu] .$$

1) Case in which Condition (H) is not satisfied:

$$S[T^*,\ q_\varepsilon]=[T^*,\ p_\varepsilon^2]S+[S,\ [T^*,\ p_\varepsilon^2]]$$
$$=2[T^*,\ p_\varepsilon]p_\varepsilon S+[p_\varepsilon,\ [T^*,\ p_\varepsilon]]S+[S,\ [T^*,\ p_\varepsilon^2]]$$
$$=2[T^*,\ p_\varepsilon]Sp_\varepsilon+2[T^*,\ p_\varepsilon][p_\varepsilon,\ S]+[p_\varepsilon,\ [T^*,\ p_\varepsilon]]S+[S,\ [T^*,\ p_\varepsilon^2]] .$$

Thus

$$\mathcal{R}e\ (Ru,\ S[T^*,\ q_\varepsilon]Tu)\leqq L_6(\|u\|_{r,\Sigma}\,\|p_\varepsilon Tu\|_{1,\Sigma}+\|u\|_{r,\Sigma}^2)$$

with $L_6=L_6(M_r,\ N_{r+1})$. By the similar argument as above

$$\mathcal{R}e\ (\delta u,\ [T^*,\ q_\varepsilon]Tu)_\sigma\leqq L_7(\|u\|_{r,\Sigma}\,\|p_\varepsilon Tu\|_{1,\Sigma}+\|u\|_{r,\Sigma}^2)$$

with $L_7=L_7(M_r)$. Hence

$$\mathcal{R}e\ P_1[u,\ [T^*,\ q_\varepsilon]Tu]\leqq L_8(\|u\|_{r,\Sigma}\,\|p_\varepsilon Tu\|_{1,\Sigma}+\|u\|_{r,\Sigma}^2) \tag{4.3}$$

with $L_8=L_8(M_r,\ N_{r+1})$

2) Case in which Condition (H) is satisfied: Put

$$v=[T^*,\ q_\varepsilon]Tu .$$

It follows from Condition (H) that, denoting by P_1^0 and δ^0 the principal parts of the form P_1 and δ, we have

$$\mathcal{R}e\ P_1[u,\ v]=\mathcal{R}e\ P_1^0[u,\ v]+\mathcal{R}e\ \{P_1[u,\ v]-P_1^0[u,\ v]\} \tag{4.4}$$
$$=\frac{1}{2}\{P_1^0[u,\ v]+P_1^0[v,\ u]\}+(v,\ \delta^0u)-(\delta^0v,\ u)$$
$$+\mathcal{R}e\ \{P_1[u,\ v]-P_1^0[u,\ v]\} .$$

(i) $(Ru, Sv)+(Rv, Su)=(Ru, [T^*, q_\varepsilon]TSu)$

$$+(Ru, [S, [T^*, q_\varepsilon]T]u)+([T^*, q_\varepsilon]TRu, Su)+([R, [T^*, q_\varepsilon]T]u, Su)$$

$$=(Ru, [T^*, q_\varepsilon]TSu)+(Ru, T^*[q_\varepsilon, T]Su)+O(\|u\|_{r,\Sigma}^2)\,.$$

Here

$$\begin{aligned}[T^*, q_\varepsilon]T+T^*[q_\varepsilon, T] &= [\zeta D^\rho, q_\varepsilon]D^\rho\zeta+\zeta D^\rho[q_\varepsilon, D^\rho\zeta] \\ &= \zeta[D^\rho, q_\varepsilon]D^\rho\zeta+\zeta D^\rho[q_\varepsilon, D^\rho]\zeta \\ &= \zeta([D^\rho, q_\varepsilon]D^\rho+D^\rho[q_\varepsilon, D^\rho])\zeta \\ &= \zeta(D^\rho[q_\varepsilon, D^\rho]-[q_\varepsilon, D^\rho]D^\rho)\zeta \\ &= \zeta[D^\rho, [q_\varepsilon, D^\rho]]\zeta\,.\end{aligned}$$

Hence

$$|(Ru, Sv)-(Rv, Su)|\leqq L_9(N_{r+1})\|u\|_{r,\Sigma}^2\,.$$

(ii) $(\delta^0 u, v)_\sigma+(\delta^0 v, u)_\sigma=(\delta^0[T^*, q_\varepsilon]Tu, u)_\sigma+(T^*[q_\varepsilon, T]\delta^0 u, u)_\sigma\,.$

