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Anti-locality of certain functions of the Laplace operator

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§ 1. Introduction.

In connection with non relativistic approximation of the relativistic quantum theory, I. Segal and R. Goodman [4] showed that the fractional power $(m^2I-\Delta)^\lambda$ (λ; non-integral number) of $(m^2I-\Delta)$ is anti-local in $L^2(E_n)$ when the space dimension n is odd, where the anti-locality means that if f and $(m^2I-\Delta)^\lambda f$ ($f\in L^2(E_n)$) vanish in some non-empty open set U in E_n, then $f(x)$ must be identically zero in E_n.

The anti-locality of the operator $(m^2I-\Delta)^{1/2}$ is also relevant to the quantum field theory, and the result of H. Reeh and S. Schlieder [2] is essentially equivalent to the anti-locality of the operator $(m^2I-\Delta)^{1/2}$ (on E_n). Recently K. Masuda [1] generalized the result of H. Reeh and S. Schlieder, and showed that $(-T)^{1/2}$ is anti-local in $L^2(\Omega)$ where $T=\sum_{j,k=1}^{n}\frac{\partial}{\partial x_j}a_{jk}(x)\frac{\partial}{\partial x_k}+a(x)$ is an elliptic operator associated with Dirichlet condition.

The purpose of the present paper is to show that the operator $(m^2I-\Delta)^\lambda$ has the anti-local property even if the space dimension is even.

Let A be the differential operator $\sum_{j,k=1}^{n}a_{jk}(D_j-b_j)(D_k-b_k)$ where $D_j=-i\frac{\partial}{\partial x_j}$, $\{a_{jk}\}$ is a constant positive definite symmetric matrix and $\{b_j\}$ is a constant real vector. For any function $h\in C^\infty([0,\infty))$ which has polynomial growth with its derivatives, we define the operator $h(A)$ by

$$h(A)f=\mathcal{F}^{-1}\circ h(\sum_{j,k=1}^{n}a_{jk}(\xi_j-b_j)(\xi_k-b_k))\circ\mathcal{F}(f)\,,\qquad f\in\mathcal{S}'(E_n)$$

where $\mathcal{S}'(E_n)$ is the space of temperate distributions, and $\mathcal{F}$ is the Fourier transform on $\mathcal{S}'(E_n)$. Our result is the following

THEOREM. *Let the functian $h(t)\in C^\infty([0,\infty))$ have polynomial growth with its derivatives, and let $q(t)$ be the composition $h(t^2)$ of h and the function: $t\to t^2$. Suppose that the function $q(t)$ has the following properties:*

(i) *$q(t)$ is real analytic in (R,∞) for some $R>0$, and the restriction $q|(R,\infty)$ onto (R,∞) of $q(t)$ can be continued analytically to the domain*

$\boldsymbol{C}\backslash((-\infty, -R]\cup\{t\in\boldsymbol{C};\ |t|\leqq R\})$; *we denote the extension by* $q_1(t)$;

(ii) *There exist positive constants* C *and* N *such that* $|q_1(t)|\leqq C(1+|t|)^N$ *for all complex* t *such that* $|t|>R$ *and* $\mathrm{Im}\, t\neq 0$;

(iii) $q_1(-t)\not\equiv q_1(t)$ *in the half plane* $\{t;\ \mathrm{Im}\, t>R\}$.

Then the operator $h(A)$ *is anti-local in* $\mathcal{S}'(E_n)$, *i.e. if* f *and* $h(A)f$ $(f\in\mathcal{S}'(E_n))$ *vanish in some non-empty open set* U, *then* f *must be zero in* E_n.

In § 2, we prove the theorem. We shall show that the operator $h(A)$ is anti-local not only in $L^2(E_n)$ but also in $\mathcal{S}'(E_n)$ when n is odd. Then we shall reduce, by the method of descent, the even-dimensional case to the odd-dimensional case. In § 3, as applications we show that some operators such as $(m^2I-\Delta)^\lambda$ have the anti-locality.

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§ 2. Proof of the theorem.

