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# Generation of semi-groups of nonlinear contractions

Dedicated to Professor Asajiro Ichida on his 70th birthday

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## § 1. Introduction.

Let $X$ be a Banach space, and let $X_0$ be a subset of $X$. By a *contraction semi-group* on $X_0$ we mean a family $\{T(t);\ t\geqq 0\}$ of operators $T(t): X_0\rightarrow X_0$ satisfying the following conditions:

(1.1) $$T(t+s)=T(t)T(s) \quad \text{for} \quad t, s\geqq 0,$$

(1.2) $$\|T(t)x-T(t)y\|\leqq\|x-y\| \quad \text{for} \quad t\geqq 0 \text{ and } x, y\in X_0,$$

(1.3) $$\lim_{t\rightarrow 0+}T(t)x=T(0)x=x \quad \text{for} \quad x\in X_0.$$

We define the *infinitesimal generator* $A_0$ of $\{T(t);\ t\geqq 0\}$ (a contraction semi-group on $X_0$) by

(1.4) $$A_0x=\lim_{h\rightarrow 0+}h^{-1}(T(h)x-x)$$

whenever the limit exists. It is easy to see that $A_0$ is a dissipative operator.

Recently, in the case when both $X$ and $X^*$ (the dual of $X$) are uniformly convex, Martin [5] has characterized the infinitesimal generator $A_0$ of a contraction semi-group on $X_0$ having the property that $\overline{D(A_0)}=X_0$. The purpose of this paper is to generalize his results to the case that $X^*$ is uniformly convex. To this end we introduce the following

DEFINITION 1.1. Let $\{T(t);\ t\geqq 0\}$ be a contraction semi-group on $X_0$, and let $A_0$ be the infinitesimal generator of $\{T(t);\ t\geqq 0\}$. Define the set $\hat{D}$ by

(1.5) $$\hat{D}=\{x\in X_0;\ \|T(h)x-x\|=O(h) \text{ as } h\rightarrow 0+\}.$$

If $A$ is an extension of $A_0$ and maximal dissipative on $\hat{D}$, then $A$ is called a $(g)$-operator of $\{T(t);\ t\geqq 0\}$.

If $\{T(t);\ t\geqq 0\}$ is a contraction semi-group on $X_0$ with $D(A_0)\neq\emptyset$, then its $(g)$-operator exists by the maximal principle.

Our main results are stated as follows: Let $X^*$ be uniformly convex.

(I) *If $A$ is a $(g)$-operator of a contraction semi-group $\{T(t);\ t\geqq 0\}$ on $X_0$, then $A$ has the property* $(\mathfrak{g})$, *$A$ is demiclosed and $D(A^0)=D(A)=\hat{D}$, where $A^0$*

*is the canonical restriction of* $A$. (See Theorem 2.3.)

(II) *If* $A$ *has the property* ($\mathfrak{g}$) *and* $A$ *is demiclosed, then there exists a unique contraction semi-group* $\{T(t)\,;\,t\geqq 0\}$ *on* $\overline{D(A)}$ *such that* $D(A)\subset\hat{D}$ *and for each* $x\in D(A)$

$$(d/dt)T(t)x\in A^0T(t)x \qquad \text{for a.e. } t\geqq 0\,.$$

(See Theorem 3.3.)

(III) *Let* $X_0$ *be closed and let* $A$ *be a multi-valued operator from* $D(A)\subset X_0$ *into* $X$ *such that* $\overline{D(A)}=X_0$. *Then the following three conditions are equivalent;*

(i) $A$ *is a* $(g)$*-operator of a contraction semi-group on* $X_0$,

(ii) $A$ *has the property* ($\mathfrak{g}$) *and* $A$ *is maximal dissipative on* $X_0$,

(iii) $A$ *is a maximal element in the partially ordered set* $\mathfrak{g}(X_0)$,

*where* $\mathfrak{g}(X_0)$ *is the set of all multi-valued operators* $B$ *satisfying the assumptions in the above* (II) *with* $D(B)\subset X_0$ *and the partial order* $B_1\leqq B_2$ *is defined by* $B_1\subset B_2$ (*i.e.,* $B_2$ *is an extension of* $B_1$). (See Theorem 3.4.)

## §2. Some properties of contraction semi-groups.

By a multi-valued operator $A$ in $X$ we mean that $A$ assigns to each $x\in D(A)$ a subset $Ax\neq\emptyset$ of $X$, where $D(A)=\{x\in X;\ Ax\neq\emptyset\}$. And $D(A)$ is called the domain of $A$, and the range of $A$ is defined by $R(A)=\bigcup_{x\in D(A)}Ax$. We define $|||Ax|||=\inf\{\|x'\|\,;\ x'\in Ax\}$ for $x\in D(A)$ and $A^0x=\{x'\in Ax;\ \|x'\|=|||Ax|||\}$. $A^0$ is called the *canonical restriction* of $A$. A multi-valued operator $A$ in $X$ is said to be *demiclosed* if the following condition is satisfied; if $x_n\in D(A)$, $n=1, 2, \cdots$, $\lim_{n\to\infty}x_n=x$ and if there are $x_n'\in Ax_n$ such that $w\text{-}\lim_{n\to\infty}x_n'=x'$, then $x\in D(A)$ and $x'\in Ax$. We say $A$ is *almost demiclosed* if the above condition is satisfied except the assertion $x'\in Ax$.

Let $(x, x^*)$ denote the value of $x^*\in X^*$ at $x\in X$. A multi-valued operator $A$ in $X$ is said to be *dissipative* if for each $x, y\in D(A)$ and $x'\in Ax$, $y'\in Ay$ there exists a $\zeta^*\in F(x-y)$ such that

$$\text{(2.1)} \qquad \operatorname{Re}(x'-y', \zeta^*)\leqq 0$$

where $\operatorname{Re}(x, x^*)$ means the real part of $(x, x^*)$ and $F(x)=\{x^*\in X^*;\ (x, x^*)=\|x\|^2=\|x^*\|^2\}$ for $x\in X$. In general $F(\cdot):X\to X^*$ is multi-valued and it is called the *duality map*. It is well known that the duality map $F(\cdot)$ is single-valued and uniformly continuous in every bounded set of $X$ if $X^*$ is uniformly convex. Thus, in this case, (2.1) becomes

$$\text{(2.2)} \qquad \operatorname{Re}(x'-y', F(x-y))\leqq 0\,.$$

Let $A_i$, $i=1,2$ be multi-valued operators in $X$. $A_2$ is an extension of $A_1$, and $A_1$ is a restriction of $A_2$, in symbol $A_2 \supset A_1$, $A_1 \subset A_2$, if $D(A_1) \subset D(A_2)$ and $A_1 x \subset A_2 x$ for $x \in D(A_1)$. If $A$ is dissipative and $S$ is a subset of $X$, we say that $A$ is *maximal dissipative* on $S$ if $D(A) \subset S$ and $A$ has not any proper dissipative extension $\tilde{A}$ such that $D(\tilde{A}) \subset S$. If $A$ is dissipative and if $R(I-\lambda A)=X$ for $\lambda>0$, then $A$ is said to be *m-dissipative*.