Now

$$\begin{aligned}&\delta^0[T^*, q_\varepsilon]T+T^*[q_\varepsilon, T]\delta^0 \\ &\qquad=\delta^0([T^*, q_\varepsilon]T+T^*[q_\varepsilon, T])+[T^*[q_\varepsilon, T], \delta^0] \\ &\qquad=\delta^0\zeta[D^\rho, [q_\varepsilon, D^\rho]]\zeta+[T^*[q_\varepsilon, T], \delta^0]\,.\end{aligned}$$

Hence

$$|(\delta^0 u, v)+(\delta^0 v, u)|\leqq L_{10}(N_{r+1})\|u\|_{r,\Sigma}^2\,.$$

(iii) $|P_1[u, v]-P_1^0[u, v]|\leqq L_{11}(N_{r+1})\|u\|_{r,\Sigma}^2\,.$

Accordingly by (4.4), (i), (ii) and (iii)

$$\mathcal{R}e\, P_1[u, [T^*, q_\varepsilon]Tu]\leqq L_{12}(N_{r+1})\|u\|_{r,\Sigma}^2\,.$$

This together with (4.2) and (4.3) establishes

$$\begin{aligned}&\mathcal{R}e\,\{P_1[Tu, q_\varepsilon Tu]-P_1[u, q_\varepsilon T^*Tu]\} \\ &\qquad\leqq L_{13}(\|u\|_{r,\Sigma}\|p_\varepsilon Tu\|_{1,\Sigma}+\|u\|_{r,\Sigma}^2)\end{aligned}$$

with $L_{13}=L_{13}(M_r, N_{r+1})$ in Case 1) and $=L_{13}(M_1, N_{r+1})$ in Case 2).

Thus the proposition immediately follows from the following:

$$\begin{aligned}&P_2^\varepsilon[Tu, q_\varepsilon Tu]-P_2^\varepsilon[u, q_\varepsilon T^*Tu] \\ &\qquad=(\beta_\varepsilon bTu, Tu)_\sigma-(\beta_\varepsilon bu, T^*Tu)_\sigma=([\beta_\varepsilon b, T]u, Tu)_\sigma\,.\end{aligned}$$

Q. E. D.

PROPOSITION 4.3. *There exists a constant* $C_3>0$ *such that for every* $f\in C^\infty(\bar{\Omega})$ *and every* ε

$$\|\tilde{p}_\varepsilon T\tilde{u}_\varepsilon\|_{1,\Sigma}^2+\int_\sigma |T\tilde{u}_\varepsilon|^2 dy'$$
$$\leqq C_3(\|\tilde{u}_\varepsilon\|_{r,\Sigma}^2+\sum_{|\nu|=r-1}\|\tilde{p}_\varepsilon D^\nu \tilde{f}\|_{0,\Sigma}+\|\tilde{f}\|_{r-2+1/2,\Sigma}\|\tilde{u}_\varepsilon\|_{r+1/2,\Sigma}),$$

where u_ε *is a function introduced in Proposition* 3.3.

PROOF. In the proof we omit again the wave sign $\tilde{}$ like in the preceding one. Substituting $p_\varepsilon Tu_\varepsilon$ for u in (3.5), we obtain

$$c'\Big(\|p_\varepsilon Tu_\varepsilon\|_{1,\Sigma}^2+\int_\sigma |Tu_\varepsilon|^2 dy'\Big)\leqq \mathcal{R}e\ P^\varepsilon[p_\varepsilon Tu_\varepsilon, p_\varepsilon Tu_\varepsilon]. \tag{4.5}$$

From Proposition 4.1,

$$\mathcal{R}e\ \{P^\varepsilon[p_\varepsilon Tu_\varepsilon, p_\varepsilon Tu_\varepsilon]-P^\varepsilon[Tu_\varepsilon, q_\varepsilon Tu_\varepsilon]\}$$
$$\leqq L_{14}(\|u_\varepsilon\|_{r,\Sigma}\|p_\varepsilon Tu_\varepsilon\|_{1,\Sigma}+\|u_\varepsilon\|_{r,\Sigma}^2).$$