We may assume without loss of generality that $A=-\Delta$. In fact, we have by the change of variable

$$\begin{aligned}h(A)f(x)&=\int h\Big(\sum_{j,k=1}^{n}a_{jk}(\xi_j-b_j)(\xi_k-b_k)\Big)\hat{f}(\xi)e^{ix\cdot\xi}d\!\!\!{}^{-}\xi\\&=(\det M)^{-1/2}e^{ix\cdot b}\int h(|\xi|^2)\hat{f}(M^{-1/2}\xi+b)e^{iM^{-1/2}x\xi}d\!\!\!{}^{-}\xi\\&=e^{ix\cdot b}h(-\Delta)F(M^{-1/2}x)\end{aligned}$$

where $M=\{a_{jk}\}$, $F(x)=f(M^{1/2}x)e^{-iM^{1/2}x\cdot b}$ and $d\!\!\!{}^{-}\xi=(2\pi)^{-n}d\xi$. Hence, if f and $h(A)f$ vanish in some open set U, then F and $h(-\Delta)F$ vanish in $M^{-1/2}U$. If this implies $F=0$, then $f=0$ will follow.

We may assume also that $f\in C^\infty(E_n)\cap\mathcal{S}'(E_n)$. Let $f\in\mathcal{S}'(E_n)$ and take $\varphi\in C_0^\infty(E_n)$ such that $\int_{E_n}\varphi(x)dx=1$ and $\varphi(x)\geqq 0$. Then $f_j=f*\varphi_j\in C^\infty(E_n)$ converges to f in $\mathcal{S}'(E_n)$, where $\varphi_j(x)=j^n\varphi(jx)$. Moreover, if f and $h(A)f$ vanish in some non-empty open set, then f_j and $h(A)f_j=h(A)f*\varphi_j$ both vanish in some open set. If this implies $f_j=0$, then $f=0$, will follow.

The case $n=1$. Since $h\Big(-\frac{d^2}{dx^2}\Big)$ is translation invariant, it suffices to prove that if f and $h\Big(-\frac{d^2}{dx^2}\Big)f$ $(f\in\mathcal{S}'(E_1))$ vanish in $(-\delta, \delta)$, then $f=0$. We set $f_\pm(x)=Y(\pm x)f(x)$, where $Y(x)$ is Heaviside's function. We set $q_2(t)=q_1(-t)$. We first claim that

$$Hf_+(x)=e^{2xR}\frac{1}{2\pi i}\int_\delta^\infty(D_x-i)^k\Big(\frac{1}{x-y}\Big)g_+(y)dy+F_+(x)\quad\text{in } (-\infty, \delta)\qquad(1)$$

where $H=h\left(-\frac{d^2}{dx^2}\right)$, $g_+(x)=\mathcal{F}^{-1}[(\xi-i)^{-k}(q_2-q_1)(\xi-2Ri)\hat{f}_+(\xi-2Ri)]$ is an L^2-function for some positive integer k, and $F_+(x)$ is an entire function.

To this end, let us represent the distribution f_+ in the form $f_+=\sum_{\alpha,\beta=1}^{l} p_\alpha D^\beta f_{\alpha\beta}$, where p_α are polynomials in x, and $f_{\alpha\beta}\in L^1(E_1)$ such that $\mathrm{Supp}\,(f_{\alpha\beta})\subset[\delta,\infty)$, (see [3]). Using this representation of f_+, we set $f_+^j(x)=\sum_{\alpha,\beta=1}^{l} p_\alpha(x)D^\beta f_{\alpha\beta}^j(x)$ where $f_{\alpha\beta}^j$ converges to $f_{\alpha\beta}$ in $L^1(E_1)$ and $f_{\alpha\beta}^j\in C_0^\infty((0,\infty))$. Since $\hat{f}_+^j(\xi)$ can be continued analytically to the entire plane C, we have by Cauchy's theorem

