J. Math. Soc. Japan
Vol. 25, No. 3, 1973

On certain nonlinear evolution equations

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(Received May 22, 1972)

§ 1. Introduction.

We consider nonlinear evolution equations of the form

$$\frac{d}{dt} u(t)+A(t)u(t) \ni f(t, u(t)), \qquad 0 \leqq t \leqq T$$

in a real Hilbert space H. Here, for each fixed t, $A(t)$ is a (possibly) multi-valued nonlinear operator in H of the form $\partial\varphi^t$ (subdifferential of a lower semicontinuous convex function φ^t from H into $(-\infty, \infty]$, $\varphi^t \not\equiv \infty$), while $f(t, \cdot)$ is continuous from the strong to the weak topology of H and $f(t, \cdot)-\beta(t)\cdot$ is dissipative in H for some Lebesgue integrable function β on $[0, T]$.

We denote the inner product and the norm in H by $(\ ,\ )$ and $\|\ \|$ respectively. Let φ be a lower semicontinuous convex function from H into $(-\infty, \infty]$. The effective domain of φ is defined by $\{u \in H;\ \varphi(u) < \infty\}$. The subdifferential $\partial\varphi$ of φ is defined by $\partial\varphi(u) = \{w \in H;\ \varphi(v) \geqq \varphi(u)+(w, v-u)$ for all $v \in H\}$ for each $u \in H$, and the domain of the subdifferential $\partial\varphi$ is defined by $D(\partial\varphi) = \{u \in H;\ \partial\varphi(u) \neq \emptyset\}$.

Let T be a positive constant. For each $t \in [0, T]$, let φ^t be a lower semicontinuous convex function from H into $(-\infty, \infty]$ with nonvoid effective domain, and suppose that $\{\varphi^t;\ 0 \leqq t \leqq T\}$ satisfies the following three conditions:

(I) *The effective domain D of φ^t is independent of t.*

(II) *For every $r > 0$, there exist two positive constants c_r and c_r' such that*

$$|\varphi^s(u)-\varphi^t(u)| \leqq |s-t| \cdot [c_r\varphi^t(u)+c_r']$$

holds if $0 \leqq s,\ t \leqq T,\ u \in D$ and $\|u\| \leqq r$.

(III) *For some $b \in D$, b is in $D(\partial\varphi^t)$ for almost all $t \in [0, T]$ and $|||\partial\varphi^t(b)||| \equiv \min\{\|v\|;\ v \in \partial\varphi^t(b)\}$ is Lebesgue integrable in $0 \leqq t \leqq T$.* (See Corollary 3.4.)

Let f be a map of $[0, T]\times H$ into H, and suppose that f satisfies the following three conditions:

(IV) *For each fixed $u \in H$, $f(t, u)$ is strongly measurable in $0 \leqq t \leqq T$, and for each fixed $t \in [0, T]$ it is continuous from the strong to the weak topology of H.*

(V) *For some Lebesgue integrable function* β *on* $0 \leqq t \leqq T$,

$$(f(t, u)-f(t, v), u-v) \leqq \beta(t)\|u-v\|^2$$

holds for all $t \in [0, T]$ *and all* $u, v \in H$.

(VI) *For each bounded subset* H_0 *of* H *there exists a Lebesgue integrable function* γ *on* $0 \leqq t \leqq T$ *satisfying* $\|f(t, u)\|^2 \leqq \gamma(t)$ *for all* $t \in [0, T]$ *and all* $u \in H_0$.

In this paper we consider the initial value problem for the equation

$$\frac{d}{dt} u(t) + \partial\varphi^t(u(t)) \ni f(t, u(t)), \qquad 0 \leqq t \leqq T \tag{1.1}$$

with the initial condition $u(0)=a$. Here and henceforth in this paper d/dt denotes the strong differentiation with respect to t. We shall prove that under the assumptions (I) to (VI) there exists one and only one solution u of (1.1) with $u(0)=a$ for each $a \in \bar{D}$ (closure of D in H) and that $u(t)$ is in D for all $t \in (0, T]$ and $u(t)$ is in $D(\partial\varphi^t)$ for almost all $t \in [0, T]$.

When X is a Banach space, $C(I; X)$ denotes the set of all X-valued strongly continuous functions on an interval I of real numbers and $L^p(t_1, t_2; X)$ ($p \geqq 1$, $t_1 < t_2$) denotes the set of all X-valued strongly measurable functions u on (t_1, t_2) such that $\int_{t_1}^{t_2} \|u(t)\|^p dt < \infty$.

In the case of $a \in D$ we have the following theorem which will be proved in § 6.

THEOREM 1.1. *If conditions* (I) *to* (VI) *are satisfied, then for each* $a \in D$ *there exists a uniquely determined pair of functions* $u \in C([0, T]; H)$ *and* $y \in L^2(0, T; H)$ *satisfying the following two conditions:*

(i) *For all* $t \in [0, T]$, $u(t)$ *is in* D *and for almost all* $t \in [0, T]$, $u(t)$ *is in* $D(\partial\varphi^t)$ *and* $y(t)$ *is in* $\partial\varphi^t(u(t))$.

(ii) $u(t) + \int_0^t y(s) ds = a + \int_0^t f(s, u(s)) ds$ *for all* $t \in [0, T]$.

REMARK 1.2. It is easily seen from (IV) and (VI) that $f(s, u(s))$ in (ii) of Theorem 1.1 is in $L^2(0, T; H)$. Cf. [**4**, Theorem 3.5.4, (4)] and [**11**]. Note that (ii) of Theorem 1.1 holds if and only if u is strongly absolutely continuous in $[0, T]$ and satisfies $du(t)/dt + y(t) = f(t, u(t))$ a. e. in $[0, T]$ and $u(0)=a$. See [**7**, Appendix].

DEFINITION 1.3. A function u from $[0, T]$ into H is called a *strong solution* of the equation (1.1) on $[0, T]$, if u is in $C([0, T]; H)$ and if there exists an H-valued strongly measurable function y on $[0, T]$ satisfying the following two conditions:

(i) For almost all $t \in [0, T]$, $u(t)$ is in $D(\partial\varphi^t)$ and $y(t)$ is in $\partial\varphi^t(u(t))$.

(ii) For each $s \in (0, T]$, y is in $L^2(s, T; H)$ and

$$u(t)+\int_s^t y(\sigma)d\sigma = u(s)+\int_s^t f(\sigma, u(\sigma))d\sigma$$

holds when $0 < s \leqq t \leqq T$.

REMARK 1.4. If $u \in C([0, T]; H)$ is a strong solution of (1.1) on $[s, T]$ for each $s > 0$, then u is a strong solution of (1.1) on $[0, T]$. Note that $y(t) = f(t, u(t)) - du(t)/dt$ a. e. in $0 \leqq t \leqq T$.

In case a is in the closure $\bar{D}$ of D we have the following theorem.

THEOREM 1.5. *If conditions* (I) *to* (VI) *are satisfied, then for each* $a \in \bar{D}$ *there exists one and only one strong solution* u *of* (1.1) *on* $[0, T]$ *with* $u(0) = a$. *Thus, in particular,* $u(t)$ *is in* $D(\partial\varphi^t)$ *for almost all* $t \in [0, T]$. *Moreover,* $u(t)$ *is in* D *for all* $t \in (0, T]$.

A subtle feature of Theorem 1.5 (as well as Theorem 1.1) is the nice differentiability property of solutions for rather general initial elements a. In this sense our results generalize those by Brezis [**1**]. We shall prove Theorem 1.5 in § 7 by making use of a method suggested personally by Professor Y. Kōmura to whom the author wishes to express his hearty thanks.

