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On homogeneous P^N-bundles over an abelian variety

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Let $M=M(T, \pi, P^N)$ be a P^N-bundle over an abelian variety T, $G=\mathrm{Aut}^0 M$ and $H=\mathrm{Aut}^0 T$ the connected components of the complex Lie groups containing the identities of all holomorphic automorphisms of M and T respectively. Then there exists a holomorphic homomorphism π_* of G into H canonically induced by π.

M is said to be a *homogeneous bundle* if π_* is surjective. If M is a bundle defined by a homomorphism of the fundamental group Γ of T into $PGL(N)$, it is called a *flat bundle.*

In § 1, we shall prove the following proposition.

PROPOSITION. *Let M be a P^N-bundle over an abelian variety T. Then M is a homogeneous bundle if and only if it is a flat bundle.*

Let α be a homomorphism of Γ into $PGL(N)$. We call α *of finite type* if $\mathrm{Im}\,\alpha$ is a finite group. In § 2, we shall prove the following proposition.

PROPOSITION. *Let M be a flat P^N-bundle over an abelian variety T defined by a homomorphism α. If α is of finite type, then*

1) *$A\times P^N$ is a finite holomorphic covering manifold of M, where $A=C^n/\ker\alpha$,*

2) *there exists a Kähler metric canonically induced by that of $A\times P^N$ such that the corresponding Ricci curvature of M is positive semi-definite.*

A connected compact complex manifold M is called an *almost homogeneous manifold* if there exists a complex subgroup G of $\mathrm{Aut}\,M$ such that the G-orbit through some point of M contains an open subset of M.

COROLLARY. *Assume that $N+1$ is a prime number. If the bundle space of a P^N-bundle M over an abelian variety T is an almost homogeneous manifold, then there exists a flat vector bundle E over T such that M is the projection of E.*

We shall give an example of an almost homogeneous P^3-bundle over an abelian variety T which is not the projection of a flat vector bundle over T.

In § 3, we shall classify homogeneous P^2-bundles over an abelian variety T and give a necessary and sufficient condition that such a bundle space is an almost homogeneous manifold.

§ 1.

Lemma 1 ([1], Lemma 3.15). *Let G be a connected Lie group and B a closed connected normal subgroup of G. Then there exists a maximal compact subgroup K of G such that $B\cap K$ and the image of K in G/B are maximal compact subgroups of B and G/B respectively.*

Now let G be a connected complex Lie group, K a maximal compact subgroup of G and $\mathfrak{k}$ the Lie algebra of K. Denote by $\tilde{K}$ the connected complex subgroup of G corresponding to the complex Lie subalgebra $\tilde{\mathfrak{k}}=\mathfrak{k}+\sqrt{-1}\,\mathfrak{k}$. Then $\tilde{K}$ has the following property:

Lemma 2 ([5], § 2, Proposition). *There exist connected closed normal complex subgroups $\tilde{S}$ and $\tilde{Z}$ satisfying the following properties:*

1) $\tilde{K}=\tilde{S}\cdot\tilde{Z}$, $\tilde{S}\cap\tilde{Z}$ *is a finite group,*
2) $\tilde{S}$ *is semi-simple,*
3) $\tilde{Z}$ *is the connected centre of* $\tilde{K}$.

Lemma 3. *If there exists a holomorphic homomorphism π_* of a connected complex Lie group G onto a connected compact complex abelian Lie group H such that the number of connected components of the kernel π_* is finite, then there exists a holomorphic splitting μ of the induced holomorphic homomorphism π of $\mathfrak{g}$ onto $\mathfrak{h}$, that is, μ is a holomorphic homomorphism of $\mathfrak{h}$ into $\mathfrak{g}$ such that $\pi\cdot\mu=\mathrm{id.}$ on $\mathfrak{h}$, where $\mathfrak{g}$ and $\mathfrak{h}$ denote the complex Lie algebras of G and H respectively.*