$$\begin{aligned}
Hf_+^j(x) &= \int_{-\infty}^{-2R} q_2(\xi)\hat{f}_+^j(\xi)e^{ix\cdot\xi}d\xi+\int_{-2R}^{2R} q(\xi)\hat{f}_+^j(\xi)e^{ix\cdot\xi}d\xi \\
&\quad+\int_{2R}^{\infty} q_1(\xi)\hat{f}_+^j(\xi)e^{ix\cdot\xi}d\xi \\
&= \int_{-\infty}^{0} q_2(\xi-2Ri)\hat{f}_+^j(\xi-2Ri)e^{ix(\xi-2Ri)}d\xi \\
&\quad+\int_{\Gamma_2} q_2(\zeta)\hat{f}_+^j(\zeta)e^{ix\cdot\zeta}d\zeta+\int_{-2R}^{2R} q(\xi)\hat{f}_+^j(\xi)e^{ix\cdot\xi}d\xi \\
&\quad+\int_{0}^{\infty} q_1(\xi-2Ri)\hat{f}_+^j(\xi-2Ri)e^{ix(\xi-2Ri)}d\xi \\
&\quad+\int_{\Gamma_1} q_1(\zeta)\hat{f}_+^j(\zeta)e^{ix\cdot\zeta}d\zeta
\end{aligned}$$

where Γ_1 is the directed line segment from $2R$ to $-2Ri$, and the directed line segment Γ_2 goes $-2Ri$ to $-2R$.

If we define the closed curve Γ and the function q_Γ on Γ as

$$\Gamma=\Gamma_2+[-2R,2R]+\Gamma_1$$

$$q_\Gamma(\zeta)=\begin{cases} q_2(\zeta) & \text{on } \Gamma_2, \\ q(\zeta) & \text{on } [-2R,2R] \\ q_1(\zeta) & \text{on } \Gamma_1 \end{cases}$$

then we obtain by the integration by parts

$$\begin{aligned}
\int_{\Gamma_2}+\int_{-2R}^{2R}+\int_{\Gamma_1} &= \int_\Gamma \sum_{\alpha,\beta}\{p_\alpha(-2\pi D_\xi)[\zeta^\beta\hat{f}_{\alpha\beta}^j(\zeta)]\}\,q_\Gamma(\zeta)e^{ix\cdot\xi}d\zeta \\
&= \int_\Gamma \sum_{\alpha,\beta}\zeta^\beta\hat{f}_{\alpha\beta}^j(\zeta)\{p_\alpha(2\pi D_\zeta)[q_\Gamma(\zeta)e^{ix\cdot\zeta}]\}\,d\zeta \\
&\quad+e^{2xR}\sum_k\{\sum_{\alpha,\beta,\gamma,\gamma'} D^\gamma\hat{f}_{\alpha\beta}^j(-2Ri)[C_{\alpha\beta\gamma\gamma'}^{1,k}D^{\gamma'}q_1(-2Ri) \\
&\quad+C_{\alpha\beta\gamma\gamma'}^{2,k}D^{\gamma'}q_2(-2Ri)]\}\,x^k\,.
\end{aligned}$$

On the other hand, since $f_{\alpha\beta}^j$ converges to $f_{\alpha\beta}$ in $L^1(E_1)$ and $\operatorname{Supp} f_{\alpha\beta}^j \cup \operatorname{Supp} f_{\alpha\beta} \subset [0, \infty)$, we have

$$f_+^j \longrightarrow f_+ \quad \text{in } \mathcal{S}'(E_1),$$

$$\hat{f}_+^j(\zeta) \longrightarrow \hat{f}_+(\zeta) \quad \text{uniformly on the half plane } \{\zeta\,;\, \operatorname{Im}\zeta \leqq -\varepsilon\} \text{ for any } \varepsilon,$$

$$\hat{f}_{\alpha\beta}^j(\zeta) \longrightarrow \hat{f}_{\alpha\beta}(\zeta) \quad \text{uniformly on the lower half plane } \{\zeta\,;\, \operatorname{Im}\zeta \leqq 0\},$$

$$D^\gamma \hat{f}_{\alpha\beta}^j(-2Ri) \longrightarrow D^\gamma \hat{f}_{\alpha\beta}(-2Ri).$$

