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On the coincidence of two Dirichlet series associated with cusp forms of Hecke's "Neben"-type and Hilbert modular forms over a real quadratic field

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1. As the title indicates, the purpose of the present note is to consider a relation, which is analogous to the "decomposition theorem" in the case of L-functions of algebraic number fields, between the Dirichlet series of Hecke type associated with the cusp forms belonging to a Hilbert modular group over a real quadratic field and the series associated with the modular forms of "Neben"-type in Hecke's sense. It may be observed that the problem of this investigation is in the same framework of our previous paper [1] in collaboration with Doi. In fact, the principle of the proof of the result is essentially the same as that of [1]. Let N be a positive integer and ψ_N a character of $(\boldsymbol{Z}/N\boldsymbol{Z})^{\times}$ such that $\psi_N(-1)=1$. Put

$$\Gamma_0(N)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\boldsymbol{Z}) \mid c\equiv 0 \bmod N\right\}.$$

We let $\mathfrak{H}$ denote the upper half complex plane: $\mathfrak{H}=\{\tau\in\boldsymbol{C}\mid \mathrm{Im}(\tau)>0\}$. Let $S_k(\Gamma_0(N), \psi_N)$ be the space of cusp forms of weight k on $\mathfrak{H}$ such that

$$f\left(\frac{a\tau+b}{c\tau+d}\right)=\psi_N(d)(c\tau+d)^k f(\tau) \qquad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma_0(N). \tag{1.1}$$

We shall take N to be *a prime number* q throughout the present investigation and let ψ_q be a non-trivial character of order 2. (The elements of $S_k(\Gamma_0(q), \psi_q)$ are called cusp forms of "Neben"-type after Hecke.) Hereafter we denote the real quadratic field $\boldsymbol{Q}(\sqrt{q})$ by F. We assume that *k is an even positive integer, and the class number of F is one.* Note that $q\equiv 1 \bmod 4$, since $\psi_q(-1)=1$. Let $f(\tau)=\sum_{n=1}^{\infty} a_n e^{2\pi i n\tau}$ be an element of $S_k(\Gamma_0(q), \psi_q)$. Suppose that $f(\tau)$ is a common eigen-function of Hecke operators $T(n)$ for all n, and $a_1=1$. Then the Mellin transform of $f(\tau)$ defines a Dirichlet series $L(s, f)$ with the Euler product:

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$$(1.2) \qquad L(s, f)=\sum_{n=1}^{\infty} a_n n^{-s}$$
$$=(1-a_q q^{-s})^{-1} \prod_{p \neq q}(1-a_p p^{-s}+\psi_q(p) p^{k-1-2s})^{-1}.$$

Moreover if we consider $f_\rho(\tau)=\sum_{n=1}^{\infty} a_n^\rho e^{2\pi i n\tau}$ (here ρ denotes the complex conjugation of the complex number field $\boldsymbol{C}$; we shall use also "$^-$" instead of ρ in § 2) then $f_\rho(\tau)$ is contained in $S_k(\Gamma_0(q), \psi_q)$ (cf. [**3**, Theorem 7.14, p. 183]). From this $f_\rho(\tau)$ we can naturally get one more series $L(s, f_\rho)=\sum_{n=1}^{\infty} a_n^\rho n^{-s}$. Here we remark that, in the notation of [**1**], if we take $L(s, f)$ for $\varphi(s)$, then $L(s, f_\rho)$ is nothing but $\varphi(s, \psi)$ except for the q-factor. Define a sequence of numbers $\{C_\mathfrak{a}\}$ for integral ideals $\mathfrak{a}$ in F, in the following manner: For prime ideals $\mathfrak{p}$ in F, we put

$$C_\mathfrak{p}=C_{\mathfrak{p}'}=a_p \qquad \text{if } \mathfrak{p}\mathfrak{p}'=(p) \text{ and } \mathfrak{p} \neq \mathfrak{p}',$$
$$C_\mathfrak{p}=a_p^2+2p^{k-1} \qquad \text{if } \mathfrak{p}=(p),$$
$$C_\mathfrak{p}=a_p+a_p^\rho \qquad \text{if } \mathfrak{p}=(\sqrt{q}),$$

and define

$$C_\mathfrak{a}=C_{\mathfrak{p}^e}=C_\mathfrak{p} \cdot C_{\mathfrak{p}^{e-1}}-N\mathfrak{p}^{k-1} C_{\mathfrak{p}^{e-2}} \qquad \text{if } \mathfrak{a}=\mathfrak{p}^e \text{ and } \mathfrak{p} \neq(\sqrt{q}),$$
$$C_\mathfrak{a}=C_{\mathfrak{p}^e}=C_\mathfrak{p}^e \qquad \text{if } \mathfrak{a}=\mathfrak{p}^e \text{ and } \mathfrak{p}=(\sqrt{q}),$$
$$C_\mathfrak{a}=\prod_i C_{\mathfrak{p}_i^{e_i}} \qquad \text{if } \mathfrak{a}=\prod_i \mathfrak{p}_i^{e_i}.$$