We should mention a few more existing works on nonlinear evolution equations which are closely related with the present paper. When A is a multivalued maximal monotone operator in a Hilbert space, Kōmura [**7**] considered the initial value problem

$$\frac{du}{dt} + Au(t) \ni 0 \quad (t \geqq 0) \quad \text{and} \quad u(0) = a, \tag{1.2}$$

a being in the domain $D(A)$ of A, and proved the existence and uniqueness of solutions of (1.2), using the Yosida approximation of A. Kato [**5**] generalized a proof in Kōmura [**7**] when A depends on t, and showed a existence theorem for the initial value problem

$$\frac{du}{dt} + A(t)u(t) = 0 \quad (0 \leqq t \leqq T) \quad \text{and} \quad u(0) = a \in D(A(0)) \tag{1.3}$$

in a Banach space whose dual space is uniformly convex, assuming that $A(t)$ is a single-valued m-accretive operator depending on t in a certain lipschitzian sense and that $D(A(t))$ is independent of t. Crandall-Liggett [**2**] also treated (1.3) in a general Banach space, assuming among other things that $D(A(t))$ is independent of t. In this paper $A(t)$ is a multivalued maximal monotone operator of the particular form $\partial\varphi^t$ in a Hilbert space, but we do not assume that $D(A(t))$ is independent of t. In our case, however, the closure $\overline{D(\partial\varphi^t)}$ of $D(\partial\varphi^t)$ is independent of t by the assumption (I), since $\overline{D(\partial\varphi^t)}$ coincides with the closure of the effective domain D of φ^t (see [**10**]).

The author is indebted to Professor H. Fujita for valuable advice.

§ 2. Approximation of subdifferentials.

Let φ be a lower semicontinuous convex function from H into $(-\infty, \infty]$ with nonvoid effective domain. In this section we summarize for later use some properties of the Yosida approximation $\partial\varphi_\lambda$, $\lambda > 0$, of the subdifferential $\partial\varphi$. Most of these properties of $\partial\varphi_\lambda$ are essentially due to Moreau [8]. See also [1] and [10].

We can see easily that the subdifferential $\partial\varphi$ of φ is a nonempty monotone set in $H\times H$, i.e., $D(\partial\varphi)\neq\emptyset$ and $(u_1-u_2, v_1-v_2)\geqq 0$ if $v_i\in\partial\varphi(u_i)$, $i=1, 2$. Define for all $\lambda>0$ and all $u, v\in H$

$$\Phi_\lambda(u, v)=\varphi(v)+(1/2\lambda)\|u-v\|^2. \tag{2.1}$$

For each $u\in H$ and each $\lambda>0$, there exists one and only one u' in H such that $\Phi_\lambda(u, u')\leqq\Phi_\lambda(u, v)$ for all $v\in H$, since the set of all $v\in H$ such that $\Phi_\lambda(u, v)\leqq r$ is a nonempty weakly compact set in H for sufficiently large real number r, and since the function $v\rightarrow\Phi_\lambda(u, v)$ is strictly convex. This element u', which is uniquely determined by φ, λ and u, is denoted by $J_\lambda(\varphi)u$ or $J_\lambda u$.

Let $v\in H$ and $\alpha\in(0, 1]$. Since $\Phi_\lambda(u, \alpha v+(1-\alpha)J_\lambda u)>\Phi_\lambda(u, J_\lambda u)$, by using (2.1) we obtain

$$\varphi(v)+(\alpha/2\lambda)\|v-J_\lambda u\|^2\geqq\varphi(J_\lambda u)+(v-J_\lambda u, (u-J_\lambda u)/\lambda) \tag{2.2}$$

and hence, by letting $\alpha\downarrow 0$ in (2.2), we have for all $u\in H$ and all $\lambda>0$

$$(u-J_\lambda u)/\lambda\in\partial\varphi(J_\lambda u) \tag{2.3}$$

which implies that $\partial\varphi$ is maximal monotone and that $J_\lambda u=(1+\lambda\cdot\partial\varphi)^{-1}u$. We define $(\partial\varphi)_\lambda u=(u-J_\lambda u)/\lambda$. It is well-known that for $u, v\in H$ and $\lambda>0$ we have

$$\|J_\lambda u-J_\lambda v\|\leqq\|u-v\|, \tag{2.4}$$

$$\|(\partial\varphi)_\lambda u\|\leqq|||\partial\varphi(u)|||\equiv\inf\{\|w\|\,;\,w\in\partial\varphi(u)\} \tag{2.5}$$

and that $(\partial\varphi)_\lambda$ is monotone for each $\lambda>0$.

For each $u\in H$ and each $\lambda>0$ we define

$$\varphi_\lambda(u)=\varphi(J_\lambda u)+(\lambda/2)\|(\partial\varphi)_\lambda u\|^2. \tag{2.6}$$

By the definition of $J_\lambda u$ we have

$$\varphi_\lambda(u)=\Phi_\lambda(u, J_\lambda u)\leqq\Phi_\lambda(u, u)=\varphi(u). \tag{2.7}$$

It follows from (2.3) that for all $u, v\in H$ and all $\lambda>0$

$$\begin{aligned}&\Phi_\lambda(u, v)-\varphi_\lambda(u)\\&\quad=\varphi(v)-\varphi(J_\lambda u)-(v-J_\lambda u, (\partial\varphi)_\lambda u)+\frac{\|v-J_\lambda u\|^2}{2\lambda}\geqq\frac{\|v-J_\lambda u\|^2}{2\lambda}.\end{aligned} \tag{2.8}$$

For any function ψ from H into $(-\infty, \infty]$ we define its conjugate ψ^* by

$$\psi^*(u)=\sup\{(u, v)-\psi(v)\,;\ v\in H\} \qquad \text{for all} \quad u\in H.$$

If φ is a lower semicontinuous convex function from H into $(-\infty, \infty]$ with nonvoid effective domain, then so is its conjugate φ^*. It is well-known that for $u, v\in H$

$$v\in\partial\varphi(u) \text{ is equivalent to } \varphi(u)+\varphi^*(v)=(u, v). \tag{2.9}$$

We can easily verify

$$(\varphi_\lambda)^*(u)=\varphi^*(u)+(\lambda/2)\|u\|^2 \qquad \text{for all } u\in H \text{ and all } \lambda>0. \tag{2.10}$$

By (2.6), (2.10), (2.9) and (2.3) we get

$$\begin{aligned}\varphi_\lambda(u)+(\varphi_\lambda)^*((\partial\varphi)_\lambda(u)) &= \varphi(J_\lambda u)+\varphi^*((\partial\varphi)_\lambda(u))+\lambda\|(\partial\varphi)_\lambda(u)\|^2\\ &=(J_\lambda u, (\partial\varphi)_\lambda(u))+\lambda\|(\partial\varphi)_\lambda(u)\|^2\\ &=(u, (\partial\varphi)_\lambda(u)),\end{aligned}$$

and hence, by (2.9), $(\partial\varphi)_\lambda(u)\in\partial(\varphi_\lambda)(u)$. Then for all $u, v\in H$ and all $\lambda>0$ we have

$$\begin{aligned}0 &\leqq \varphi_\lambda(v)-\varphi_\lambda(u)-(v-u, (\partial\varphi)_\lambda(u))\\ &\leqq (v-u, (\partial\varphi)_\lambda(v)-(\partial\varphi)_\lambda(u))\\ &\leqq \|v-u\|^2/\lambda\end{aligned} \tag{2.11}$$

which implies that $\varphi_\lambda(u)$ is Fréchet differentiable at each $u\in H$ and the Fréchet derivative of φ_λ at u is equal to $(\partial\varphi)_\lambda(u)$. Hence $\partial(\varphi_\lambda)(u)=\{(\partial\varphi)_\lambda(u)\}$, and henceforth we will write $\partial\varphi_\lambda$ instead of $(\partial\varphi)_\lambda$.

§ 3. Continuity of $\partial\varphi_\lambda^t$ with respect to t.