We define $\langle\ ,\ \rangle_s : X \times X \to (-\infty, \infty)$ by

$$\langle x, y\rangle_s = \sup\{\operatorname{Re}(x, y^*)\,;\, y^* \in F(y)\}\,.$$

Clearly $|\langle x, y\rangle_s| \leqq \|x\|\,\|y\|$, and it is shown that the function $\langle\ ,\ \rangle_s$ is upper semicontinuous (see, for example, [**2**, Lemma 2.16]).

LEMMA 2.1. *Assume that $X$ is reflexive, and let $\{T(t)\,;\, t\geqq 0\}$ be a contraction semi-group on $X_0$. If $B$ is a dissipative operator such that $A_0 \subset B$ and $D(B) \subset \overline{D(A_0)}$, then we have*

(i) $\|T(t)x-x_0\|^2-\|x-x_0\|^2 \leqq 2\int_0^t \langle y_0, T(\tau)x-x_0\rangle_s d\tau$ *for every* $t\geqq 0$, $x\in\overline{D(A_0)}$, $x_0 \in D(B)$ *and* $y_0 \in Bx_0$,

(ii) $D(B)\subset \hat{D}$ *and* $\|T(t)x_0-x_0\| \leqq t|\!|\!| Bx_0 |\!|\!|$ *for every* $t\geqq 0$ *and* $x_0 \in D(B)$.

PROOF. (i) Let $x\in D(A_0)$, $x_0\in D(B)$ and $y_0 \in Bx_0$. Since $T(t)x$ is Lipschitz continuous in $t\geqq 0$, the reflexivity of $X$ implies that $T(t)x$ is strongly differentiable at a.e. $t\geqq 0$ and

$$(d/dt)T(t)x = A_0 T(t)x \in BT(t)x \qquad \text{for a.e. } t\geqq 0\,. \tag{2.3}$$

For a.e. $t\geqq 0$ we have that

$$(d/dt)\|T(t)x-x_0\|^2 = 2\operatorname{Re}((d/dt)T(t)x, \zeta_t^*) \qquad \text{for all } \zeta_t^* \in F(T(t)x-x_0)\,,$$

and that $\operatorname{Re}((d/dt)T(t)x-y_0, \eta_t^*) \leqq 0$ for some $\eta_t^* \in F(T(t)x-x_0)$ by (2.3) and the dissipativity of $B$. Therefore

$$(d/dt)\|T(t)x-x_0\|^2 \leqq 2\operatorname{Re}(y_0, \eta_t^*) \leqq 2\langle y_0, T(t)x-x_0\rangle_s \qquad \text{for a.e. } t\geqq 0\,.$$

Integrating this inequality on $[0, t]$ we obtain

$$\|T(t)x-x_0\|^2-\|x-x_0\|^2 \leqq 2\int_0^t \langle y_0, T(\tau)x-x_0\rangle_s d\tau\,.$$

Now let $x\in\overline{D(A_0)}$ and choose a sequence $\{x_n\}$ such that $x_n \in D(A_0)$ and $\lim_{n\to\infty} x_n = x$. By the inequality above

$$\|T(t)x_n-x_0\|^2-\|x_n-x_0\|^2 \leqq 2\int_0^t \langle y_0, T(\tau)x_n-x_0\rangle_s d\tau$$

for every $n$, $x_0 \in D(B)$ and $y_0 \in Bx_0$. Taking the limit superior as $n\to\infty$ we see from the Lebesgue convergence theorem and the upper semicontinuity of $\langle\ ,\ \rangle_s$ that

$$\|T(t)x-x_0\|^2-\|x-x_0\|^2 \leqq 2\int_0^t \langle y_0, T(\tau)x-x_0\rangle_s d\tau$$

for every $t \geqq 0$, $x_0 \in D(B)$ and $y_0 \in Bx_0$.

(ii) Since $D(B) \subset \overline{D(A_0)}$, by using (i) with $x=x_0$ we have

$$\|T(t)x_0-x_0\|^2 \leqq 2\|y_0\| \int_0^t \|T(\tau)x_0-x_0\| d\tau$$

for every $t \geqq 0$, $x_0 \in D(B)$ and $y_0 \in Bx_0$. And this inequality implies that $\|T(t)x_0-x_0\| \leqq t|||Bx_0|||$ for every $t \geqq 0$ and $x_0 \in D(B)$. Q.E.D.

LEMMA 2.2. *Assume that $X$ is reflexive, and let $A$ be a $(g)$-operator of a contraction semi-group $\{T(t); t \geqq 0\}$ on $X_0$. Then the following* (i′)—(iii′) *hold good:*

(i′) *If $x \in X_0$ and $x' = w\text{-}\lim_{n\to\infty} t_n^{-1}(T(t_n)x-x)$ for some positive sequence $\{t_n\}$ with $t_n \to 0$ as $n \to \infty$, then $x \in D(A^0)$ and $x' \in A^0 x$.*

(ii′) *$D(A^0)=D(A)=\hat{D}$ and $\lim_{t\to 0+} t^{-1}\|T(t)x-x\| = |||Ax|||$ for $x \in \hat{D}$.*

(iii′) *$A$ is maximal dissipative on $\overline{D(A)}$ and almost demiclosed.*

PROOF. Note that $A_0 \subset A$ and $D(A) \subset \hat{D} \subset \overline{D(A_0)} = \overline{D(A)}$.

(i′) We first remark that the existence of $w\text{-}\lim_{n\to\infty} t_n^{-1}(T(t_n)x-x)$ implies that $\|T(t)x-x\| = O(t)$ as $t \to 0+$, i.e., $x \in \hat{D}$ (see [**3**, Lemma 1.1]). Let $x_0 \in D(A)$ and $y_0 \in Ax_0$. In view of Lemma 2.1 (i)

$$\|T(t)x-x_0\|^2-\|x-x_0\|^2 \leqq 2\int_0^t \langle y_0, T(\tau)x-x_0\rangle_s d\tau \qquad \text{for} \quad t \geqq 0 .$$

Noting $\|T(t)x-x_0\|^2-\|x-x_0\|^2 \geqq 2\,\mathrm{Re}\,(T(t)x-x, \zeta^*)$ for any $\zeta^* \in F(x-x_0)$, we have

$$\mathrm{Re}\,(T(t)x-x, \zeta^*) \leqq \int_0^t \langle y_0, T(\tau)x-x_0\rangle_s d\tau \qquad \text{for} \quad t \geqq 0 . \tag{2.4}$$

Since $\langle\ ,\ \rangle_s$ is upper semicontinuous, for any $\varepsilon > 0$ there is a $\delta > 0$ such that $\langle y_0, T(\tau)x-x_0\rangle_s < \langle y_0, x-x_0\rangle_s + \varepsilon$ for $0 \leqq \tau < \delta$. We see from (2.4) that if $0 < t < \delta$ then

$$\mathrm{Re}\,(t^{-1}(T(t)x-x), \zeta^*) \leqq \langle y_0, x-x_0\rangle_s + \varepsilon .$$

This implies that

$$\mathrm{Re}\,(x', \zeta^*) \leqq \langle y_0, x-x_0\rangle_s \qquad \text{for any} \quad \zeta^* \in F(x-x_0) .$$

We note here that there is an $\eta^* \in F(x-x_0)$ such that $\langle y_0, x-x_0\rangle_s = \mathrm{Re}\,(y_0, \eta^*)$ because $F(x-x_0)$ is compact in the weak* topology of $X^*$. Consequently

$$\mathrm{Re}\,(x'-y_0, \eta^*) \leqq 0 \qquad \text{for some} \quad \eta^* \in F(x-x_0) ;$$

so it follows from the maximal dissipativity on $\hat{D}$ of $A$ that

(2.5) $$x \in D(A) \quad \text{and} \quad x' \in Ax .$$

Then by Lemma 2.1 (ii)

$$\lim_{t\to 0+} t^{-1}\|T(t)x-x\| \leqq |||Ax||| .$$

(Note that the existence of the above limit follows from [3, Lemma 1.1].) But $|||Ax||| \leqq \|x'\| \leqq \lim_{n\to\infty} t_n^{-1}\|T(t_n)x-x\| = \lim_{t\to 0+} t^{-1}\|T(t)x-x\|$. Therefore

(2.6) $$|||Ax||| = \|x'\| = \lim_{t\to 0+} t^{-1}\|T(t)x-x\| .$$

Combining this with (2.5) we have that $x \in D(A^0)$ and $x' \in A^0 x$.