Proof. Let I be the kernel of π_*. Then we have $G/I\simeq H$. First, assume that I is connected. Then, since H is compact, there exists a maximal compact subgroup K of G such that $\pi_*(K)=H$, by Lemma 1. Therefore, the restriction $\pi_*|\tilde{K}$ of π_* to $\tilde{K}$ is a holomorphic surjection of $\tilde{K}$ onto H. Moreover, by Lemma 2, there exist connected closed complex subgroups $\tilde{S}$ and $\tilde{Z}$ of $\tilde{K}$ satisfying $\tilde{K}=\tilde{S}\cdot\tilde{Z}$. Since $\tilde{S}$ is semi-simple and H is abelian, π_* induces a holomorphic surjection $\tilde{\pi}_*$ of $\tilde{Z}$ onto H. Thus we have a holomorphic homomorphism $\tilde{\pi}$ of $\tilde{\mathfrak{z}}$ onto $\mathfrak{h}$ induced by $\tilde{\pi}_*$, where $\tilde{\mathfrak{z}}$ is the complex Lie subalgebra corresponding to $\tilde{Z}$. Moreover, since $\tilde{\mathfrak{z}}$ and $\mathfrak{h}$ are both complex abelian Lie algebras, a complex linear splitting of the complex linear mapping $\tilde{\pi}$ of $\tilde{\mathfrak{z}}$ onto $\mathfrak{h}$ as complex vector spaces defines a holomorphic splitting μ of $\tilde{\pi}$ as complex Lie algebras. This is the desired one. Thus we complete the proof in this case.

Next we shall prove the general case. By our assumption, I/I^0 is a finite group, where I^0 denotes the connected component of I containing the identity. Therefore, G/I^0 is a finite holomorphic covering group of $G/I\simeq H$. Thus G/H^0 is also compact. Put $H'=G/H^0$. We have an exact sequence of complex Lie groups:

$$(1) \qquad 0 \longrightarrow H^0 \longrightarrow G \longrightarrow H' \longrightarrow 0\,.$$

Now the previous arguments may be applied to the sequence (1). Hence we have a holomorphic splitting μ' of $\pi' : \mathfrak{g} \to \mathfrak{h}' \to 0$. But, since $\mathfrak{h}' \simeq \mathfrak{h}$, μ' can be considered as a holomorphic splitting of $\pi : \mathfrak{g} \to \mathfrak{h} \to 0$. This completes the proof.

Let $M = M(T, \pi, P^N)$ be a P^N-bundle over an abelian variety T, G and H the connected components of the complex Lie groups of all holomorphic automorphisms of M and T containing the identities respectively. Then there exists a holomorphic homomorphism π_* of G into H canonically induced by π ([7], Satz 1.3).

LEMMA 5. *$M = M(T, \pi, P^N)$ is a flat bundle if and only if there exists a connected complex abelian Lie subgroup A of G such that the restriction $\pi_*|A$ of π_* to A is a holomorphic covering homomorphism of A onto H.*

PROOF. Let $M = C^n \times_\alpha P^N$ be a flat bundle defined by a homomorphism α of the fundamental group Γ of T into $PGL(N)$. For an arbitrary element w of C^n, the mapping w of $C^n \times P^N$ onto itself defined by $(z, \xi) \to (z+w, \xi)$ induces a holomorphic automorphism of M. By this operation, C^n can be considered as a complex Lie group of holomorphic automorphisms of M. Moreover, two elements w and w' of C^n induce the same operation on M if and only if $w \equiv w'$ (mod. $\ker \alpha$). Hence $A = C^n/\ker \alpha$ can be considered as a connected complex abelian subgroup of G and, by this construction, the restriction $\pi_*|A$ of π_* to A is a holomorphic covering homomorphism of A onto H.

Conversely, assume that there exists a connected complex abelian subgroup A of G satisfying the condition described in Lemma 5. Let $\tilde{\Gamma}$ be the kernel of $\pi_*|A$ of A onto H. Then, for a fixed point $o \in T$, $\tilde{\Gamma}$ can be considered as a group of holomorphic automorphisms of the fibre $\pi^{-1}(o) \simeq P^N$. Thus there exists a homomorphism $\tilde{\alpha}$ of $\tilde{\Gamma}$ into $PGL(N)$ corresponding to the operation of $\tilde{\Gamma}$ on $\pi^{-1}(o)$. Moreover, M is clearly the bundle defined by $\alpha = \tilde{\alpha}(\sigma|\Gamma)$, where σ is the canonical projection of C^n onto A. This completes the proof.

