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Π_1^1 sets of sets, hyperdegrees and related problems[1)]

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Let A and B be sets of integers. $A \leqq_H B$ denotes the relation that A is hyperarithmetic in B. Then the following relation (denoted by $=_H$) is an equivalence relation between A and B: $A \leqq_H B \wedge B \leqq_H A$. Each equivalence class with respect to $=_H$ is called a *hyperdegree.* A set A of natural numbers has a hyperdegree $\boldsymbol{d}$ if and only if A belongs to the equivalence class $\boldsymbol{d}$. Let $\mathcal{A} \subset 2^{\omega}$. $\mathcal{A}$ has *property* (S) if and only if for *every* hyperdegree $\boldsymbol{d}$ there exists a member A of $\mathcal{A}$ whose hyperdegree is $\boldsymbol{d}$. G. E. Sacks has shown that if $\mathcal{A}$ is a hyperarithmetic subset of 2^{ω} then either $\mathcal{A}$ or $2^{\omega}-\mathcal{A}$ has property (S). In this paper we shall show the following theorems each of which implies Sack's Theorem:

THEOREM A. *Let $\mathcal{A}$ be a Π_1^1 set of sets of natural numbers. If $\mathcal{A}$ has positive measure then $\mathcal{A}$ has property (S)*[3)].

THEOREM B. *Let $\mathcal{A}$ be a non-meager Π_1^1 set of sets of natural numbers. Then $\mathcal{A}$ has property (S).*

Theorem A is proved by a measure-theoretic method as in [8] (§ 2), whereas Theorem B by a forcing method as in Feferman [1], Hinman [3] and Jockusch [unpublished] (§§ 3, 4).

In § 5, we shall relativize the theory developed in §§ 3-4. As applications of relativization, we shall discuss, in § 7, some generalization of Theorems A and B, and show, in § 6, the following theorem:

THEOREM C. *Let $\mathcal{A}$ be a set of sets which has the property of Baire. If $\mathcal{A}$ is not meager, then $\mathcal{A}$ contains an infinitely many elements which have the same Turing degree.*

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2) The following Theorems A and B were obtained by being stimulated by Jackusch's proof of Sack's Theorem on Borel Determinateness. I wish to thank Professor C. G. Jockusch for lending me his unpublished note. Also I wish to thank Professor G. Takeuti for his kind advice.

3) Theorem A was independently obtained by G.E. Sacks (private communication).

§ 1. Preliminaries.

Let 2^ω be the set of sets of natural numbers. We denote members of 2^ω by α, β and γ (instead of uppercase Roman letters, because we would like to use α, β, and γ as functions, too; i.e., each member α of 2^ω is identified with the representing function of α, thus: $\alpha(i)=0 \longleftrightarrow i \in \alpha$. Let Σ be the set of all finite binary sequences, where a sequence consisting of 0's and 1's only is called a binary sequence. Let σ and τ be variables on Σ. If $\sigma=\langle a_0, a_1, \cdots, a_{k-1}\rangle$ then we write $\sigma(i)=a_i$ for $i<k$ and $lh(\sigma)=k$. We identify σ with the set $\{\alpha \subset \omega \mid (\forall i)[i<lh(\sigma) \rightarrow \alpha(i)=\sigma(i)]\}$ as well as with the sequence number $\prod p_i^{\sigma(i)+1}[i<lh(\sigma)]$, where p_i is the i^{th} prime number. For $\alpha \in 2^\omega$, $\bar{\alpha}(k)$ denotes the finite sequence $\langle \alpha(0), \alpha(1), \cdots, \alpha(k-1)\rangle$. Similarly for $\sigma \in \Sigma$ with $k \leqq lh(\sigma)$. Let 2^ω have a topology by taking Σ as the basic open sets (the Cantor space) and let it have the infinite product measure μ by associating the equiprobable measure with 2 [**5**; §§ 11, 13]. $\sigma \subset \tau$ means that the finite sequence τ (not necessarily properly) extends the finite sequence σ, in other words the basic open set τ is contained in the basic open set σ. $\sigma * \langle i_0, i_1, \cdots, i_k\rangle$ denotes the sequence obtained from σ by extending by the sequence $\langle i_0, i_1, \cdots, i_k\rangle$. Let T be a *tree*, i.e., T is a subset of Σ such that if $\sigma \in T$ and $\tau \subset \sigma$ then $\tau \in T$. Then $[T]$ denotes the set of all path through T, where α is called a path through T if $\alpha \in 2^\omega$ and for each $i \in \omega$ $\bar{\alpha}(i) \in T$. Two members σ and τ of Σ are said to be *incompatible* if neither $\sigma \subset \tau$ nor $\tau \subset \sigma$. A tree T is called *perfect* if every member of T has at least two incompatible extensions in T. It is well-known that a tree T is perfect if and only if $[T]$ is a perfect subset of the Cantor space 2^ω.

The following lemma is already known by several persons.

LEMMA 1.1. *Let T be a hyperarithmetic subset of Σ. If T is a perfect tree then $[T]$ has property (S).*

PROOF. For $\sigma \in T$, let $g_0(\sigma)$ and $g_1(\sigma)$ be two incompatible extensions of σ in T such that they are of the least length among such pairs of extensions. That is, let

$$P(\tau, \tau', i) \longleftrightarrow \tau, \tau' \in T \wedge lh(\tau)=lh(\tau') \wedge \bar{\tau}(lh(\tau)-1)=\bar{\tau}'(lh(\tau')-1)$$
$$\wedge\, \tau(lh(\tau)-1)=i \wedge \tau'(lh(\tau')-1)=1-i\,.$$

Then for $i=0$ and 1

$$g_i(\sigma)=(\mu\tau)(\exists\tau')[\sigma \subset \tau, \tau' \wedge P(\tau, \tau', i) \wedge (\forall\rho)(\forall\rho')$$
$$\{\sigma \subset \rho, \rho' \wedge P(\rho, \rho', i) \longrightarrow lh(\tau) \leqq lh(\rho)\}]\,.$$

Since T is a perfect tree, $g_i(\sigma)$ is well-defined for $\sigma \in T$ and for $i=0, 1$. Since T is hyperarithmetic, so is g_i.

Let $\boldsymbol{d}$ be a given hyperdegree and let $\delta \in \boldsymbol{d}$. For this δ we define $\{\sigma_n | n \in \omega\}$ as follows:

$$\sigma_0 = \langle\ \rangle = \text{the empty sequence, and } \sigma_{n+1} = g_{\delta(n)}(\sigma_n).$$

Then for each n, σ_{n+1} is a proper extension of σ_n. Therefore $\{\sigma_n | n \in \omega\}$ determines a unique path through T. Let α be the path through T. Since σ_n is hyperarithmetic in δ, we have $\alpha \leqq_H \delta$.