(ii′) Let $x \in \hat{D}$. Since $\|t^{-1}(T(t)x-x)\| = O(1)$ as $t\to 0+$, the reflexivity of $X$ implies that there is a positive sequence $\{t_n\}$ with $t_n \to 0$ as $n\to\infty$ and an $x' \in X$ such that

$$x' = w\text{-}\lim_{n\to\infty} t_n^{-1}(T(t_n)x-x) .$$

By virtue of (i′) we obtain that $x \in D(A^0)$ and $|||Ax||| = \lim_{t\to 0+} t^{-1}\|T(t)x-x\|$. Thus $\hat{D} \subset D(A^0)$.

(iii′) To show that $A$ is maximal dissipative on $\overline{D(A)}$, let $\tilde{A}$ be a dissipative operator such that $A \subset \tilde{A}$ and $D(\tilde{A}) \subset \overline{D(A)}$. Then by virtue of Lemma 2.1 (ii) we obtain that $D(\tilde{A}) \subset \hat{D}$. This implies $\tilde{A} = A$, since $A$ is maximal dissipative on $\hat{D}$. Thus $A$ is maximal dissipative on $\overline{D(A)}$. Next, to prove the almost demiclosedness of $A$, let $\{x_n\}$ and $\{x_n'\}$ be sequences such that $x_n \in D(A)$, $x_n' \in Ax_n$, $\lim_{n\to\infty} x_n = x$ and $w\text{-}\lim_{n\to\infty} x_n' = x'$. By Lemma 2.1 (ii)

$$\|T(t)x_n - x_n\| \leqq t|||Ax_n||| \leqq t\|x_n'\| \leqq Mt \qquad \text{for} \quad t \geqq 0 ;$$

and hence $\|T(t)x-x\| \leqq Mt$ for $t \geqq 0$, where $M>0$ is a constant such that $\|x_n'\| \leqq M$ for all $n \geqq 1$. This shows that $x \in \hat{D} = D(A)$ and hence $A$ is almost demiclosed. Q. E. D.

DEFINITION 2.1. Let $A$ be a multi-valued operator in $X$. We say that $A$ has the property $(\mathfrak{g})$, if the following conditions (I) and (II) are fulfilled:

(I) $A$ is maximal dissipative on $D(A)$.

(II) For each $x \in D(A)$ and $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon, x) > 0$ with $\delta \leqq \varepsilon$ and a triple $(f_\delta, g_\delta, h_\delta)$ of functions such that

($\text{II}_1$) $f_\delta : [0, \delta] \to X$ is strongly absolutely continuous on $[0, \delta]$ and strongly differentiable at a. e. $t \in [0, \delta]$; moreover $f_\delta(0) = x$, $f_\delta(\delta) \in D(A)$ and $|||Af_\delta(\delta)||| \leqq e^{\varepsilon\delta}|||Ax|||$,

($\text{II}_2$) $g_\delta : [0, \delta) \to D(A)$ satisfies

$$\|g_\delta(t) - f_\delta(t)\| \leqq \varepsilon e^{\varepsilon\delta}(|||Ax|||+\varepsilon) \qquad \text{for} \quad t \in [0, \delta) ,$$

($\text{II}_3$) $h_\delta : [0, \delta) \to X$ satisfies $h_\delta(t) \in Ag_\delta(t)$ for $t \in [0, \delta)$, $\|h_\delta(t)\| \leqq e^{\varepsilon\delta}|||Ax|||$

for $t\in[0, \delta)$ and

$$\|(d/dt)f_{\delta}(t)-h_{\delta}(t)\|\leqq\varepsilon \quad \text{for a.e. } t\in[0, \delta).$$

REMARK. If $X$ is reflexive, then the strong differentiability of $f_{\delta}$ (in $(\mathrm{II}_1)$) is superfluous since every strongly absolutely continuous function is strongly differentiable almost everywhere.

THEOREM 2.3. *Let $A$ be a (g)-operator of a contraction semi-group $\{T(t); t\geqq 0\}$ on $X_0$. If $X$ is reflexive, then $A$ has the property* ($\mathfrak{g}$), *and $A$ is maximal dissipative on $\overline{D(A)}$ and almost demiclosed, and moreover $D(A^0)=D(A)=\hat{D}$ and $A_0\subset A^0$. Furthermore $A$ is demiclosed if $X^*$ is uniformly convex.*

PROOF. At first let $X$ be reflexive. By virtue of Lemma 2.2, it suffices to show that $A$ has the property ($\mathfrak{g}$). Obviously $A$ is maximal dissipative on $D(A)$.

Let $x\in D(A)$ $(=\hat{D})$. Then $T(t)x$ is Lipschitz continuous in $t\geqq 0$, $T(t)x\in\hat{D}=D(A)$ for all $t\geqq 0$ and there exists a null set $N\subset[0, \infty)$ such that

$$(d/dt)T(t)x=A_0T(t)x\in AT(t)x \quad \text{for all } t\in[0, \infty)\setminus N. \tag{2.7}$$

Moreover

$$|||AT(t)x|||\leqq|||Ax||| \quad \text{for all } t\geqq 0. \tag{2.8}$$

Indeed, by Lemma 2.2 (ii′),

$$|||AT(t)x|||=\lim_{h\to 0+}h^{-1}\|T(h)T(t)x-T(t)x\| \leqq\lim_{h\to 0+}h^{-1}\|T(h)x-x\|=|||Ax|||.$$

Let $\varepsilon>0$ and set $\delta=\varepsilon$. And put $f_{\delta}(t)=g_{\delta}(t)=T(t)x$ for $0\leqq t\leqq\delta$, and define $h_{\delta}$ by

$$h_{\delta}(t)=A_0T(t)x \quad \text{for } t\in[0, \delta)\setminus N$$
$$=\text{an element in } A^0T(t)x \text{ for } t\in[0, \delta)\cap N.$$

It is now easy to see that $f_{\delta}$, $g_{\delta}$ and $h_{\delta}$ satisfy the conditions $(\mathrm{II}_1)$—$(\mathrm{II}_3)$ in Definition 2.1. Therefore $A$ has the property ($\mathfrak{g}$).

Next let $X^*$ be uniformly convex. Then $A$ is demiclosed, since $A$ is maximal dissipative on $D(A)$ and $A$ is almost demiclosed. (See [**4**, Lemma 3.7].) Q.E.D.

We give some examples of operators having the property ($\mathfrak{g}$).