In this section we will show that, under the assumptions (I) and (II) stated in § 1, $\partial\varphi_\lambda^t(u)$ satisfies a Hölder condition with exponent 1/2 in $0\leqq t\leqq T$ for each $u\in H$ and each $\lambda>0$.

LEMMA 3.1. *Let φ^1 and φ^2 be two lower semicontinuous convex functions from H to $(-\infty, \infty]$ with nonvoid effective domains. Then*

$$\varphi_\lambda^1(u)-\varphi_\lambda^2(u)\leqq\varphi^1(J_\lambda(\varphi^2)u)-\varphi^2(J_\lambda(\varphi^2)u)-(\lambda/2)\|\partial\varphi_\lambda^1(u)-\partial\varphi_\lambda^2(u)\|^2 \tag{3.1}$$

for all $\lambda>0$ and all $u\in H$.

PROOF. We write $J_\lambda^i u=J_\lambda(\varphi^i)u$ for $i=1, 2$. Using (2.6) and $\partial\varphi_\lambda^1(u)\in\partial\varphi^1(J_\lambda^1 u)$ we obtain

$$\begin{aligned}&[\varphi_\lambda^1(u)-\varphi_\lambda^2(u)]-[\varphi^1(J_\lambda^2 u)-\varphi^2(J_\lambda^2 u)]\\ &\qquad=\varphi^1(J_\lambda^1 u)-\varphi^1(J_\lambda^2 u)+(\lambda/2)(\|\partial\varphi_\lambda^1(u)\|^2-\|\partial\varphi_\lambda^2(u)\|^2)\end{aligned}$$

$$= \varphi^1(J_\lambda^1 u) - \varphi^1(J_\lambda^2 u) - (\partial\varphi_\lambda^1 u, J_\lambda^1 u - J_\lambda^2 u)$$
$$+ (\lambda/2)[\|\partial\varphi_\lambda^1(u)\|^2 - \|\partial\varphi_\lambda^2(u)\|^2 - 2(\partial\varphi_\lambda^1(u), \partial\varphi_\lambda^1(u) - \partial\varphi_\lambda^2(u))]$$
$$\leqq -(\lambda/2)\|\partial\varphi_\lambda^1(u) - \partial\varphi_\lambda^2(u)\|^2 ,$$

which proves (3.1).

LEMMA 3.2. *Let* φ^1 *and* φ^2 *be two lower semicontinuous convex functions from* H *to* $(-\infty, \infty]$ *with nonvoid effective domains, and let* $x \in H$, $\lambda > 0$, $\delta > 0$, $\varepsilon > 0$, $\varepsilon' > 0$ *and* $\eta \in (0, 1)$. *Suppose that the following three conditions are satisfied:*

(i) *If* $u \in H$ *and* $\|u - J_\lambda(\varphi^1)x\| \leqq \delta$, *then* $\varphi^2(u) \geqq (1-\eta)\varphi^1(u) - \varepsilon$.

(ii) $\varphi^2(J_\lambda(\varphi^1)x) \leqq \varphi^1(J_\lambda(\varphi^1)x) + \varepsilon'$.

(iii) $\delta^2/(2\lambda) \geqq \varepsilon + \varepsilon' + \eta \cdot \varphi_\lambda^1(x) (\equiv \alpha)$.

Then $\|\partial\varphi_\lambda^1(x) - \partial\varphi_\lambda^2(x)\|^2 < 2\alpha/\lambda$.

PROOF. We write $\Phi^i(u) = \varphi^i(u) + \|u - x\|^2/(2\lambda)$ and $x_i = J_\lambda(\varphi^i)x$ for $i = 1, 2$ and for $u \in H$. By (2.6) and (2.8) we have

$$\Phi^i(u) \geqq \Phi^i(x_i) + \|u - x_i\|^2/(2\lambda) \qquad \text{for } i = 1, 2 \text{ and for } u \in H, \tag{3.2}$$

and (i) and (ii) give respectively

$$\Phi^2(u) \geqq (1-\eta)\Phi^1(u) - \varepsilon , \qquad \text{if } \|u - x_1\| \leqq \delta \tag{3.3}$$

and

$$\Phi^2(x_1) \leqq \Phi^1(x_1) + \varepsilon' . \tag{3.4}$$

We will show $\|x_1 - x_2\| \leqq \delta$. Suppose $\|x_1 - x_2\| > \delta$. Define $\theta = \delta/\|x_1 - x_2\|$ and $x_\theta = (1-\theta)x_1 + \theta x_2$, then $0 < \theta < 1$ and $\|x_\theta - x_1\| = \delta$. By using (3.2), (3.3), (3.4) and the convexity of Φ^2 we have

$$\begin{aligned}(1-\eta)\Phi^1(x_1) - \varepsilon &\leqq (1-\eta)\Phi^1(x_\theta) - \varepsilon \\ &\leqq \Phi^2(x_\theta) \leqq (1-\theta)\Phi^2(x_1) + \theta\Phi^2(x_2) \\ &\leqq (1-\theta)[\Phi^1(x_1) + \varepsilon'] + \theta\Big[\Phi^1(x_1) + \varepsilon' - \frac{\|x_1 - x_2\|^2}{2\lambda}\Big] \\ &= \Phi^1(x_1) + \varepsilon' - \frac{\theta}{2\lambda}\|x_1 - x_2\|^2\end{aligned}$$

which shows

$$\frac{\delta^2}{2\lambda\theta} = \frac{\theta}{2\lambda}\|x_1 - x_2\|^2 \leqq \varepsilon + \varepsilon' + \eta \cdot \Phi^1(x_1) = \alpha , \tag{3.5}$$

since $\Phi^1(x_1) = \varphi_\lambda^1(x)$. Combining (iii) and (3.5) we have $\theta \geqq 1$, which is a contradiction. Hence $\|x_1 - x_2\| \leqq \delta$.

Using (3.2), (3.3) and (3.4) we obtain

$$\Phi^2(x_2) + \varepsilon \geqq (1-\eta)\Phi^1(x_2) \geqq (1-\eta)[\Phi^1(x_1) + \|x_1 - x_2\|^2/(2\lambda)]$$

and

$$\Phi^1(x_1)+\varepsilon' \geqq \Phi^2(x_2)+\|x_1-x_2\|^2/(2\lambda)\,.$$

Adding these two inequalities together, we get

$$\varepsilon+\varepsilon'+\eta\cdot\Phi^1(x_1) \geqq \frac{2-\eta}{2\lambda}\|x_1-x_2\|^2$$

and so

$$\|\partial\varphi^1_\lambda(x)-\partial\varphi^2_\lambda(x)\|^2=\|x_1-x_2\|^2/\lambda^2 \leqq 2\alpha/\lambda(2-\eta) < 2\alpha/\lambda\,.$$

This completes the proof.

PROPOSITION 3.3. *Suppose that* $\{\varphi^t;\ 0\leqq t\leqq T\}$ *satisfies condition* (I) *in* § 1, *and that for each* $r>0$ *there exists a real constant* c'_r *and a nonnegative function* $h(s,t)$ *on* $0\leqq s,\ t\leqq T$ *such that*

$$|\varphi^s(u)-\varphi^t(u)| \leqq h(s,t)[\varphi^t(u)+c'_r] \tag{3.6}$$

for all $s,t\in[0,T]$ *and all* $u\in D$ *with* $\|u\|\leqq r$. *If, for each fixed* $t\in[0,T]$, $h(s,t)$ *is continuous in* $0\leqq s\leqq T$ *and* $h(t,t)=0$, *then* $\partial\varphi^t_\lambda(x)$ *is continuous in* $0\leqq t\leqq T$ *for each* $\lambda>0$ *and each* $x\in H$. *Moreover, if* h *is of the form*

$$h(s,t)=c|s-t|^\alpha \qquad (0\leqq s,\ t\leqq T) \tag{3.7}$$

with constants $c>0$ *and* $\alpha\in(0,1]$, *then* $\partial\varphi^t_\lambda(x)$ *satisfies a Hölder condition with exponent* $\alpha/2$ *in* $0\leqq t\leqq T$ *for each* $\lambda>0$ *and each* $x\in H$.