Here we denote by "$'$" the conjugation of $F/\boldsymbol{Q}$. Then we have, for a constant $\sigma>0$,

$$L(s, f) L(s, f_\rho)=\sum_\mathfrak{a} C_\mathfrak{a} N\mathfrak{a}^{-s}, \qquad \operatorname{Re} s>\sigma,$$

by means of the relations of Hecke operators. We remark that our Dirichlet series $L(s, f) L(s, f_\rho)$ is, by Shimura [**3**, Theorem 7.25, p. 194], closely connected with the zeta function of a certain abelian variety associated with a cusp form of "Neben"-type, of level q. Now let ξ_m $(m \in \boldsymbol{Z})$ be a Grössen-character of F which is defined by

$$\xi_m(\mathfrak{a})=|\alpha / \alpha'|^{\frac{m\pi i}{\log \varepsilon_0}},$$

for every ideal $\mathfrak{a}=(\alpha)$ in F, where ε_0 is the fundamental unit in F, $\varepsilon_0>1$. Put

$$D(s, f, \xi_m)=\sum_\mathfrak{a} \xi_m(\mathfrak{a}) C_\mathfrak{a} N\mathfrak{a}^{-s}$$

and

$$D^*(s, f, \xi_m)=q^s(2\pi)^{-2s} \Gamma(s+im\kappa) \Gamma(s-im\kappa) D(s, f, \xi_m),$$

where $\kappa = \pi/\log \varepsilon_0$.

THEOREM. *In the above situation, $D(s, f, \xi_m)$ converges absolutely for $\mathrm{Re}(s) > \sigma_1$ with a suitable constant $\sigma_1 > 0$ and can be expressed in the form of Euler product*

$$D(s, f, \xi_m) = \prod_{\mathfrak{p}} (1 - \xi_m(\mathfrak{p}) C_{\mathfrak{p}} N\mathfrak{p}^{-s} + \xi_m(\mathfrak{p})^2 N\mathfrak{p}^{k-1-2s})^{-1};$$

$D^*(s, f, \xi_m)$ can be continued holomorphically to the whole s-plane, and is bounded in every vertical strip $\sigma < \mathrm{Re}(s) < \sigma'$. Moreover, it satisfies a functional equation

$$D^*(s, f, \xi_m) = D^*(k-s, f, \xi_m).$$

Now the Mellin transform of $L(s, f)L(s, f_\rho)$, as a series over F in the above sense, is given by the following form** on $\mathfrak{H} \times \mathfrak{H}$:

$$h(\tau_1, \tau_2) = \sum_{\mu \in \mathfrak{o}_+/E_+} C_{(\mu)} \sum_{\nu \in E_+} e^{\frac{2\pi i}{\sqrt{q}}(\varepsilon_0 \nu \mu \tau_1 - \varepsilon_0' \nu' \mu' \tau_2)}. \tag{1.3}$$

Here $\mathfrak{o}$ denotes the ring of integers of F and E the group of all units of $\mathfrak{o}$. "$+$" denotes the set of all totally positive elements in each set. Thus, by the same argument as [**1**, § 3], our theorem shows that $h(\tau_1, \tau_2)$ satisfies

$$h\left(\frac{-1}{\tau_1}, \frac{-1}{\tau_2}\right) = \tau^k \tau_2^k h(\tau_1, \tau_2). \tag{1.4}$$

The author would like to mention here one remark on the previous work [**1**]. There, we have considered a Dirichlet series $\varphi(s)$ associated to a cusp form of "Haupt"-type with respect to $SL_2(\boldsymbol{Z})$ and have showed that the corresponding function $h(\tau_1, \tau_2)$ to $\varphi(s) \cdot \varphi(s, \chi)$ admits a transformation formula of Hilbert modular type (see [**1**, (3.3.2), p. 13]). We can also obtain the same formula as (1.4) for $h(\tau_1, \tau_2)$ in [**1**] by a minor change of variables,

$$(\tau_1, \tau_2) \longrightarrow \left(\frac{\tau_1}{\varepsilon\sqrt{D}}, \frac{-\tau_2}{\varepsilon'\sqrt{D}}\right).$$

As a direct consequence of our theorem, $h(\tau_1, \tau_2)$ (both in the present case and in the previous case of [**1**]) is a Hilbert cusp form of weight k with respect to $SL_2(\mathfrak{o})$ when $\mathfrak{o}$ is a Euclidean domain.