EXAMPLE 2.1. Let $A$ be maximal dissipative on $D(A)$ and satisfy the following condition (a):

(a) For each $x\in D(A)$ and $\varepsilon>0$ there is a $\delta=\delta(\varepsilon, x)>0$ with $\delta\leqq\varepsilon$ and there are $x_{\delta}\in D(A)$ and $x_{\delta}'\in Ax_{\delta}$ satisfying the following

$(a_1)$ $\|x_{\delta}-\delta x_{\delta}'-x\|\leqq\delta\varepsilon$,

$(a_2)$ $\|x_{\delta}'\|\leqq e^{\varepsilon\delta}|||Ax|||$.

Then $A$ has the property $(\mathfrak{g})$.

PROOF. Let $x\in D(A)$ and $\varepsilon>0$. We now define a triple $(f_\delta, g_\delta, h_\delta)$, where $f_\delta:[0,\delta]\to X$, $g_\delta:[0,\delta)\to D(A)$, $h_\delta:[0,\delta)\to X$, by

$$f_\delta(t)=\delta^{-1}[(\delta-t)x+tx_\delta] \qquad \text{for} \quad t\in[0,\delta],$$

$$g_\delta(t)=x_\delta \text{ and } h_\delta(t)=x'_\delta \qquad \text{for} \quad t\in[0,\delta).$$

It follows from $(a_2)$ that

$$|||Af_\delta(\delta)|||=|||Ax_\delta|||\leqq\|x'_\delta\|\leqq e^{\varepsilon\delta}|||Ax|||.$$

And then $(II_1)$ is satisfied. By $(a_1)$ and $(a_2)$ we have

$$\begin{aligned}\|g_\delta(t)-f_\delta(t)\|&=\delta^{-1}(\delta-t)\|x_\delta-x\|\leqq\|x_\delta-x\|\leqq\delta\varepsilon+\delta\|x'_\delta\|\\&\leqq\varepsilon(\varepsilon+e^{\varepsilon\delta}|||Ax|||)\leqq\varepsilon e^{\varepsilon\delta}(|||Ax|||+\varepsilon) \qquad \text{for} \quad t\in[0,\delta).\end{aligned}$$

So $(II_2)$ is satisfied. Since $(d/dt)f_\delta(t)=\delta^{-1}(x_\delta-x)$ for $t\in(0,\delta)$, we see from $(a_1)$ that

$$\|(d/dt)f_\delta(t)-h_\delta(t)\|=\|\delta^{-1}(x_\delta-x)-x'_\delta\|\leqq\varepsilon \qquad \text{for} \quad t\in(0,\delta).$$

Therefore $(II_3)$ is also satisfied. Q. E. D.

EXAMPLE 2.2. (1) If $A$ is maximal dissipative on $D(A)$ and if for each $x\in D(A)$ there is a sequence $\{\lambda_n\}$ of positive numbers such that

$$\lambda_n\to0 \text{ as } n\to\infty \text{ and } R(I-\lambda_nA)\supset\{x\} \text{ for each } n,$$

then $A$ has the property $(\mathfrak{g})$.

(2) Every $m$-dissipative operator has the property $(\mathfrak{g})$.

PROOF. To prove (1) it suffices to show that the condition (a) in Example 2.1 is satisfied. Let $x\in D(A)$. It follows from our assumption that for each $n$ there are $x_{\lambda_n}\in D(A)$ and $x'_{\lambda_n}\in Ax_{\lambda_n}$ such that

$$x_{\lambda_n}-\lambda_nx'_{\lambda_n}=x. \tag{2.9}$$

And we have

$$\begin{aligned}\|x'_{\lambda_n}\|&=\lambda_n^{-1}\|x_{\lambda_n}-x\|\\&=\lambda_n^{-1}\|(I-\lambda_nA)^{-1}x-x\|\leqq|||Ax||| \qquad \text{for all} \quad n.\end{aligned} \tag{2.10}$$

For any $\varepsilon>0$, choose a $\lambda_n$ with $\lambda_n\leqq\varepsilon$ and put $\delta=\lambda_n$. Now, $(a_1)$ and $(a_2)$ (in Example 2.1) follow from (2.9) and (2.10) respectively. Since every $m$-dissipative operator $A$ is maximal dissipative on $D(A)$, (2) follows from (1). Q. E. D.

EXAMPLE 2.3. Let $A$ be maximal dissipative on $D(A)$ and satisfy the following condition:

(b) For each $x\in D(A)$ and $\varepsilon>0$ there is a $\delta=\delta(\varepsilon,x)>0$ with $\delta\leqq\varepsilon$ and there are $x_\delta\in D(A)$ and $x'_{(\delta)}\in A^0x$ such that (b$_1$) $\|x+\delta x'_{(\delta)}-x_\delta\| \leqq \delta\varepsilon$,

(b$_2$) $|||Ax_\delta||| \leqq e^{\varepsilon\delta}|||Ax|||$.

Then $A$ has the property ($\mathfrak{g}$).

PROOF. Let $x \in D(A)$ and $\varepsilon>0$. We define a triple $(f_\delta, g_\delta, h_\delta)$ of functions by

$$f_\delta(t)=\delta^{-1}[(\delta-t)x+tx_\delta] \quad \text{for} \quad t\in[0, \delta],$$

$$g_\delta(t)=x \quad \text{for} \quad t\in[0, \delta),$$

$$h_\delta(t)=x'_{(\delta)} \quad \text{for} \quad t\in[0, \delta).$$

Then it is obvious that $f_\delta$, $g_\delta$ and $h_\delta$ satisfy the conditions (II$_1$)—(II$_3$) in Definition 2.1. Q.E.D.

REMARKS. Let both $X$ and $X^*$ be uniformly convex. It is known that $A^0$ is single-valued and $D(A^0)=D(A)$ if $A$ is maximal dissipative on $D(A)$ (see [**4**, Lemma 3.10]). Then we remark the following:

1°) $A$ given in Example 2.3 is nothing else but an operator having the property $(G)_0^0$ in the sense of Martin [**5**, Definition 2.2].

2°) Let $A$ be maximal dissipative on $D(A)$. $A$ satisfies the condition (b) in Example 2.3 if and only if $A$ satisfies the condition (a) with $x'_\delta=A^0x_\delta$ in Example 2.1.

## § 3. Generation of contraction semi-groups.

Throughout this section it is assumed that $X^*$ is *uniformly convex*. We start from the following

LEMMA 3.1. *Suppose that $A$ has the property* ($\mathfrak{g}$) *and $A$ is almost demiclosed. Let $x\in D(A)$, $T>0$ and $\varepsilon>0$ where $\varepsilon<T$. Then there exists a triple $(u, g, h)$ where $u:[0, T]\to X$, $g:[0, T)\to D(A)$, and $h:[0, T)\to X$, satisfying the properties*

( i ) $u(0)=x$,

( ii ) $\|u(t)-u(s)\| \leqq |t-s|\{e^{\varepsilon T}|||Ax|||+\varepsilon\}$ *for* $t, s\in[0, T]$,

(iii) $u(T)\in D(A)$ *and* $|||Au(T)||| \leqq e^{\varepsilon T}|||Ax|||$,

(iv) $\|u(t)-g(t)\| \leqq \varepsilon e^{\varepsilon T}(|||Ax|||+\varepsilon)$ *for* $t\in[0, T)$,

( v ) $h(t)\in Ag(t)$ *for* $t\in[0, T)$, $\|h(t)\| \leqq e^{\varepsilon T}|||Ax|||$ *for* $t\in[0, T)$ *and* $\|(d/dt)u(t)-h(t)\| \leqq \varepsilon$ *for a.e.* $t\in[0, T)$.