Hence, letting $j\to\infty$ we have

$$\begin{aligned} Hf_+(x) = e^{2xR}\mathcal{F}^{-1}[Y(\xi)q_1(\xi-2Ri)\hat{f}_+(\xi-2Ri)] \\ +e^{2xR}\mathcal{F}^{-1}[Y(-\xi)q_2(\xi-2Ri)\hat{f}_+(\xi-2Ri)]+F_+(x), \end{aligned} \tag{2}$$

where

$$\begin{aligned} F_+(x) = \int_\Gamma \sum_{\alpha,\beta=1}^{l} \zeta^\beta \hat{f}_{\alpha\beta}(\zeta)\{p_\alpha(2\pi D_\zeta)[q_\Gamma(\zeta)e^{ix\cdot\zeta}]\}\, d\zeta \\ +e^{2xR}\sum_k \{\sum_{\alpha,\beta,\gamma,\gamma'} D^\gamma \hat{f}_{\alpha\beta}(-2Ri)[C_{\alpha\beta\gamma\gamma'}^{1,k} D^{\gamma'}q_1(-2Ri)+C_{\alpha\beta\gamma\gamma'}^{2,k} D^{\gamma'}q_2(-2Ri)]\}\, x^k. \end{aligned}$$

Next we investigate the first two terms of the right hand side in (2). Since $f_+\in\mathcal{S}'(E_1)$ vanishes in $(-\infty, \delta)$, the function $e^{i\delta\zeta}\hat{f}_+(\zeta-2Ri)$ is analytic in the half plane $\{\zeta\,;\, \operatorname{Im}\zeta<2R\}$ and of at most polynomial growth at infinity, which implies the functions $e^{i\delta\zeta}[(\zeta-i)^{-k}q_j(\zeta-2Ri)\hat{f}_+(\zeta-2Ri)]$ $(j=1, 2)$ belong to the Hardy class for some positive integer k. Hence the L^2-functions

$$\mathcal{F}^{-1}[(\xi-i)^{-k}q_j(\xi-2Ri)\hat{f}_+(\xi-2Ri)](x) \qquad (j=1, 2)$$

vanish in $(-\infty, \delta)$. Hence we have in $(-\infty, \delta)$

$$\begin{aligned} & e^{2xR}\mathcal{F}^{-1}[Y(\xi)q_1(\xi-2Ri)\hat{f}_+(\xi-2Ri)]+e^{2xR}\mathcal{F}^{-1}[Y(-\xi)q_2(\xi-2Ri)\hat{f}_+(\xi-2Ri)] \\ &= e^{2xR}(D_x-i)^k\{\mathcal{F}^{-1}[(\xi-i)^{-k}q_1(\xi-2Ri)\hat{f}_+(\xi-2Ri)] \\ &\quad +\mathcal{F}^{-1}[Y(-\xi)(\xi-i)^{-k}(q_2-q_1)(\xi-2Ri)\hat{f}_+(\xi-2Ri)]\} \\ &= e^{2xR}(D_x-i)^k\{\mathcal{F}^{-1}[Y(-\xi)]*\mathcal{F}^{-1}[(\xi-i)^{-k}(q_2-q_1)(\xi-2Ri)\hat{f}_+(\xi-2Ri)]\} \\ &= e^{2xR}\frac{1}{2\pi i}\int_\delta^\infty (D_x-i)^k\Big(\frac{1}{x-y}\Big)g_+(y)dy. \end{aligned}$$

This proves the claim (1).

In a similar way we obtain

$$Hf_-(x) = e^{-2xR}\frac{1}{2\pi i}\int_{-\infty}^{-\delta}(D_x+i)^k\Big(\frac{1}{x-y}\Big)g_-(y)dy+F_-(x) \qquad \text{in } (-\delta, \infty)$$

where $g_-(x)=\mathcal{F}^{-1}[(\xi+i)^{-k}(q_2-q_1)(\xi+2Ri)\hat{f}_-(\xi+2Ri)]$ is an L^2-function and $F_-(x)$ is an entire function defined in the same way as $F_+(x)$. Then we have in $(-\delta, \delta)$

$$Hf(x) = e^{2xR}G_+(x)+e^{-2xR}G_-(x)+F_+(x)+F_-(x)$$

where $G_{\pm}(x)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}(D_x\mp i)^k\left(\frac{1}{x-y}\right)g_{\pm}(y)dy.$