Thus with the aid of (4.5) and Proposition 4.2 we obtain, by using the Cauchy inequality,

$$\|p_\varepsilon Tu_\varepsilon\|_{1,\Sigma}^2+\int_\sigma |Tu_\varepsilon|^2 dy'\leqq L_{15}(\|u_\varepsilon\|_{r,\Sigma}^2+|P^\varepsilon[u_\varepsilon, q_\varepsilon T^*Tu_\varepsilon]|). \tag{4.6}$$

Now from (3.1) in Proposition 3.3

$$P^\varepsilon[u_\varepsilon, q_\varepsilon T^*Tu_\varepsilon]=(Jf, q_\varepsilon T^*Tu_\varepsilon)\qquad \Big(J=\Big|\frac{\partial x}{\partial y}\Big|\Big)$$
$$=(Jf, q_\varepsilon\zeta D^\rho Tu_\varepsilon)=(f, Jq_\varepsilon\zeta D^{\rho'}DTu_\varepsilon)\qquad (D^\rho=D^{\rho'}D,\ |\rho'|=r-1)$$
$$=(p_\varepsilon D^{\rho'}f, J\zeta p_\varepsilon DTu_\varepsilon)+(f, [Jq_\varepsilon\zeta, D^{\rho'}]DTu_\varepsilon)$$
$$=(\zeta p_\varepsilon D^{\rho'}f, JD(p_\varepsilon Tu_\varepsilon))+(\zeta p_\varepsilon D^{\rho'}f, J[p_\varepsilon, D]Tu_\varepsilon)$$
$$+(f, [Jq_\varepsilon\zeta, D^{\rho'}]DTu_\varepsilon).$$

Hence

$$|P^\varepsilon[u_\varepsilon, q_\varepsilon T^*Tu_\varepsilon]|\leqq L_{16}\{\|\zeta p_\varepsilon D^{\rho'}f\|_{0,\Sigma}(\|p_\varepsilon Tu_\varepsilon\|_{1,\Sigma}+\|Tu_\varepsilon\|_{0,\Sigma})$$
$$+\|f\|_{r-2+1/2,\Sigma}\|Tu_\varepsilon\|_{1/2,\Sigma}\}.$$

This together with (4.6) guarantees the proposition. Q. E. D.

PROPOSITION 4.4. *There exists a constant* $C_4>0$ *such that for every* $f\in C^\infty(\bar{\Omega})$ *and every* ε

$$\sum_{|\nu|=k}\|pD^\nu u_\varepsilon\|_{0,\Omega}^2+\|u_\varepsilon\|_{k-1/2,\Omega}^2\leqq C_4\Big(\sum_{|\nu|=k-2}\|p_\varepsilon D^\nu f\|_{0,\Omega}^2+\|f\|_{k-2-1/2,\Omega}^2\Big).$$

PROOF. With the aid of Proposition 4.3 we have for $j=1, 2, \cdots, n$

$$\|\tilde{p}_\varepsilon D_j T\tilde{u}_\varepsilon\|_0^2+\int_\sigma |T\tilde{u}_\varepsilon|^2 dy'$$
$$\leqq L_{17}(\|u_\varepsilon\|_{r,\Omega}^2+\sum_{|\nu|\leqq r-1}\|p_\varepsilon D^\nu f\|_{0,\Omega}^2+\|f\|_{r-2+1/2,\Omega}\|u_\varepsilon\|_{r+1/2,\Omega})$$
$$(=L_{17}F).$$

Thus

$$\sum_{\substack{|\rho|=r\\ \rho_m=0}}\Big(\sum_{|\nu|=1}\|\tilde{p}_\varepsilon D^\rho D^\nu(\widetilde{\zeta u_\varepsilon})\|_{0,\Sigma}^2+\int_\sigma|D^\rho(\widetilde{\zeta u_\varepsilon})|^2dy'\Big)\leqq L_{18}F. \tag{4.7}$$

It now follows from (3.2) and the ellipticity of A that $D_m^2(\widetilde{\zeta u_\varepsilon})$ is expressed by a linear combination of the terms

$$\begin{cases} D_nD_j(\widetilde{\zeta u_\varepsilon}), \quad D_jD_k(\widetilde{\zeta u_\varepsilon}) \quad (1\leqq j,\, k\leqq m-1), \\ D_j(\widetilde{\zeta u_\varepsilon}) \quad (1\leqq j\leqq m), \\ \widetilde{\zeta u_\varepsilon}, \quad \widetilde{\zeta f}, \quad \widetilde{[A_\lambda, \zeta]u_\varepsilon}. \end{cases}$$