PROOF. We use Martin's technique [**5**]. Let $\mathcal{P}$ denote the family of all triples $(v, p, q)$ where $v:[0, c]\to X$, $p:[0, c)\to D(A)$, and $q:[0, c)\to X$, $0<c\leqq T$, satisfy each of the properties (i)—(v) of the lemma with $T$ replaced by $c$, $u$ replaced by $v$, $g$ replaced by $p$, and $h$ replaced by $q$. If $(v_i, p_i, q_i)\in\mathcal{P}$ $(i=1, 2)$ and each $v_i$ is defined on $[0, c_i]$, we write

$$(v_1, p_1, q_1)\leqq(v_2, p_2, q_2)$$

whenever $c_1 \leqq c_2$, $v_1(t)=v_2(t)$ for $t \in [0, c_1]$, and $p_1(t)=p_2(t)$ and $q_1(t)=q_2(t)$ for $t \in [0, c_1)$. Since $A$ has the property $(\mathfrak{g})$, there is a triple $(f_\delta, g_\delta, h_\delta)$ satisfying $(\mathrm{II}_1)$—$(\mathrm{II}_3)$ in Definition 2.1. It is clear that $(f_\delta, g_\delta, h_\delta) \in \mathcal{P}$ and $\mathcal{P}$ is a partially ordered set by the relation "$\leqq$". Let $Q=\{(v_\alpha, p_\alpha, q_\alpha);\ \alpha \in \Lambda\}$, $\Lambda$ being some indexing set, be a totally ordered subset of $\mathcal{P}$ where $v_\alpha$ is defined on $[0, c_\alpha]$ for each $\alpha$. Let $\bar{c}=\sup\{c_\alpha;\ \alpha \in \Lambda\}$. If $\bar{c}=c_\alpha$ for some $\alpha \in \Lambda$, then $(v_\alpha, p_\alpha, q_\alpha)$ is an upper bound for $Q$. Assume now that $c_\alpha < \bar{c}$ for all $\alpha \in \Lambda$, and for each $t \in [0, \bar{c})$ define $v(t)=v_\alpha(t)$, $p(t)=p_\alpha(t)$ and $q(t)=q_\alpha(t)$ whenever $t<c_\alpha$. Then $v, p$ and $q$ are well defined on $[0, \bar{c})$, and it is easy to see that $v, p$ and $q$ satisfy the properties (i), (ii), (iv) and (v) on $[0, \bar{c})$. By $\|v(t)-v(s)\| \leqq |t-s|\{e^{\varepsilon\bar{c}}|||Ax|||+\varepsilon\}$ for $t, s \in [0, \bar{c})$ the limit $\lim_{t\to\bar{c}-0} v(t)$ exists, and we define $v(\bar{c})$ by $v(\bar{c})=\lim_{t\to\bar{c}-0} v(t)$. Then the triple $(v, p, q)$ with $v$ extended to $[0, \bar{c}]$ has the properties (i), (ii), (iv) and (v) (with $T$ replaced by $\bar{c}$). To show that $v(\bar{c}) \in D(A)$ and $|||Av(\bar{c})||| \leqq e^{\varepsilon\bar{c}}|||Ax|||$ (i.e., $v$ satisfies the property (iii)), choose a sequence $\{(v_n, p_n, q_n)\}$ in $Q$ such that $v_n$ is defined on $[0, c_n]$ and $\lim_{n\to\infty} c_n=\bar{c}$. Then $v(\bar{c})=\lim_{n\to\infty} v(c_n)=\lim_{n\to\infty} v_n(c_n)$ and $|||Av_n(c_n)||| \leqq e^{\varepsilon\bar{c}}|||Ax|||$. It follows from the demiclosedness of $A$ that $v(\bar{c}) \in D(A)$ and $|||Av(\bar{c})||| \leqq e^{\varepsilon\bar{c}}|||Ax|||$ (see [**4**, Lemma 3.8]). Thus $(v, p, q)$ is an element of $\mathcal{P}$, and it is clear that $(v, p, q)$ is an upper bound for $Q$. Therefore, by Zorn's lemma, $\mathcal{P}$ has a maximal element $(u, g, h)$.

We now show that $u$ is defined on $[0, T]$. Suppose, for contradiction, that $u$ is defined on $[0, c]$ and $c<T$. Take an $\eta>0$ such that $c+\eta \leqq T$. Since $u(c) \in D(A)$ and $A$ has the property $(\mathfrak{g})$, there exists a $\delta>0$ with $\delta \leqq \min(\varepsilon, \eta)$ and a triple $(u_\delta, g_\delta, h_\delta)$ of functions $u_\delta: [0, \delta] \to X$, $g_\delta: [0, \delta) \to D(A)$, and $h_\delta: [0, \delta) \to X$, satisfying the following (3.1)—(3.3);

(3.1) $u_\delta$ is strongly absolutely continuous on $[0, \delta]$, $u_\delta(0)=u(c)$, $u_\delta(\delta) \in D(A)$ and $|||Au_\delta(\delta)||| \leqq e^{\varepsilon\delta}|||Au(c)|||$,

(3.2) $\|g_\delta(t)-u_\delta(t)\| \leqq \varepsilon e^{\varepsilon\delta}(|||Au(c)|||+\varepsilon)$ for $t \in [0, \delta)$,

(3.3) $h_\delta(t) \in Ag_\delta(t)$ for $t \in [0, \delta)$, $\|h_\delta(t)\| \leqq e^{\varepsilon\delta}|||Au(c)|||$ for $t \in [0, \delta)$ and $\|(d/dt)u_\delta(t)-h_\delta(t)\| \leqq \varepsilon$ for a.e. $t \in [0, \delta)$.

Put $d=c+\delta$ $(\leqq T)$, and define a triple $(u_0, g_0, h_0)$ by

$$\begin{aligned} u_0(t) &= u(t) && \text{for } 0 \leqq t \leqq c \\ &= u_\delta(t-c) && \text{for } c \leqq t \leqq d, \\ g_0(t) &= g(t) && \text{for } 0 \leqq t < c \\ &= g_\delta(t-c) && \text{for } c \leqq t < d, \end{aligned}$$

$$h_0(t)=h(t) \qquad \text{for} \quad 0\leqq t<c$$
$$\quad =h_\delta(t-c) \qquad \text{for} \quad c\leqq t<d\,.$$

We see that $(u_0, g_0, h_0)$ is an element of $\mathcal{P}$. Indeed it is clear that $u_0$ satisfies the property (i). Since $u_0(d)=u_\delta(\delta)\in D(A)$ and

$$|||Au_0(d)|||=|||Au_\delta(\delta)|||\leqq e^{\varepsilon\delta}|||Au(c)|||\leqq e^{\varepsilon d}|||Ax|||$$

by (3.1), $u_0$ satisfies the property (iii). Also, if $t\in[c, d)$, then

$$\|u_0(t)-g_0(t)\|=\|u_\delta(t-c)-g_\delta(t-c)\|$$
$$\leqq \varepsilon e^{\varepsilon\delta}(|||Au(c)|||+\varepsilon)\leqq \varepsilon e^{\varepsilon\delta}(e^{\varepsilon c}|||Ax|||+\varepsilon)\leqq \varepsilon e^{\varepsilon d}(|||Ax|||+\varepsilon)\,.$$