Now, in general, for a given $\gamma \in [T]$ we shall define a function f_γ such that $f_\gamma(n)$ represents the height of n^{th} branching point (i. e., node) of γ *in* T. Namely

$$\begin{cases} f_\gamma(0) = (\mu k)(\exists \sigma, \tau \in T)[lh(\sigma) = lh(\tau) = k+1 \wedge \bar{\gamma}(k) = \bar{\sigma}(k) = \bar{\tau}(k) \\ \qquad \wedge\ \gamma(k) = \sigma(k) \neq \tau(k)], \\ f_\gamma(n+1) = (\mu k)(\exists \sigma, \tau \in T)[lh(\sigma) = lh(\tau) = k+1 > f_\gamma(n)+1 \\ \qquad \wedge\ \bar{\gamma}(k) = \bar{\sigma}(k) = \bar{\tau}(k) \wedge \gamma(k) = \sigma(k) \neq \tau(k)]. \end{cases}$$

Clearly f_γ is hyperarithmetic in γ, since T is. Taking the above α as γ we have $\delta(n) = \alpha(f_\alpha(n))$, which implies $\delta \leqq_H \alpha$. Thus $\alpha =_H \delta$ and $\alpha \in [T]$. This proves the lemma.

DEFINITION 1.2. ([4], [5]). Let O be the ordinal system of notations defined by Kleene. For $e \in O$, $z \in \omega$ and $\alpha \subset \omega$, we define the relation $z \in H^\alpha(e)$ inductively as follows:

(1) $e = 1$. $z \in H^\alpha(e) \longleftrightarrow z \in \alpha$,

(2) $e = 2^x \neq 1$. $z \in H^\alpha(e) \longleftrightarrow (\exists y) T_1^{H^{\alpha}(x)}(z, z, y)$,

(3) $e = 3 \cdot 5^x$. $z \in H^\alpha(e) \longleftrightarrow (z)_1 <_O e \wedge (z)_0 \in H^\alpha((z)_1)$.

For $e \in O$ and $z \in \omega$, let $\mathscr{H}(z, e) = \{\alpha \subset \omega | z \in H^\alpha(e)\}$.

For Definition 3.1 below we rewrite $\mathscr{H}(z, 2^x)$ as follows:

$$\alpha \in \mathscr{H}(z, 2^x) \longleftrightarrow (\exists \sigma)[\sigma \in \Sigma \wedge T_1^1(\sigma, z, z) \wedge (\forall n)\{n < lh(\sigma) \longrightarrow$$
$$(\sigma(n) = 0 \longrightarrow \alpha \in \mathscr{H}(n, x)) \wedge (\sigma(n) = 1 \longrightarrow \alpha \notin \mathscr{H}(n, x))\}].$$

A subset $\mathscr{A}$ of 2^ω is called *hyperarithmetic* if there exists a number $e \in O$ and a number $z \in \omega$ such that $\mathscr{A} = \mathscr{H}(z, e)$ or $\mathscr{A} = 2^\omega - \mathscr{H}(z, e)$. We say $\langle 0, z, e\rangle$ the index of $\mathscr{H}(z, e)$ and $\langle 1, z, e\rangle$ the index of $2^\omega - \mathscr{H}(z, e)$. It is well-known that $\mathscr{A}$ is hyperarithmetic if and only if $\mathscr{A}$ is Δ_1^1 (Souslin-Kleene).

§ 2. Measure-theoretic case.

LEMMA 2.1. *Let $\mathscr{A}$ be a subset of 2^ω whose measure is positive. Then there exists a basic open set σ such that $\mu(\mathscr{A} \cap \sigma * \langle 0\rangle) > 0$ and $\mu(\mathscr{A} \cap \sigma * \langle 1\rangle) > 0$.*

Proof is obvious.

LEMMA 2.2. *Let $\mathcal{A}$ be a subset of 2^{ω}. If $\mathcal{A}$ is closed, hyperarithmetic and has positive measure, then there is a hyperarithmetic, perfect tree T such that $[T]\subset\mathcal{A}$.*

PROOF. We shall hyperarithmetically construct an infinite binary system $\{\sigma_{i_0 i_1\cdots i_n}\}$ of basic open sets such that for all $\langle i_0, i_1, \cdots, i_n\rangle\in\Sigma$

(1) $\mu(\mathcal{A}\cap\sigma_{i_0 i_1\cdots i_n})>0$,

(2) $\sigma_{i_0 i_1\cdots i_{n-1}}\subset\sigma_{i_0 i_1\cdots i_{n-1} j}$ for $j=0, 1$; and

(3) $\sigma_{i_0 i_1\cdots i_{n-1} 0}$ and $\sigma_{i_0 i_1\cdots i_{n-1} 1}$ are incompatible.

Stage 0. Let σ be the least $\sigma\in\Sigma$ (as a natural number) such that

$$\mu(\mathcal{A}\cap\sigma*\langle j\rangle)>0 \qquad \text{for } j=0, 1$$

(by Lemma 2.1). Then we define $\sigma_i=\sigma*\langle i\rangle$ for $i=0, 1$.

Stage $n+1$. Suppose that for each binary sequence $\langle i_0, i_1, \cdots, i_{n-1}\rangle$ of length n, $\sigma_{i_0 i_1\cdots i_{n-1}}$ is defined so that the conditions (1)-(3) are satisfied with $n-1$ instead of n. Let $\sigma'_{i_0 i_1\cdots i_{n-1}}$ be the least $\sigma\in\Sigma$ such that

$$\mu(\mathcal{A}\cap\sigma_{i_0 i_1\cdots i_{n-1}}\cap\sigma*\langle j\rangle)>0 \qquad \text{for } j=0, 1,$$

by Lemma 2.1. Then we define

$$\sigma_{i_0 i_1\cdots i_{n-1} j}=\sigma'_{i_0 i_1\cdots i_{n-1}}*\langle j\rangle \qquad \text{for } j=0, 1.$$

Clearly $\sigma_{i_0 i_1\cdots i_{n-1}}\subset\sigma'_{i_0 i_1\cdots i_{n-1}}$. So we have (1). And also clearly the above process is infinite. Let $T=\{\tau\in\Sigma\,|\,\tau$ is an initial segment of some sequence $\sigma_{i_0 i_1\cdots i_n}\}$. Then T is a perfect tree. By [7; Corollary 4 with parameter(s)][4)], if $D(\alpha, x)$ is a hyperarithmetic predicate on $2^{\omega}\times\omega$, then $\lambda xy[\mu(\{\alpha\,|\,D(\alpha, x)\})>2^{-y}]$ is also hyperarithmetic. So the entire process above is hyperarithmetic, and hence T is a hyperarithmetic set.