Since $G_+(x)$ and $G_-(x)$ are analytically continued to the domain $C\backslash[\delta, \infty)$ and $C\backslash(-\infty, -\delta]$ respectively and since $e^{\pm 2xR}G_{\pm}(x)=-e^{\mp 2xR}G_{\mp}(x)-F_+(x)-F_-(x)$ in $(-\delta, \delta)$ by the assumption that $Hf(x)$ vanishes in $(-\delta, \delta)$, $G_{\pm}(x)$ are analytically continued to the entire plane C. Hence we have

$$0=\lim_{\varepsilon\downarrow 0}(G_+(x-i\varepsilon)-G_+(x+i\varepsilon))=(D-i)^k g_+(x) \qquad \text{in } \mathcal{S}'(E_1)$$

$$0=\lim_{\varepsilon\downarrow 0}(G_-(x-i\varepsilon)-G_-(x+i\varepsilon))=(D+i)^k g_-(x) \qquad \text{in } \mathcal{S}'(E_1),$$

which imply that

$$(q_2-q_1)(\xi-2Ri)\hat{f}_+(\xi-2Ri)=0, \qquad \text{for any } \xi\in E_1^*$$

$$(q_2-q_1)(\xi+2Ri)\hat{f}_-(\xi+2Ri)=0, \qquad \text{for any } \xi\in E_1^*,$$

and hence $f=0$ by the assumption (iii). q. e. d.

Now we turn to the general case. We set $H_n=h(-\Delta)$, acting on $\mathcal{S}'(E_n)$, and let $\mathcal{F}_n$ denote Fourier transform on $\mathcal{S}'(E_n)$.

The case n is odd. We shall first demonstrate the anti-locality on radially symmetric functions. To this end we use the following

LEMMA (Segal-Goodman [4] Lemma 3). *Let $f\in C^{\infty}(E_n)\cap\mathcal{S}'(E_n)$ be a radially symmetric function vanishing in a neighborhood of* 0. *Let the operator D be defined as $Df=\mathcal{F}_1^{-1}\circ|\xi|^{n-2}\circ\mathcal{F}_n(f)$. If $n=2k+1$, then the following properties* (i), (ii), (iii) *hold.*

(i) *There exist constants $C_{\alpha\beta}$ such that*

$$Df=\sum_{\substack{\alpha\leqq k\\ \beta\leqq k-1}}C_{\alpha\beta}r^{\alpha}\cdot\left(\frac{1}{i}\frac{\partial}{\partial r}\right)^{\beta}f.$$

(ii) $H_1Df=DH_nf$.

(iii) *If $Df=0$, then $f=0$.*

PROOF. We first observe that (i) holds for $f\in C_0^{\infty}(E_n)$ thanks to Segal-Goodman [4]. Let $f\in C^{\infty}(E_n)\cap\mathcal{S}'(E_n)$ and take $\varphi\in C_0^{\infty}(E_1)$ such that $\varphi(r)=1$ when $|r|\leqq 1$. Then $f_j(r)=f(r)\varphi\left(\frac{r}{j}\right)\in C_0^{\infty}(E_n)$, and Df_j and $\sum_{\alpha,\beta}C_{\alpha\beta}r^{\alpha}\left(\frac{1}{i}\frac{\partial}{\partial r}\right)^{\beta}f_j$ both converge to Df and $\sum_{\alpha,\beta}C_{\alpha\beta}r^{\alpha}\left(\frac{1}{i}\frac{\partial}{\partial r}\right)^{\beta}f$ in $\mathcal{S}'(E_1)$ respectively. Since (i) holds for f_j, we conclude that it holds also for f.