§ 2. Proof of Theorem.

Firstly we note that, from the definition of $C_{\mathfrak{a}}$, $|C_{\mathfrak{a}}| < c_1 N\mathfrak{a}^{c_2}$ for every $\mathfrak{a}$ with suitable positive constants c_1, c_2. Therefore $D(s, f, \xi_m)$ converges absolutely and can be expressed in the form of an Euler product

** As for this, see [**1**, § 3, pp. 10-13].

$$D(s, f, \xi_m)=\prod_{\mathfrak{p}}(1-\xi_m(\mathfrak{p})C_{\mathfrak{p}}N\mathfrak{p}^{-s}+\xi_m(\mathfrak{p})^2N\mathfrak{p}^{k-1-2s})^{-1}$$

by means of the definition of $C_{\mathfrak{a}}$, for $\mathrm{Re}(s)>\sigma_1$ with a suitable constant $\sigma_1>0$. Now we shall give the lemma which is essential to prove the functional equation for $D^*(s, f, \xi_m)$. To state it, we define a Dirichlet series L_{ξ_m} by

$$L_{\xi_m}(s)=\sum_{\mathfrak{a}}\xi_m(\mathfrak{a})N\mathfrak{a}^{-s}=\sum_{n=1}^{\infty}t_n n^{-s}$$

with $t_n=\sum_{N\mathfrak{a}=n}\xi_m(\mathfrak{a})$ and put

$$\zeta_q(s)=\sum_{(n,q)=1}n^{-s}.$$

LEMMA 1. *Notation being the same as above, we have*

$$(2.1)\qquad (1-a_q^{\rho}q^{-s})D(s, f, \xi_m)=\zeta_q(2s-k+1)\sum_{n=1}^{\infty}a_n t_n n^{-s}\qquad \textit{for}\quad \mathrm{Re}\, s>\sigma_2$$

with a suitable constant $\sigma_2>0$.

PROOF. By (1.2) and the definition of t_n, $\sum_{n=1}^{\infty}a_n t_n n^{-s}$ is equal to $\prod_p(\sum_{\nu=0}^{\infty}a_{p^\nu}t_{p^\nu}p^{-\nu s})$, where p ranges over all prime numbers. It can be easily seen that both sides of (2.1) have the same q-factors. Therefore we have to prove, for a prime $p\neq q$,

$$\Big(\sum_{\nu=0}^{\infty}a_{p^\nu}t_{p^\nu}p^{-\nu s}\Big)\prod_{\mathfrak{p}|p}(1-\xi_m(\mathfrak{p})C_{\mathfrak{p}}N\mathfrak{p}^{-s}+\xi(\mathfrak{p})^2N\mathfrak{p}^{k-1-2s})=1-p^{k-1-2s}.$$

In fact, we can check it by virtue of the formula

$$a_p\cdot a_{p^\nu}=a_{p^{\nu+1}}+\psi(p)p^{k-1}a_{p^{\nu-1}}$$

which is well-known for the Hecke operators in Neben-type (for details see [1, the proof of Lemma 2.3, p. 5]).

By Lemma 1, our task is to prove a functional equation for

$$(1-a_q^{\rho}q^{-s})^{-1}\zeta_q(2s+1-k)\sum_{n=1}^{\infty}a_n t_n n^{-s}.$$

To do it, we make use of a method of Rankin as in [1, 2.4]. For this purpose, we take again a real analytic automorphic function (due to Maass) $g(\tau, \xi_m)$ attached to the L-function of F:

$$g(\tau, \xi_m)=\sum_{\substack{\mu\in\mathfrak{o}/E_+\\ \mu\neq 0}}\xi_m(\mu)y^{1/2}K_{im\kappa}(2\pi|N\mu|y)e^{2\pi iN\mu x},\qquad \tau(=x+iy)\in\mathfrak{H}$$