Hence, operating $\tilde{p}_\varepsilon D^\rho(|\rho|=r-1,\ \rho_m=0)$ to $D_m^2(\widetilde{\zeta u_\varepsilon})$ and using (4.7), we get

$$\sum_{\substack{|\rho|=r-1\\ \rho_m=0}}\sum_{|\nu|=2}\|\tilde{p}_\varepsilon D^\rho D^\nu(\widetilde{\zeta u_\varepsilon})\|_{0,\Sigma}^2\leqq L_{19}F.$$

Similarly, operating $\tilde{p}_\varepsilon D^\rho D_m$ $(|\rho|=r-2,\ \rho_m=0)$ to $D_m^2(\widetilde{\zeta u_\varepsilon})$ we obtain

$$\sum_{\substack{|\rho|=r-2\\ \rho_m=0}}\sum_{|\nu|=3}\|\tilde{p}_\varepsilon D^\rho D^\nu(\widetilde{\zeta u_\varepsilon})\|_{0,\Sigma}^2\leqq L_{20}F.$$

Successive repetition of this process gives us

$$\sum_{|\nu|=r+1}\|\tilde{p}_\varepsilon D^\nu(\widetilde{\zeta u_\varepsilon})\|_{0,\Sigma}^2+\sum_{\substack{|\rho|=r\\ \rho_m=0}}\int_\sigma|D^\rho\widetilde{\zeta u_\varepsilon}|^2dy'\leqq L_{21}F.$$

By the use of the partition of unity of $\bar{\Omega}$ we obtain, noting $0<p(x)\leqq p_\varepsilon(x)$ in Ω,

$$\sum_{|\nu|=r+1}\|p_\varepsilon D^\nu u_\varepsilon\|_{0,\Omega}^2+\|u_\varepsilon\|_{r,\Gamma}^2\leqq L_{22}F. \tag{4.8}$$

Here we shall use the coercive estimate:

$$\|u\|_{r+1/2,\Omega}^2\leqq \text{const.}\,(\|A_\lambda u\|_{r-2+1/2,\Omega}^2+\|u\|_{r,\Gamma}^2), \qquad u\in C_\infty(\bar{\Omega}),$$

and the interpolation inequality: For any $\delta>0$ there exists a constant $C_\delta>0$ such that

$$\|u\|_{r,\Omega}^2\leqq\delta\|u\|_{r+1/2,\Omega}^2+C_\delta\|u\|_{0,\Omega}^2, \qquad u\in C^\infty(\bar{\Omega}).$$

With the aid of (3.3), (4.8) and the Cauchy inequality as well as the above two inequalities, we can conclude the proposition, since $p\leqq p_\varepsilon$ in Ω and r was arbitrarily fixed so that $r\leqq k-1$. Q.E.D.

Let s be an integer such that $s\geqq 1$. By $H^s(\Omega\,;p)$ we denote the Banach space obtained by the completion of $C^\infty(\bar{\Omega})$ with respect to the norm $\|\cdot\,;p\|_s$ defined by

$$\|u\,;p\|_{s,\Omega}^2=\sum_{|\nu|=s}\|pD^\nu u\|_{0,\Omega}^2+\|u\|_{s-1/2,\Omega}^2.$$

For $s\geqq 3$, we set

$$\begin{cases} H_B^s(\Omega;p)=\{u\in H^s(\Omega;p);\ Bu=0 \quad \text{on} \quad \Gamma\} \\ H_{B^*}^s(\Omega;p)=\{u\in H^s(\Omega;p);\ B^*u=0 \quad \text{on} \quad \Gamma\}\,. \end{cases}$$

PROPOSITION 4.5. 1) *For* $s\geqq 1$,

$$\begin{cases} H^s(\Omega)\subset H^s(\Omega;p)\subset H^{s-1/2}(\Omega) \\ H^s(\Omega;p)\subset V'\,. \end{cases}$$

2) *The space* $H_B^s(\Omega;p)$ *and* $H_{B^*}^s(\Omega;p)$ *are closed subspaces of* $H^s(\Omega;p)$ *when* $s\geqq 3$.

3) *For* $s\geqq 3$,

$$\begin{cases} H_B^s(\Omega;p)\subset V \\ H_{B^*}^s(\Omega;p)\subset V\,. \end{cases}$$

Here the injections are all continuous.