PROPOSITION 1. *Let M be a P^N-bundle over an abelian variety. Then M is a homogeneous bundle if and only if it is a flat bundle.*

PROOF. Let $M = M(T, \pi, P^N)$ be a homogeneous bundle. Let $G = \mathrm{Aut}^0 M$ and $H = \mathrm{Aut}^0 T$ be the connected components of complex Lie groups of all holomorphic automorphisms containing the identities of M and T respectively. By a theorem ([2], Theorem 8), M has a Hodge metric. Moreover, since the irregularity of M equals the complex dimension of T, T can be considered as the Albanese manifold of M. Therefore the component group I/I^0 of the kernel I of the holomorphic homomorphism π_* of G onto H is a finite group

([3], § 8, Proposition). Thus, there exists a holomorphic splitting μ of $\pi: \mathfrak{g} \to \mathfrak{h}$, by Lemma 4. Denote by A the complex abelian subgroup of G corresponding to the complex abelian Lie algebra $\mu(\mathfrak{h})$. It is easily proved that A satisfies the condition described in Lemma 5. Thus M is a flat bundle.

The converse is trivial by Lemma 5. This completes the proof.

§ 2. Let Γ be a free abelian group of finite rank and α a homomorphism of Γ into $PGL(N)$. α is said to be *of finite type* if $\operatorname{Im} \alpha$ is a finite group.

LEMMA 6. *If α is of finite type, then every element of $\alpha(\Gamma)$ can be represented by a unitary matrix.*

PROOF. Let $\{\gamma_1, \cdots, \gamma_g\}$ be a system of generators of Γ and $\alpha(\gamma_i) = p(A_i)$, $A_i \in GL(N+1)$, for $i = 1, 2, \cdots, g$, where p is the canonical projection of $GL(N+1)$ onto $PGL(N)$.

We can choose $\{A_i\}$ such that they satisfy the following conditions:

1) for any i, there exists a positive integer m_i such that $A_i^{m_i} = \mathrm{id.}$,

2) for any pair (i, j), there exists a non zero complex number ρ_{ij} such that

$$A_i A_j = \rho_{ij} A_j A_i .$$

Since

$$\det A_i \det A_j = (\rho_{ij})^{N+1} \det A_j \det A_i ,$$

we have $(\rho_{ij})^{N+1} = 1$.

Denote by ρ a primitive $(N+1)$-th root of 1. Set

$$\tilde{\Gamma} = \{\rho^k A_1^{e_1} \cdots A_g^{e_g} ; \ 0 \leqq k \leqq N, \ 0 \leqq e_i \leqq m_i - 1\}$$

and

$$\tilde{\Gamma}' = \{\rho^k ; \ 0 \leqq k \leqq N\} .$$

Then we have a central extension of abstract groups:

$$0 \longrightarrow \tilde{\Gamma}' \longrightarrow \tilde{\Gamma} \xrightarrow{p \mid \tilde{\Gamma}} \alpha(\Gamma) \longrightarrow 0 .$$

Therefore $\tilde{\Gamma}$ is a finite nilpotent subgroup of $GL(N+1)$ of class 2. It is well-known that a representation of a finite group is equivalent to that of unitary matrices. Thus we have the Lemma.

PROPOSITION 2. *Let $M = C^n \times_\alpha P^N$ be a flat P^N-bundle over an abelian variety T defined by α. If α is of finite type, then $A = C^n / \ker \alpha$ is an abelian variety and we have*

1) *$A \times P^N$ is a finite holomorphic covering manifold of M,*

2) *there exists a Hodge metric on M canonically induced by that of $A \times P^N$ such that the Ricci curvature $R(M)$ of M is positive semi-definite.*

PROOF. 1) is clear by Lemma 5.

Since α is of finite type, every element of $\alpha(\Gamma)$ can be represented by a unitary matrix, by Lemma 6. Since $A\times P^N$ has the standard Hodge metric with $R(A\times P^N)\geqq 0$ and (a translation on A)$\times$(a projective transformation defined by a unitary matrix) is an isometry with respect to the above metric, M has a Hodge metric canonically induced by that of $A\times P^N$. Moreover, it is clear that $R(M)\geqq 0$. This completes the proof.

Now let $M=C^n\times_\alpha P^N$ be a flat P^N-bundle over an abelian variety T defined by a homomorphism α of the fundamental group Γ of T into $PGL(N)$. Set $G=\mathrm{Aut}^0 M$, $A=C^n/\ker\alpha$ and $I=\ker\pi_*$. Moreover, denote by $\tilde{I}$ the group consisting of holomorphic automorphisms Φ of M satisfying $\pi_*\Phi=\mathrm{id.}$ on T.

PROPOSITION 3. *$\tilde{I}$ contains the centralizer $C(\alpha(\Gamma))$ of $\alpha(\Gamma)$ in $PGL(N)$.*

Moreover, if α is of finite type, then $\tilde{I}$ is isomorphic to $C(\alpha(\Gamma))$.