PROOF. Let $\lambda>0$ and $x\in H$. Set $x_t=J_\lambda(\varphi^t)x$ for $0\leqq t\leqq T$, and fix an arbitrary $t\in[0,T]$. Let $r>\|x_t\|$. Since by (3.6)

$$\varphi^s(u)\geqq[1-h(s,t)]\varphi^t(u)-c'_r h(s,t)$$

and

$$\varphi^s(x_t)\leqq\varphi^t(x_t)+h(s,t)[\varphi^t(x_t)+c'_r]$$

for $s\in[0,T]$ and $u\in D$ with $\|u\|\leqq r$, it follows from Lemma 3.2 that

$$\|\partial\varphi^s_\lambda(x)-\partial\varphi^t_\lambda(x)\|^2\leqq(4/\lambda)h(s,t)[\varphi^t(x_t)+c'_r+(\lambda/4)\|\partial\varphi^t_\lambda(x)\|^2] \tag{3.8}$$

if s is sufficiently close to t. By (3.8), $\partial\varphi^s_\lambda(x)$ is continuous at $s=t$ and hence, by the arbitrariness of t, it is continuous in $0\leqq s\leqq T$.

Therefore we may take r so large that $\|x_t\|=\|x-\lambda\partial\varphi^t_\lambda(x)\|<r$ for all $t\in[0,T]$. Using Lemma 3.1 and (3.6), we obtain for all $s,t\in[0,T]$

$$\varphi^s_\lambda(x)-\varphi^t_\lambda(x)\leqq\varphi^s(x_t)-\varphi^t(x_t)\leqq h(s,t)[\varphi^t(x_t)+c'_r]$$

which implies that $\varphi^s_\lambda(x)$ is bounded from above in $0\leqq s\leqq T$. Thus, since $\varphi^s(x_s)\leqq\varphi^s_\lambda(x)$ by (2.6), $\varphi^s(x_s)$ is bounded from above in $0\leqq s\leqq T$. Hence by (3.8) there exists an $M>0$ independent of t and satisfying

$$\|\partial\varphi^s_\lambda(x)-\partial\varphi^t_\lambda(x)\|\leqq Mh(s,t)^{1/2} \tag{3.9}$$

whenever s is sufficiently close to $t\in[0,T]$. If h is of the form (3.7), then (3.9) implies that $\partial\varphi^t_\lambda(x)$ satisfies a Hölder condition with exponent $\alpha/2$ in

$0 \leqq t \leqq T$. The proof is completed.

For every nonempty closed convex subset K of H, K^0 denotes the uniquely determined element u in K such that $\|u\| = \inf\{\|v\|;\ v \in K\}$.

COROLLARY 3.4. *Assume that conditions* (I) *and* (II) *in* §1 *are satisfied. Let* $b \in D$. *If* b *is in* $D(\partial\varphi^t)$ *for a. a.* $t \in [0, T]$, *then* $\partial\varphi^t(b)^0$ *is strongly measurable in* $0 \leqq t \leqq T$.

PROOF. Since by Proposition 3.3 $\partial\varphi^t_{1/n}(b)$ is continuous in $0 \leqq t \leqq T$ for each positive integer n and by [**3**, Theorem 2.3, (a)] it converges to $\partial\varphi^t(b)^0$ as $n \to \infty$ for a. a. $t \in [0, T]$, it follows from [**4**, Theorem 3.5.4, (3)] that $\partial\varphi^t(b)^0$ is strongly measurable in $0 \leqq t \leqq T$.

§4. Uniqueness of solutions.

In this section we suppose that f satisfies conditions (IV) and (V). First we shall show an *a priori* estimate for strong solutions of the equation (1.1).

LEMMA 4.1. *If* u_1 *and* u_2 *are two strong solutions of* (1.1) *on* $[0, T]$, *then*

$$\|u_1(t) - u_2(t)\| \leqq \exp\Big(\int_s^t \beta(\sigma)d\sigma\Big)\|u_1(s) - u_2(s)\| \qquad \text{for} \quad 0 \leqq s \leqq t \leqq T. \tag{4.1}$$

PROOF. By Definition 1.3 and by condition (IV), $y_i(t) = f(t, u_i(t)) - du_i(t)/dt \in \partial\varphi^t(u_i(t))$ a. e. in $[0, T]$ for $i = 1, 2$. Then, by using the monotonicity of $\partial\varphi^t$ and condition (V), we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_1(t) - u_2(t)\|^2 \\ &= ([f(t, u_1(t)) - f(t, u_2(t))] - [y_1(t) - y_2(t)],\ u_1(t) - u_2(t)) \\ &\leqq \beta(t)\|u_1(t) - u_2(t)\|^2 \end{aligned} \tag{4.2}$$

for a. a. $t \in [0, T]$. Since $\|u_1(t) - u_2(t)\|$ is continuous in $0 \leqq t \leqq T$ and absolutely continuous in $s \leqq t \leqq T$ for each $s \in (0, T]$, we deduce (4.1) from (4.2).

COROLLARY 4.2. *If* u_1 *and* u_2 *are two strong solutions of* (1.1) *on* $[0, T]$ *satisfying* $u_1(0) = u_2(0)$, *then* $u_1(t) = u_2(t)$ *holds for all* $t \in [0, T]$.

This corollary proves the uniqueness part of Theorems 1.1 and 1.5, because u in Theorem 1.1 is trivially a strong solution of (1.1).

§5. Equations in Banach spaces.

The purpose of this section is to prepare an existence theorem for solutions of nonlinear evolution equations of a certain class, which can be applied to the approximate equations for (1.1) with $\partial\varphi^t$ replaced by $\partial\varphi^t_\lambda$. However, we shall give a somewhat general version of the result, which seems to be of an independent interest.

Let X be a real Banach space and suppose its dual space X^* is uniformly convex. $\| \ \|$ denotes the norms in X and X^*. (x, x^*) denotes the value of $x^* \in X^*$ at $x \in X$. Let F be the duality map of X into X^*; for each $x \in X$, Fx is the uniquely determined element x^* of X^* such that $(x, x^*) = \|x\|^2 = \|x^*\|^2$. It is known that F is uniformly continuous in any bounded subset of X. See Kato [**5**].

Let f be a map of $[0, T] \times X$ into X, where T is a positive constant. We consider the equation

$$\frac{d}{dt} u(t) = f(t, u(t)) \tag{5.1}$$

in $0 \leqq t \leqq T$ with the initial condition $u(0) = a \in X$. For the completeness of this paper we shall show the local and global existence of solutions of (5.1) in Theorems 5.1 and 5.3 respectively. Cf. Murakami [**9**, Remark 2, p. 157] and Kato [**5**, **6**].

THEOREM 5.1. *Suppose that*

(i) *For each fixed* $x \in X$, $f(t, x)$ *is strongly measurable in* $0 \leqq t \leqq T$, *and for each fixed* $t \in [0, T]$ *it is continuous from the strong to the weak topology of* X.

