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Another characterization of the small Janko group

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Recently, considerable interest has arisen in primitive permutation groups of small rank in which the stabiliser of a point has one or more multiply transitive constituents. (See [1], [2], [4], and many other papers.) Livingstone [6] constructed the small Janko simple group of order 175 560 as a permutation group of this type, with degree 266 and rank 5; the stabiliser of a point has doubly transitive constituents of degree 11 and 12. It follows from theorem 1 of [4] that a primitive group with two doubly transitive suborbits of different sizes has rank at least 5; here, such a group of rank 5, in which the subdegrees differ by 1, is shown to be isomorphic to the Janko group.

THEOREM. *Let G be a primitive rank* 5 *permutation group of degree n on Ω. Suppose that, for $a\in\Omega$, G_a is doubly transitive on $\Gamma(a)$ and $\Delta(a)$, where $|\Gamma(a)|=v$, $|\Delta(a)|=v+1$. Then $v=11$, $n=266$, and $G\cong J_1$ (the simple group of order* 175 560).

I use the notation and results of [2]. By Section 1 of [2] and Theorem 1 of [4], Γ and Δ are self-paired. By Theorem 2.2 of [2], $\Gamma\circ\Gamma$ and $\Delta\circ\Delta$ are single suborbits with $|(\Gamma\circ\Gamma)(a)|=v(v-1)/k$, $|(\Delta\circ\Delta)(a)|=v(v+1)/l$, with $k\leqq(1/2)(v-1)$, $l\leqq(1/2)v$. If $\Gamma\circ\Gamma=\Delta\circ\Delta$, then $(v+1)k=(v-1)l$, so $(1/2)(v+1)|l$, which is impossible. So $\Gamma\circ\Gamma\neq\Delta\circ\Delta$.

$(v, v+1)=1$, so if $c\in\Gamma(a)$ then G_{ac} is transitive on $\Delta(a)$ ([7], 17.3). This means that $\Gamma\circ\Delta$ is a single suborbit, with $|(\Gamma\circ\Delta)(a)|=v(v+1)/m$, where m is the number of points in $\Gamma(a)\cap\Delta(b)$ for $b\in(\Gamma\circ\Delta)(a)$. If $m>1$, let c, d be two points in this intersection. Then $a\in\Gamma(c)\cap\Gamma(d)$, $b\in\Delta(c)\cap\Delta(d)$, and so (c, d) lies both in $\Gamma\circ\Gamma$ and $\Delta\circ\Delta$. This contradicts the fact that these are different suborbits; so $m=1$. Since G has rank 5, $\Gamma\circ\Delta$ must be equal to one of Γ, Δ, $\Gamma\circ\Gamma$, or $\Delta\circ\Delta$, and it is too large to be any of the first three. So $\Gamma\circ\Delta=\Delta\circ\Delta$, and $l=1$.

Let q and r be rational numbers such that, for $e\in(\Delta\circ\Delta)(a)$, $|(\Gamma\circ\Gamma)(a)\cap\Delta(e)|=q(v-1)$, $|(\Gamma\circ\Gamma)(a)\cap\Gamma(e)|=r(v-1)$. All the intersection numbers [5] involving Γ and Δ can be expressed in terms of v, k, q, and r. (See Fig. 1.) Note that the diameter of the graph corresponding to Γ is 4, and so is maximal (as defined in [5]). If B, C, D, E denote the basis matrices corresponding to Γ, $\Gamma\circ\Gamma$, $\Delta\circ\Delta$, and Δ respectively, we can read off from Fig. 1 the relations given below for the commuting symmetric matrices B, C, D, E.