Hence the property (iv) is satisfied. It follows from (3.3) that the property (v) is satisfied. Since

$$\|u_\delta(s_2)-u_\delta(s_1)\|\leqq\int_{s_1}^{s_2}\|(d/ds)u_\delta(s)\|ds$$
$$\leqq(s_2-s_1)(e^{\varepsilon\delta}|||Au(c)|||+\varepsilon) \qquad \text{for} \quad 0\leqq s_1\leqq s_2\leqq\delta$$

(by (3.3)), we have

$$\|u_0(t)-u_0(s)\|=\|u_\delta(t-c)-u_\delta(s-c)\|$$
$$\leqq|t-s|(e^{\varepsilon\delta}|||Au(c)|||+\varepsilon)\leqq|t-s|(e^{\varepsilon d}|||Ax|||+\varepsilon)$$

for $s, t\in[c, d]$. The fact that $u_0$ satisfies the property (ii) now follows easily, and we have that $(u_0, g_0, h_0)\in\mathcal{P}$. But obviously $(u, g, h)\leqq(u_0, g_0, h_0)$ and $(u, g, h)\neq(u_0, g_0, h_0)$, which contradicts to the fact that $(u, g, h)$ is a maximal element in $\mathcal{P}$. Thus, $u$ must be defined on $[0, T]$; and $(u, g, h)$ is the desired triple. Q. E. D.

THEOREM 3.2. *Suppose that $A$ has the property* ($\mathfrak{g}$) *and $A$ is almost demiclosed. Let $T>0$ be arbitrarily fixed. Then for each $x\in D(A)$ there is a unique function $u(t; x):[0, T]\to X$ satisfying*

(i) $\|u(t; x)-u(s; x)\|\leqq|||Ax|||\cdot|t-s|$ *for* $t, s\in[0, T]$,

(ii) $u(t; x)\in D(A)$ *for all* $t\in[0, T]$ *and*

$$\text{(3.4)} \qquad \begin{cases} u(0; x)=x \\ (d/dt)u(t; x)\in Au(t; x) \qquad \textit{for a.e.} \quad t\in[0, T]\,. \end{cases}$$

PROOF. Let $x\in D(A)$, and let $\{\varepsilon_n\}$ be a sequence such that $0<\varepsilon_n\leqq 1$ for each $n$ and $\varepsilon_n\to 0$ as $n\to\infty$. By virtue of Lemma 3.1, for each $n$ there exists a triple $(u_n(\cdot\,; x), g_n(\cdot), h_n(\cdot))$ where $u_n(\cdot\,; x):[0, T]\to X$, $g_n(\cdot):[0, T)\to D(A)$, and $h_n(\cdot):[0, T)\to X$ satisfy the following (3.5)—(3.10);

$$\text{(3.5)} \qquad u_n(0; x)=x\,,$$

$$\text{(3.6)} \qquad \|u_n(t; x)-u_n(s; x)\|\leqq|t-s|\{e^{\varepsilon_n T}|||Ax|||+\varepsilon_n\} \qquad \text{for} \quad t, s\in[0, T]\,,$$

(3.7) $u_n(T; x) \in D(A)$ and $|||Au_n(T; x)||| \leqq e^{\varepsilon_n T}|||Ax|||$,

(3.8) $\|u_n(t; x)-g_n(t)\| \leqq \varepsilon_n e^{\varepsilon_n T}(|||Ax|||+\varepsilon_n)$ for $t \in [0, T)$,

(3.9) $h_n(t) \in Ag_n(t)$ and $\|h_n(t)\| \leqq e^{\varepsilon_n T}|||Ax|||$ for $t \in [0, T)$,

(3.10) $\|(d/dt)u_n(t; x)-h_n(t)\| \leqq \varepsilon_n$ for a. e. $t \in [0, T)$.

By (3.9), (3.10) and the dissipativity of $A$, for a. e. $t \in [0, T)$

$$\begin{aligned}
&(d/dt)\|u_n(t; x)-u_m(t; x)\|^2 \\
&\quad = 2\,\mathrm{Re}\,((d/dt)u_n(t; x)-(d/dt)u_m(t; x),\ F(u_n(t; x)-u_m(t; x))) \\
&\quad = 2\,\mathrm{Re}\,(\{(d/dt)u_n(t; x)-h_n(t)\}-\{(d/dt)u_m(t; x)-h_m(t)\},\ F(u_n(t; x)-u_m(t; x))) \\
&\qquad +2\,\mathrm{Re}\,(h_n(t)-h_m(t),\ F(u_n(t; x)-u_m(t; x))-F(g_n(t)-g_m(t))) \\
&\qquad +2\,\mathrm{Re}\,(h_n(t)-h_m(t),\ F(g_n(t)-g_m(t))) \\
&\quad \leqq 2(\varepsilon_n+\varepsilon_m)\|u_n(t; x)-u_m(t; x)\| \\
&\qquad +4e^T\|F(u_n(t; x)-u_m(t; x))-F(g_n(t)-g_m(t))\|\cdot|||Ax|||.
\end{aligned}$$

(3.5), (3.6) and (3.8) imply that the set

$$\{u_n(t; x)-u_m(t; x),\ g_n(t)-g_m(t);\ n, m=1, 2, \cdots, 0 \leqq t \leqq T\}$$

is bounded. Let $B$ be a bounded set containing the bounded set above. Since $F(\cdot)$ is uniformly continuous on $B$, for any $\varepsilon>0$ there is a $\delta=\delta_\varepsilon>0$ such that if $y, z \in B$ and $\|y-z\|<\delta$ then $\|F(y)-F(z)\|\cdot|||Ax|||<\varepsilon/(8e^T)$. Choose an integer $n_0=n_0(\varepsilon)>0$ such that if $n \geqq n_0$ then $\varepsilon_n e^T(|||Ax|||+1)<\delta/2$ and $M\varepsilon_n<\varepsilon/8$, where $M=\sup\{\|z\|;\ z \in B\}$. Let $n, m \geqq n_0$. Then, by (3.8), $\|(u_n(t; x)-u_m(t; x))-(g_n(t)-g_m(t))\| \leqq (\varepsilon_n+\varepsilon_m)e^T(|||Ax|||+1)<\delta$ for all $t \in [0, T)$; and hence

$$4e^T\|F(u_n(t; x)-u_m(t; x))-F(g_n(t)-g_m(t))\|\cdot|||Ax|||<\varepsilon/2$$

for all $t \in [0, T)$. Moreover

$$2(\varepsilon_n+\varepsilon_m)\|u_n(t; x)-u_m(t; x)\| \leqq 2M(\varepsilon_n+\varepsilon_m)<\varepsilon/2 \quad \text{for all} \quad t \in [0, T].$$

Therefore, if $n, m \geqq n_0$ then

$$(d/dt)\|u_n(t; x)-u_m(t; x)\|^2<\varepsilon \quad \text{for a. e.} \quad t \in [0, T]$$

and then

$$\|u_n(t; x)-u_m(t; x)\|^2 \leqq \varepsilon t \leqq \varepsilon T \quad \text{for all} \quad t \in [0, T].$$

Consequently, $\lim_{n\to\infty} u_n(t; x)$ exists uniformly on $[0, T]$.