For the proof of (ii), we have only to observe that

$$H_1Df=\mathcal{F}_1^{-1}\circ h(|\xi|^2)\circ|\xi|^{n-2}\circ\mathcal{F}_n(f)=\mathcal{F}_1^{-1}\circ|\xi|^{n-2}\circ h(|\xi|^2)\circ\mathcal{F}_n(f)=DH_nf.$$

Finally, if $Df=0$, then $|\xi|^{n-2}\mathcal{F}_nf=0$, and so $\mathrm{Supp}(\mathcal{F}_nf)=\{0\}$, from which it follows that f is a polynomial. On the other hand, since f vanishes near the origin, we have $f=0$. q. e. d.

The Lemma together with the anti-locality of H_1 implies that H_n is anti-local on radially symmetric functions. Indeed, if f and $H_n f$ vanish near the origin, then Df and $H_1(Df)=D(H_n f)$ vanish near the origin. Hence $Df=0$, and so $f=0$.

Now, following Segal-Goodman [**4**], we shall reduce the problem to the radially symmetric case. Let $\tilde{f}_x(r)=\int_{|\omega|=1} f(x+r\omega)d\omega$ be the integral of f over the sphere of radius r about x in E_n. Since H_n commutes with translations and rotations, it follows that $H_n\tilde{f}_x=\widetilde{(H_n f)}_x$. Thus, if f and $H_n f$ vanish in a neighborhood of 0, then $\tilde{f}_x(r)\equiv 0$ for all x in a neighborhood of 0. Set $u(x,t)=(\partial/\partial t)^{n-2}\int_0^t \tilde{f}_x(r)(t^2-r^2)^{(n-3)/2}r\,dr$. Then u satisfies the wave equation $\Box u=0$, with initial data $u(x,0)=0$, $\frac{\partial u}{\partial t}(x,0)=C\cdot f(x)$. But $u(x,t)=0$ for x near 0 and for all t, which implies that $u=0$, and hence $f=0$.

The case n is even. We show first the equality,

$$H_{n+1}(f\otimes 1)=H_n f\otimes 1\,, \qquad {}^\forall f\in\mathcal{S}'(E_n)\,. \tag{3}$$

If $\hat{f}(\xi')$ has compat support, then we have

$$\begin{aligned} H_{n+1}(f\otimes 1) &= \mathcal{F}_{n+1}^{-1}[h(|\xi'|^2+\xi_{n+1}^2)\cdot\hat{f}(\xi')\otimes 2\pi\delta(\xi_{n+1})] \ (\delta \text{ being Dirac's function})\\ &= \langle \hat{f}(\xi')\otimes 2\pi\delta(\xi_{n+1}),\ h(|\xi'|^2+\xi_{n+1}^2)(2\pi)^{-n-1}\cdot e^{ix\cdot\xi}\rangle\\ &= \langle \hat{f}(\xi'),\ h(|\xi'|^2)\cdot(2\pi)^{-n}e^{ix'\cdot\xi'}\rangle\\ &= \mathcal{F}_n^{-1}[h(|\xi'|^2)\hat{f}(\xi')]\\ &= H_n f\otimes 1\,. \end{aligned}$$

If $\hat{f}$ has not compact support, we can establish the equality by approximation. Let $f_j=\mathcal{F}_n^{-1}\left[\varphi\left(\frac{\xi'}{j}\right)\hat{f}(\xi')\right]$, where $\varphi\in C_0^\infty(E_n)$ such that $\varphi(\xi')=1$ when $|\xi'|\leqq 1$. Then $\hat{f}_j(\xi')$ has compact support, and $H_{n+1}(f_j\otimes 1)$ and $H_n f_j\otimes 1$ both converge to $H_{n+1}(f\otimes 1)$ and $H_n f\otimes 1$ in $\mathcal{S}'(E_{n+1})$ respectively. Hence the equality (3) holds for any $f\in\mathcal{S}'(E_n)$.

Suppose $f\in\mathcal{S}'(E_n)$ and $H_n f$ vanish in a neighborhood of 0. Then $F=f\otimes 1$ and $H_{n+1}F=H_n f\otimes 1$ vanish in a neighborhood of 0, which implies $F=0$, and hence $f=0$.