Now let $\alpha\in[T]$. Then by (2) and (3) there is a $\beta\in 2^{\omega}$ and a function h from ω into ω such that

$$lh(\sigma_{\beta(0)\beta(1)\cdots\beta(h(n))})>n \quad\text{and}\quad \bar{\alpha}(n)=\bar{\sigma}_{\beta(0)\beta(1)\cdots\beta(h(n))}(n).$$

By (1) $\mu(\mathcal{A}\cap\sigma_{\beta(0)\beta(1)\cdots\beta(h(n))})>0$. Therefore α is a limiting point of $\mathcal{A}$, and hence $\alpha\in\mathcal{A}$ because $\mathcal{A}$ is a closed set. Thus $[T]\subset\mathcal{A}$. This proves Lemma 2.2.

LEMMA 2.3. *If $\mathcal{A}$ is a Π_1^1 set of positive measure, then $\mathcal{A}$ contains a closed, hyperarithmetic subset $\mathcal{B}$ which has also positive measure.*

For proof see [8; Theorems 1 and B].

By Lemmas 2.2 and 2.3 we obtain the following theorem:

THEOREM A. *If $\mathcal{A}$ is a Π_1^1 subset of 2^{ω} which has positive measure, then*

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4) Although it was obtained for the case of the space ω^{ω}, it also holds for the space 2^{ω}.

$\mathcal{A}$ *has property* (S).

Now let $\mathcal{A}$ be a hyperarithmetic subset of 2^ω. Then either $\mathcal{A}$ or $2^\omega-\mathcal{A}$ has positive measure, because $\mathcal{A}$ is measurable. Therefore either $\mathcal{A}$ or $2^\omega-\mathcal{A}$ has property (S). This is Sacks' Theorem stated at the beginning of this paper.

§ 3. Forcing relation.

To prove Theorem B we shall define a forcing relation as follows:

DEFINITION 3.1. σ *forces* $\langle j, z, e\rangle$ (denoted by $\sigma \Vdash \langle j, z, e\rangle$) if and only if $\sigma \in \Sigma$, $j \in 2$, $z \in \omega$, $e \in O$ and the following conditions are satisfied:

(1) If $e=1$, $\sigma \Vdash \langle 0, z, e\rangle \longleftrightarrow lh(\sigma) > z \wedge \sigma(z)=1$.

(2) If $e=2^d \neq 1$, $\sigma \Vdash \langle 0, z, e\rangle \longleftrightarrow (\exists\tau)[\tau \in \Sigma \wedge T_1^1(\tau, z, z) \wedge (\forall n)\{n < lh(\tau) \rightarrow [\tau(n)=0 \rightarrow \sigma \Vdash \langle 0, n, d\rangle] \wedge [\tau(n)=1 \rightarrow \sigma \Vdash \langle 1, n, d\rangle]\}]$.

(3) If $e=3\cdot 5^i$, $\sigma \Vdash \langle 0, z, e\rangle \longleftrightarrow (z)_1 <_O e \wedge \sigma \Vdash \langle 0, (z)_0, (z)_1\rangle$.

(4) $\sigma \Vdash \langle 1, z, e\rangle \longleftrightarrow (\forall\tau)[\tau \supset \sigma \rightarrow \text{not } (\tau \Vdash \langle 0, z, e\rangle)]$.

DEFINITION 3.2. Let I be the set $\{\langle j, z, e\rangle \mid j \in 2,\ z \in \omega,\ e \in O\}$. Suppose $J \subset I$ and $\alpha \in 2^\omega$. Then α is *generic* for J if and only if for every $\langle j, z, e\rangle \in J$ there exists a number k such that $\bar{\alpha}(k) \Vdash \langle j, z, e\rangle$ or $\bar{\alpha}(k) \Vdash \langle 1-j, z, e\rangle$. Let $I(e)=\{\langle j, z, d\rangle : |d| < |e|\}$ for $e \in O$, where $|e|$ is the ordinal represented by e, and let $\mathcal{G}(e)=\{\alpha \in 2^\omega \mid \alpha \text{ is generic for } I(e)\}$. We call a member of $\mathcal{G}(e)$ an e-generic element.

LEMMA 3.3. *For each* $e \in O$, *the relation* "$\langle j, z, d\rangle \in I(e) \wedge \sigma \Vdash \langle j, z, d\rangle$" *is hyperarithmetic with respect to* σ, j, z *and* d.

PROOF. By [**6**], for an $e \in O$ $I(e)$ is hyperarithmetic (uniformly in e). So this lemma follows from the same technique as in Feferman [**1**; Proof of Theorem 2.9].

LEMMA 3.4. *For any* $\sigma \in \Sigma$ *and any index* $\langle j, z, e\rangle$:

(i) *not* $(\sigma \Vdash \langle j, z, e\rangle$ *and* $\sigma \Vdash \langle 1-j, z, e\rangle)$,

(ii) $(\forall\tau)[\tau \supset \sigma \wedge \sigma \Vdash \langle j, z, e\rangle \rightarrow \tau \Vdash \langle j, z, e\rangle]$,

(iii) $(\exists\tau)[\tau \supset \sigma \wedge \{\tau \Vdash \langle j, z, e\rangle$ *or* $\tau \Vdash \langle 1-j, z, e\rangle\}]$,

(iv) *if* $\alpha \in 2^\omega$, *then* $(\exists k)[\bar{\alpha}(k) \Vdash \langle 0, z, 1\rangle]$ *if and only if* $\alpha(z)=0$.

PROOF. Similar to [**1**; Theorems 2.3 and 2.4].

LEMMA 3.5. *If* $\alpha \in \mathcal{G}(e)$ *and if* $\langle j, y, d\rangle \in I(e)$, *then* $(\exists k)[\bar{\alpha}(k) \Vdash \langle j, y, d\rangle$ *if and only if* $\alpha \in \mathcal{H}(y, d)$ *or* $\alpha \notin \mathcal{H}(y, d)$ *according as* $j=0$ *or* 1.

PROOF. (1) $\langle j, y, d\rangle = \langle 0, y, 1\rangle$. This case is (iv) in Lemma 3.4.

