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On homogeneous Kähler manifolds of solvable Lie groups

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Introduction

Let M be a connected homogeneous complex manifold on which a connected Lie group G acts transitively as a group of holomorphic transformations. We assume that M admits a G-invariant volume element v. If v has an expression

$$v = i^n F(z, \bar{z}) dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n$$

in a local coordinate system $\{z_1, \cdots, z_n\}$, then the G-invariant hermitian form

$$h = \sum_{i,j} \frac{\partial^2 \log F(z, \bar{z})}{\partial z_i \partial \bar{z}_j} dz_i d\bar{z}_j$$

is called the canonical hermitian form of M. If M carries a G-invariant Kähler metric and if v is the volume element determined by this metric, the Ricci tensor of the Kähler manifold is equal to $-h$. From now on, M is assumed to be a homogeneous Kähler manifold unless otherwise specified. The canonical hermitian form h plays an important role in the investigation of homogeneous Kähler manifolds, and results in this direction are the following:

(i) If G is a semi-simple Lie group, then h is non-degenerate and the number of negative squares of h is equal to the difference between the dimension of a maximal compact subgroup of G and the dimension of the isotropy subgroup of G at a point of M [8].

(ii) If G is a unimodular Lie group and if h is non-degenerate, then G is a semi-simple Lie group [2].

(iii) h is negative definite if and only if G is a compact semi-simple Lie group [8], [**11**].

In [**13**], E. B. Vinberg, S. G. Gindikin, I. I. Pjateckii-Šapiro studied the structure of J-algebras. The J-algebra of a homogeneous bounded domain is proper in their sense. They proved the following:

(iv) Every proper J-algebra is isomorphic to the J-algebra of a homogeneous Siegel domain of the second kind. Since this domain is holomorphically isomorphic to a homogeneous bounded domain, the canonical hermitian form h of a proper J-algebra is positive definite. Moreover there exists a solvable Lie group which acts simply transitively on the homogeneous bounded domain.

Further they developed the theory of Kähler algebras and they proved [1], [14].

(v) If a connected simply connected Kähler manifold M admits a simply transitive solvable and splittable Lie group G, then M is a holomorphic fibre bundle whose base space is a homogeneous bounded domain and whose fibre is a locally flat homogeneous Kähler manifold. (A Lie group G is said to be splittable if the adjoint operator $\mathrm{ad}(X)$ has only real eigenvalues for any element X in the Lie algebra of G.)

We assume that the canonical hermitian form of M is positive definite and that M admits a transitive solvable Lie group. Then the corresponding Kähler algebra of M is a proper J-algebra and hence the universal covering manifold of M is holomorphically isomorphic to a homogeneous bounded domain.

Now, we shall denote by $\hat{G}$ the identity component of the group of all holomorphic transformations of M leaving h invariant. If h is non-degenerate, the group $\hat{G}$ is a Lie group acting on M as a Lie transformation group. In [3], Hano proved

(vi) Let M be a homogeneous complex (not necessarily Kähler) manifold with non-degenerate canonical hermitian form. Then the adjoint group $\mathrm{Ad}_{\hat{G}}(\hat{G})$ of $\hat{G}$ is the identity component of a real algebraic group in $GL(\hat{\mathfrak{g}}, \boldsymbol{R})$, where $\hat{\mathfrak{g}}$ is the Lie algebra of $\hat{G}$.

In the present paper, we show that the positive definiteness of h follows from its non-degeneracy provided that a (not necessarily splittable) solvable Lie group acts on M simply transitively. Precisely speaking, we prove the following theorem:

THEOREM. *Let M be a connected simply connected homogeneous Kähler manifold with non-degenerate canonical hermitian form h. If M admits a simply transitive solvable Lie group, then the canonical hermitian form h is positive definite and hence M is holomorphically isomorphic to a homogeneous bounded domain.*

We denote by $I^0(M)$ the identity component of the group of all isometries of M. If the canonical hermitian form is non-degenerate, then $I^0(M)\subset\hat{G}$ [6], and the center of a transitive subgroup of $I^0(M)$ is discrete. Moreover, if M is a homogeneous bounded domain, then we have $\hat{G}=I^0(M)$ [4] and the isotropy subgroup of $\hat{G}$ at a point $o\in M$ is a maximal compact subgroup of $\hat{G}$, and the center of the group $\hat{G}$ is reduced to the identity [5].

As immediate applications of our theorem and (vi), we have

COROLLARY 1. *Let M be a connected homogeneous Kähler manifold with non-degenerate canonical hermitian form. We assume that M admits a transitive Lie group G whose isotropy subgroup K at a point o of M is a maximal compact subgroup of G and that the group $\hat{G}$ coincides with $I^0(M)$ and of finite center. Then M is holomorphically isomorphic to a homogeneous bounded domain.*

COROLLARY 2. *Let M be a connected simply connected homogeneous Kähler manifold with non-degenerate canonical hermitian form. We assume that a point $o \in M$ has no conjugate point and that the group $\hat{G}$ coincides with $I^0(M)$ and of finite center. Then M is holomorphically isomorphic to a homogeneous bounded domain.*

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§ 1. Preliminaries

A. A $2m$-dimensional real vector space V is called a symplectic space if there exists a skew symmetric bilinear form σ on V and a linear endomorphism J of V satisfying the following conditions; for $u, v \in V$

$$J^2u = -u\,,$$
$$\sigma(Ju, Jv) = \sigma(u, v)\,,$$
$$\sigma(Ju, u) > 0\,, \qquad u \neq \mathbf{0}\,.$$

In this case V may be regarded as a complex vector space with the complex structure J, which we shall denote by $\tilde{V}$. Moreover,

$$(u, v) = \sigma(Ju, v) + i\sigma(u, v) \qquad (u, v \in \tilde{V})$$

is a positive definite hermitian form on this complex vector space $\tilde{V}$.