PROOF. 1) is easily proved by the above definition and (1.9). 2) follows from (1.3). With the aid of Remark 1 in § 3 we may prove 3). Q. E. D.

PROPOSITION 4.6. *If* $f\in C^\infty(\bar\Omega)$, *then* Gf *is contained in* $H_B^k(\Omega;p)$ *and* $A_\lambda Gf=f$ *in* Ω. *Moreover it follows that there exists a constant* $C_5>0$ *such that for every* $f\in C^\infty(\bar\Omega)$

$$\|Gf;p\|_k\leqq C_5\|f;p\|_{k-2}\,. \tag{4.9}$$

PROOF. Let $f\in C^\infty(\bar\Omega)$. Then from Proposition 4.4 it follows that $\|u_\varepsilon;p\|_k$ is bounded with respect to ε, for p_ε is bounded on Ω. Making use of the theorem of Banach-Sacks, we can find a decreasing sequence $\{\varepsilon_j\}$, converging to zero, such that

$$v_n=\frac{u_{\varepsilon_1}+\cdots+u_{\varepsilon_n}}{n}$$

converges to some u in $H^k(\Omega;p)$ by the norm $\|\cdot;p\|_k$. By the same argument as in Proposition 3.8, we can assert that $u=Gf\in H_B^k(\Omega;p)$ and $A_\lambda Gf=f$ in Ω. To complete the proof we must show (4.9). Now the use of Proposition 4.4 gives us

$$\begin{aligned} \|v_n;p\|_k &\leqq \frac{1}{n}\sum_{j=1}^n \|u_{\varepsilon_j};p\|_k \\ &\leqq \frac{C_4}{n}\sum_{j=1}^n \Big(\sum_{|\nu|=k-2}\|p_{\varepsilon_j}D^\nu f\|_{0,\Omega}^2+\|f\|_{k-2-1/2,\Omega}^2\Big)^{1/2}\,. \end{aligned}$$

Letting n tend to infinity we obtain (4.9). Q. E. D.

THEOREM 6. *Let* k *be an integer such that* $k\geqq 3$. *Suppose that* $\alpha\in\mathscr{B}^k(\Gamma)$ *when Condition* (H) *is satisfied, that* $\sqrt{\alpha}\in\mathscr{B}^k(\Gamma)$ *when Condition* (H) *is not* *satisfied, and that Conditions* 2 *and* 3 *hold. Then the Green kernel* G *stated in Theorem* 1 *is also an isomorphism of* $H^{k-2}(\Omega;p)$ *onto* $H_B^k(\Omega;p)$ *with* $G^{-1}=A_\lambda$.

PROOF. The proof may be done in a way quite similar to one of Theorem 5. To do so, we have only to notice that if $u\in H^s(\Omega;p)$ with $s\geqq 2$, then $Du\in H^{s-1}(\Omega;p)$ and the mapping $u\to Du$ is continuous. Q. E. D.

THEOREM 6*. *Under the same supposition as in Theorem* 6, *it follows that the Green kernel* $\tilde{G}$ *stated in Theorem* 1* *is also an isomorphism of* $H^{k-2}(\Omega;p)$ *onto* $H_{B^*}^k(\Omega;p)$ *with* $G^{*-1}=A_\lambda^*$.

The proof is just the same as one of Theorem 6.

COROLLARY. *Every weak solution of Problem* $[A,f]$ (*resp.* $[A^*,g]$) *which belongs to* V *is in* $H_B^k(\Omega;p)$ (*resp.* $H_{B^*}^k(\Omega;p)$), *if* $f\in H^{k-2}(\Omega;p)$ (*resp.* $g\in H^{k-2}(\Omega;p)$).

PROOF. Let $u\in V$ be a weak solution of $[A,f]$ with $f\in H^{k-2}(\Omega;p)$, i. e. from Proposition 2.1, we assume that the u satisfies

$$Q[u,v]=(f,v)\,,\qquad v\in V\,.$$

Accordingly

$$Q_\lambda[u,v]=(f+\lambda u,v)\,,\qquad v\in V\,.$$

Thus $u=G(f+\lambda u)$. Hence by Theorem 6 we have $u\in H_B^3(\Omega;p)$ and $A_\lambda u=f+\lambda u$, since $f+\lambda u\in H^1(\Omega;p)$. If $k-2>1$, we can assert, by the same argument as above, $u\in H_B^4(\Omega;p)$. After repeating this process, we finally obtain $u\in H_B^k(\Omega;p)$. Q. E. D.