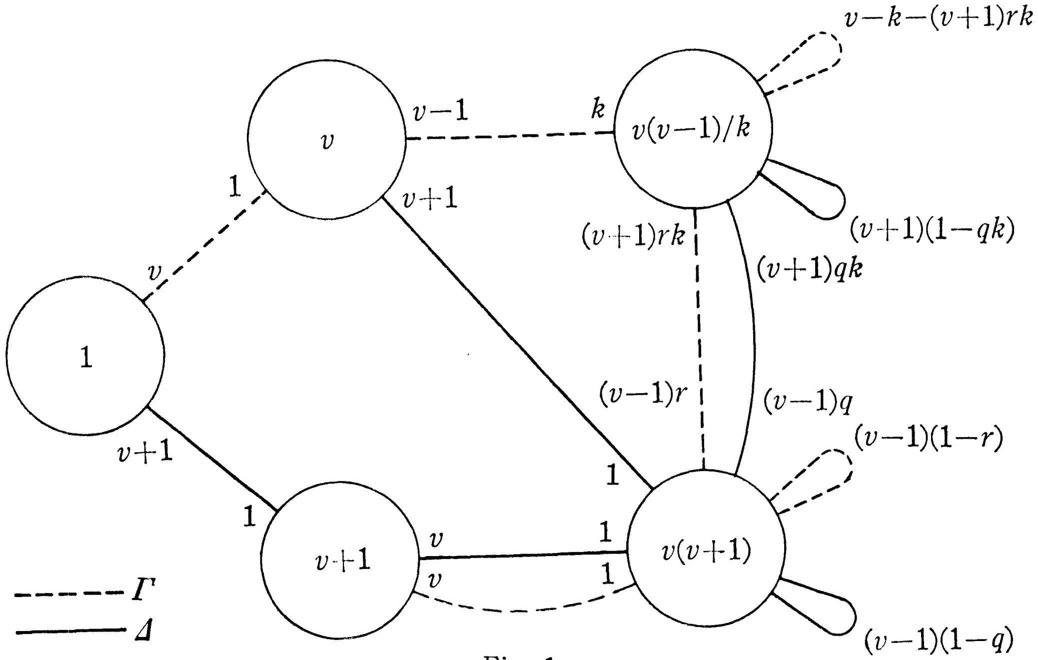


Fig. 1.

$$B^2 = vI + kC,$$
$$BE = D,$$
$$E^2 = (v+1)I + D,$$
$$ED = vE + (v-1)(1-q)D + qk(v+1)C + (v+1)B,$$
$$EC = q(v-1)D + (v+1)(1-qk)C.$$

Then

$$E(E-B) = (v+1)I,$$
$$E^2 - B^2 = I + D - kC,$$

so

$$(v+1)(E+B) = E(I+D-kC)$$
$$= E + vE + (v-1)(1-q)D + qk(v+1)C + (v+1)B$$
$$-qk(v-1)D - k(v+1)(1-qk)C,$$
$$(1-q-qk)((v-1)D - k(v+1)C) = 0.$$

So

$$1 - q - qk = 0,$$
$$q = 1/(1+k).$$

Since $q(v-1)$ and $qk(v+1)$ are integers, we have $1+k\,|\,v-1$ and $1+k\,|\,v+1$, so $1+k\,|\,2$, $k=1$, $q=1/2$, and v is odd. Also, $r(v-1)$ and $r(v+1)$ are integers, and $0 < r(v+1) < v+1$; so $r = 1/2$.

$(\Delta \circ \Delta)(a)$ is isomorphic, as G_a-space, to the set of ordered pairs of distinct points of $\Delta(a)$. (To the pair (d_1, d_2) corresponds the unique point in $\Delta(d_1) \cap \Gamma(d_2)$.) Let N be the kernel of the action of G_a on $\Delta(a)$. N act trivially on $(\Delta \circ \Delta)(a)$, and hence on the whole of Ω (since the Δ-graph is connected); that is, $N = 1$.