For a real linear transformation f of V, we put

$$f^+(u) = (1/2)(f(u) - Jf(Ju))\,,$$
$$f^-(u) = (1/2)(f(u) + Jf(Ju))\,, \qquad u \in V\,.$$

Then, we have $f = f^+ + f^-$, $f^+J = Jf^+$, and $f^-J = -Jf^-$. Let $u_1, u_2, \cdots, u_m$ be an orthonormal basis of $\tilde{V}$ with respect to the hermitian form (u, v). Put

$$f^+(u_j) = \sum_{k=1}^{m} a_{kj} u_k\,, \qquad a_{kj} \in \boldsymbol{C}\,,$$

$$f^-(u_j) = \sum_{k=1}^{m} b_{kj} u_k\,, \qquad b_{kj} \in \boldsymbol{C}\,.$$

Identifying $\tilde{V}$ and $\boldsymbol{C}^m$ by means of the map $u=\sum_{j=1}^{m} z_j u_j \to z={}^t(z_1, z_2, \cdots, z_m)$, f may be considered as a map $z\to Az+B\bar{z}$, where $A=(a_{kj})$, $B=(b_{kj})$ and $\bar{z}={}^t(\bar{z}_1, \cdots, \bar{z}_m)$. We denote this map f by $f=(A, B)$. If we have $f=(A', B')$ with respect to another orthonormal basis of $\tilde{V}$, then there exists a unitary matrix U such that

$$A'=UA{}^t\bar{U}, \qquad B'=UB{}^tU.$$

A real linear transformation f of V is said to be symplectic if

$$\sigma(f(u), v)+\sigma(u, f(v))=0$$

for $u, v\in V$. It is easy to see that $f=(A, B)$ is symplectic if and only if A is a skew-hermitian matrix and B is a symmetric matrix. By a simple calculation, we see that if f is symplectic, then $\mathrm{Tr} f=0$.

Now, let f be a real linear transformation of V which commutes with J. For real numbers α, β, we put

$$V_{(\alpha+i\beta)}=\{u\in V\,;\,(f-(\alpha+\beta J))^m u=0 \text{ for some } m\}$$

and

$$V_{[\alpha]}=\sum_{\beta} V_{(\alpha+i\beta)}.$$

Then, we have $V=\sum_{\alpha} V_{[\alpha]}$ and $V_{[\alpha]}$ is the largest subspace of V on which the real parts of the eigenvalues of f are equal to α.

B. We denote by M a connected Kähler manifold on which a Lie group G acts simply transitively as a group of holomorphic isometries. Let (I, g) be the Kähler structure on M, i.e. I is a G-invariant complex structure tensor on M and g is a G-invariant Kähler metric on M.

Let $\mathfrak{g}$ be the Lie algebra of all left invariant vector fields on G, and let π be the canonical projection from G onto M defined by $\pi(a)=a\cdot o$, for $a\in G$, where o is a fixed point of M. Let π_e denote the differential of π at the identity e of G, and let X_e, I_o and g_o be the values of $X\in\mathfrak{g}$, I and g at e and o respectively. Then there exist a linear endomorphism J of $\mathfrak{g}$ and a skew symmetric bilinear form ρ on $\mathfrak{g}$ such that

$$\pi_e(JX)_e=I_o(\pi_e X_e),$$

$$\rho(X, Y)=g_o(\pi_e X_e, I_o\pi_e Y_e),$$

for $X, Y\in\mathfrak{g}$. Then $(\mathfrak{g}, J, \rho)$ satisfies the following properties ([1], [8]):

(K.1) $$J^2X=-X;$$

(K.2) $$[JX, JY]=J[JX, Y]+J[X, JY]+[X, Y];$$

(K.3) $$\rho(JX, JY)=\rho(X, Y);$$

(K.4) $$\rho(JX, X) > 0, \qquad X \neq 0;$$

(K.5) $$\rho([X, Y], Z) + \rho([Y, Z], X) + \rho([Z, X], Y) = 0;$$

where $X, Y, Z \in \mathfrak{g}$.
$(\mathfrak{g}, J, \rho)$ will be called the normal Kähler algebra of M.

It is known that the canonical hermitian form h of a homogeneous Kähler manifold M has the following expression due to J. L. Koszul [8].

Putting

(1.1) $$\eta(X, Y) = h_0(\pi_e X_e, \pi_e Y_e),$$

and

(1.2) $$\psi(X) = \mathrm{Tr}_{\mathfrak{g}}(\mathrm{ad}(JX) - J\,\mathrm{ad}(X)),$$

we have

(1.3) $$\eta(X, Y) = (1/2)\psi([JX, Y]),$$

for $X, Y \in \mathfrak{g}$. The form η satisfies the following properties:

(1.4) $$\eta(X, Y) = \eta(Y, X),$$

(1.5) $$\eta(JX, JY) = \eta(X, Y),$$

for $X, Y \in \mathfrak{g}$.

Now, the following lemma is due to [1].

LEMMA 1. *For $E, X, Y \in \mathfrak{g}$,*

$$\begin{aligned}\frac{d}{dt}\rho(&\exp t\,\mathrm{ad}(JE)X,\ \exp t\,\mathrm{ad}(JE)Y)\\ &= \rho(JE,\ \exp t\,\mathrm{ad}(JE)[X, Y]).\end{aligned}$$

PROOF. By the property (K.5) of the Kähler algebra,

$$\begin{aligned}\frac{d}{dt}\rho(&\exp t\,\mathrm{ad}(JE)X,\ \exp t\,\mathrm{ad}(JE)Y)\\ &= \rho([JE, \exp t\,\mathrm{ad}(JE)X],\ \exp t\,\mathrm{ad}(JE)Y)\\ &\quad + \rho(\exp t\,\mathrm{ad}(JE)X,\ [JE, \exp t\,\mathrm{ad}(JE)Y])\\ &= \rho(JE, [\exp t\,\mathrm{ad}(JE)X,\ \exp t\,\mathrm{ad}(JE)Y])\\ &= \rho(JE,\ \exp t\,\mathrm{ad}(JE)[X, Y]).\end{aligned}$$

Q. E. D.

Now, let $\mathfrak{g}$ be a real Lie algebra and let $\mathfrak{k}$ be a subalgebra of $\mathfrak{g}$. Suppose that there exist a linear endomorphism J of $\mathfrak{g}$ and a 1-form ω on $\mathfrak{g}$, which satisfy the following conditions:

(J.1) $$J\mathfrak{k} \subset \mathfrak{k}, \ J^2 X \equiv -X \pmod{\mathfrak{k}};$$

(J.2) $$[W, JX] \equiv J[W, X] \pmod{\mathfrak{k}};$$

(J.3) $$[JX, JY] \equiv J[JX, Y]+J[X, JY]+[X, Y] (\operatorname{mod} \mathfrak{k});$$

(J.4) $$\omega([W, X])=0;$$

(J.5) $$\omega([JX, JY])=\omega([X, Y]);$$

(J.6) $$\omega([JX, X])>0, \quad X \notin \mathfrak{k};$$

where $X, Y \in \mathfrak{g}$, $W \in \mathfrak{k}$.
Then, $(\mathfrak{g}, \mathfrak{k}, J, \omega)$ will be called a J-algebra.