REMARK 1. Let $q(t)\in C^\infty(E_1)$ have polynomial growth at infinity with its derivatives. We see from the proof of the case $n=1$ that if there exist analytic functions $q_1\in\mathcal{O}(\boldsymbol{C}\backslash((-\infty,-R]\cup\{t;\,|t|\leqq R\}))$ and $q_2\in\mathcal{O}(\boldsymbol{C}\backslash([R,\infty)\cup\{t;\,|t|\leqq R\}))$ with polynomial growth such that

$$q_1|(R,\infty)=q|(R,\infty)\,,\qquad q_2|(-\infty,-R)=q|(-\infty,-R)\,,$$

$$q_1(t)\not\equiv q_2(t)\qquad \text{in the half planes } \{t;\,\mathrm{Im}\,t<-R\} \text{ and } \{t;\,\mathrm{Im}\,t>R\}\,,$$

then the convolution operator $q\left(\frac{1}{i}\frac{d}{dx}\right)$ is anti-local in $\mathcal{S}'(E_1)$.

§ 3. Examples.

As the applications of the theorem we present some operators which have the anti-locality.

EXAMPLE 1. The operator $(m^2I-\Delta)^\lambda$ (λ; non-integral complex number) is anti-local in $\mathcal{S}'(E_n)$.

In fact, we set $h(t)=(m^2+t)^\lambda=e^{\mathrm{Log}(m^2+t)}$ on $[0,\infty)$, where Log is the principal branch of the logarithm. If we set $q_1(t)=e^{\lambda(\mathrm{Log}(t+mi)+\mathrm{Log}(t-mi))}$, then the assumptions (i) and (ii) in the theorem are satisfied. Since we have

$$q_1(-t)=e^{-2\lambda\pi i}q_1(t)\not\equiv q_1(t) \qquad \text{in the half plane } \{t;\ \mathrm{Im}\,t>m+1\},$$

the assumption (iii) in the theorem is also satisfied.

REMARK 2. The inverse of the local operator has not, in general, the anti-local property. For example, the operator $(m^2I-\Delta)^{-n}$ is not anti-local. Indeed, let $g\in C_0^\infty(E_n)$ be $g\not\equiv 0$, then $f=(m^2I-\Delta)^n g\in C_0^\infty(E_n)$ and $f\not\equiv 0$. Since f and $(m^2I-\Delta)^{-n}f$ have compact support, f must be identically zero if $(m^2I-\Delta)^{-n}$ had the anti-locality. This is a contradiction.

EXAMPLE 2. Let $p(t)=a_0t^m+a_1t^{m-1}+\cdots+a_m$ be a polynomial with complex coefficients with the property;

$$-\pi<\arg p(t)<\pi \quad \text{and} \quad p(t)\neq 0 \qquad \text{for any } t\geqq 0.$$

Set $h(t)=(p(t))^\lambda$. Then the operator $h(A)=(a_0A^m+\cdots+a_m)^\lambda$ is anti-local in $\mathcal{S}'(E_n)$ when $m\lambda$ is a non-integral complex number.

Since $p(t)$ is a polynomial, there exists positive constant R such that $p(t^2)\neq 0$, in $\{t\in \boldsymbol{C};\ |t|>R\}$. Hence the function $e^{\lambda\,\mathrm{Log}\,p(t^2)}|(R,\infty)$ can be continued analitically to the domain $\boldsymbol{C}\backslash((-\infty,R]\cup\{t;\ |t|\leqq R\})$, which we donote by $q_1(t)$. Since we have $q_1(-t)=e^{-2m\lambda\pi i}q_1(t)\not\equiv q_1(t)$ in the half plane $\{t;\ \mathrm{Im}\,t>R\}$, the assumption (iii) in the theorem is also satisfied.

EXAMPLE 3. Set $h(t)=\mathrm{Log}\,p(t)$, where p is the polynomial stated above. Then the operator $h(A)=\mathrm{Log}\,(a_0A^m+\cdots+a_m)$ is anti-local in $\mathcal{S}'(E_n)$.

For the proof, we have only to note that $q_1(-t)=q_1(t)-2m\pi i$ for all t such that $\mathrm{Im}\,t>R$.