PROOF. Let Φ be an element of $\tilde{I}$. Since $C^n\times P^N$ is a holomorphic covering manifold of M, Φ induces a holomorphic automorphism of $C^n\times P^N$, which we denote by $(z,\xi)\to(\varphi_1(z,\xi),\varphi_2(z,\xi))$. For a fixed $z\in C^n$, $\xi\to\varphi_1(z,\xi)$ defines a holomorphic mapping of P^N into C^n, hence $\varphi_1(z_1,\xi)$ is a constant mapping, in other words, $\varphi_1(z,\xi)=\varphi_1(z)$ is independent of $\xi\in P^N$. Moreover, since $\pi_*\Phi=\mathrm{id.}$ on T, there exists an element $\gamma\in\Gamma$ such that $\varphi_1(z)=z+\gamma$, for arbitrary $z\in C^n$.

Next we may assume that $\varphi_2(z,\xi)=\varphi_2(z)\xi$, for arbitrary $z\in C^n$ and $\xi\in P^N$, where φ_2 is a holomorphic mapping of C^n into $PGL(N)$. Moreover the condition that $\pi_*\Phi=\mathrm{id.}$ on T implies that

$$(2)\qquad \varphi_2(z+\gamma')=\alpha(-\gamma')\varphi_2(z)\alpha(\gamma')\,,\quad \text{for arbitrary } z\in C^n \text{ and } \gamma'\in\Gamma\,.$$

Hence, if $\varphi_2(z)=\varphi_2$ is a constant element in $C(\alpha(\Gamma))$, (2) is always satisfied, in other words, $\mathrm{id.}\times\varphi_2$ belongs to $\tilde{I}$, for $\varphi_2\in C(\alpha(\Gamma))$.

Next, assume that α is of finite type. Then $A=C^n/\ker\alpha$ is an abelian variety. Since φ_2 can be considered as a holomorphic mapping of A into $PGL(N)$ and $PGL(N)$ is a Stein manifold, φ_2 is a constant mapping. Therefore, by (2), φ_2 is contained in $C(\alpha(\Gamma))$. Thus we get

$$\{(\mathrm{id.}\times\varphi_2),\ \varphi_2\in C(\alpha(\Gamma))\}\simeq\tilde{I}\,.$$

This completes the proof.

A homomorphism α of Γ into $PGL(N)$ is said to be *non proper*, if there exists a homomorphism $\tilde{\alpha}$ of Γ into $GL(N+1)$ such that $\alpha=p\tilde{\alpha}$, where p is the canonical projection of $GL(N+1)$ onto $PGL(N)$.

PROPOSITION 4. *Assume that $N+1$ is a prime number. If α is proper, then*

1) *α is of finite type and $\alpha(\Gamma)\simeq Z_{N+1}\times Z_{N+1}$,*

2) *$C(\alpha(\Gamma))=\alpha(\Gamma)$.*

COROLLARY 1. *Assume that $N+1$ is a prime number. If α is proper, then*

$M=C^n\times_\alpha P^N$ is not an almost homogeneous manifold.

COROLLARY 2. *Assume that $N+1$ is a prime number. If a P^N-bundle M over an abelian variety T is an almost homogeneous manifold, then there exists a flat vector bundle E over T such that $M=$* proj. *E.*

Let $A^r(\alpha)$ be a complex r-square matrix of the form:

$$A^r(\alpha)=\begin{pmatrix} \alpha, & 1, & & & \\ & \alpha, & 1, & & \\ & & \ddots & \ddots & \\ & 0 & & \ddots & 1 \\ & & & & \alpha \end{pmatrix}, \qquad \alpha\in C^*.$$

Denote by $\Delta(r,s;\rho)$ the set of complex $r\times s$-matrices of the form:

$$\begin{pmatrix} a_1, & a_2, & \cdots\cdots\cdots, & a_s \\ & \rho a_1, & \cdots\cdots\cdots, & \rho a_{s-1} \\ & & \ddots & \vdots \\ & 0 & & \rho^{s-1}a_1 \end{pmatrix} \quad \text{for} \quad r\geqq s,$$

$$\begin{pmatrix} a_1, & a_2, & \cdots\cdots\cdots, & a_r \\ & \rho a_1, & \cdots\cdots\cdots, & \rho a_{r-1} \\ & & \ddots & \vdots \\ & 0 & & \rho^{r-1}a_1 \end{pmatrix} \quad \text{for} \quad r< s.$$

LEMMA 7. *For given $A^r(\alpha)$, $A^s(\beta)$ and $\rho\in C^*$, a complex $r\times s$-matrix B satisfying*

$$A^r(\alpha)B=\rho BA^s(\beta) \tag{3}$$

is the following form:

1) *if $\alpha\neq\rho\beta$, then $B=(0)$,*
2) *if $\alpha=\rho\beta$, then B is contained in $\Delta(r,s;\rho)$.*

PROOF. We may assume that $r\geqq s$. If $r=s=1$, then the above statements are clear.