Let $(\mathfrak{g}, \mathfrak{k}, J, \omega)$ be a J-algebra and let $\mathfrak{g}'$ be a subalgebra such that

$$J\mathfrak{g}' \subset \mathfrak{g}'+\mathfrak{k}.$$

We can then define a linear endomorphism J' of $\mathfrak{g}'$ so that $JX' \equiv J'X' (\operatorname{mod} \mathfrak{k})$, for $X' \in \mathfrak{g}'$. We define a 1-form ω' on $\mathfrak{g}'$ as the restriction of ω on $\mathfrak{g}'$ and we put $\mathfrak{k}'=\mathfrak{k} \cap \mathfrak{g}'$. It is easy to see that $(\mathfrak{g}', \mathfrak{k}', J', \omega')$ is a J-algebra. It is called a J-subalgebra of $(\mathfrak{g}, \mathfrak{k}, J, \omega)$.

A J-algebra $(\mathfrak{g}, \mathfrak{k}, J, \omega)$ is said to be proper, if it satisfies the following condition (P):

(P) Every compact semi-simple J-subalgebra of $(\mathfrak{g}, \mathfrak{k}, J, \omega)$ is contained in $\mathfrak{k}$.

A J-algebra $(\mathfrak{g}, \mathfrak{k}, J, \omega)$ will be called normal if $\mathfrak{k}=0$, and we denote it by $(\mathfrak{g}, J, \omega)$.

We know then the following theorem (cf. Introduction (iv)):

Let $(\mathfrak{g}, J, \omega)$ be a normal J-algebra and let $\mathfrak{g}$ be a solvable Lie algebra. Then this J-algebra is proper, and is isomorphic to the J-algebra of a homogeneous bounded domain.

§ 2. Statement of Theorem

In this section, we shall state our theorem and sketch the proof.

THEOREM. *Let M be a connected Kähler manifold on which a connected solvable Lie group G acts simply transitively as a group of holomorphic isometries. Let $(\mathfrak{g}, J, \rho)$ be the normal Kähler algebra of M. If the canonical hermitian form h of M is non-degenerate, then we get the decomposition*

(2.1) $$\mathfrak{g}=\sum_{k=1}^{m} \mathfrak{g}_k$$

of $\mathfrak{g}$ into direct sum of vector spaces with the following properties:

1) *$\mathfrak{g}_k$ is a J-invariant subalgebra in which there exist an element $E_k \in \mathfrak{g}_k$ and a subspace $\mathfrak{p}_k$ with the following properties:*

(2.2) $$\mathfrak{g}_k=\{JE_k\}+\{E_h\}+\mathfrak{p}_k;$$

(2.3) $$J\mathfrak{p}_k \subset \mathfrak{p}_k;$$

$$[JE_k, E_k] = E_k\,; \tag{2.4}$$

$$[JE_k, \mathfrak{p}_k] \subset \mathfrak{p}_k\,. \tag{2.5}$$

Moreover the real parts of the eigenvalues of ad(JE_k) *on* $\mathfrak{p}_k$ *are equal to* 1/2, *and*

$$[E_k, \mathfrak{p}_k] = \{0\}\,; \tag{2.6}$$

$$[\mathfrak{p}_k, \mathfrak{p}_k] \subset \{E_k\}\,. \tag{2.7}$$

2) *Put*

$$\mathfrak{g}^{k+1} = \mathfrak{g}_{k+1} + \mathfrak{g}_{k+2} + \cdots + \mathfrak{g}_m\,, \tag{2.8}$$

then, we have

$$[JE_k, \mathfrak{g}^{k+1}] \subset \mathfrak{g}^{k+1}\,. \tag{2.9}$$

Moreover the real parts of the eigenvalues of ad(JE_k) *on* $\mathfrak{g}^{k+1}$ *are equal to* 0, *and*

$$[E_k, \mathfrak{g}^{k+1}] = \{0\}\,; \tag{2.10}$$

$$[\mathfrak{p}_k, \mathfrak{g}^{k+1}] \subset \mathfrak{p}_k\,. \tag{2.11}$$

3) *The form* η *defined by* (1.2) *and* (1.3) *is positive definite on* $\mathfrak{g}$ *and the factors of the decomposition* $\mathfrak{g} = \sum_{k=1}^{m} (\{JE_k\} + \{E_k\} + \mathfrak{p}_k)$ *are mutually orthogonal with respect to this form* η.

Under the same assumption of Theorem, $(\mathfrak{g}, J, \psi)$ becomes a solvable normal J-algebra, since η is positive definite on $\mathfrak{g}$ (cf. § 1. B). Therefore it is isomorphic to a J-algebra of a homogeneous bounded domain (cf. Introduction (iv) and § 1. B). Hence we have our theorem in the introduction.

As for the proof of Theorem, put $\mathfrak{g}^1 = \mathfrak{g}$. We shall show by induction on n that there exists a decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \sum_{k=1}^{n-1} \mathfrak{g}_k + \mathfrak{g}^n \tag{2.12}$$

with the following properties:

1) For each $k = 1, 2, \cdots, n-1$, there exist an element $E_k \in \mathfrak{g}_k$ and a subspace $\mathfrak{p}_k \subset \mathfrak{g}_k$ with the properties as stated in Theorem 1).

2) Put $\mathfrak{g}^{k+1} = \sum_{l=k+1}^{n-1} \mathfrak{g}_l + \mathfrak{g}^n$. Then $\mathfrak{g}^{k+1}$ has the properties as stated in Theorem 2).

3) The form η which is non-degenerate on $\mathfrak{g}$ is positive definite on $\sum_{k=1}^{n-1} \mathfrak{g}_k$ and the factors of the decomposition $\mathfrak{g} = \sum_{k=1}^{n-1} (\{JE_k\} + \{E_k\} + \mathfrak{p}_k) + \mathfrak{g}^n$ are mutually orthogonal with respect to this form η.