Let us now define $u(\cdot\,; x)$ by

(3.11) $$u(t; x)=\lim_{n\to\infty} u_n(t; x) \quad \text{for} \quad t \in [0, T].$$

It follows from (3.5) and (3.6) that $u(0;x)=x$ and

$$\|u(t;x)-u(s;x)\| \leqq |||Ax|||\cdot|t-s| \qquad \text{for} \quad t, s\in[0,T]. \tag{3.12}$$

We want to show

$$u(t;x)\in D(A) \qquad \text{for all} \quad t\in[0,T]. \tag{3.13}$$

In fact, $u(T;x)\in D(A)$ is obtained from (3.7) and almost demiclosedness of $A$. Let $0\leqq t<T$. By (3.8), $\|g_n(t)-u(t;x)\|\leqq\|g_n(t)-u_n(t;x)\|+\|u_n(t;x)-u(t;x)\| \leqq \varepsilon_n e^{\varepsilon_n T}(|||Ax|||+\varepsilon_n)+\|u_n(t;x)-u(t;x)\|\to 0$ as $n\to\infty$. Moreover $|||Ag_n(t)|||\leqq \|h_n(t)\|\leqq e^{\varepsilon_n T}|||Ax|||$. Since $A$ is almost demiclosed, we obtain that $u(t;x)\in D(A)$.

Let $x_0\in D(A)$ and $y_0\in Ax_0$. Then for a.e. $t\in[0,T]$

$$\begin{aligned}
&(d/dt)\|u_n(t;x)-x_0\|^2\\
&\quad=2\,\mathrm{Re}\,((d/dt)u_n(t;x),\ F(u_n(t;x)-x_0))\\
&\quad=2\,\mathrm{Re}\,((d/dt)u_n(t;x)-h_n(t),\ F(u_n(t;x)-x_0))\\
&\qquad+2\,\mathrm{Re}\,(h_n(t),\ F(u_n(t;x)-x_0)-F(g_n(t)-x_0))\\
&\qquad+2\,\mathrm{Re}\,(h_n(t),\ F(g_n(t)-x_0))\\
&\quad\leqq 2\varepsilon_n\|u_n(t;x)-x_0\|+2e^{\varepsilon_n T}|||Ax|||\cdot\|F(u_n(t;x)-x_0)-F(g_n(t)-x_0)\|\\
&\qquad+2\,\mathrm{Re}\,(y_0,\ F(g_n(t)-x_0))
\end{aligned}$$

by (3.9), (3.10) and the dissipativity of $A$. It follows from (3.8) that $\lim_{n\to\infty}\|F(u_n(t;x)-x_0)-F(g_n(t)-x_0)\|=0$ uniformly on $[0,T)$; thus for any $\varepsilon>0$ there is an integer $n_0=n_0(\varepsilon)>0$ such that

$$(d/dt)\|u_n(t;x)-x_0\|^2\leqq 2\,\mathrm{Re}\,(y_0,\ F(g_n(t)-x_0))+\varepsilon$$

for a.e. $t\in[0,T]$ if $n\geqq n_0$. Integrating this inequality on $[s,t]$ we have

$$\begin{aligned}
&\|u_n(t;x)-x_0\|^2-\|u_n(s;x)-x_0\|^2\\
&\quad\leqq 2\int_s^t \mathrm{Re}\,(y_0,\ F(g_n(\tau)-x_0))d\tau+\varepsilon T.
\end{aligned}$$

Letting $n\to\infty$ we have

$$\begin{aligned}
&\|u(t;x)-x_0\|^2-\|u(s;x)-x_0\|^2\\
&\quad\leqq 2\int_s^t \mathrm{Re}\,(y_0,\ F(u(\tau;x)-x_0))d\tau \qquad \text{for} \quad 0\leqq s\leqq t\leqq T.
\end{aligned}$$

Noting that $2\,\mathrm{Re}\,(u(t;x)-u(s;x),\ F(u(s;x)-x_0))\leqq\|u(t;x)-x_0\|^2-\|u(s;x)-x_0\|^2$, we obtain

(3.14) $$\operatorname{Re}(u(t;x)-u(s;x), F(u(s;x)-x_0)) \leqq \int_s^t \operatorname{Re}(y_0, F(u(\tau;x)-x_0))d\tau$$

for any $x_0 \in D(A)$, $y_0 \in Ax_0$ and $0 \leqq s \leqq t \leqq T$. Hence if $u(t;x)$ is strongly differentiable at $t>0$, then

$$\operatorname{Re}((d/dt)u(t;x), F(u(t;x)-x_0)) \leqq \operatorname{Re}(y_0, F(u(t;x)-x_0))$$

for all $x_0 \in D(A)$ and $y_0 \in Ax_0$, and then

$$(d/dt)u(t;x) \in Au(t;x)$$

by (3.13) and the maximal dissipativity on $D(A)$ of $A$. Since $u(t;x)$ is strongly differentiable for a.e. $t \in [0, T]$, we see that

(3.15) $$(d/dt)u(t;x) \in Au(t;x) \qquad \text{for a.e.} \quad t \in [0, T].$$

Finally, the dissipativity of $A$ implies that $u(t;x)$ is a unique function satisfying (i) and (ii). Q.E.D.

THEOREM 3.3. *Suppose that $A$ has the property* ($\mathfrak{g}$) *and $A$ is almost demiclosed. Then there exists a unique contraction semi-group $\{T(t);t \geqq 0\}$ on $\overline{D(A)}$ such that for each $x \in D(A)$*

(i) $\|T(t)x-T(s)x\| \leqq |||Ax||| \cdot |t-s|$ *for* $t, s \geqq 0$,

(ii) $T(t)x \in D(A)$ *for all* $t \geqq 0$ *and*

$$(d/dt)T(t)x \in A^0T(t)x \qquad \textit{for a.e.} \quad t \geqq 0.$$

PROOF. For each $T>0$ and $x \in D(A)$, let $u_T(t;x)$ be the function obtained in Theorem 3.2. It is easy to see that if $t, s \geqq 0$ and $t+s \in [0, T]$ then

(3.16) $$u_T(t+s;x) = u_T(t; u_T(s;x)),$$

and that if $0 < T_1 \leqq T_2$ then

(3.17) $$u_{T_2}(t;x) = u_{T_1}(t;x) \qquad \text{for} \quad t \in [0, T_1].$$

Let us define $T(t)$ $(t \geqq 0)$ on $D(A)$ by

$$T(t)x = u_T(t;x) \qquad \text{for} \quad x \in D(A)$$

whenever $0 \leqq t \leqq T$. We see from (3.17) that $T(t)$, $t \geqq 0$ are well defined. And then it is clear that (i) is satisfied, $T(t)x \in D(A)$ for all $t \geqq 0$ and

(3.18) $$\begin{cases} T(0)x = x \\ (d/dt)T(t)x \in AT(t)x \qquad \text{for a.e.} \quad t \geqq 0. \end{cases}$$

Moreover by (3.16)