Next we assume that $r>s=1$. Let $B=(b_{i1})$, then (3) is equivalent to

$$\alpha b_{i1}+b_{i+11}=\rho\beta b_{i1} \quad \text{for} \quad 1\leqq i\leqq r-1, \tag{3.1$'$}$$

$$\alpha b_{r1}=\rho\beta b_{r1}. \tag{3.2$'$}$$

Therefore, if $\alpha\neq\rho\beta$, then we get $B=(0)$ and if $\alpha=\rho\beta$, then we get ${}^tB=(b_{11},0,\cdots,0)$, where b_{11} is an arbitrary complex number.

Now we assume that $r\geqq s>1$. Let $B=(b_{ij})$, then (3) is equivalent to

(3.1)″ $\quad \alpha b_{i1}+b_{i+11}=\rho\beta b_{i1} \quad$ for $1\leqq i<r$,

(3.2)″ $\quad \alpha b_{r1}=\rho\beta b_{r1}$,

(3.3)″ $\quad \alpha b_{ij}+b_{i+1j}=\rho b_{ij-1}+\rho\beta b_{ij} \quad$ for $i\neq r$ and $1<j\leqq s$,

(3.4)″ $\quad \alpha b_{rj}=\rho b_{rj-1}+\rho\beta b_{rj} \quad$ for $1<j\leqq s$.

1) Assume that $\alpha\neq\rho\beta$. Since we have $b_{r1}=0$ by (3.2)″, the r-th column vector is zero. If we assume that the k-th column vector is zero for $k=i+1,\cdots,r$, then (3.1)″ and (3.3)″ are equivalent to

$$\alpha b_{ij}=\rho b_{ij-1}+\rho\beta b_{ij} \quad \text{for} \quad 1<j\leqq s,$$

$$\alpha b_{i1}=\rho\beta b_{i1} \quad \text{for} \quad 1\leqq i<r.$$

Thus the i-th column vector is also zero. Hence we have $B=(0)$ by the induction method on i.

2) Assume that $\alpha=\rho\beta$. Then we have that, by (3.1)″, b_{11} is arbitrary and $b_{i1}=0$, for $i>1$. If we assume that, for a fixed j, b_{1j} is arbitrary, $b_{ij}=\rho b_{i-1\,j-1}$, for $1<i\leqq j$ and $b_{ij}=0$, for $i>j$, then we have that b_{1j+1} is arbitrary, $b_{i+1\,j+1}=\rho b_{ij}$, for $1\leqq i\leqq j$ and $b_{i+1\,j+1}=0$, for $i>j$.

Thus we complete the proof by the induction method on j.

PROOF OF PROPOSITION 4. Now we assume that $N+1$ is a prime number. Let γ be one of generators of Γ satisfying $\alpha(\gamma)\neq$ id. and $\alpha(\gamma)=p(A)$, $A\in GL(N+1)$. Denote by $\{\alpha_1,\cdots,\alpha_k\}$ all distinct eigen values of A. If there exist $B\in GL(N+1)$ and $\rho\in C^*$, $\rho\neq 1$, satisfying $AB=\rho BA$, then, by Lemma 7, A, B and ρ must satisfy the following conditions:

$$k=N+1$$

and

$$\rho\alpha_i=\alpha_{i+1} \quad \text{for} \quad 1\leqq i\leqq N,$$

$$\rho\alpha_{N+1}=\alpha_1.$$

Thus we may assume that

$$A=\begin{bmatrix}\rho & & & 0\\ & \rho^2 & & \\ & & \ddots & \\ 0 & & & \rho^{N+1}\end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix} & & & b_1\\ b_2 & & 0 & \\ & \ddots & & \\ 0 & & b_{N+1} & \end{bmatrix}$$

where ρ is a $(N+1)$-th root of 1, $\rho\neq 1$ and $b_1, b_2,\cdots,b_{N+1}$ are non zero complex numbers.