Now, our theorem will follow by this inductive process, since we shall have $\mathfrak{g}^{m+1}=\{0\}$ for a certain m.

Suppose we have a decomposition (2.12) for an integer n. Then $\mathfrak{g}_k$ $(1\leqq k\leqq n-1)$ and $\mathfrak{g}^n$ are clearly J-invariant solvable subalgebras of $\mathfrak{g}$.

Define

$$\sigma_k(X, Y)=\psi([X, Y]), \tag{2.13}$$

for $X, Y\in\mathfrak{p}_k$ $(1\leqq k\leqq n-1)$. Then $(\mathfrak{p}_k, J, \sigma_k)$ becomes a symplectic space (cf. §1. A). A representation f_k of $\mathfrak{g}^n$ in $\mathfrak{p}_k$ is defined by

$$f_k(X)U=[X, V] \tag{2.14}$$

where $X\in\mathfrak{g}^n$, $U\in\mathfrak{p}_k$, and f_k is symplectic, i.e. $f_k(X)$ is a symplectic transformation of $\mathfrak{p}_k$ for $X\in\mathfrak{g}^n$. Indeed, for $X\in\mathfrak{g}^n$, U, $V\in\mathfrak{p}_k$, we have

$$[X, [U, V]]=[[X, U], V]+[U, [X, V]].$$

By (2.7) and (2.10), the left side of the equation is equal to 0, and hence

$$\psi([[X, U], V])+\psi([U, [X, V]])=0.$$

Therefore $f_k(X)$ is symplectic.

§3. Proof of Theorem: Existence of E_n in $\mathfrak{g}^n$

We shall prove the following.

PROPOSITION 1. *Suppose we have a decomposition* (2.12) *with the properties given there for an integer* $n\geqq 1$. *Then, there exists a non-zero element* E_n *in* $\mathfrak{g}^n$ *such that*

$$[X, E_n]=\lambda(X)E_n, \qquad \lambda(X)\in\boldsymbol{R}, \qquad \text{for } X\in\mathfrak{g}^n;$$

$$[JE_n, E_n]=E_n.$$

In the first place, we show

LEMMA 2. *A real solvable Lie algebra* $\mathfrak{g}$ *contains a commutative ideal of dimension* 1 *or* 2 *spanned by the elements* E, F *such that for* $X\in\mathfrak{g}$

$$\begin{aligned}[X, E]&=\lambda(X)E+\mu(X)F,\\ [X, F]&=-\mu(X)E+\lambda(X)F,\end{aligned} \tag{3.1}$$

where λ, μ *are linear functions on* $\mathfrak{g}$.

PROOF. Let $\mathfrak{g}$ be a real solvable Lie algebra and let $\mathfrak{g}^c=\{X+iY; X, Y\in\mathfrak{g}\}$ be its complexification. By Lie's theorem, there exists a non-zero element Z in $\mathfrak{g}^c$ such that $[W, Z]=k(W)Z$ holds for all $W\in\mathfrak{g}^c$ with $k(W)\in\boldsymbol{C}$.

Let E (resp. F) denote the real part (resp. imaginary part) of Z. Then for any $X\in\mathfrak{g}$,

$$(3.2)\qquad \begin{aligned}[X, E] &= \lambda(X)E+\mu(X)F\,,\\ [X, F] &= -\mu(X)E+\lambda(X)F\,,\end{aligned}$$

hold, where λ, μ are linear functions on $\mathfrak{g}$.

Let $\mathfrak{r}=\{E, F\}$ be the real vector subspace spanned by the elements E, F. If E, F are linearly dependent, $\mathfrak{r}$ is a one-dimensional ideal of $\mathfrak{g}$. If E, F are linearly independent, $\mathfrak{r}$ is a two-dimensional ideal of $\mathfrak{g}$ satisfying the above conditions. For, since $[E, E]=0$, we get $\lambda(E)=\mu(E)=0$ by the first relation in (3.2), which implies $[E, F]=0$ by the second relation in (3.2) and then $\mathfrak{r}$ is a commutative ideal. Q. E. D.

We shall now prove that $\mathfrak{g}^n$ contains a one-dimensional ideal. In view of Lemma 2, it is sufficient to prove that $\mathfrak{g}^n$ contains no two-dimensional ideal $\mathfrak{r}=\{E, F\}$ as in Lemma 2.

Let $\mathfrak{r}=\{E, F\}$ be such an ideal of $\mathfrak{g}^n$. By a simple calculation, we have

$$(3.3)\qquad [[\mathfrak{g}^n, \mathfrak{g}^n], \mathfrak{r}]=\{0\}\,.$$

LEMMA 3. *Let X be an element of $\mathfrak{g}^n$. If $\phi([X, C])=0$ for all $C\in\mathfrak{r}$, then $[X, C]=0$ for all $C\in\mathfrak{r}$.*

PROOF. First we note that $\phi\neq 0$ on $\mathfrak{r}$. In fact, if $\phi=0$ on $\mathfrak{r}$, $\eta(Y, D)=\phi([JY, D])=0$ for $Y\in\mathfrak{g}^n$, $D\in\mathfrak{r}$. Since η is non-degenerate on $\mathfrak{g}^n$, it follows $\mathfrak{r}=\{0\}$, which is a contradiction. Therefore $\phi\neq 0$ on $\mathfrak{r}$. Since $[X, E]=\lambda(X)E+\mu(X)F$, $[X, F]=-\mu(X)E+\lambda(X)F$, we have $\lambda(X)\phi(E)+\mu(X)\phi(F)=0$, $-\mu(X)\phi(E)+\lambda(X)\phi(F)=0$. As $\phi(E)\neq 0$ or $\phi(F)\neq 0$, $\lambda(X)^2+\mu(X)^2=0$ holds, and hence we have $\lambda(X)=\mu(X)=0$, which implies $[X, C]=0$ for all $C\in\mathfrak{r}$. Q. E. D.