with $\kappa=\pi/\log\varepsilon_0$. Then we have

$$(2.2)\qquad \int_0^{\infty}\int_{-1/2}^{1/2}f(\tau)\overline{g(\tau, \xi_m)}y^{s-3/2}dx\,dy$$
$$=\sqrt{\pi}(4\pi)^{-s}\frac{\Gamma(s+im\kappa)\Gamma(s-im\kappa)}{\Gamma(s+1/2)}\sum_{n=1}^{\infty}a_n t_n n^{-s}\qquad \text{for}\quad \mathrm{Re}(s)>\sigma_3$$

with a suitable positive constant σ_3 (see [1, Lemma 2.5]). Now we put for a positive integer l

$$G(s,\tau,l)=\sum_{c,d}{}'\frac{(c\tau+d)^l}{|c\tau+d|^{2s+l}}\,, \tag{2.3}$$
$$\Lambda(s,\tau,l)=\left(\frac{y}{\pi}\right)^s\Gamma\left(s+\frac{l}{2}\right)G(s,\tau,l)\,.$$

Here the prime on the summation symbol means to omit the term (0, 0) and c, d range over all integers. Here we recall the function

$$D^*(s,f,\xi_m)=q^s(2\pi)^{-2s}\Gamma(s+im\kappa)\Gamma(s-im\kappa)D(s,f,\xi_m)\,,$$

which is defined in §1. Then, by means of the property of $g(\tau,\xi_m)$ (see [1, (2.4.3)]) and (1.1), we have

$$\begin{aligned}(2.4)\qquad &(1-a_q^\rho q^{-s})D^*(s,f,\xi_m)\\ &=4^{-1}\pi^{-1/2}q^{(k-1)/2}\iint_{\mathcal{D}_{\Gamma_0(q)}} y^{k/2-2}f(\tau)\overline{g(\tau,\xi_m)}\\ &\quad\cdot\left[\Lambda\left(s-\frac{k}{2}+\frac{1}{2},q\tau,k\right)-q^{-(s-k/2+1/2)}\Lambda\left(s-\frac{k}{2}+\frac{1}{2},\tau,k\right)\right]dx\,dy\\ &\qquad\qquad(\mathrm{Re}\,(s)>\sigma_3)\,,\end{aligned}$$

where $\mathcal{D}_{\Gamma_0(q)}$ denotes a fundamental domain for $\Gamma_0(q)$. Though we did not mention explicitly in [1] that the integral on the right hand side of (2.4) is absolutely convergent for any s, and, as a function in s, bounded in every vertical strip $\sigma<\mathrm{Re}\,(s)<\sigma'$, these facts can be proved by an argument similar to Shimura [4, Lemma 3.3] with our $g(\tau,\xi_m)$ in place of $g(z)$ there. Obviously, $1-a_q^\rho q^{-s}$ is meromorphic in the whole s-plane, hence $D^*(s,f,\xi_m)$ is continued meromorphically to the whole s-plane. In the later discussion, we actually know that $D^*(s,f,\xi_m)$ can be continued to a holomorphic function in the whole s-plane and is bounded in every vertical strip. Now, the functional equation for $D^*(s,f,\xi_m)$ is reduced to that of $\Lambda(s,\tau,l)$ except for the first factor of the left hand side of (2.4). Now let us simplify the second term in the integration of the right hand side of (2.4) by the transformation $\tau\mapsto\omega(q)\tau=\dfrac{-1}{q\tau}$. In fact, from the definition of $\Lambda(s,\tau,l)$ we have

$$\Lambda\left(s,\frac{-1}{\tau},l\right)=\left(\frac{|\tau|}{\tau}\right)^l\Lambda(s,\tau,l)\,, \tag{2.5}$$

and by Hecke [3, Satz 61, p. 896] we can put

$$f\left(\frac{-1}{q\tau}\right)=\lambda_q q^{1/2}\tau^k f_\rho(\tau) \tag{2.6}$$

with a constant $\lambda_q \in \boldsymbol{C}$, $|\lambda_q| = q^{(k-1)/2}$. By (2.5) and (2.6), (2.4) can be expressed as

$$(2.7) \qquad (1-a_q^\rho q^{-s})D^*(s, f, \xi_m) = 4^{-1}\pi^{-k/2}q^{(k-1)/2} \iint_{\mathcal{D}\ {}_0(q)} y^{k/2-2}(f(\tau)-\lambda_q q^{-s} f_\rho(\tau)) \cdot \overline{g(\tau, \xi_m)} \Lambda\Big(s-\frac{k}{2}+\frac{1}{2}, q\tau, k\Big)\, dx\, dy\,.$$