(2) $\langle j, y, d\rangle = \langle 0, y, 2^i\rangle$. $(\exists k)[\bar{\alpha}(k) \Vdash \langle 0, y, 2^i\rangle] \longleftrightarrow (\exists k)(\exists\sigma)[\sigma \in \Sigma \wedge T_1^1(\sigma, y, y) \wedge (\forall n)\{n < lh(\sigma) \rightarrow (\sigma(n)=0 \rightarrow \bar{\alpha}(k) \Vdash \langle 0, n, i\rangle) \wedge (\sigma(n)=1 \rightarrow \bar{\alpha}(k) \Vdash \langle 1, n, i\rangle)\}] \longleftrightarrow (\exists\sigma)[\sigma \in \Sigma \wedge T_1^1(\sigma, y, y) \wedge (\forall n)\{n < lh(\sigma) \rightarrow (\sigma(n)=0 \rightarrow \alpha \in \mathcal{H}(n, i)) \wedge (\sigma(n)=1 \rightarrow \alpha \notin \mathcal{H}(n, i))\}]$, [by induction hypothesis; use Lemma 3.4 (ii) for $\leftarrow$], $\longleftrightarrow \alpha \in \mathcal{H}(y, 2^i)$.

(3) $\langle j, y, d\rangle = \langle 0, y, 3\cdot 5^i\rangle$. $(\exists k)[\bar{\alpha}(k) \Vdash \langle 0, y, 3\cdot 5^i\rangle] \longleftrightarrow (\exists k)[(y)_1 <_O 3\cdot 5^i \wedge \bar{\alpha}(k) \Vdash \langle 0, (y)_0, (y)_1\rangle] \longleftrightarrow (y_1 <_O 3\cdot 5^i \wedge \alpha \in \mathcal{H}((y)_0, (y)_1)$ [by induction hypothesis], $\longleftrightarrow \alpha \in \mathcal{H}(y, 3\cdot 5^i)$.

(4) $\langle j, y, d\rangle = \langle 1, y, d\rangle$. $(\exists k)[\bar{\alpha}(k) \Vdash \langle 1, y, d\rangle] \longleftrightarrow (\exists k)(\forall \tau)[\tau \supset \bar{\alpha}(k) \rightarrow$ not $(\tau \Vdash \langle 0, y, d\rangle)] \longrightarrow$ not $(\alpha \in \mathcal{H}(y, d))$ [by taking $\tau = \bar{\alpha}(k)$ and by induction hypothesis]. Conversely, suppose $\alpha \notin \mathcal{H}(y, d)$. By induction hypothesis there is no k such that $\bar{\alpha}(k) \Vdash \langle 0, y, d\rangle$. Since α is generic for $I(e)$, $(\exists k)[\bar{\alpha}(k) \Vdash \langle 0, y, d\rangle$ or $\bar{\alpha}(k) \Vdash \langle 1, y, d\rangle]$. These imply $(\exists k)[\bar{\alpha}(k) \Vdash \langle 1, y, d\rangle]$.

DEFINITION 3.6. For any subset $\mathcal{A}$ of 2^ω,

(i) $\mathcal{A}$ is *nowhere dense* if and only if $\mathcal{A}$ is dense in no basic open set.

(ii) $\mathcal{A}$ is *meager* (first Baire Category) if and only if $\mathcal{A}$ is a countable union of nowhere dense sets.

(iii) $\mathcal{A}$ is *co-meager* if and only if $2^\omega - \mathcal{A}$ is meager.

LEMMA 3.7. *Let $\mathcal{A}$ and $\mathcal{G}$ be subsets of 2^ω. If $\mathcal{A}$ is non-meager and $\mathcal{G}$ is co-meager then $\mathcal{A} \cap \mathcal{G}$ is not empty; in fact it is dense in some basic open set.*

Proof is clear.

§ 4. Baire category case.

LEMMA 4.1 [**9**]. *If $\mathcal{A}$ is a non-meager Π_1^1 subset of 2^ω, then $\mathcal{A}$ contains a hyperarithmetic subset $\mathcal{B}$ which is non-meager.*

PROOF. Let $S \subset 2^\omega \times \mathrm{Seq}$ be a *recursive* sieve determining the given Π_1^1 set $\mathcal{A}$, where Seq is the set of all finite sequences of natural numbers. Then

$$\alpha \in \mathcal{A} \longleftrightarrow S^{<\alpha>} \text{ is well-ordered by the Kleene-Brouwer ordering}$$

where $S^{<\alpha>} = \{u \mid \langle \alpha, u\rangle \in S\}$. Let $\mathcal{A}_\nu$ be the ν-th constituent of $\mathcal{A}$ with respect to S for each countable ordinal ν. Let $\mathcal{A}^* = \cup \{\mathcal{A}_\nu \mid \nu \geqq \omega_1\}$, where ω_1 is the first nonrecursive ordinal. Then

$$\begin{aligned} \alpha \in \mathcal{A}^* &\longrightarrow S^{<\alpha>} \text{ represents an ordinal } \geqq \omega_1, \\ &\longrightarrow \text{There is a recursive-in-}\alpha \text{ ordinal } \geqq \omega_1, \\ &\longrightarrow \omega_1^\alpha > \omega_1. \end{aligned}$$

Since $\{\alpha \in 2^\omega \mid \omega_1^\alpha > \omega_1\}$ is meager by Thomason [**10**; § 2.2], $\mathcal{A}^*$ is also meager. Therefore there exists a $\nu_0 < \omega_1$ such that $\mathcal{A}_{\nu_0}$ is not meager. It is well-known that for each $\nu < \omega_1$, $\mathcal{A}_\nu$ is a hyperarithmetic set. So $\mathcal{B} = \mathcal{A}_{\nu_0}$ is the desired.

LEMMA 4.2. *Let $\mathcal{B}$ be a hyperarithmetic set, and let $\mathcal{B} = \mathcal{H}(z, e)$ or $2^\omega - \mathcal{H}(z, e)$ for some $z \in \omega$ and $e \in O$. If $\mathcal{B} \cap \mathcal{G}(2^e)$ is not empty, then there is a hyperarithmetic, perfect tree T such that $[T] \subset \mathcal{B}$.*

PROOF. Let $\{\langle 0, z_n, e_n\rangle, \langle 1, z_n, e_n\rangle \mid n \in \omega\}$ be an enumeration of the members of $I(2^e)$. By [6], we may assume that the enumeration is hyperarithmetic. Suppose $\mathscr{B}=\mathscr{H}(z, e)$ and $\mathscr{B}\cap\mathscr{G}(2^e)$ is not empty. We shall hyperarithmetically construct an infinite binary system $\{\sigma_{i_0 i_1\cdots i_n}\}$ of basic open sets satisfying the following conditions:

(1) $\sigma_{i_0 i_1\cdots i_{n-1}j}\supset\sigma_{i_0 i_1\cdots i_{n-1}}$ for $j=0, 1$,

(2) $\sigma_{i_0 i_1\cdots i_n}\Vdash\langle 0, z, e\rangle$, and

(3) either $\sigma_{i_0 i_1\cdots i_n}\Vdash\langle 0, z_n, e_n\rangle$ or $\sigma_{i_0 i_1\cdots i_n}\Vdash\langle 1, z_n, e_n\rangle$.