Put $G_a = H$, $\Delta(a) = U$, $\Gamma(a) = V$, and $(1/2)(v-1) = t$. I shall show that H, U, V, t satisfy the hypotheses of Theorem 2 of [3]; these are that H is a permutation group on $U \cup V$ with orbits U and V, with $|U| = 2(t+1)$, $|V| = 2t+1$, and

(i) H is doubly transitive on U;

(ii) for $u_1, u_2 \in U$, $H_{u_1 u_2}$ has two orbits of size t in $U - \{u_1, u_2\}$ which are isomorphic as $H_{u_1 u_2}$-spaces;

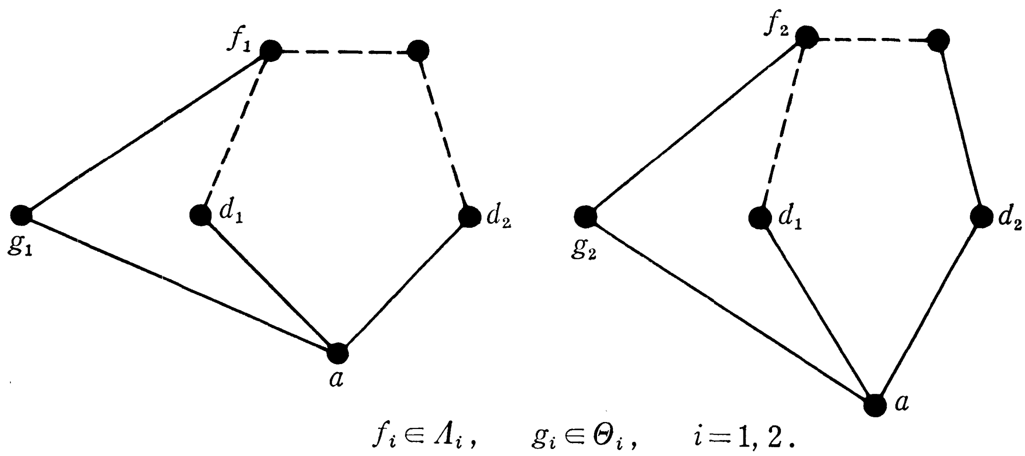
(iii) for $u \in U$, $U - \{u\}$ and V are isomorphic H_u-spaces.

The first hypothesis follows from our assumptions.

For $e \in (\Delta \circ \Delta)(a)$, all G_{ae}-orbits in $(\Gamma \circ \Gamma)(a)$ have size divisible by t ([7], 17.3). In particular, G_{ae} is transitive on $\Gamma(e) \cap (\Gamma \circ \Gamma)(a)$ and $\Delta(e) \cap (\Gamma \circ \Gamma)(a)$. Equivalently, for $d_1, d_2 \in \Delta(a)$, $G_{ad_1d_2}$ is transitive on $\Lambda_1 = \Gamma(d_1) \cap (\Gamma \circ \Gamma)(d_2)$ and on $\Lambda_2 = \Gamma(d_1) \cap (\Gamma \circ \Delta)(d_2)$. Putting

$$\Theta_i = \bigcup_{f_i \in \Lambda_i} \Delta(f_i) \cap \Delta(a), \qquad i = 1, 2,$$

Θ_1 and Θ_2 are orbits of $G_{ad_1d_2}$ in $\Delta(a) - \{d_1, d_2\}$, both of length t; and $\Theta_1 \cap \Theta_2 = \emptyset$, since if $f_i \in \Lambda_i$ $(i = 1, 2)$ then $f_2 \in (\Gamma \circ \Gamma)(f_1)$, and so $\Delta(f_1) \cap \Delta(f_2) = \emptyset$. (See Fig. 2.)



$f_i \in \Lambda_i$, $\quad g_i \in \Theta_i$, $\quad i = 1, 2$.

Fig. 2.

We must show that the stabiliser (in $G_{ad_1d_2}$) of a point $g_1 \in \Theta_1$ fixes a point in Θ_2. Equivalently, we must show that $G_{ad_1d_2g_1}$ fixes d_1', d_2' and g_2', where g_2' lies in the orbit Θ_2 with respect to d_1' and d_2'.