Here we note that $\omega(q)^{-1}\mathcal{D}_{\Gamma_0(q)}$ is a fundamental domain for

$$\begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix}^{-1} \Gamma_0(q) \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix} = \Gamma_0(q)\,.$$

Taking $f_\rho(\tau)$ instead of $f(\tau)$, by the same procedure as above, we have

$$(2.8) \qquad (1-a_q q^{-s})D^*(s, f_\rho, \xi_m) = 4^{-1}\pi^{-k/2}q^{(k-1)/2} \iint_{\mathcal{D}_{\Gamma_0(q)}} y^{k/2-2}(f_\rho(\tau)-\lambda_q^\rho q^{-s} f(\tau)) \cdot \overline{g(\tau, \xi_m)} \Lambda\Big(s-\frac{k}{2}+\frac{1}{2}, q\tau, k\Big)\, dx\, dy\,.$$

By the property of a_p in the definition of C_a (see [3, (7.7.1), p. 198]), it can be easily verified that

$$(2.9) \qquad D^*(s, f, \xi_m) = D^*(s, f_\rho, \xi_m)\,.$$

Therefore, combining (2.7) and (2.8), we have

$$(2.10) \qquad \Big(\frac{1-a_q^\rho q^{-s}+\lambda_q q^{-s}-\lambda_q a_q q^{-2s}}{1-q^{k-1-2s}}\Big) D^*(s, f, \xi_m)$$
$$= 4^{-1}\pi^{-k/2}q^{(k-1)/2} \iint_{\mathcal{D}\ {}_0(q)} y^{k/2-2} f(\tau)\overline{g(\tau, \xi_m)} \Lambda\Big(s-\frac{k}{2}+\frac{1}{2}, q\tau, k\Big)\, dx\, dy\,.$$

In the next step, we are going to prove

$$(2.11) \qquad \lambda_q = a_q^\rho\ {}^{***}\,.$$

If we have done it, the first factor of the left hand side of (2.10) becomes the constant 1. Then the functional equation for $D^*(s, f, \xi_m)$ is reduced exactly to that of $\Lambda(\tau, s, l)$. To prove $\lambda_q = a_q^\rho$, put

$$P(s) = \frac{1-a_q^\rho q^{-s}+\lambda_q q^{-s}-\lambda_q a_q q^{-2s}}{1-q^{k-1-2s}}\,.$$

We observe that $P(s)$ does not depend on ξ_m and that the right hand side

*** T. Miyake showed us a direct proof for this fact different from ours. The author would like to express his hearty thanks for his permission to include his lemma in the present paper.

of (2.10) is invariant under the transformation $s\to k-s$ by means of the functional equation

$$\Lambda(s,\tau,l)=\Lambda(1-s,\tau,l).$$

On the other hand, if we take $\xi_m=1=$ identity character, we already know the functional equation

$$D^*(s,f,1)=D^*(k-s,f,1).$$

Thus $P(k-s)=P(s)$, hence one can easily obtain $\lambda_q=a_q^\rho$.

As we have indicated in the footnote***, here we shall give another proof of (2.11) in the following lemma. To state it, let us use the following notation. For a holomorphic function $f(\tau)$ on $\mathfrak{H}$ and for a real matrix $\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ with positive determinant, we define

$$\left(f\Big|\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right)(\tau)=(c\tau+d)^{-k}f\left(\frac{a\tau+b}{c\tau+d}\right).$$

LEMMA 2 (Miyake). *The notation being as above, let* $f(\tau)=\sum_{n=1}^{\infty}a_ne^{2\pi in\tau}$ *be an element of* $S_k(\Gamma_0(q),\psi_q)$, *and assume that* $f(\tau)$ *is an eigen-function of Hecke operators* $T(n)$ *for all* n, *and* $a_1=1$. *Put* $f_\rho(\tau)=\sum_{n=1}^{\infty}a_n^\rho e^{2\pi in\tau}$. *Then we have*

$$\left(f\Big|\begin{pmatrix} 0 & 1 \\ -q & 0\end{pmatrix}\right)(\tau)=q^{-k+1/2}a_q^\rho f_\rho(\tau).$$

PROOF. Recall here the definition of $T(q)$ for $f(\tau)\in S_k(\Gamma_0(q),\psi_q)$:

$$f\,|\,T(q)=\sum_{a=0}^{q-1}f\Big|\begin{pmatrix} 1 & a \\ 0 & q\end{pmatrix}.$$