Stage 0. Let β be an element of $\mathscr{B}\cap\mathscr{G}(2^e)$. Then by Lemma 3.5 there exists a number k such that $\bar{\beta}(k)\Vdash\langle 0, z, e\rangle$. Let $\sigma=\bar{\beta}(k)$.

Stage $m+1$. Suppose that $\sigma_{i_0 i_1\cdots i_{m-1}}$'s are defined for all binary sequences $\langle i_0, i_1, \cdots, i_{m-1}\rangle$ of length m (if $m=0$ it is σ) such that (1)-(3) hold for $n=m-1$.

Let $\sigma'_{i_0 i_1\cdots i_{m-1}}$ be the least $\tau\in\Sigma$ (as a natural number) such that

$$\tau\supset\sigma_{i_0 i_1\cdots i_{m-1}} \quad\text{and}\quad [\tau\Vdash\langle 0, z_m, e_m\rangle \quad\text{or}\quad \tau\Vdash\langle 1, z_m, e_m\rangle].$$

(Existence of such a τ follows from Lemma 3.4 (iii).) Define $\sigma_{i_0 i_1\cdots i_{m-1}j}=\sigma'_{i_0 i_1\cdots i_{m-1}}*\langle j\rangle$ for $j=0, 1$. Then clearly (1) holds for $n=m$. And by Lemma 3.4 (ii) (2) and (3) with $n=m$ hold, too.

Since $I(2^e)$ is infinite, the above process is infinite. Let $T=\{\tau\in\Sigma\mid\tau$ is an initial segment of some $\sigma_{i_0 i_1\cdots i_m}\}$. Then, clearly T is a perfect tree. Since the entire process of the definition of $\sigma_{i_0 i_1\cdots i_n}$'s is hyperarithmetic (by Lemma 3.3), it follows that T is a hyperarithmetic set. Now let $\alpha\in[T]$. Then there is a $\beta\in 2^\omega$ and a function h from ω into ω such that

$$lh(\sigma_{\beta(0)\beta(1)\cdots\beta(h(n))})>n \quad\text{and}\quad \bar{\alpha}(n)=\bar{\sigma}_{\beta(0)\beta(1)\cdots\beta(h(n))}(n).$$

So, by (3) $\alpha\in\mathscr{G}(2^e)$. Therefore by (2) and Lemma 3.5 $\alpha\in\mathscr{H}(z, e)$. Thus $[T]\subset\mathscr{H}(z, e)=\mathscr{B}$.

Similarly for the case $\mathscr{B}=2^\omega-\mathscr{H}(z, e)$.

COROLLARY 4.3. *If $\mathscr{H}(z, e)$ (or $2^\omega-\mathscr{H}(z, e)$) is countable, then it contains no 2^e-generic elements.*

LEMMA 4.4. *For each $e\in O$, $\mathscr{G}(e)$ is co-meager.*

Proof is entirely similar to that of Hinman [3; Lemma 9]. Note that $I(e)$ is countable.

By previous Lemma 4.1, 3.7, 4.4, 4.2 and 1.1 we can obtain the following theorem:

THEOREM B. *If $\mathscr{A}$ is a non-meager Π_1^1 set, then $\mathscr{A}$ has property (S).*

§ 5. Relativization.

We relativize several definitions and lemmas in the preceding sections.

DEFINITION 5.1. For $\alpha, \xi\in 2^\omega$ and $e\in O^\xi$ we define $H^{\alpha,\xi}(e)$ as follows:

1°) $e=1$. $H^{\alpha,\xi}(1)=\{2n \mid \alpha(n)=0\} \cup \{2n+1 \mid \xi(n)=0\}$.

2°) $e=2^d \wedge d \neq 0$. $H^{\alpha,\xi}(e)=(H^{\alpha,\xi}(d))'$

$$= \{z \mid (\exists \tau \in \Sigma)[T_1^1(\tau, z, z) \wedge (\forall n)_{n<lh(\tau)}\{(\tau(n)=0 \longrightarrow n \in H^{\alpha,\xi}(d)) \wedge (\tau(n)=1 \longrightarrow n \notin H^{\alpha,\xi}(d))\}]\}.$$

3°) $e=3\cdot 5^i$. $H^{\alpha,\xi}(e)=\{\langle u, v\rangle \mid v <_\xi e \wedge u \in H^{\alpha,\xi}(v)\}$,

where we simply write $<_\xi$ instead of $<_{O^\xi}$.

DEFINITION 5.2. $\mathcal{H}^\xi(z, e)=\{\alpha \in 2^\omega \mid z \in H^{\alpha,\xi}(e)\}$ for $e \in O^\xi$ and $z \in \omega$. Then $\mathcal{A}$ is a Borel set if and only if there are element $\xi \in 2^\omega$ (called a Borel code), a number $z \in \omega$ and an $e \in O^\xi$ such that

$$\mathcal{A}=\mathcal{H}^\xi(z, e) \quad \text{or} \quad \mathcal{A}=2^\omega-\mathcal{H}^\xi(z, e).$$

For example, $\mathcal{H}^\xi(z, 1)=\{\alpha \mid (\exists n)_{n\leq z}[z=2n \wedge \alpha(n)=0 \,.\vee.\, z=2n+1 \wedge \xi(n)=0]\}$.

DEFINITION 5.3. Let π be an element of 2^ω. For $\sigma \in \Sigma$, $\pi(\sigma)(\in \Sigma)$ is defined by

$$\pi(\sigma)(z)=\begin{cases} \sigma(z) & \text{if } z<lh(\sigma) \wedge \pi(z)=0 \\ 1-\sigma(z) & \text{if } z<lh(\sigma) \wedge \pi(z)=1, \end{cases}$$

$$lh(\pi(\sigma))=lh(\sigma).$$

For $\alpha \in 2^\omega$, $\pi(\alpha)(\in 2^\omega)$ is defined by

$$\pi(\alpha)(z)=\begin{cases} \alpha(z) & \text{if } \pi(z)=0, \\ 1-\alpha(z) & \text{if } \pi(z)=1. \end{cases}$$

LEMMA 5.4. (i) $\pi(\pi(\sigma))=\sigma$ *for* $\sigma \in \Sigma$; (ii) $\pi(\pi(\alpha))=\alpha$ *for* $\alpha \in 2^\omega$.