We put

$$\mathfrak{r}^0=\{C\in\mathfrak{r}\,;\ \phi([JC, E])=\phi([JC, F])=0\}\,.$$

Then, the following three cases are possible:

$$\dim\mathfrak{r}^0=0\,,\quad \dim\mathfrak{r}^0=1\quad \text{or}\quad \dim\mathfrak{r}^0=2\,.$$

We shall show that $\dim\mathfrak{r}^0\neq 0$. Suppose $\dim\mathfrak{r}^0=0$. Then η is non-degenerate on $\mathfrak{r}$ and hence there exists a unique non-zero element $A\in\mathfrak{r}$ such that $\phi([JA, C])=\phi(C)$, for all $C\in\mathfrak{r}$. When $A=\alpha E+\beta F$ ($\alpha, \beta\in\boldsymbol{R}$), we put $B=-\beta E+\alpha F$. Then A, B is a base of $\mathfrak{r}$ such that for $X\in\mathfrak{g}^n$

$$(3.4)\qquad \begin{aligned}[X, A] &= \lambda'(X)A+\mu'(X)B\,,\\ [X, B] &= -\mu'(X)A+\lambda'(X)B\,,\end{aligned}$$

where λ' and μ' are linear functions on $\mathfrak{g}^n$. Now, for $C\in\mathfrak{r}$, we have by (1.5), (3.3),

$$\begin{aligned}\psi([J[JA, A], C]) &= -\psi([[JA, A], JC]) \\ &= \psi([[A, JC], JA]) + \psi([[JC, JA], A]) \\ &= -\psi([A, JC]) \\ &= \psi([JA, C]) \\ &= \psi(C).\end{aligned}$$

Thus we get $[JA, A]=A$, and so $[JA, B]=B$ by (3.4). When we put $[JB, A] =\lambda_0 A+\mu_0 B$, λ_0, $\mu_0\in \boldsymbol{R}$, then $[JB, B]=-\mu_0 A+\lambda_0 B$ by (3.4). Hence by (K.2)

$$\begin{aligned}[[JA, JB], A] &= [J[JA, B]+J[A, JB], A] \\ &= [JB-\lambda_0 JA-\mu_0 JB, A] \\ &= -\lambda_0[JA, A]+(1-\mu_0)[JB, A] \\ &= -\lambda_0\mu_0 A+\mu_0(1-\mu_0)B.\end{aligned}$$

Since $[[JA, JB], A]=0$ by (3.3), we have $\mu_0=0$ or $\mu_0=1$ and $\lambda_0=0$. In the case $\mu_0=0$, put $\gamma=\psi(B)$, $\delta=-\psi(A)$. Then, since $\psi\neq 0$ on $\mathfrak{r}$, at least one of γ, δ is not zero and $\gamma\psi(A)+\delta\psi(B)=0$. Let $X=\gamma A+\delta B$. Then $X\neq 0$ and we have

$$\psi([JA, X])=\psi(\gamma A+\delta B)=\gamma\psi(A)+\delta\psi(B)=0,$$

$$\psi([JB, X])=\psi(\gamma\lambda A+\delta\lambda B)=\lambda(\gamma\psi(A)+\delta\psi(B))=0.$$

Since η is non-degenerate on $\mathfrak{r}$, it follows that $X=0$, which is a contradiction. Suppose $\mu_0=1$ and $\lambda_0=0$. Then we have by (K.2),

$$[JA, JB]=J[JA, B]+J[A, JB]=JB-JB=0.$$

From this and (K.5), it follows

$$\rho([JA, JB], B)+\rho([JB, B], JA)+\rho([B, JA], JB)=0,$$

$$\rho(JA, A)+\rho(JB, B)=0,$$

which is a contradiction since $\rho(JA, A)>0$ and $\rho(JB, B)>0$. Thus we have shown that $\dim\mathfrak{r}^0=0$ is impossible.

We show that $\dim\mathfrak{r}^0\neq 1$. Suppose $\dim\mathfrak{r}^0=1$ and let A be a non-zero element in $\mathfrak{r}^0$. When $A=\alpha E+\beta F$ (α, $\beta\in\boldsymbol{R}$), we put $B=-\beta E+\alpha F$. Then A, B form a basis of $\mathfrak{r}$ such that for $X\in\mathfrak{g}^n$

$$\begin{aligned}&[X, A]=\lambda'(X)A+\mu'(X)B \\ &[X, B]=-\mu'(X)A+\lambda'(X)B,\end{aligned} \tag{3.5}$$

where λ' and μ' are linear functions on $\mathfrak{g}^n$. Since $\psi([JA, A])=\psi([JA, B])=0$, we have $\psi([JA, C])=0$ for all $C\in\mathfrak{r}$ and it follows by Lemma 3 that $[JA, A] =0$ and $[JA, B]=0$. From this and (3.3), we have

$$
\begin{aligned}
0 &= \psi([[JA, JB], A]) \\
&= \psi([J[JA, B]+J[A, JB], A]) \\
&= \psi([J[A, JB], A]), \\
0 &= \psi([[JA, JB], B]) \\
&= \psi([J[JA, B]+J[A, JB], B]) \\
&= \psi([J[A, JB], B]).
\end{aligned}
$$

Hence there exists a real number λ_0 such that $[JB, A]=\lambda_0 A$. Then $[JB, B]=\lambda_0 B$ by (3.5). Since $B \notin \mathfrak{r}^0$, it follows that either $\psi([JB, B])=\lambda_0\psi(B)$ or $\psi([JB, A])=\lambda_0\psi(A)$ is not zero. Therefore $\lambda_0 \neq 0$. On the other hand, we have by (K.2) and (K.5)

$$
\begin{aligned}
0 &= \rho([JA, JB], A)+\rho([JB, A], JA)+\rho([A, JA], JB) \\
&= \rho(J[JA, B], A)+\rho(J[A, JB], A)+\rho([JB, A], JA)+\rho([A, JA], JB) \\
&= -2\lambda_0\rho(JA, A).
\end{aligned}
$$

Since $\rho(JA, A)>0$, we get $\lambda_0=0$, which is a contradiction. Thus $\dim \mathfrak{r}^0=1$ does not occur.