Let $a \in X$ *and* $\rho > 0$, *and suppose that there exists two Lebesgue integrable functions* β *and* γ *on* $0 \leqq t \leqq T$ *satisfying*

$$(f(t, x) - f(t, y), F(x-y)) \leqq \beta(t) \|x-y\|^2 \tag{5.2}$$

and

$$\|f(t, x)\| \leqq \gamma(t) \tag{5.3}$$

respectively, whenever $0 \leqq t \leqq T$, $\|x-a\| < \rho$ *and* $\|y-a\| < \rho$. *Then for some* $r \in (0, T]$ *there exists one and only one* u *in* $C([0, r]; X)$ *such that for all* $t \in [0, r]$

$$u(t) = a + \int_0^t f(s, u(s)) ds\,. \tag{5.4}$$

REMARK 5.2. For $v \in C([t_1, t_2]; X)$ $(0 \leqq t_1 < t_2 \leqq T)$, $f(t, v(t))$ is strongly measurable in $[t_1, t_2]$ by (i), and it is Bochner integrable in $[t_1, t_2]$ by (5.3) if $\|v(t) - a\| \leqq \rho$ for all t. Cf. Remark 1.2. Since X is reflexive, (5.4) holds for $t \in [0, r]$ if and only if u is strongly absolutely continuous in $[0, r]$ and satisfies (5.1) for a. a. $t \in [0, r]$ and $u(0) = a$. See Kōmura [**7**, Appendix].

PROOF OF THEOREM 5.1. Let $r \in (0, T]$ such that $\int_0^r \gamma(t) dt \leqq \rho$. For each integer $n > 1/r$ we define u_n as

$$u_n(t) = \begin{cases} a & (t \leqq 1/n) \\ a + \int_{1/n}^t f(s, u_n(s-1/n)) ds & (1/n < t \leqq r)\,. \end{cases} \tag{5.5}$$

Clearly $\|u_n(t)-a\| \leqq \rho$. Let $x_{mn}(t)=u_m(t)-u_n(t)$, $y_{mn}(t)=u_m(t-1/m)-u_n(t-1/n)$ and $B(t)=\int_0^t \beta(s)ds$. We obtain

$$(5.6) \qquad e^{-2B(t)}\|x_{mn}(t)\|^2$$
$$\leqq 2\int_0^t e^{-2B(s)}[2\gamma(s)\|Fx_{mn}(s)-Fy_{mn}(s)\|-\beta(s)(\|x_{mn}(s)\|^2-\|y_{mn}(s)\|^2)]ds$$

for $t\in[0, r]$. Because it is easily seen from (5.5), (5.2) and (5.3) that the derivative of the left-hand side of (5.6) with respect to t is not greater than the derivative of the right-hand side for almost all t (cf. Kato [**5**, Proof of Lemma 4.3]). Since by (5.5) and (5.3) $\|x_{mn}(t)-y_{mn}(t)\|\to 0$ as $m, n\to\infty$, it follows from (5.6) that u_n converges to a u in $C([0, r]; X)$ as $n\to\infty$, and by letting $n\to\infty$ in (5.5) we obtain (5.4).

We conclude the proof by noting that the uniqueness of solutions is easily seen by a standard argument similar to § 4.

THEOREM 5.3. *Suppose that* $f(t, x)$ *satisfies condition* (i) *of Theorem* 5.1 *and the following three conditions:*

(ii) *For some Lebesgue integrable function* β *on* $0\leqq t\leqq T$ (5.2) *holds for all* $t\in[0, T]$ *and all* $x, y\in X$.

(iii) *For each bounded subset* X_0 *of* X *there exists a Lebesgue integrable function* γ *on* $0\leqq t\leqq T$ *satisfying* (5.3) *for all* $t\in[0, T]$ *and all* $x\in X_0$.

(iv) *For some* b *in* X, $f(t, b)$ *is in* $L^2(0, T; X)$.

Then for each $a\in X$ *there exists one and only one* u *in* $C([0, T]; X)$ *such that* (5.4) *holds for* $0\leqq t\leqq T$.

PROOF. By Theorem 5.1 there exists at least a u in $C([0, t_1); X)$ such that (5.4) holds for $0\leqq t<t_1$, where $t_1\in(0, T]$. By using (ii) we have for almost all $t\in[0, t_1]$

$$\exp(2B(t)+t)\frac{d}{dt}(\exp(-2B(t)-t)\|u(t)-b\|^2)$$
$$=2(f(t, u(t)), F(u(t)-b))-(2\beta(t)+1)\|u(t)-b\|^2$$
$$\leqq \|f(t, b)\|^2,$$

from which by means of (iv) it follows easily that u is bounded in $0\leqq t<t_1$. Hence, by (i) and (iii), $f(t, u(t))$ is in $L^1(0, t_1; X)$ and, by (5.4), $u(t)$ converges strongly to an a_1 in X as $t\uparrow t_1$. If $t_1<T$, it is readily seen from Theorem 5.1 that the solution u can be continued to the right of t_1. Thus the solution u has the continuation to the whole interval $[0, T]$.

Since the uniqueness of solutions follows from Theorem 5.1, we complete the proof.

§ 6. Proof of Theorem 1.1.

We assume that conditions (I) to (VI) are satisfied. Let $\lambda > 0$. Since $\partial\varphi_\lambda^t(u)$ satisfies a Lipschitz condition with respect to u in H with Lipschitz constant $1/\lambda$ (cf. (2.4)) and a Hölder condition with exponent 1/2 in $0 \leqq t \leqq T$ for each fixed $u \in H$ by virtue of (I) and (II) (see Proposition 3.3), it is easily seen that (IV) and (VI) are satisfied with $f(t, u)$ replaced by $f(t, u) - \partial\varphi_\lambda^t(u)$. By (V) and by the monotonicity of $\partial\varphi_\lambda^t$ we have

$$([f(t, u) - \partial\varphi_\lambda^t(u)] - [f(t, v) - \partial\varphi_\lambda^t(v)], u - v) \leqq \beta(t)\|u - v\|^2 \tag{6.1}$$

for all $t \in [0, T]$ and all $u, v \in H$. Then it follows from Theorem 5.3 that for each $a \in H$ there exists one and only one H-valued strongly absolutely continuous function u_λ on $[0, T]$ satisfying

$$\frac{d}{dt} u_\lambda(t) + \partial\varphi_\lambda^t(u_\lambda(t)) = f(t, u_\lambda(t)) \quad \text{a.e. in } 0 \leqq t \leqq T \tag{6.2}$$

and $u_\lambda(0) = a$. We shall write $y_\lambda(t) = \partial\varphi_\lambda^t(u_\lambda(t))$. Note that y_λ is in $C([0, T]; H)$. We set

$$B(t) = \int_0^t \beta(s)ds .$$

LEMMA 6.1. *$\{u_\lambda(t);\ \lambda > 0,\ 0 \leqq t \leqq T\}$ is bounded in H.*

PROOF. For almost all $t \in [0, T]$, by using (6.1) and (6.2), we obtain

$$\begin{aligned} \frac{d}{dt} \|u_\lambda(t) - b\|^2 &= 2([f(t, u_\lambda) - f(t, b)] - [y_\lambda - \partial\varphi_\lambda^t(b)] + f(t, b) - \partial\varphi_\lambda^t(b), u_\lambda - b) \\ &\leqq 2\beta(t)\|u_\lambda(t) - b\|^2 + 2(\|f(t, b)\| + \|\partial\varphi_\lambda^t(b)\|)\|u_\lambda(t) - b\| , \end{aligned}$$

and hence

$$\frac{d}{dt}\|u_\lambda(t) - b\| \leqq \beta(t)\|u_\lambda(t) - b\| + \|f(t, b)\| + \|\partial\varphi_\lambda^t(b)\| \tag{6.3}$$

(cf. [5, p. 515]). Since $u_\lambda(0) = a$, (6.3) gives for $0 \leqq t \leqq T$

$$\|u_\lambda(t) - b\| \leqq e^{B(t)}\|a - b\| + \int_0^t e^{B(t) - B(s)}(\|f(s, b)\| + \|\partial\varphi_\lambda^s(b)\|)ds ,$$

whose right-hand side is bounded in $0 \leqq t \leqq T$ and $\lambda > 0$ because by (III) and (2.5) we have

$$\int_0^T \|\partial\varphi_\lambda^s(b)\|ds \leqq \int_0^T |||\partial\varphi^s(b)|||\,ds < \infty .$$

Our conclusion follows.