By the assumption, we can put

$$f\,|\,T(q)=t_qf,$$

$$f\Big|\begin{pmatrix} 0 & 1 \\ -q & 0\end{pmatrix}=cf_\rho$$

for some constants t_q and c. Now, we have

$$\begin{aligned} f\Big|T(q)\Big|\begin{pmatrix} 0 & 1 \\ -q & 0\end{pmatrix} &=\sum_{a=0}^{q-1}f\Big|\begin{pmatrix} 1 & a \\ 0 & q\end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix} q & 0 \\ 0 & 1\end{pmatrix} \\ &=\sum_{\substack{a=1 \\ ab+1\equiv 0 \bmod q}}^{q-1}f\Big|\begin{pmatrix} -a & q^{-1}(ab+1) \\ -q & -b\end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & q\end{pmatrix}\begin{pmatrix} q & 0 \\ 0 & 1\end{pmatrix} \\ &\quad +f\Big|\begin{pmatrix} 0 & 1 \\ -q & 0\end{pmatrix}\begin{pmatrix} q & 0 \\ 0 & 1\end{pmatrix},\end{aligned}$$

with a fixed integer b satisfying $ab+1\equiv 0 \bmod q$. Since $\begin{pmatrix} -a & q^{-1}(ab+1) \\ -q & -b\end{pmatrix}\in\Gamma_0(q)$, we get

$$f\left|T(q)\right|\begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix} = \sum_{a=1}^{q-1} \psi_q(b) q^{-k} \sum_{n=1}^{\infty} e^{2\pi i n(\tau+b/q)} + c \sum_{n=1}^{\infty} a_n^{\rho} e^{2\pi i q n \tau}.$$

On the other hand,

$$f\left|T(q)\right|\begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix} = t_q c f_{\rho} = t_q c \sum_{n=1}^{\infty} a_n^{\rho} e^{2\pi i n \tau}.$$

Comparing the term $e^{2\pi i \tau}$ and $e^{2\pi i q \tau}$ of both expansions of $f\left|T(q)\right|\begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix}$, we get

$$\left(\sum_{b=1}^{q-1} \psi_q(b) e^{2\pi i b/q}\right) q^{-k} = t_q c,$$

and

$$c = t_q c a_q^{\rho}.$$

Here we have $\sum_{b=1}^{q-1} \psi_q(b) e^{2\pi i b/q} = \sqrt{q}$ and from the definition, $a_q = q^{k-1} t_q$. Hence $a_q a_q^{\rho} = q^{k-1}$ and $c = q^{-k+1/2} t_q^{-1} = q^{-k+1/2} a_q^{\rho}$.

Coming back to the proof of Theorem, the above argument tells us

$$D^*(s, f, \xi_m) = D^*(k-s, f, \xi_m) \tag{2.12}$$

for every $m \in \boldsymbol{Z}$. The next step is to show that $D^*(s, f, \xi_m)$ is holomorphic and bounded in the whole s-place. As we remarked in (2.4), $(1-a_q^{\rho} q^{-s}) D^*(s, f, \xi_m)$ is continued to a holomorphic function in the whole s-plane. By changing f and f_{ρ} in the above discussion of (2.4) and by (2.9), $(1-a_q q^{-s}) D^*(s, f, \xi_m)$ is also holomorphic in the whole s-plane. Moreover, by the functional equation (2.12), $(1-a_q^{\rho} q^{-(k-s)}) D^*(s, f, \xi_m)$ and $(1-a_q q^{-(k-s)}) D^*(s, f, \xi_m)$ are also holomorphic. Therefore, the poles of $D^*(s, f, \xi_m)$ are contained in the set of common zeros of $1-a_q^{\rho} q^{-s}$, $1-a_q q^{-s}$, $1-a_q^{\rho} q^{-(k-s)}$ and $1-a_q q^{-(k-s)}$. However, $1-a_q^{\rho} q^{-s}$ and $1-a_q q^{-(k-s)}$ have no common zeros since $|a_q| = q^{(k-1)/2}$. Now, for a given vertical strip $U = \{s \mid \sigma < \mathrm{Re}(s) < \sigma'\}$, we can find two subsets V_1, V_2 such that $U = V_1 \cup V_2$ and $(1-a_q^{\rho} q^{-s})^{-1}$ (resp. $(1-a_q q^{-(k-s)})^{-1}$) is bounded in V_1 (resp. V_2), since each function is periodic and has no common zeros. Thus $D^*(s, f, \xi_m)$ is bounded in U. Therefore it is holomorphic in the whole s-plane.

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