Appendix.

LEMMA A.1. *Let* f *be in* $\mathscr{B}^2(R^m)$ *such that* $f(x)\geqq 0$ *in* R^m. *Then*

$$\text{(A.1)}\qquad \left|\frac{\partial f}{\partial x_j}(x)\right|^2\leqq 2K_jf(x)\,,\qquad x\in R^m\,,$$

where

$$K_j=\sup_{x\in R^m}\left|\frac{\partial^2 f}{\partial x_j^2}(x)\right|.$$

PROOF. We first prove (A.1) when $m=1$, i. e.

$$\text{(A.2)}\qquad |f'(x)|^2\leqq 2\sup_{t\in R}|f''(t)|\cdot f(x)\,,\qquad x\in R\,.$$

It is trivial at $x=x_0$ such that $f'(x_0)=0$. We hence assume $f'(x_0)\neq 0$. Clearly $f(x_0)>0$. By the Taylor expansion formula,

$$f(x_0-h)=f(x_0)-hf'(x_0)+\frac{h^2}{2}f''(x_0-\theta h)\,,$$

with $0<\theta<1$. Here, putting

$$h=2f(x_0)/f'(x_0)\,,$$

we obtain, by using the fact $f(x_0-h)\geqq 0$,

$$f(x_0)\leqq \frac{1}{2}\left(\frac{2f(x_0)}{f'(x_0)}\right)^2 f''(x_0-2\theta f(x_0)/f'(x_0)),$$

from which it follows

$$f'(x_0)^2\leqq 2f''(x_0-2\theta f(x_0)/f'(x_0))f(x_0).$$

This proves (A.2).

Now we shall prove in general case. Let $f\in\mathcal{B}^2(R^m)$ be such that $f(x)\geqq 0$ in R^m. We consider $f(x_1,\cdots,x_m)$ as a function of a variable x_j, freezing the remainder. Applying (A.2), we obtain, for every $x\in R^m$,

$$\left|\frac{\partial f}{\partial x_j}(x_1,\cdots,x_m)\right|^2\leqq 2\sup_{t\in R}\left|\frac{\partial^2 f}{\partial x_j^2}(x_1,\cdots,t,\cdots,x_m)\right|f(x_1,\cdots,x_m)$$
$$\leqq 2K_j f(x).$$

Q. E. D.

Let Ω and Γ be the same as in Introduction and let there be given a function $h(x)$ on Γ such that $h\in\mathcal{B}^k(\Gamma)$, $k\geqq 0$, and $0\leqq h(x)\leqq 1$ on Γ. We can then easily find a function $H(x)$ in $\mathcal{B}^k(R^m)$ (an extension of h on the whole space R^m) such that

$$\text{(A.3)}\qquad \begin{cases} 0\leqq H(x)\leqq 1 & \text{on } R^m \\ H(x)>0 & \text{in } \Omega \\ H(x)=h(x) & \text{on } \Gamma \\ \operatorname{supp}[H] \text{ is compact.} \end{cases}$$

Let $f(t)\in C^\infty(R)$ such that $f(t)=0$ for $t\geqq 0$ and $=e^{1/t}$ for $t<0$. Using this, we define a function $\zeta(x)$ in $C_0^\infty(R^m)$ by

$$\zeta(x)=f(|x|-1)/\int f(|x|-1)dx$$

and set

$$\rho_\delta(x)=\delta^{-m}\zeta\left(\frac{x}{\delta}\right),\qquad \delta>0.$$

It then follows that $\rho_\delta(x)\geqq 0$ in R^m, $\rho_\delta(x)=0$ for $|x|\geqq\delta$ and

$$\int_{R^m}\rho_\delta(x)dx=1.$$

LEMMA A.2. *Let $\alpha\in\mathcal{B}^k(\Gamma)$, $k\geqq 1$, be such that $0\leqq\alpha\leqq 1$ on Γ and $q(x)$ be an extension of α on R^m satisfying* (A.3). *Then a mollifier q_ε, $\varepsilon>0$, of q defined by the convolution*

$$q_\varepsilon=(q(x)+\varepsilon)*\rho_{\varepsilon/L}\qquad \left(L=\sum_{j=1}^m\sup_{x\in R^m}\left|\frac{\partial q}{\partial x_j}(x)\right|\right)$$

is in $C^\infty(R^m)$ and uniformly convergent to $q(x)$ on R^m, and its derivatives of orders up to k are all uniformly bounded on R^m with respect to ε. Moreover it follows that $\varepsilon \leqq q_\varepsilon(x) \leqq 1+\varepsilon$ and $q_\varepsilon \geqq q$ on R^m for every $\varepsilon > 0$.