Denote by K the subgroup of $PGL(N)$ generated by $p(A)$ and $p(B)$. We can easily prove that $K\simeq Z_{N+1}\times Z_{N+1}$ and $C(K)=K$.

Since α is proper, $\alpha(\Gamma)$ is not cyclic. Hence the arguments described above show that $\alpha(\Gamma)\simeq Z_{N+1}\times Z_{N+1}$ and $C(\alpha(\Gamma))=\alpha(\Gamma)$.

Corollaries 1 and 2 are clear by Propositions 3 and 4.

REMARK. If $N+1$ is not a prime number, then there exists an almost homogeneous P^N-bundle M over an abelian variety T which is not the projection of a flat vector bundle E over T.

Let $\{\gamma_1, \cdots, \gamma_{2n}\}$ be a system of generators of the fundamental group Γ of T, where n is the complex dimension of T. Let α be a proper homomorphism of Γ into $PGL(3)$ defined by

$$\alpha(\gamma_1)=p(A), \qquad \alpha(\gamma_2)=p(B)$$

and

$$\alpha(\gamma_i)=\mathrm{id.} \quad \text{for} \quad 3\leqq i\leqq 2n,$$

where

$$A=\begin{bmatrix} 1 & & & \\ & 1 & 0 & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix} 0 & & 1 & 1 \\ & & & 1 \\ 1 & -1 & & \\ & 1 & 0 & \end{bmatrix}.$$

Set

$$K=\Big\{p(D)\in PGL(3);$$

$$D=\begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix},\ D_2=\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} D_1 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ D_1\in GL(2)\Big\}.$$

It is clear that $K\subseteq C(\alpha(\Gamma))$ and K acts on P^3 almost transitively. Hence $M=C^n\times_\alpha P^3$ is an almost homogeneous manifold which is not the projection of a flat vector bundle over T.

§3. Let Γ be the fundamental group of an abelian variety T. A homomorphism $\tilde{\alpha}$ of Γ into $GL(N+1)$ is said to be a *unipotent representation* (*a special unipotent representation*) if $\tilde{\alpha}$ is equivalent to a homomorphism of Γ into N ($\tilde{N}$), where N ($\tilde{N}$) is a subgroup of $GL(N+1)$ consisting of matrices of the form:

$$\begin{bmatrix} 1 & & & \\ & 1 & & * \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & a_1 & a_2 & a_N \\ & 1 & a_1 & \\ & & \ddots & a_1 \\ 0 & & & 1 \end{bmatrix}\right).$$

LEMMA 8. *An indecomposable unipotent representation $\tilde{\alpha}$ of Γ into $GL(3)$ is equivalent to one of the following:*

1) *a special unipotent* $\begin{bmatrix} 1 & a & b \\ & 1 & a \\ 0 & & 1 \end{bmatrix}$, *where* $a: \Gamma \to C$ *is non trivial,*

2) $\begin{bmatrix} 1 & 0 & b \\ & 1 & a \\ 0 & & 1 \end{bmatrix}$, *where* a *and* b *are homomorphisms of* Γ *into* C *which are linearly independent over* C,

3) $\begin{bmatrix} 1 & a & b \\ & 1 & 0 \\ 0 & & 1 \end{bmatrix}$, *where* a *and* b *are homomorphisms of* Γ *into* C *which are linearly independent over* C.

PROOF. Let $\tilde{\alpha} = \begin{bmatrix} 1 & a_1 & b \\ & 1 & a_2 \\ 0 & & 1 \end{bmatrix}$, then a_1 and a_2 are homomorphisms of Γ into C which are linearly dependent over C. If a_1 and a_2 are both non trivial, then there exists a non zero complex number λ such that $a_2 = \lambda a_1$. Therefore we have

$$\tilde{\alpha} \sim \begin{bmatrix} \lambda & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix} \begin{bmatrix} 1 & a_1 & b \\ & 1 & a_2 \\ 0 & & 1 \end{bmatrix} \begin{bmatrix} 1/\lambda & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix} = \begin{bmatrix} 1 & a_2 & b \\ & 1 & a_2 \\ 0 & & 1 \end{bmatrix}.$$

This is the case 1).