From now on we use the notation $J_\lambda^t u$ instead of $J_\lambda(\varphi^t)u$.

LEMMA 6.2. *If H_0 is a bounded subset of H, then $\{J_\lambda^t u;\ 0 \leqq t \leqq T,\ 0 < \lambda \leqq 1,\ u \in H_0\}$ is also bounded in H.*

PROOF. Let $0 \leqq t \leqq T$, $\lambda, \mu > 0$ and $u \in H$, then

$$J_\mu^t u = J_\lambda^t\left(\frac{\lambda}{\mu} u + \frac{\mu-\lambda}{\mu} J_\mu^t u\right) \tag{6.4}$$

(see e. g. [2, Lemma 1.2, (iv)]). It is easily seen from (6.4) and (2.4) that

$$\|J_\lambda^t u - J_1^t u\| \leqq (1-\lambda)\|u - J_1^t u\| \tag{6.5}$$

when $0 < \lambda \leqq 1$. Let v be a fixed element in H_0. By using (2.4), the triangle inequality and (6.5) we get

$$\|J_\lambda^t u\| \leqq \|J_\lambda^t v\| + \|u-v\| \leqq (1-\lambda)\|v - J_1^t v\| + \|J_1^t v\| + \|u-v\|,$$

which proves the lemma since $J_1^t v$ is bounded in $0 \leqq t \leqq T$ by Proposition 3.3.

We assume that a is in D.

LEMMA 6.3. *$\varphi_\lambda^t(u_\lambda(t))$ is bounded in $0 \leqq t \leqq T$ and $0 < \lambda \leqq 1$.*

PROOF. Let r be the supremum of $\|J_\mu^t u_\lambda(s)\|$ in $0 \leqq s$, $t \leqq T$, $0 < \mu \leqq 1$ and $\lambda > 0$. By Lemmas 6.1 and 6.2, r is finite. Let $\lambda > 0$ and $u \in H$ such that $r \geqq \|J_\lambda^s u\|$ for all $s \in [0, T]$. Using Lemma 3.1, (II) and (2.6), we obtain

$$\begin{aligned} |\varphi_\lambda^t(u) - \varphi_\lambda^s(u)| &\leqq \max\{\varphi^t(J_\lambda^s u) - \varphi^s(J_\lambda^s u),\ \varphi^s(J_\lambda^t u) - \varphi^t(J_\lambda^t u)\} \\ &\leqq |t-s| \cdot [c_r \max\{\varphi^s(J_\lambda^s u),\ \varphi^t(J_\lambda^t u)\} + c_r'] \\ &\leqq |t-s| \cdot [c_r \max\{\varphi_\lambda^s(u),\ \varphi_\lambda^t(u)\} + c_r'], \end{aligned} \tag{6.6}$$

which implies, in particular, the continuity of $\varphi_\lambda^t(u)$ with respect to t. Using (2.11) and (6.6) we obtain

$$\begin{aligned} &|\varphi_\lambda^s(u_\lambda(s)) - \varphi_\lambda^t(u_\lambda(t))| \\ &\quad \leqq |\varphi_\lambda^s(u_\lambda(s)) - \varphi_\lambda^s(u_\lambda(t))| + |\varphi_\lambda^s(u_\lambda(t)) - \varphi_\lambda^t(u_\lambda(t))| \\ &\quad \leqq |(y_\lambda(s), u_\lambda(s) - u_\lambda(t))| + \|u_\lambda(s) - u_\lambda(t)\|^2/\lambda \\ &\qquad + |t-s| \cdot [c_r \max\{\varphi_\lambda^t(u_\lambda(t)),\ \varphi_\lambda^s(u_\lambda(t))\} + c_r'], \end{aligned}$$

from which we get easily the absolute continuity of $\varphi_\lambda^t(u_\lambda(t))$ in $0 \leqq t \leqq T$ by the strong absolute continuity of u_λ and the boundedness of y_λ.

For almost all $t \in [0, T]$, by using (2.11) and (6.2), we get

$$\begin{aligned} \frac{\partial}{\partial s}\varphi_\lambda^t(u_\lambda(s))|_{s=t} &= (y_\lambda(t), u_\lambda'(t)) \\ &= (f(t, u_\lambda(t)) - u_\lambda'(t), u_\lambda'(t)) \\ &\leqq (1/4)\|f(t, u_\lambda(t))\|^2, \end{aligned}$$

which implies by Lemma 6.1 and (VI) the existence of a Lebesgue integrable function γ on $[0, T]$ independent of λ and satisfying

$$\frac{\partial}{\partial s}\varphi_\lambda^t(u_\lambda(s))|_{s=t} \leqq \gamma(t) \qquad \text{for almost all } t \in [0, T].$$

Hence, using (6.6), we have for almost all $t \in [0, T]$

$$\frac{d}{dt}\varphi_\lambda^t(u_\lambda(t)) = \lim_{s\to t}\frac{1}{s-t}[\varphi_\lambda^s(u_\lambda(s)) - \varphi_\lambda^t(u_\lambda(s))] + \frac{\partial}{\partial s}\varphi_\lambda^t(u_\lambda(s))|_{s=t}$$

$$\leqq c_r\varphi_\lambda^t(u_\lambda(t)) + c_r' + \gamma(t),$$

which gives

(6.7) $$\varphi_\lambda^t(u_\lambda(t)) \leqq e^{c_r t}\varphi_\lambda^0(a) + \int_0^t e^{c_r(t-s)}[c_r' + \gamma(s)]ds \quad \text{for all} \quad t \in [0, T].$$

Since by (2.7)

(6.8) $$\varphi_\lambda^0(a) \leqq \varphi^0(a) < \infty,$$

it follows from (6.7) that $\varphi_\lambda^t(u_\lambda(t))$ is bounded from above in $0 \leqq t \leqq T$ and $0 < \lambda \leqq 1$.

For $v \in H$ with $\|v\| \leqq r$ and for $t \in [0, T]$, by (II), we get

$$\frac{1}{1+tc_r}[\varphi^0(v) - tc_r'] \leqq \varphi^t(v),$$

which shows that $\varphi^t(v)$ is bounded from below in $0 \leqq t \leqq T$ and $\|v\| \leqq r$ because $\varphi^0(v)$ is bounded from below in $\|v\| \leqq r$. Therefore, since $\varphi_\lambda^t(u_\lambda(t)) \geqq \varphi^t(J_\lambda^t(u_\lambda(t)))$ by (2.6), $\varphi_\lambda^t(u_\lambda(t))$ is bounded from below in $0 \leqq t \leqq T$ and $0 < \lambda \leqq 1$. The proof is complete.

LEMMA 6.4. *$\{y_\lambda;\ 0 < \lambda \leqq 1\}$ is bounded in $L^2(0, T; H)$.*

PROOF. Let r be the same number as in the proof of the preceding lemma. Let $T' \in (0, T)$ and $\lambda \in (0, 1]$. By using (2.11), (6.2) and (6.6) we obtain

$$\begin{aligned}&\int_0^{T'}(y_\lambda(t), f(t, u_\lambda(t)) - y_\lambda(t))dt \\ &= \lim_{h\downarrow 0}\frac{1}{h}\int_0^{T'}[\varphi_\lambda^t(u_\lambda(t+h)) - \varphi_\lambda^t(u_\lambda(t))]dt \\ &= \varphi_\lambda^{T'}(u_\lambda(T')) - \varphi_\lambda^0(a) - \lim_{h\downarrow 0}\frac{1}{h}\int_0^{T'}[\varphi_\lambda^{t+h}(u_\lambda(t+h)) - \varphi_\lambda^t(u_\lambda(t+h))]dt \\ &\geqq \varphi_\lambda^{T'}(u_\lambda(T')) - \varphi_\lambda^0(a) - \int_0^{T'}[c_r\varphi_\lambda^t(u_\lambda(t)) + c_r']dt,\end{aligned}$$

from which, by using Lemma 6.3, we get

(6.9) $$\int_0^T \|y_\lambda(t)\|^2 dt - \int_0^T (y_\lambda(t), f(t, u_\lambda(t)))dt \leqq \varphi_\lambda^0(a) + C$$

for some real constant C independent of λ. Since, by (VI) and Lemma 6.1, $\int_0^T \|f(t, u_\lambda(t))\|^2 dt$ is bounded in $\lambda > 0$, it follows from (6.8) and (6.9) that $\int_0^T \|y_\lambda(t)\|^2 dt$ is bounded in $0 < \lambda \leqq 1$.