LEMMA 5.5. *Let* $\sigma, \rho \in \Sigma$. *Then* (i) $\pi(\rho) \supset \pi(\sigma)$ *if and only if* $\rho \supset \sigma$. (ii) $\pi(\rho) \supset \sigma$ *if and only if* $\rho \supset \pi(\sigma)$.

DEFINITION 5.6. For $\xi, \pi \in 2^\omega$, $z \in \omega$ and $e \in O^\xi$, we define $\pi(\pm\mathcal{H}^\xi(z, e))$ as follows:

$$\pi(\mathcal{H}^\xi(z, e))=\{\alpha \in 2^\omega \mid z \in H^{\pi(\alpha),\xi}(e)\},$$

$$\pi(-\mathcal{H}^\xi(z, e))=-\pi(\mathcal{H}^\xi(z, e)),$$

where we write $-\mathcal{H}$ for $2^\omega-\mathcal{H}$.

It directly follows from Definition 5.6 that

LEMMA 5.7. *For* $e \in O^\xi$

$$\alpha \in \pi(\pm\mathcal{H}^\xi(z, e)) \quad \textit{if and only if} \quad \pi(\alpha) \in \pm\mathcal{H}^\xi(z, e).$$

DEFINITION 5.8. For $j=0, 1$; $\xi, \pi \in 2^\omega$, $\sigma \in \Sigma$, $z \in \omega$ and $e \in O^\xi$, we define $\sigma \Vdash^\xi \langle j, z, e\rangle$ and $\sigma \Vdash^\xi_\pi \langle j, z, e\rangle$ as follows:

1°) $e=1.\quad \sigma \Vdash_\pi^\xi \langle 0, z, 1\rangle \longleftrightarrow z < lh(\sigma) \wedge (\exists n)_{n<z}$

$[z=2n \wedge n < lh(\sigma) \wedge \pi(\sigma)(n)=0 \ .\vee.\ z=2n+1 \wedge \xi(n)=0]$.

2°) $e=2^d \wedge d \neq 0.\quad \sigma \Vdash_\pi^\xi \langle 0, z, e\rangle \longleftrightarrow (\exists \tau \in \Sigma)$

$[T_1^1(\tau, z, z) \wedge (\forall n)_{n<lh(\tau)} \{(\tau(n)=0 \longrightarrow \sigma \Vdash_\pi^\xi \langle 0, n, d\rangle)$

$\wedge (\tau(n)=1 \longrightarrow \sigma \Vdash_\pi^\xi \langle 1, n, d\rangle)\}]$.

3°) $e=3\cdot 5^i.\quad \sigma \Vdash_\pi^\xi \langle 0, z, e\rangle \longleftrightarrow (z)_1 <_\xi e \wedge \sigma \Vdash_\pi^\xi \langle 0, (z)_0, (z)_1\rangle$.

4°) $\sigma \Vdash_\pi^\xi \langle 1, z, e\rangle \longleftrightarrow (\forall \tau)[\tau \supset \sigma \longrightarrow \text{not } \tau \Vdash_\pi^\xi \langle 0, z, e\rangle]$.

5°) $\sigma \Vdash^\xi \langle j, z, e\rangle \longleftrightarrow \sigma \Vdash_{\lambda x[0]}^\xi \langle j, z, e\rangle$,

where $\lambda z[0]$ denotes the constant 0-function and $j=0, 1$.

Then we have

LEMMA 5.9. *Under the same condition as in Definition* 5.8,

$$\pi(\sigma) \Vdash_\pi^\xi \langle j, z, e\rangle \quad \textit{if and only if} \quad \sigma \Vdash^\xi \langle j, z, e\rangle.$$

PROOF.

1°) $e=1.\quad \pi(\sigma) \Vdash_\pi^\xi \langle 0, z, 1\rangle$

$\longleftrightarrow (\exists n)_{n\leq z}[z=2n \wedge n < lh(\pi(\sigma)) \wedge \pi(\pi(\sigma))(n)=0$

$.\vee.\ z=2n+1 \wedge \xi(n)=0]$

$\longleftrightarrow \sigma \Vdash^\xi \langle 0, z, 1\rangle$.

2°) $e=2^d \wedge d \neq 0.\quad \pi(\sigma) \Vdash_\pi^\xi \langle 0, z, e\rangle$

$\longleftrightarrow (\exists \tau \in \Sigma)[T_1^1(\tau, z, z) \wedge (\forall n)_{n<lh(\tau)} \{(\tau(n)=0 \longrightarrow$

$\pi(\sigma) \Vdash_\pi^\xi \langle 0, n, d\rangle) \wedge (\tau(n)=1 \longrightarrow \pi(\sigma) \Vdash_\pi^\xi \langle 1, n, d\rangle)\}]$

$\longleftrightarrow (\exists \tau \in \Sigma)[T_1^1(\tau, z, z) \wedge (\forall n)_{n<lh(\tau)} \{(\tau(n)=0 \longrightarrow \sigma \Vdash^\xi \langle 0, n, d\rangle)$

$\wedge (\tau(n)=1 \longrightarrow \sigma \Vdash^\xi \langle 1, n, d\rangle)\}]$ (by the induction hypothesis)

$\longleftrightarrow \sigma \Vdash^\xi \langle 0, z, e\rangle$.

3°) $e=3\cdot 5^i.\quad \pi(\sigma) \Vdash_\pi^\xi \langle 0, z, e\rangle$

$\longleftrightarrow (z)_1 <_\xi e \wedge \pi(\sigma) \Vdash_\pi^\xi \langle 0, (z)_0, (z)_1\rangle$

$\longleftrightarrow (z)_1 <_\xi e \wedge \sigma \Vdash^\xi \langle 0, (z)_0, (z)_1\rangle$ (by the induction hypothesis)

$\longleftrightarrow \sigma \Vdash^\xi \langle 0, z, e\rangle$.