Next we assume that a_1 and a_2 are both trivial. Then we have

$$\tilde{\alpha} = \begin{bmatrix} 1 & 0 & b \\ & 1 & 0 \\ 0 & & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & b \\ & 1 & 0 \\ 0 & & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & b & 0 \\ & 1 & 0 \\ 0 & & 1 \end{bmatrix}.$$

But this contradicts to the fact that $\tilde{\alpha}$ is indecomposable. Thus one of $\{a_i\}$ is non trivial. Moreover, if a_1 is trivial and a_2 and b are linearly dependent over C, then there exists a complex number λ such that $b = \lambda a_2$. And we have

$$\tilde{\alpha}=\begin{pmatrix}1 & 0 & b\\ & 1 & a_2\\ 0 & & 1\end{pmatrix}\sim\begin{pmatrix}1 & -\lambda & 0\\ & 1 & 0\\ 0 & & 1\end{pmatrix}\begin{pmatrix}1 & 0 & b\\ & 1 & a_2\\ 0 & & 1\end{pmatrix}\begin{pmatrix}1 & \lambda & 0\\ & 1 & 0\\ 0 & & 1\end{pmatrix}$$

$$=\begin{pmatrix}1 & 0 & 0\\ & 1 & a_2\\ 0 & & 1\end{pmatrix}.$$

This contradicts to the fact that $\tilde{\alpha}$ is indecomposable. Thus if a_1 is trivial, a_2 and b are linearly independent over C. This completes the proof.

Denote by M_α the P^2-bundle over T defined by a homomorphism α of Γ into $PGL(2)$ and by $E_{\tilde{\alpha}}$ the vector bundle over T defined by a homomorphism $\tilde{\alpha}$ of Γ into $GL(3)$.

LEMMA 9 ([4], Théorème 3). *For an indecomposable flat vector bundle E over a complex torus, there exist a flat line bundle L and a unipotent representation $\tilde{\alpha}$ of the fundamental group of the complex torus such that E is equivalent to $L\otimes E_{\tilde{\alpha}}$.*

PROPOSITION 5. *A homogeneous P^2-bundle M over an abelian variety T is equivalent to one of the following:*

1) *M_α, where α is proper,*

2) proj. $E_{\tilde{\alpha}}$,

a) *$\tilde{\alpha}=\tilde{\alpha}_1\oplus\tilde{\alpha}_2\oplus\tilde{\alpha}_3$, where $\tilde{\alpha}_i$ is a homomorphism of Γ into C^*, for* $i=1, 2, 3$,

b) *$\tilde{\alpha}=\tilde{\alpha}_1\oplus\tilde{\alpha}_2$, where $\tilde{\alpha}_1$ is a homomorphism of Γ into C^* and $\tilde{\alpha}_2$ is an indecomposable unipotent representation of Γ of degree* 2,

c) *$\tilde{\alpha}$ is an indecomposable unipotent representation of degree* 3 *of the forms* 1), 2) *or* 3) *described in Lemma* 8.

Moreover, M is an almost homogeneous manifold if and only if M is equivalent to one of 2, a), 2, b), 2, c, 1) *and* 2, c, 2).

PROOF. If α is proper, then, by Corollary 1 of Proposition 4, M_α is not almost homogeneous.

Next we assume that α is non proper. Then there exists a flat vector bundle $E_{\tilde{\alpha}}$ such that $M=\text{proj.}\,E_{\tilde{\alpha}}$.

Decompose $\tilde{\alpha}$ into indecomposable components, then we have three cases a), b) and c). Moreover, we can easily prove that, for the cases a), b), c, 1) and c, 2), the corresponding manifold $M=\text{proj.}\,E_{\tilde{\alpha}}$ is almost homogeneous ([6]).

Therefore, to prove Proposition 5, it is sufficient to show that, for the case c, 3), the corresponding manifold $M=\text{proj.}\,E_{\tilde{\alpha}}$ is not almost homogeneous.

Let $\mathfrak{U}=\{U_i\}$ be a sufficiently fine open covering of T and $\left\{ f_{ij}=\begin{pmatrix} 1 & a_{ij} & b_{ij} \\ & 1 & 0 \\ 0 & & 1 \end{pmatrix} \right\}$ a system of transition functions of the bundle $M=\text{proj.}\, E_{\tilde{\alpha}}$ with respect to $\mathfrak{U}$, where $\tilde{\alpha}$ is of type c, 3).