PROOF OF THEOREM 1.1. We shall prove the existence of a $u \in C([0, T]; H)$

and a $y \in L^2(0, T; H)$ satisfying (i) and (ii) of Theorem 1.1. When $\lambda, \mu > 0$, by using (6.2), (V), (2.3) and the monotonicity of $\partial\varphi^t$ we have for almost all $t \in [0, T]$

$$\begin{aligned}\frac{d}{dt}\|u_\lambda(t)-u_\mu(t)\|^2 &= 2([f(t, u_\lambda)-f(t, u_\mu)]-[y_\lambda(t)-y_\mu(t)],\ u_\lambda(t)-u_\mu(t)) \\ &\leqq 2\beta(t)\|u_\lambda(t)-u_\mu(t)\|^2-2(y_\lambda(t)-y_\mu(t),\ \lambda y_\lambda(t)-\mu y_\mu(t)) \\ &= 2\beta(t)\|u_\lambda(t)-u_\mu(t)\|^2-(\lambda+\mu)\|y_\lambda(t)-y_\mu(t)\|^2-(\lambda-\mu)(\|y_\lambda(t)\|^2-\|y_\mu(t)\|^2),\end{aligned}$$

from which by using $u_\lambda(0)=u_\mu(0)=a$ we obtain for $0 \leqq t \leqq T$

$$e^{-2B(t)}\|u_\lambda(t)-u_\mu(t)\|^2+(\lambda+\mu)\int_0^t e^{-2B(s)}\|y_\lambda(s)-y_\mu(s)\|^2 ds \leqq (\mu-\lambda)\int_0^t e^{-2B(s)}(\|y_\lambda(s)\|^2-\|y_\mu(s)\|^2)ds. \tag{6.10}$$

By (6.10) and by Lemma 6.4, $\int_0^t e^{-2B(s)}\|y_\lambda(s)\|^2 ds$ is nondecreasing as $\lambda \downarrow 0$ and bounded in $0 < \lambda \leqq 1$, and hence it converges to a real number as $\lambda \downarrow 0$. Then it follows from (6.10) again that, as $\lambda \downarrow 0$, u_λ converges to a u in $C([0, T]; H)$ and y_λ converges to a y in $L^2(0, T; H)$. Since we can choose a sequence of real numbers $\lambda_n \downarrow 0$ such that $\lim_{n\to\infty} \partial\varphi_{\lambda_n}^t(u_{\lambda_n}(t)) = \lim_{n\to\infty} y_{\lambda_n}(t) = y(t)$ a. e. in $0 \leqq t \leqq T$, we have

$$u(t) \in D(\partial\varphi^t) \quad \text{and} \quad y(t) \in \partial\varphi^t(u(t)) \qquad \text{for a. a. } t \in [0, T]. \tag{6.11}$$

See, e. g., [**5**, Lemma 2.5, (b)]. Since u is continuous, (6.11) implies that $u(t)$ is in the closure of $D(\partial\varphi^t)$ for all $t \in [0, T]$, and moreover,

$$\lim_{\lambda\downarrow 0} J_\lambda^t u_\lambda(t) = u(t) \qquad \text{for all} \quad t \in [0, T]. \tag{6.12}$$

In fact, $J_\lambda^t u(t) \to u(t)$ as $\lambda \downarrow 0$ (see [**3**, Theorem 2.3, (b)]) and hence we obtain (6.12) by using (2.4). Using (6.12), (2.6), Lemma 6.3 and the lower semicontinuity of φ^t we get

$$\varphi^t(u(t)) \leqq \liminf_{\lambda\downarrow 0} \varphi^t(J_\lambda^t u_\lambda(t)) \leqq \liminf_{\lambda\downarrow 0} \varphi_\lambda^t(u_\lambda(t)) < \infty,$$

which implies $u(t) \in D$ for all $t \in [0, T]$. Thus we have proved that u and y satisfy condition (i). Condition (ii) follows from the relation

$$u_\lambda(t)+\int_0^t y_\lambda(s)ds = a+\int_0^t f(s, u_\lambda(s))ds, \qquad 0 \leqq t \leqq T,$$

by making $\lambda \downarrow 0$.

Since the uniqueness part of the theorem has been proved already in § 4, the proof is complete.

§ 7. Proof of Theorem 1.5.

We shall prove Theorem 1.5. Suppose conditions (I) to (VI) are satisfied, and let $a \in \bar{D}$. Choose $a_n \in D (n=1, 2, \cdots)$ such that $\|a_n - a\| \to 0$ as $n \to \infty$, and let u_n be a strong solution of (1.1) on $[0, T]$ with $u_n(0) = a_n$ (see Theorem 1.1). By (4.1), u_n converges to a u in $C([0, T]; H)$ as $n \to \infty$. If $u(s) \in D$ for some $s \in (0, T)$, then u is a strong solution of (1.1) on $[s, T]$ and $u(t)$ is in D for all $t \in [s, T]$. Because by Theorem 1.1 there exists a strong solution v of (1.1) on $[s, T]$ with $v(s) = u(s)$ and $v(t)$ is in D for all $t \in [s, T]$, and hence by using (4.1) we get $u(t) = \lim_{n \to \infty} u_n(t) = v(t) \in D$ for all $t \in [s, T]$. Therefore, if $u(t) \in D$ for all $t \in (0, T]$, it follows from Remark 1.4 that u is a strong solution of (1.1) on $[0, T]$ with $u(0) = a$. Since the uniqueness part of Theorem 1.5 has been proved already in § 4, it remains to show $u(t) \in D$ for all $t \in (0, T]$.

Suppose there exists a $t_0 \in (0, T]$ such that $u(t_0) \notin D$. Then $u(t) \notin D$ for all $t \in [0, t_0]$, and by the lower semicontinuity of φ^t we have

$$\liminf_{n \to \infty} \varphi^t(u_n(t)) \geqq \varphi^t(u(t)) = \infty \qquad \text{for all} \quad t \in [0, t_0]\,. \tag{7.1}$$

For $\lambda > 0$ let $u_{n\lambda}$ be a strong solution of (6.2) on $[0, T]$ with $u_{n\lambda}(0) = a_n$. Since, by (6.12), $\lim_{\lambda \downarrow 0} J_\lambda^t u_{n\lambda}(t) = u_n(t)$, we have

$$\liminf_{\lambda \downarrow 0} \varphi_\lambda^t(u_{n\lambda}(t)) \geqq \liminf_{\lambda \downarrow 0} \varphi^t(J_\lambda^t u_{n\lambda}(t)) \geqq \varphi^t(u_n(t)) \tag{7.2}$$

for all $t \in [0, T]$ and all $n \geqq 1$.