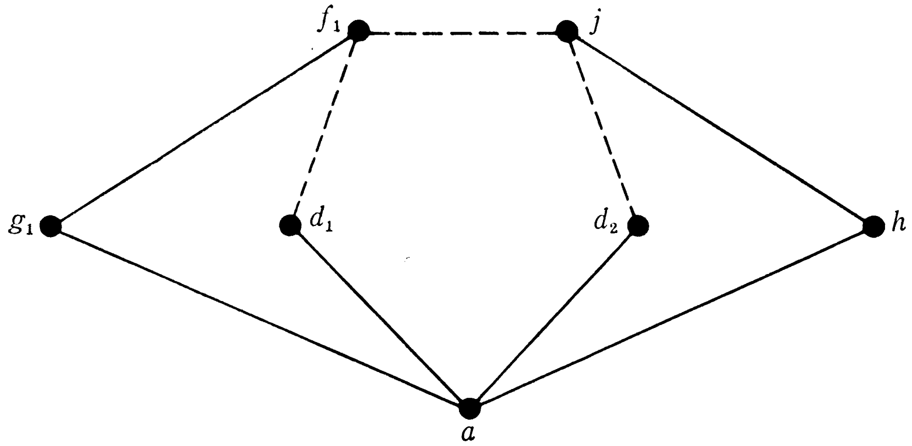


Fig. 3.

There is a unique point $h \in \Delta(j) \cap \Delta(a)$, which is fixed by $G_{ad_1d_2g_1}$ (Fig. 3); put $d_1' = d_1$, $d_2' = h$, $g_2' = g_1$. So hypothesis (ii) holds.

Finally, for $d \in \Delta(a)$, G_{ad} is transitive on $\Delta(a) - \{d\}$ and on $\Gamma(a)$. We must show that the stabiliser of a point in one of these sets fixes a point in the other. If $d' \in \Delta(a) - \{d\}$, there is a unique point $e \in \Delta(d) \cap \Gamma(d')$, and a unique point $c \in \Gamma(a) \cap \Delta(e)$; then $G_{add'}$ fixes c.

Theorem 2 of [**3**] shows that only five groups satisfy the hypotheses listed earlier. Of these five, two fail to satisfy the further condition that H is doubly transitive on V. We conclude that $t = 2$, 3, or 5, and $H \cong PSL(2, 2t+1)$.

The cases $t = 2$ and $t = 3$ can be eliminated by applying Sylow's theorem to the primes 31 and 19 respectively, or by the methods of D. G. Higman [**5**]. For $t = 5$, the small Janko group J_1 satisfies the hypotheses of the theorem; its uniqueness follows from Livingstone's construction [**6**] or the characterisation of J_1 by its order.

REMARKS 1. Some of these arguments can be applied in more general situations. For example, if G is a primitive rank 5 group in which G_a is doubly transitive on $\Gamma(a)$ and $\Delta(a)$, where $|\Gamma(a)| = v$, $|\Delta(a)| = w > v$, and $\Gamma \circ \Gamma \neq \Delta \circ \Delta$, then $w = vl + 1$ for some integer l, the other subdegrees are $v(v-1)$ and $v(vl+1)$, and all intersection numbers can be computed in terms of v and l.

2. The small Janko group is the only example known to the author of a primitive rank 5 group with two self-paired doubly transitive constituents. Two examples of primitive rank 5 groups with paired doubly transitive constituents are a group of degree 27 with a regular normal subgroup and a group of degree 144 isomorphic to the Mathieu group M_{12}.

3. The proof given in this paper is intended to show that results can be obtained about groups of moderate rank, under suitable assumptions,

without resorting to computation of the eigenvalues of the incidence matrices and their multiplicities (as outlined in [5]). The referee, however, has pointed out that the conclusion $v=11$ can be obtained by using these methods and obtaining fairly precise estimates for the eigenvalues and multiplicities, as was done by E. Bannai and T. Ito for Moore graphs ("The non-existence of certain Moore graphs", unpublished). This method (which, however, requires quite lengthy calculations) should also be applicable to the cases mentioned in remark 1, at least for small values of l; but it is the author's belief that it must be combined with the combinatorial analysis of doubly transitive groups as used in this paper and [3] to give the strongest results.

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