Suppose now $\dim \mathfrak{r}^0=2$. Since $\psi([JC, E])=0$, $\psi([JC, F])=0$ for any $C\in\mathfrak{r}$, it follows by Lemma 3 that $[JC, E]=[JC, F]=0$ for all $C\in\mathfrak{r}$ and hence we have

$$
[J\mathfrak{r}, \mathfrak{r}]=\{0\}. \tag{3.6}
$$

Now, we put

$$
\mathfrak{p}=\{P\in\mathfrak{g}^n\,;\ [P, E]=[JP, E]=0\}. \tag{3.7}
$$

Then, clearly

$$
J\mathfrak{p}\subset\mathfrak{p}. \tag{3.8}
$$

Moreover we see

$$
\operatorname{ad}(JE)\mathfrak{p}\subset\mathfrak{p}, \quad \text{and} \tag{3.9}
$$

$$
\operatorname{ad}(JE)J=J\operatorname{ad}(JE) \quad \text{on } \mathfrak{p}. \tag{3.10}
$$

Indeed, since $[JE, JP]=J[JE, P]+J[E, JP]+[E, P]=J[JE, P]$ for $P\in\mathfrak{p}$, we have $\operatorname{ad}(JE)J=J\operatorname{ad}(JE)$ on $\mathfrak{p}$. Since $[[JE, P], E]=0$ and $[J[JE, P], E]=[[JE, JP], E]=0$ by (3.3), we have $\operatorname{ad}(JE)\mathfrak{p}\subset\mathfrak{p}$.

Now, for $X\in\mathfrak{g}^n$, we get by (3.3), (3.6)

$$
[[JE, X], E]=0,
$$

$$[J[JE, X], E]=[[JE, JX]-J[E, JX]-[E, X], E]$$
$$=[[JE, JX], E]$$
$$=0,$$

which implies that $[JE, X]\in\mathfrak{p}$, and hence we have

(3.11) $$\operatorname{ad}(JE)\mathfrak{g}^n\subset\mathfrak{p}.$$

Let $P\in\mathfrak{p}$. We have

$$\rho(JE, [JE, P])=-\rho(E, J[JE, P])$$
$$=-\rho(E, [JE, JP])$$
$$=\rho(JE, [JP, E])+\rho(JP, [E, JE])$$
$$=0,$$

and it follows that for $X\in\mathfrak{g}^n$

(3.12) $$\rho(JE, \operatorname{ad}(JE)^2X)=0.$$

Applying Lemma 1 and (3.12), we have for $X, Y\in\mathfrak{g}^n$

$$\frac{d^3}{dt^3}\rho(\exp t\operatorname{ad}(JE)X, \exp t\operatorname{ad}(JE)Y)$$
$$=\frac{d^2}{dt^2}\rho(JE, \exp t\operatorname{ad}(JE)[X, Y])$$
$$=\rho(JE, \operatorname{ad}(JE)^2\exp t\operatorname{ad}(JE)[X, Y])$$
$$=0.$$

Hence we may put

(3.13) $$\rho(\exp t\operatorname{ad}(JE)X, \exp t\operatorname{ad}(JE)Y)=at^2+bt+c,$$

where a, b and c are real numbers not depending on t.

Now, let $\alpha+i\beta$ $(\alpha, \beta\in\boldsymbol{R})$ be an eigenvalue of $\operatorname{ad}(JE)$ on $\mathfrak{p}$. Since $\operatorname{ad}(JE)J=J\operatorname{ad}(JE)$ on $\mathfrak{p}$ by (3.10), there exists a non-zero element $P\in\mathfrak{p}$ such that $\operatorname{ad}(JE)P=(\alpha+\beta J)P$, and hence $\exp t\operatorname{ad}(JE)P=\exp t(\alpha+\beta J)P$. Therefore we have by (K.3) and (3.10),

$$\rho(\exp t\operatorname{ad}(JE)JP, \exp t\operatorname{ad}(JE)P)$$
$$=\rho(J\exp t\operatorname{ad}(JE)P, \exp t\operatorname{ad}(JE)P)$$
$$=\rho(J\exp t(\alpha+\beta J)P, \exp t(\alpha+\beta J)P)$$
$$=\rho(\exp t(\alpha+\beta J)JP, \exp t(\alpha+\beta J)P)$$
$$=e^{(\alpha+i\beta)t}\overline{e^{(\alpha+i\beta)t}}\rho(JP, P)$$
$$=e^{2\alpha t}\rho(JP, P).$$

From this and (3.13), we have

$$at^2+bt+c=e^{2\alpha t}\rho(JP, P).$$

Since $\rho(JP, P)>0$, it follows $\alpha=0$. Thus the real parts of the eigenvalues of $\mathrm{ad}(JE)$ on $\mathfrak{p}$ are equal to 0.

Now, we put

$$\psi_n(X)=\mathrm{Tr}_{\mathfrak{g}^n}(\mathrm{ad}(JX)-J\,\mathrm{ad}(X)) \qquad \text{for } X\in\mathfrak{g}^n. \tag{3.14}$$

From the facts that $\mathrm{ad}(JE)\mathfrak{g}^n\subset\mathfrak{p}$ (3.11) and the real parts of the eigenvalues of $ad(JE)$ on $\mathfrak{p}$ are equal to 0, it follows that $\mathrm{Tr}_{\mathfrak{g}^n}\mathrm{ad}(JE)=\mathrm{Tr}_{\mathfrak{p}}\mathrm{ad}(JE)=0$. On the other hand, by (3.6) and $J\,\mathrm{ad}(E)\mathfrak{g}^n\subset J\mathfrak{r}$, we have $\mathrm{Tr}_{\mathfrak{g}^n}J\,\mathrm{ad}(E)=\mathrm{Tr}_{J\mathfrak{r}}J\,\mathrm{ad}(E)=0$. These imply that

$$\psi_n(E)=0.$$

Taking F instead of E, we have

$$\psi_n(F)=0.$$

Thus we know

$$\psi_n=0 \qquad \text{on } \mathfrak{r}. \tag{3.15}$$

Now, for $X\in\mathfrak{g}^n$, $P\in\mathfrak{p}_k$, we have $[X, JE_k]\in\mathfrak{g}^{k+1}$, $[X, E_k]=0$, $[X, P]\in\mathfrak{p}_k$ and $\mathrm{Tr}_{\mathfrak{p}_k}f_k(X)=0$, where f_k is a symplectic representation of $\mathfrak{g}^n$ in $\mathfrak{p}_k$ (2.14). It follows that for $X\in\mathfrak{g}^n$

$$\begin{aligned}\psi(X)&=\mathrm{Tr}_{\mathfrak{g}}(\mathrm{ad}(JX)-J\,\mathrm{ad}(X))\\&=\sum_{k=1}^{n-1}\mathrm{Tr}_{\mathfrak{p}_k}(f_k(JX)-Jf_k(X))+\psi_n(X)\\&=-\sum_{k=1}^{n-1}\mathrm{Tr}_{\mathfrak{p}_k}Jf_k(X)+\psi_n(X).\end{aligned} \tag{3.16}$$