EXAMPLE 4. Let $\varphi\in C^\infty(E_1)$ be $\varphi(t)=0$ on $(-\infty,1)$ and $\varphi(t)=1$ on $(2,\infty)$. Set $h(t)=\varphi(t)\sum_{j=-N_1}^{N_2}a_jt^{j\nu}$. Then the operator $h(A)$ is anti-local in $\mathcal{S}'(E_n)$ when $j\nu$ is a non-integral complex number for some j, for which $a_j\neq 0$.

We have $q_1(-t)=\sum_{j=-N_1}^{N_2}a_je^{-2j\nu\pi i}e^{2j\nu\,\mathrm{Log}\,t}\not\equiv q_1(t)$ in the half plane $\{t;\ \mathrm{Im}\,t>1\}$.

Hence the assumption in the theorem is satisfied.

REMARK 3. Even if the function $h(t)$ has singularity in a compact set, we can show the anti-locality of $h(A)$ on some function space. For example, the operator $(-\Delta)^{\lambda}$ is anti-local in $L^2(E_n)$ for a non-integral complex number λ with $\mathrm{Re}\,\lambda \geqq 1/2$.

PROOF. Since the function $q(\xi)=(\xi^2)^{\lambda}$ is differentiable, we can show with minor modification that $\left(-\left(\frac{d}{dx}\right)^2\right)^{\lambda}$ is anti-local in $L^{2,1}(E_1)=\{g\,;\,g(x)(1+x^2)^{-1/2} \in L^2(E_1)\}$. Since $Df\in L^2(E_1)\subset L^{2,1}(E_1)$ for any $f\in L^2(E_n)$, the operator $(-\Delta)^{\lambda}$ is anti-local in $L^2(E_n)$ if the space dimension n is odd. In the case n is even, it suffices to prove that $D\tilde{F}_{x_0}(r)\in L^{2,1}(E_1)$ for $F=f\otimes 1$, where $f\in C^{\infty}(E_{2k})$ is in L^2 with its derivatives and vanishes near the origin. We have

$$\int_0^{\infty} r^{2k-2}\left|\left(\frac{d}{dr}\right)^{\gamma}\tilde{F}_{x_0}(r)\right|^2 dr$$

$$=\int_0^{\infty} r^{2k-2}\left|\int \sum_{\alpha\beta} d_{\alpha\beta}D^{\alpha}F(r\omega+x_0)\omega^{\beta}d\omega\right|^2 dr$$

$$\leqq C_1\int_0^{\infty} r^{2k-2}dr\int \sum_{\alpha}|D^{\alpha}F(r\omega+x_0)|^2 d\omega$$

$$\leqq C_2+C_3\int_1^{\infty}\frac{r^{3/2}}{r^2}dr\int \sum_{\alpha}|D^{\alpha}F(r\omega+x_0)|^2\frac{r^{2k}}{\{1+(r\omega_n)^2\}^{3/4}}d\omega \quad (n=2k+1)$$

$$\leqq C_2+C_4\int_{E_n}\sum_{\alpha}|D^{\alpha}F(x)|^2\frac{1}{(1+x_n^2)^{3/4}}dx$$

$$=C_2+C_4\int_{-\infty}^{\infty}\frac{dx_n}{(1+x_n^2)^{3/4}}\int_{E_{n-1}}\sum_{\alpha}|D^{\alpha}f(x')|^2dx'<\infty\,.$$

Hence $D\tilde{F}_{x_0}(r)=\sum_{\substack{\alpha\leqq k\\ \beta\leqq k-1}} C_{\alpha\beta}r^{\alpha}\left(\frac{1}{i}\frac{\partial}{\partial r}\right)^{\beta}\tilde{F}_{x_0}(r)$ is in $L^{2,1}(E_1)$. q. e. d.

As a corollary we obtain that the Riesz transform $Rf=(R_1f,\cdots,R_mf)$ is anti-local in $L^2(E_n)$. Indeed, if f and Rf both vanish in some non-empty open set, then $f=0$, since $\sum_{j=1}^{n} D_jR_jf=(-\Delta)^{1/2}f$.

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