Let π be the bundle projection of M onto T and $\Phi \in \ker \pi_*$. Then there exists a system $\{\varphi_i\}$ of holomorphic mappings φ_i of U_i into $GL(3)$ satisfying

$$\varphi_i f_{ij} = \rho_{ij} f_{ij} \varphi_j \qquad \text{for} \quad U_i \cap U_j = \emptyset\,, \tag{4}$$

where ρ_{ij} is a holomorphic mapping of $U_i \cap U_j$ into C^*. Moreover, by (4), $\rho=\{\rho_{ij}\}$ defines a system of transition functions of a complex line bundle L over T. Since, by (4), $\det \varphi_i \det f_{ij}=(\rho_{ij})^3 \det f_{ij} \det \varphi_j$, we have $L^3=1$.

Conversely, if there exists a system $\{\varphi_i\}$ satisfying (4), then we can construct an element Φ of $\operatorname{Aut} M$ satisfying $\pi_*\Phi=\text{id.}$ on T.

LEMMA 10. *Let L be a non trivial complex line bundle over T. If there exists a positive integer m such that $L^m=1$, then we have $H^0(T, \Omega(L))=0$, where $\Omega(L)$ denotes the sheaf of germs of holomorphic sections of L.*

PROOF. Let $\{\rho_{ij}\}$ be a system of transition functions of L with respect to a sufficiently fine open covering $\mathfrak{U}=\{U_i\}$ of T. Let $h=\{h_i\} \in H^0(T, \Omega(L))$, then we have $h_i=\rho_{ij}h_j$, for every pair (i,j) satisfying $U_i \cap U_j \neq \emptyset$. Since $L^m=1$, we may assume that $\rho_{ij}{}^m=1$, for every pair (i,j). Therefore $h^m=h^m{}_i$ is a global holomorphic function on T. Thus h_i is a constant on U_i, for every i. Since L is non trivial, we have $h=0$.

For the case c, 3), (4) is equivalent to

$$a^i{}_{31}=\rho_{ij}a^j{}_{31}\,, \tag{4.1}$$

$$a^i{}_{31}a_{ij}+a^i{}_{32}=\rho_{ij}a^j{}_{32}\,, \tag{4.2}$$

$$a^i{}_{31}b_{ij}+a^i{}_{33}=\rho_{ij}a^j{}_{33}\,, \tag{4.3}$$

$$a^i{}_{21}=\rho_{ij}a^j{}_{21}\,, \tag{4.4}$$

$$a^i{}_{21}a_{ij}+a^i{}_{22}=\rho_{ij}a^j{}_{22}\,, \tag{4.5}$$

$$a^i{}_{21}b_{ij}+a^i{}_{23}=\rho_{ij}a^j{}_{23}\,, \tag{4.6}$$

$$a^i{}_{11}=\rho_{ij}(a^j{}_{11}+a^j{}_{21}a_{ij}+a^j{}_{31}b_{ij})\,, \tag{4.7}$$

$$a^i{}_{11}a_{ij}+a^i{}_{12}=\rho_{ij}(a^j{}_{11}+a^j{}_{21}a_{ij}+a^j{}_{31}b_{ij})\,, \tag{4.8}$$

$$a^i{}_{11}b_{ij}+a^i{}_{13}=\rho_{ij}(a^j{}_{13}+a^j{}_{23}a_{ij}+a^j{}_{33}b_{ij})\,, \tag{4.9}$$

where $\varphi_i=(a^i{}_{AB})$.

If $\rho=\{\rho_{ij}\}$ is non trivial, then, by Lemma 10, (4.1), (4.2) and (4.3), we get $a^i{}_{31}=a^i{}_{32}=a^i{}_{33}=0$. If ρ is trivial, then $a^i{}_{31}=a^j{}_{31}$ is a constant by (4.1).

Since $\{a_{ij}\}$ and $\{b_{ij}\}$ are non trivial C-bundles, we have $a_{31}=\{a^i{}_{31}\}=0$ and $a_{32}=\{a^i{}_{32}\}$ and $a_{33}=\{a^i{}_{33}\}$ are constants by (4.2) and (4.3). The similar arguments show that, for every i, $\varphi_i=\begin{pmatrix} a & b & c \\ & a & 0 \\ 0 & & a \end{pmatrix}$ is a constant element in $GL(3)$. Thus we have

$$\ker \pi_* \simeq I=\left\{\begin{pmatrix} a & b & c \\ & a & 0 \\ 0 & & a \end{pmatrix}\in GL(3)\,;\ a\in C^*,\ b, c\in C\right\}.$$

It is easily proved that I does not act on P^2 almost transitively. Hence M is not an almost homogeneous manifold. This completes the proof of proposition.

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