Put $f_k(E)=(A, B)$, $f_k(JE)=(C, D)$ in the sense of § 1.A. Since $[f_k(JE), f_k(E)]=f_k([JE, E])=0$, it becomes that

$$CA-AC+D\bar{B}-B\bar{D}=0. \tag{3.17}$$

By (K.2), we have

$$[J, f_k(JE)-(1/2)[J, f_k(E)]]=0. \tag{3.18}$$

From this and $J=(i, 0)$, we know

$$D=iB. \tag{3.19}$$

Since B is a symmetric matrix, there exists a unitary matrix U such that

$$UB^tU=\begin{pmatrix} \lambda_1 & & & & & 0 \\ & \ddots & & & & \\ & & \lambda_r & & & \\ & & & \lambda_{r+1} & & \\ & & & & \ddots & \\ 0 & & & & & \lambda_l \end{pmatrix}$$

where $\lambda_i>0$ for $1\leqq i\leqq r$, $\lambda_i=0$ for $r+1\leqq i\leqq l$. Hence we may assume that

$$B=\begin{pmatrix} \lambda_1 & & & & & 0 \\ & \ddots & & & & \\ & & \lambda_r & & & \\ & & & \lambda_{r+1} & & \\ & & & & \ddots & \\ 0 & & & & & \lambda_l \end{pmatrix} \tag{3.20}$$

where $\lambda_i>0$ for $1\leqq i\leqq r$, $\lambda_i=0$ for $r+1\leqq i\leqq l$. Since $D=iB$, we have

$$CA-AC+2iB^2=0. \tag{3.21}$$

Taking the trace of the both sides of the formula (3.21), it follows that

$$2i\sum_{k=1}^{l}\lambda_k^2=0,$$

which implies that $B=0$, and hence $f_k(E)=(A, 0)$. From this we know that

$$f_k(E)J=Jf_k(E) \qquad \text{on } \mathfrak{p}_k. \tag{3.22}$$

Using (3.15), (3.16) and (3.22), we have

$$\begin{aligned} \psi([E, X]) &= -\sum_{k=1}^{n-1}\mathrm{Tr}_{\mathfrak{p}_k}Jf_k([E, X])+\psi_n([E, X]) \\ &= -\sum_{k=1}^{n-1}\mathrm{Tr}_{\mathfrak{p}_k}J[f_k(E), f_k(X)] \\ &= -\sum_{k=1}^{n-1}\mathrm{Tr}_{\mathfrak{p}_k}(Jf_k(E)f_k(X)-Jf_k(X)f_k(E)) \\ &= -\sum_{k=1}^{n-1}\mathrm{Tr}_{\mathfrak{p}_k}(f_k(E)Jf_k(X)-Jf_k(X)f_k(E)) \\ &= -\sum_{k=1}^{n-1}\mathrm{Tr}_{\mathfrak{p}_k}[f_k(E), Jf_k(X)] \\ &=0, \qquad \text{for } X\in\mathfrak{g}^n. \end{aligned} \tag{3.23}$$

Since η is non-degenerate on $\mathfrak{g}^n$, it follows that $E=0$, which is a contradiction. Thus we have known that $\mathfrak{g}^n$ contains a one-dimensional ideal $\mathfrak{r}$ of $\mathfrak{g}^n$.

Now, let E be a non-zero element of $\mathfrak{r}$ and suppose that $[JE, E]=0$. Put

$$\mathfrak{p}=\{P\in\mathfrak{g}^n;\ [P, E]=[JP, E]=0\}.$$

Then, by the argument as used above, we see the followings. First $\mathrm{ad}(JE)\mathfrak{g}^n \subset \mathfrak{p}$, and the real parts of the eigenvalues of $\mathrm{ad}(JE)$ on $\mathfrak{p}$ are equal to 0, and $\psi([E, X])=0$ for any $X \in \mathfrak{g}^n$. This is a contradiction, because η is non-degenerate on $\mathfrak{g}^n$. Therefore $[JE, E] \neq 0$.

Putting $E_n = \lambda E$ with a non-zero constant λ, we have

$$[JE_n, E_n] = E_n . \tag{3.24}$$

Thus Proposition 1 is proved.

§4. Proof of Theorem (continued): Decomposition of $\mathfrak{g}^n$

PROPOSITION 2. *Let E_n be an element in $\mathfrak{g}^n$ as in Proposition 1. Then, we get the decomposition*

$$\mathfrak{g}^n = \{JE_n\} + \{E_n\} + \mathfrak{p}_n + \mathfrak{g}^{n+1} \tag{4.1}$$

of $\mathfrak{g}^n$ into the direct sum of vector spaces with the following properties:

1) $\mathfrak{g}_n = \{JE_n\} + \{E_n\} + \mathfrak{p}_n$ *is a J-invariant subalgebra such that*

$$J\mathfrak{p}_n \subset \mathfrak{p}_n ; \tag{4.2}$$

$$[JE_n, E_n] = E_n ; \tag{4.3}$$

$$[JE_n, \mathfrak{p}_n] \subset \mathfrak{p}_n . \tag{4.4}$$

Moreover the real parts of the eigenvalues of $\mathrm{ad}(JE_n)$ *on* $\mathfrak{p}_n$ *are equal to* 1/2, *and*

$$[E_n, \mathfrak{p}_n] = 0 ; \tag{4.5}$$

$$[\mathfrak{p}_n, \mathfrak{p}_n] \subset \{E_n\} . \tag{4.6}$$

2) $\mathfrak{g}^{n+1}$ *is a J-invariant subalgebra such that*

$$[JE_n, \mathfrak{g}^{n+1}] \subset \mathfrak{g}^{n+1} . \tag{4.7}$$

Moreover the real parts of the eigenvalues of $\mathrm{ad}(JE_n)$ *on* $\mathfrak{g}^{n+1}$ *are equal to* 0, *and*

$$[E_n, \mathfrak{g}^{n+1}] = 0 ; \tag{4.8}$$

$$[\mathfrak{p}_n, \mathfrak{g}^{n+1}] \subset \mathfrak{p}_n . \tag{4.9}$$

3) *The form η is positive definite on $\mathfrak{g}_n$ and the factors of the decomposition $\mathfrak{g}_n = \{JE_n\} + \{E_n\} + \mathfrak{p}_n$ are mutually orthogonal with respect to this form η. Further, the form η is non-degenerate on $\mathfrak{g}^{n+1}$.*