4°) $\pi(\sigma) \Vdash_\pi^\xi \langle 1, z, e\rangle$

$\longleftrightarrow (\forall \rho)[\rho \supset \pi(\sigma) \longrightarrow \text{not } \rho \Vdash_\pi^\xi \langle 0, z, e\rangle]$

$\longleftrightarrow (\forall \rho)[\rho \supset \pi(\sigma) \longrightarrow \text{not } \pi(\pi(\rho)) \Vdash_\pi^\xi \langle 0, z, e\rangle]$ (by Lemma 5.4)

$$\longleftrightarrow (\forall\rho)[\rho \supset \pi(\sigma) \longrightarrow \text{not } \pi(\rho) \Vdash^{\xi} \langle 0, z, e\rangle]$$

(by the induction hypothesis)

$$\longrightarrow (\forall\tau)[\tau \supset \sigma \longrightarrow \text{not } \tau \Vdash^{\xi} \langle 0, z, e\rangle]$$

$$\longleftrightarrow \sigma \Vdash^{\xi} \langle 1, z, e\rangle .$$

Conversely, if $(\forall\tau)[\tau \supset \sigma \rightarrow \text{not } \tau \Vdash^{\xi} \langle 0, z, e\rangle]$ then by the induction hypothesis

$$(\forall\tau)[\tau \supset \sigma \longrightarrow \text{not } \pi(\tau) \Vdash^{\xi}_{\pi} \langle 0, z, e\rangle]$$

$$\longrightarrow (\forall\rho)[\rho \supset \pi(\sigma) \longrightarrow \text{not } \rho \Vdash^{\xi}_{\pi} \langle 0, z, e\rangle]$$

$$\longrightarrow \pi(\sigma) \Vdash^{\xi}_{\pi} \langle 1, z, e\rangle .$$

LEMMA 5.10. *Under the same condition as in Definition* 5.8, *for given* $\sigma \in \Sigma$

(i) *not* $(\sigma \Vdash^{\xi}_{\pi} \langle 0, z, e\rangle$ *and* $\sigma \Vdash^{\xi}_{\pi} \langle 1, z, e\rangle)$,

(ii) $(\forall\tau \supset \sigma)[\sigma \Vdash^{\xi}_{\pi} \langle j, z, e\rangle \longrightarrow \tau \Vdash^{\xi}_{\pi} \langle j, z, e\rangle]$,

(iii) $(\exists\tau \supset \sigma)[\tau \Vdash^{\xi}_{\pi} \langle 0, z, e\rangle$ *or* $\tau \Vdash^{\xi}_{\pi} \langle 1, z, e\rangle]$.

PROOF is similar to that of Lemma 3.4. We show only Case 1°) of (ii). So, suppose $\tau \supset \sigma$ and $\sigma \Vdash^{\xi}_{\pi} \langle 0, z, 1\rangle$. Then there is an n such that $z = 2n \wedge lh(\sigma) > n \wedge \pi(\sigma)(n) = 0$ or $z = 2n+1 \wedge \xi(n) = 0$. By Lemma 5.5 $\pi(\tau)(n) = 0$ if $\pi(\sigma)(n) = 0$. So we have $\tau \Vdash^{\xi}_{\pi} \langle 0, z, 1\rangle$.

DEFINITION 5.11. For $e \in O^{\xi}$, α is $\langle \pi, \xi, e\rangle$-generic if and only if for every $|d|_{\xi} < |e|_{\xi}$ and $z \in \omega$

$$(\exists k)[\bar{\alpha}(k) \Vdash^{\xi}_{\pi} \langle 0, z, d\rangle \quad \text{or} \quad \bar{\alpha}(k) \Vdash^{\xi}_{\pi} \langle 1, z, d\rangle],$$

where we write $|e|_{\xi}$ for $|e|_{O^{\xi}}$. Let $\mathcal{G}(\pi, \xi, e) = \{\alpha \in 2^{\omega} \mid \alpha \text{ is } \langle \pi, \xi, e\rangle\text{-generic}\}$ for $e \in O^{\xi}$.

LEMMA 5.12. *Suppose* $e \in O^{\xi}$. *If* α *is* $\langle \pi, \xi, 2^{e}\rangle$*-generic, then there is a number* k *such that*

$$\bar{\alpha}(k) \Vdash^{\xi}_{\pi} \langle j, z, e\rangle \longleftrightarrow \alpha \in \pi((-1)^{j}\mathcal{H}^{\xi}(z, e)) .$$

PROOF.

1°) $e = 1$. $(\exists k)[\bar{\alpha}(k) \Vdash^{\xi}_{\pi} \langle 0, z, 1\rangle]$

$$\longleftrightarrow (\exists k)(\exists n)[z = 2n \wedge n < k \wedge \pi(\bar{\alpha}(k))(n) = 0$$

$$\text{or} \quad z = 2n+1 \wedge \xi(n) = 0]$$

$$\longleftrightarrow (\exists n)[z = 2n \wedge \pi(\alpha)(n) = 0 \quad \text{or} \quad z = 2n+1 \wedge \xi(n) = 0]$$

$$\longleftrightarrow \pi(\alpha) \in \mathcal{H}^{\xi}(z, 1)$$

$$\longleftrightarrow \alpha \in \pi(\mathcal{H}^{\xi}(z, 1)) \qquad \text{by Lemma 5.7.}$$

2°-4°) Similar to Lemma 3.5.

LEMMA 5.13. *For each* $e \in O^{\xi}$, $\mathcal{G}(\pi, \xi, e)$ *is co-meager.*

PROOF. Similar to Lemma 4.4 using (iii) of Lemma 5.10.

§ 6. Anti-chain with respect to $\leqq_T$.

In this section we shall show

THEOREM C. *If* $\mathcal{A}(\subseteqq 2^{\omega})$ *has the property of Baire and if* $\mathcal{A}$ *is not meager, then* $\mathcal{A}$ *contains an infinitely many elements which have the same Turing degree.*

This theorem has the following corollary:

DEFINITION 6.1. Let $\mathcal{A} \subseteqq 2^{\omega}$. $\mathcal{A}$ is called an anti-chain with respect to $\leqq_T$ if every distinct two elements of $\mathcal{A}$ have incomparable Turing degrees, i.e., if $\alpha, \beta \in \mathcal{A} \wedge \alpha \neq \beta \longrightarrow \alpha \nleqq_T \beta \wedge \beta \nleqq_T \alpha$.

It is known that there exists an anti-chain with respect to $\leqq_T$ whose cardinality is $2^{\aleph_0}$. Then

COROLLARY 6.2. *If* $\mathcal{A}$ *has the property of Baire and if* $\mathcal{A}$ *is an anti-chain with respect to* $\leqq_T$, *then* $\mathcal{A}$ *is meager.*

Immediately after we had obtained the Corollary, C. Jockusch obtained a stronger result:

THEOREM (Jockusch). *If* $\mathcal{A}$ *has the property of Baire and if the Turing degrees of the elements of* $\mathcal{A}$ *form an anti-chain with respect to their natural ordering, then* $\mathcal{A}$ *is meager.*

Jockusch does not use any forcing method. Also he pointed out that the theorem remains true when 'the property of Baire' is replaced by 'measurable' and 'meager' by 'measure zero'. Later he extended his theorem to a very general form. However, since our Theorem C is not contained in Jockusch's Theorem, we shall give a proof of Theorem C as an application of forcing method. It is similar to Feferman [**1**].