By Lemma 6.2, $r \equiv \sup \{\|J_\lambda^t u_{n\lambda}(t)\|;\ n \geqq 1,\ 0 < \lambda \leqq 1,\ 0 \leqq t \leqq T\}$ is finite, because the proof of Lemma 6.1 shows the boundedness of $\{u_{n\lambda}(t);\ n \geqq 1,\ \lambda > 0,\ 0 \leqq t \leqq T\}$ in H. Let m be a positive integer such that

$$\exp\Big(\int_0^T |\beta(s)|\, ds\Big) \|a_n - a_m\| \leqq 1 \qquad \text{if} \quad n \geqq m\,, \tag{7.3}$$

and set $M = \sup \{|\varphi_\mu^s(u_{m\mu}(s))|;\ 0 \leqq s \leqq T,\ 0 < \mu \leqq 1\}$. By Lemma 6.3, M is finite. We define

$$g_{n\lambda}(t) = e^{-c_r t} \varphi_\lambda^t(u_{n\lambda}(t)) - M \tag{7.4}$$

for $n \geqq 1$, $\lambda > 0$ and $t \in [0, T]$.

An argument similar to the proof of Egorov's Theorem [**11**, p. 16] shows that by (7.1) and (7.2) we can choose a positive integer $n_0 \geqq m$, a Lebesgue measurable set E in $[0, t_0]$, of measure greater than $t_0/2$, and a sequence $\{k_n\}_{n=n_0}^\infty$ of positive integers such that

$$g_{n,1/k}(t) > 0 \quad \text{if} \ n \geqq n_0\,,\ k \ (\text{integer}) \geqq k_n \ \text{and} \ t \in E\,. \tag{7.5}$$

For each $n \geqq 1$ and each $\lambda > 0$ we can show similarly to the proof of Lemma 6.3

$$\frac{d}{dt}\varphi_\lambda^t(u_{n\lambda}(t)) \leqq c_r\varphi_\lambda^t(u_{n\lambda}(t))+c_r' +(\partial\varphi_\lambda^t(u_{n\lambda}(t)),\ u_{n\lambda}'(t))$$
$$= c_r\varphi_\lambda^t(u_{n\lambda}(t))+c_r' - \|\partial\varphi_\lambda^t(u_{n\lambda}(t))\|^2+(\partial\varphi_\lambda^t(u_{n\lambda}(t)),\ f(t,\ u_{n\lambda}(t)))$$

a. e. in $0\leqq t\leqq T$ and hence, making use of (7.4), we get

$$e^{c_r t}\frac{d}{dt}g_{n\lambda}(t) \leqq c_r' - \|\partial\varphi_\lambda^t(u_{n\lambda}(t))\|^2+(\partial\varphi_\lambda^t(u_{n\lambda}(t)),\ f(t,\ u_{n\lambda}(t))) \tag{7.6}$$
$$\leqq -\frac{1}{2}\|\partial\varphi_\lambda^t(u_{n\lambda}(t))\|^2+\gamma(t)e^{c_r t}, \qquad \text{a. e. in } 0\leqq t\leqq T,$$

where we have used (VI) and γ is a nonnegative Lebesgue integrable function on $[0, T]$ independent of $n\geqq 1$ and $\lambda>0$. Since by using (4.1), (7.3) and (7.4) we obtain

$$\begin{aligned}\|\partial\varphi_\lambda^t(u_{n\lambda}(t))\| &\geqq (\partial\varphi_\lambda^t(u_{n\lambda}(t)),\ u_{n\lambda}(t)-u_{m\lambda}(t))\\ &\geqq \varphi_\lambda^t(u_{n\lambda}(t))-\varphi_\lambda^t(u_{m\lambda}(t))\\ &= e^{c_r t}(g_{n\lambda}(t)+M)-\varphi_\lambda^t(u_{m\lambda}(t))\\ &\geqq e^{c_r t}g_{n\lambda}(t)\end{aligned}$$

for $t\in[0, T]$, $n\geqq m$ and $0<\lambda\leqq 1$, it follows from (7.5) and (7.6) that

$$\frac{d}{dt}g_{n,1/k}(t) \leqq -\frac{1}{2}g_{n,1/k}(t)^2\chi_E(t)+\gamma(t) \qquad \text{a. e. in } 0\leqq t\leqq T \tag{7.7}$$

for all $n\geqq n_0$ and all $k\geqq k_n$, where χ_E is the characteristic function of E.

In order to complete the proof we prepare a lemma. Let $S>0$ and let e and γ be two nonnegative Lebesgue integrable functions on $0\leqq t\leqq S$. We set

$$c(t)=\int_0^t \gamma(s)ds \qquad \text{for } 0\leqq t\leqq S$$

and

$$h_\alpha(t)=2\alpha\Big/\Big(2-\alpha\int_t^S e(s)ds\Big) \qquad \text{for real } \alpha \text{ and } t\in[0, S] \tag{7.8}$$

when the denominator in the right-hand side of (7.8) does not vanish.

LEMMA 7.1. *Let g be a real-valued absolutely continuous function on $[0, S]$ satisfying*

$$\frac{d}{dt}g(t) \leqq -\frac{1}{2}g(t)^2e(t)+\gamma(t) \qquad \text{a. e. in } 0\leqq t\leqq S. \tag{7.9}$$

Let $\alpha>0$. If $g(S)>\alpha+c(S)$, then $g(t)>h_\alpha(t)+c(t)$ whenever $\int_t^S e(s)ds<2/\alpha$ and $0\leqq t\leqq S$.

PROOF. Suppose that, for some $t_1\in[0, S]$, $\int_{t_1}^S e(s)ds<2/\alpha$ and $g(t_1)\leqq h_\alpha(t_1)+c(t_1)$ hold simultaneously. Then, since $g(S)>\alpha+c(S)=h_\alpha(S)+c(S)$, there exists a $t_2\in[t_1, S)$ such that

(7.10) $$g(t_2)=h_\alpha(t_2)+c(t_2)$$

and such that $g(t)>h_\alpha(t)+c(t)$ for all $t\in(t_2, S]$. Hence we have

(7.11) $$g(t)>h_\alpha(t)>0\,, \qquad t_2<t\leqq S\,.$$

By using (7.10), (7.8), (7.9) and (7.11) we obtain

$$\begin{aligned}\alpha-g(S)+c(S)&=\int_{t_2}^{S}\frac{d}{dt}[h_\alpha(t)-g(t)+c(t)]dt\\&\geqq-\frac{1}{2}\int_{t_2}^{S}[h_\alpha(t)^2-g(t)^2]e(t)dt\\&\geqq 0\end{aligned}$$

which is a contradiction. The proof is complete.

END OF THE PROOF OF THEOREM 1.5. We define $c(t)=\int_0^t\gamma(s)ds$ for $0\leqq t\leqq T$. By (7.1) and (7.2) we can take $n\geqq n_0$ and $k\geqq k_n$ such that $g_{n,1/k}(t_0)>c(t_0)+4/t_0$. Let $4/t_0<\alpha<g_{n,1/k}(t_0)-c(t_0)$. Then by (7.7) and Lemma 7.1 we have

(7.12) $$g_{n,1/k}(t)>c(t)+2\alpha\Big/\Big(2-\alpha\int_t^{t_0}\chi_E(s)ds\Big)$$

if $\int_t^{t_0}\chi_E(s)ds<2/\alpha$ and $0\leqq t\leqq t_0$. Since $\int_0^{t_0}\chi_E(s)ds>t_0/2>2/\alpha$, we can find the greatest number t_1 in $(0, t_0)$ such that

$$\int_{t_1}^{t_0}\chi_E(s)ds=2/\alpha\,,$$

and hence by (7.12) we obtain

$$\begin{aligned}g_{n,1/k}(t_1)&=\lim_{t\downarrow t_1}g_{n,1/k}(t)\\&\geqq c(t_1)+\lim_{t\downarrow t_1}2\alpha\Big/\Big(2-\alpha\int_t^{t_0}\chi_E(s)ds\Big)\\&=\infty\end{aligned}$$

which contradicts $g_{n,1/k}(t_1)<\infty$. The proof is complete.

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