PROOF. The first half of the lemma is well-known. So we only prove the latter half. For $\varepsilon > 0$,

$$q_\varepsilon(x)-q(x)=\int_{|y|\leqq\varepsilon/L}\{q(x-y)+\varepsilon-q(x)\}\rho_{\varepsilon/L}(y)dy\,.$$

Thus

$$|q(x-y)-q(x)|\leqq L|y|\,.$$

Hence

$$q(x-y)+\varepsilon-q(x)\geqq 0 \qquad \text{if} \quad |y|\leqq \varepsilon/L\,.$$

This assert $q_\varepsilon(x)-q(x)\geqq 0$ for all $x\in R^m$. From the fact $\varepsilon\leqq q(x)+\varepsilon\leqq 1+\varepsilon$ for all $x\in R^m$, it follows $\varepsilon\leqq q_\varepsilon(x)\leqq 1+\varepsilon$ for all $x\in R^m$ and all $\varepsilon>0$.

Q. E. D.

REMARK 1. If we set on R^m

$$\text{(A.4)} \qquad \begin{cases} p(x)=\sqrt{q(x)} \\ p_\varepsilon(x)=\sqrt{q_\varepsilon(x)}\,, \end{cases}$$

it follows from Lemma A.1 that $p_\varepsilon(x)$ is uniformly convergent to $p(x)$ on R^m, and that its derivatives are uniformly bounded on R^m with respect to $\varepsilon>0$ if $k\geqq 2$, and moreover $\sqrt{\varepsilon}\leqq p_\varepsilon(x)\leqq\sqrt{1+\varepsilon}$ and $p_\varepsilon(x)\geqq p(x)$ on R^m for every $\varepsilon>0$. In fact we have

$$\frac{\partial p_\varepsilon}{\partial x_j}(x)=-\frac{\partial q_\varepsilon}{\partial x_j}(x)/p_\varepsilon(x)$$

and

$$\left|\frac{\partial q_\varepsilon}{\partial x_j}(x)\right|^2\leqq 2\sup_{y\in R^m}\left|\frac{\partial^2 q}{\partial x_j^2}(y)\right|\cdot q_\varepsilon(x)\,, \qquad x\in R^m\,.$$

REMARK 2. Let $\sqrt{\alpha}\in\mathscr{B}^k(\Gamma)$, $k\geqq 1$. Then we can assert the following in quite parallel with Lemma A.2: A mollifier p_ε, $\varepsilon>0$, of p (an extension of $\sqrt{\alpha}$ on R^m satisfying (A.3)) defined by

$$p_\varepsilon=(p(x)+\eta)*\rho_{\eta/M} \qquad \left(\eta=\sqrt{1+\varepsilon}-1,\ M=\sum_{j=1}^m\sup_{x\in R^m}\left|\frac{\partial p}{\partial x_j}(x)\right|\right)$$

is in $C^\infty(R^m)$ and uniformly convergent to $p(x)$ on R^m, and its derivatives of orders up to k are all uniformly bounded on R^m with respect to $\varepsilon>0$. Moreover it follows that $\sqrt{1+\varepsilon}-1\leqq p_\varepsilon(x)\leqq\sqrt{1+\varepsilon}$ and $p_\varepsilon(x)\geqq p(x)$ on R^m for every $\varepsilon>0$.

If we set on R^m

$$\text{(A.5)} \qquad \begin{cases} q(x)=p(x)^2 \\ q_\varepsilon(x)=p_\varepsilon(x)^2\,, \end{cases}$$

the $q_\varepsilon(x)$ has the same property as that defined in Lemma A.2 with $(\sqrt{1+\varepsilon}-1)^2 \leqq q_\varepsilon(x) \leqq 1+\varepsilon$ instead of $\varepsilon \leqq q_\varepsilon(x) \leqq 1+\varepsilon$ on R^m.

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Yoshio KATO
Mathematical Institute
Nagoya University
Furo-cho, Chikusa-ku
Nagoya, Japan