DEFINITION 6.3.

$$\pi_k(n) = \begin{cases} 0 & \text{if } n \neq k \\ 1 & \text{if } n = k. \end{cases}$$

As a corollary of Lemmas 3.7 and 5.13 we obtain

LEMMA 6.4. *Let* $e \in O^{\xi}$. *If* $\mathcal{A}$ *is not meager, then* $\mathcal{A} \cap \mathcal{G}(\lambda x[0], \xi, e) \cap \bigcap_{k=0}^{\infty} \mathcal{G}(\pi_k, \xi, e)$ *is not empty, in fact it is somewhere dense.*

LEMMA 6.5. *Let* $e \in O^{\xi}$. *If* $\alpha \in \pm\mathcal{H}^{\xi}(z, e) \cap \mathcal{G}(\lambda x[0], \xi, 2^e) \cap \bigcap_{k=0}^{\infty} \mathcal{G}(\pi_k, \xi, 2^e)$, *then for some* k

$$\pi_k(\alpha) \in \pm\mathcal{H}^{\xi}(z, e).$$

PROOF. Suppose $\alpha \in (-1)^j \mathcal{H}^{\xi}(z, e) \cap \mathcal{G}(\lambda x[0], \xi, 2^e) \cap \bigcap_{k=0}^{\infty} \mathcal{G}(\pi_k, \xi, 2^e)$. Then for some k $\bar{\alpha}(k) \Vdash^{\xi} \langle j, z, e\rangle$. Take π_k. By Lemma 5.9 $\pi_k(\bar{\alpha}(k)) \Vdash^{\xi}_{\pi_k} \langle j, z, e\rangle$. Since each number of the domain of $\bar{\alpha}(k)$ is less than k, $\pi_k(\bar{\alpha}(k)) = \bar{\alpha}(k)$. So, $\bar{\alpha}(k) \Vdash^{\xi}_{\pi_k} \langle j, z, e\rangle$. Since α is $\langle \pi_k, \xi, 2^e\rangle$-generic, by Lemma 5.12 $\alpha \in \pi_k((-1)^j \mathcal{H}^{\xi}(z, e))$. Therefore by Lemma 5.7, $\pi_k(\alpha) \in (-1)^j \mathcal{H}^{\xi}(z, e)$.

COROLLARY 6.6. *Let* $e \in O^{\xi}$. *If* $\alpha \in (-1)^j \mathcal{H}^{\xi}(z, e) \cap \mathcal{G}(\lambda x[0], \xi, 2^e) \cap \bigcap_{k=0}^{\infty} \mathcal{G}(\pi_k, \xi, 2^e)$, *then* $(-1)^j \mathcal{H}^{\xi}(z, e)$ *contains an infinitely many elements which have the same Turing degree.*

PROOF. By the proof of Lemma 6.5 and (ii) of Lemma 5.10, for all $m \geqq k$ $\pi_m(\alpha) \in (-1)^j \mathcal{H}^{\xi}(z, e)$. Clearly if $m \neq n$, $\pi_m(\alpha) \neq \pi_n(\alpha)$. And $\pi_m(\alpha)$ has the same Turing degree as α.

PROOF OF THEOREM C. Suppose that $\mathcal{A}$ has the property of Baire and $\mathcal{A}$ is not meager. Then there exist a nonempty open set E and two meager sets M_1, M_2 such that

$$\mathcal{A} = (E - M_1) \cup M_2 .$$

Since M_1 is contained in some meager F_σ set M_0, $\mathcal{A} \supseteq E - M_0$. Since E is not empty, $E - M_0$ is not meager. Therefore $\mathcal{A}$ contains a non-meager Π^0_2-in-ξ subset $\mathcal{B}$ for some code $\xi \in 2^\omega$. So, there is a $z \in \omega$ and an $e \in O^{\xi}$ such that $\mathcal{B} = -\mathcal{H}^{\xi}(z, e)$. By Lemma 6.4 $\mathcal{B} \cap \mathcal{G}(\lambda x[0], \xi, 2^e) \cap \bigcap_{k=0}^{\infty} \mathcal{G}(\pi_k, \xi, 2^e)$ is not empty. Therefore by Corollary 6.6, $\mathcal{B}$ has an infinitely many elements which have the same Turing degree, and hence so does $\mathcal{A}$.

§7. A generalization of Theorems A and B.

Let Γ be a class of sets which is closed under the hyperarithmetic operations. Let $\mathcal{A}$ be a subset of $\omega^k \times (2^\omega)^{l+1}$ which is in Γ. If $\mathcal{B} = \{\langle x_1, \cdots, x_k, \alpha_1, \cdots, \alpha_l\rangle \mid \langle x_1, \cdots, x_k, \alpha_1, \cdots, \alpha_l, \beta\rangle \in \mathcal{A}\}$ for some $\beta \in 2^\omega$, we say that $\mathcal{B}$ is Γ in β.

DEFINITION 7.1. Let $\mathcal{A} \subseteq 2^\omega$. $\mathcal{A}$ is called a Γ-Borel set (or a Γ-$\boldsymbol{\Pi}^1_1$ set) if there exists a $\beta \in \Gamma$ such that $\mathcal{A}$ is hyperarithmetic in β (or Π^1_1 in β).

We assume that Γ satisfies the following conditions. If α is Γ in β and β is Γ in γ then α is Γ in γ. Then we can define Γ degrees in the same way as Turing degrees.

DEFINITION 7.2. Let $\mathcal{A} \subseteq 2^\omega$. $\mathcal{A}$ has property (S^Γ) if for every Γ degree $\boldsymbol{d}$ there exists a member α of $\mathcal{A}$ whose Γ degree is $\boldsymbol{d}$.

Using results in §5 and Lemma 6.4 both with $\pi = \lambda x[0]$, we can prove:

THEOREM A^Γ. *If* $\mathcal{A}$ *is a* Γ-$\boldsymbol{\Pi}^1_1$ *set which has positive measure, then* $\mathcal{A}$ *has property* (S^Γ).

THEOREM B^Γ. *If* $\mathcal{A}$ *is a non-meager* Γ-$\boldsymbol{\Pi}^1_1$ *set, then* $\mathcal{A}$ *has property* (S^Γ).

Thus, for example, if $\mathcal{A}$ is a *constructible* $\boldsymbol{\Pi}^1_1$ set (i.e., if $\mathcal{A}$ is a $\boldsymbol{\Pi}^1_1$ set with a *constructible* code) and if $\mathcal{A}$ is of positive measure or not meager, then for every degree of nonconstructibility $\boldsymbol{d}$ there exists a member of $\mathcal{A}$ whose degree of nonconstructibility is $\boldsymbol{d}$. Here of course 'constructible' means Gödel's one [2].

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