(3.19) $$T(t+s) = T(t)T(s) \qquad \text{for} \quad t, s \geqq 0.$$

By (i) and (3.19),

$$\|T(t+h)x-T(t)x\| \leqq h|||AT(t)x||| \quad \text{for } t, h>0;$$

and hence

$$\|(d/dt)T(t)x\| \leqq |||AT(t)x||| \quad \text{for a.e. } t \geqq 0.$$

Combining this with (3.18) we have

$$(d/dt)T(t)x \in A^0T(t)x \quad \text{for a.e. } t \geqq 0.$$

Since

$$(d/ds)\|T(s)x-T(s)y\|^2$$
$$=2\,\mathrm{Re}\,((d/ds)T(s)x-(d/ds)T(s)y,\ F(T(s)x-T(s)y)) \leqq 0$$

for a.e. $s \geqq 0$, we have

$$\|T(t)x-T(t)y\|^2-\|x-y\|^2=\int_0^t (d/ds)\|T(s)x-T(s)y\|^2 ds \leqq 0,$$

i.e.,

$$\|T(t)x-T(t)y\| \leqq \|x-y\| \quad \text{for } t \geqq 0. \tag{3.20}$$

Therefore $\{T(t); t \geqq 0\}$ is a unique contraction semi-group on $D(A)$ having the properties (i) and (ii). And this semi-group can be extended to the desired contraction semi-group $\{T(t); t \geqq 0\}$ on $\overline{D(A)}$. Q.E.D.

REMARK. The above contraction semi-group $\{T(t); t \geqq 0\}$ on $\overline{D(A)}$ satisfies the following inequality:

$$\limsup_{t\to 0+} \mathrm{Re}\,(t^{-1}(T(t)x-x),\ F(x-x_0)) \leqq \mathrm{Re}\,(y_0,\ F(x-x_0)) \tag{3.21}$$

for every $x_0 \in D(A)$, $y_0 \in Ax_0$ and $x \in \overline{D(A)}$.

In fact, we obtain from (3.14) that

$$\mathrm{Re}\,(T(t)x-x,\ F(x-x_0)) \leqq \int_0^t \mathrm{Re}\,(y_0,\ F(T(\tau)x-x_0))d\tau$$

for every $x_0 \in D(A)$, $y_0 \in Ax_0$, $t \geqq 0$ and $x \in D(A)$. And it is clear that this inequality remains true for each $x \in \overline{D(A)}$; and hence (3.21) holds good.

Let $X_0$ be a subset of $X$. To characterize $(g)$-operator of contraction semi-group on $X_0$ we introduce the following set $\mathfrak{g}(X_0)$ of multi-valued operators. By $\mathfrak{g}(X_0)$ we mean the set of all multi-valued operators $A$ in $X$ such that $D(A) \subset X_0$, $A$ is almost demiclosed and $A$ has the property $(\mathfrak{g})$. For $A_i \in \mathfrak{g}(X_0)$, $i=1, 2$, we write $A_1 \leqq A_2$ if $A_1 \subset A_2$. It is clear that "$\leqq$" gives a partial ordering of $\mathfrak{g}(X_0)$.

THEOREM 3.4. *Suppose that $X_0$ is a closed subset of $X$ and $A$ is a multi-valued operator in $X$ such that $\overline{D(A)}=X_0$. The following three conditions are mutually equivalent:*

(i) *$A$ is a $(g)$-operator of a contraction semi-group on $X_0$.*

(ii) *$A$ has the property $(\mathfrak{g})$ and $A$ is maximal dissipative on $X_0$.*

(iii) *A is a maximal element in the partially ordered set* $\mathfrak{g}(X_0)$.

PROOF. We proved already in Theorem 2.3 that (i) implies (ii). Suppose (ii). Since $A$ is maximal dissipative on $\overline{D(A)}$, it is demiclosed (see [4, Lemma 3.7]) and hence $A \in \mathfrak{g}(X_0)$. Let $\tilde{A} \in \mathfrak{g}(X_0)$ be an element such that $A \leqq \tilde{A}$. Then $A \subset \tilde{A}$, $D(\tilde{A}) \subset X_0$ and $\tilde{A}$ is dissipative. Therefore we have $\tilde{A} = A$, because $A$ is maximal dissipative on $X_0$. This means that $A$ is maximal element in $\mathfrak{g}(X_0)$; so (ii) implies (iii).

Finally we show that (iii) implies (i). To this end, assume that $A$ is a maximal element in $\mathfrak{g}(X_0)$. By virtue of Theorem 3.3 there exists a contraction semi-group $\{T(t)\,;\, t \geqq 0\}$ on $X_0$ $(=\overline{D(A)})$ such that (3.21) holds and

$$(3.22) \qquad \|T(t)x - T(s)x\| \leqq |||Ax||| \cdot |t-s| \qquad \text{for} \quad x \in D(A) \text{ and } t, s \geqq 0\,.$$

Let $A_0$ be the infinitesimal generator of $\{T(t)\,;\, t \geqq 0\}$ on $X_0$. It follows from the inequality (3.21) that if $x \in D(A_0)$ then

$$(3.23) \qquad \mathrm{Re}\,(A_0 x - y_0,\, F(x - x_0)) \leqq 0 \qquad \text{for every} \quad x_0 \in D(A) \text{ and } y_0 \in A x_0\,.$$

We now define an operator $A_1$ with $D(A_1) = D(A) \cup D(A_0)$ by

$$\begin{aligned} A_1 x &= Ax && \text{if} \quad x \in D(A) \setminus D(A_0) \\ &= A_0 x && \text{if} \quad x \in D(A_0) \setminus D(A) \\ &= Ax \cup \{A_0 x\} && \text{if} \quad x \in D(A) \cap D(A_0)\,. \end{aligned}$$

We see from (3.23) that $A_1$ is dissipative; and moreover $D(A_1) \subset \hat{D} \equiv \{x \in X_0\,;\, \|T(t)x - x\| = O(t) \text{ as } t \to 0+\}$ by (3.22). Take an $\tilde{A}_1$ such that $\tilde{A}_1 \supset A_1$ and $\tilde{A}_1$ is maximal dissipative on $\hat{D}$. Since $\tilde{A}_1 \supset A_0$, $\tilde{A}_1$ is a $(g)$-operator of $\{T(t)\,;\, t \geqq 0\}$ on $X_0$ and hence $\tilde{A}_1 \in \mathfrak{g}(X_0)$ by Theorem 2.3. Clearly $\tilde{A}_1 \geqq A$. Since $A$ is a maximal element in $\mathfrak{g}(X_0)$, we obtain $A = \tilde{A}_1$; so $A$ is a $(g)$-operator of a contraction semi-group on $X_0$. Q. E. D.

## References


[1] H. Brezis, On a problem of T. Kato, Comm. Pure Appl. Math., **24** (1971), 1-6.

[2] M. G. Crandall and T. M. Liggett, Generation of semi-groups of nonlinear transformations on general Banach spaces, Amer. J. Math., **93** (1971), 265-298.

[3] M. G. Crandall and A. Pazy, Semi-groups of nonlinear contractions and dissipative sets, J. Funct. Anal., **3** (1969), 376-418.

[4] T. Kato, Accretive operators and nonlinear evolution equations in Banach spaces, Proc. Symp. in Pure Math., **18**, Part I, Amer. Math. Soc., Providence, Rhode Island, 138-161.

[5] R. H. Martin, Jr., Generating an evolution system in a class of uniformly convex Banach spaces, J. Funct. Anal., **11** (1972), 62-76.

[6] I. Miyadera, Some remarks on semi-groups of nonlinear operators, Tôhoku Math. J., **23** (1971), 245-258.

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