PROOF. For the convenience of notation, we denote the element E_n by E. We put

$$\mathfrak{p} = \{P \in \mathfrak{g}^n ; [P, E] = [JP, E] = 0\} . \tag{4.10}$$

Then, we have

$$J\mathfrak{p}\subset\mathfrak{p},\ \mathrm{ad}(JE)\mathfrak{p}\subset\mathfrak{p}, \tag{4.11}$$

$$\mathrm{ad}(JE)J=J\,\mathrm{ad}(JE) \quad \text{on } \mathfrak{p}, \tag{4.12}$$

$$\mathfrak{g}^n=\{JE\}+\{E\}+\mathfrak{p}. \tag{4.13}$$

Indeed, (4.11) and (4.12) can be shown in the same way as (3.8), (3.9) and (3.10). Since $\{E\}$ is a one-dimensional ideal of $\mathfrak{g}^n$, we get $[X, E]=\alpha(X)E$, $[JX, E]=\beta(X)E$ for $X\in\mathfrak{g}^n$, where α, β are linear functions on $\mathfrak{g}^n$. It is easily seen that $P=X-\alpha(X)JE-\beta(X)E$ belongs to $\mathfrak{p}$ for any $X\in\mathfrak{g}^n$.

LEMMA 4. *The real parts of the eigenvalues of* $\mathrm{ad}(JE)$ *on* $\mathfrak{p}$ *are equal to* 0 *or* 1/2.

PROOF. By Lemma 1, we have for $P\in\mathfrak{p}$,

$$\begin{aligned}\frac{d}{dt}\rho(\exp t\,\mathrm{ad}(JE)E,\ \exp t\,\mathrm{ad}(JE)P)\\ &=\rho(JE,\ \exp t\,\mathrm{ad}(JE)[E, P])\\ &=0.\end{aligned}$$

Since $\exp t\,\mathrm{ad}(JE)E=e^tE$, this implies that

$$\rho(E,\ \exp t\,\mathrm{ad}(JE)P)=a'e^{-t}$$

where a' is a constant determined by P and independent of t. We have then

$$\begin{aligned}\rho(JE,\ \exp t\,\mathrm{ad}(JE)P)&=-\rho(E,\ J\exp t\,\mathrm{ad}(JE)P)\\ &=-\rho(E,\ \exp t\,\mathrm{ad}(JE)JP)\\ &=ae^{-t}\end{aligned}$$

where a is the constant determined by JP. Since any element X in $\mathfrak{g}^n$ is expressed in the form $X=\lambda JE+\mu E+P$, where λ, $\mu\in\boldsymbol{R}$ and $P\in\mathfrak{p}$ (4.13), we have

$$\begin{aligned}\rho(JE,\ \exp t\,\mathrm{ad}(JE)X)&=\rho(JE,\ \lambda JE+\mu e^tE+\exp t\,\mathrm{ad}(JE)P)\\ &=\mu\rho(JE,\ E)e^t+\rho(JE,\ \exp t\,\mathrm{ad}(JE)P)\\ &=ae^{-t}+be^t\end{aligned}$$

where a, b are constant independent of t. This fact and Lemma 1 imply that for X, $Y\in\mathfrak{g}^n$

$$\begin{aligned}\frac{d}{dt}\rho(\exp t\,\mathrm{ad}(JE)X,\ \exp t\,\mathrm{ad}(JE)Y)\\ &=\rho(JE,\ \exp t\,\mathrm{ad}(JE)[X, Y])\\ &=ae^{-t}+be^t.\end{aligned}$$

Hence we obtain

(4.14) $$\rho(\exp t\,\mathrm{ad}\,(JE)X,\ \exp t\,\mathrm{ad}\,(JE)Y)=ae^{-t}+be^{t}+c\,,$$

where a, b and c are constant independent of t. Let $\alpha+i\beta$ ($\alpha, \beta\in \boldsymbol{R}$) be an eigenvalue of $\mathrm{ad}\,(JE)$ on $\mathfrak{p}$. Since $\mathrm{ad}\,(JE)J=J\,\mathrm{ad}\,(JE)$ on $\mathfrak{p}$, there exists a non-zero element P in $\mathfrak{p}$ such that $\mathrm{ad}\,(JE)P=(\alpha+\beta J)P$. Hence we have

$$\begin{aligned}
&\rho(\exp t\,\mathrm{ad}\,(JE)JP,\ \exp t\,\mathrm{ad}\,(JE)P)\\
&\quad=\rho(J\exp t\,\mathrm{ad}\,(JE)P,\ \exp t\,\mathrm{ad}\,(JE)P)\\
&\quad=\rho(J\exp t(\alpha+\beta J)P,\ \exp t(\alpha+\beta J)P)\\
&\quad=\rho(\exp t(\alpha+\beta J)JP,\ \exp t(\alpha+\beta J)P)\\
&\quad=e^{(\alpha+i\beta)t}\overline{e^{(\alpha+i\beta)t}}\rho(JP,\ P)\\
&\quad=e^{2\alpha t}\rho(JP,\ P)\,.
\end{aligned}$$

Therefore

$$e^{2\alpha t}\rho(JP,\ P)=ae^{-t}+be^{t}+c\,.$$

This implies that $\alpha=0$ or $1/2$ or $-1/2$, since $\rho(JP, P)>0$. We put

(4.15) $$\mathfrak{p}_{(\alpha+i\beta)}=\{P\in\mathfrak{p};\ (\mathrm{ad}\,(JE)-(\alpha+\beta J))^{m}P=0 \text{ for some integer } m>0\}\,.$$

Then we have

(4.16) $$\mathfrak{p}=\sum_{\alpha+i\beta}\mathfrak{p}_{(\alpha+i\beta)}\,,$$

where α is equal to 0 or $1/2$ or $-1/2$. Let P be a non-zero element in $\mathfrak{p}_{(\alpha+i\beta)}$. Then there exists a positive integer m such that $(\mathrm{ad}\,(JE)-(\alpha+\beta J))^{m}P=0$. Hence we have

$$\begin{aligned}
\exp t\,\mathrm{ad}\,(JE)P&=\exp t(\alpha+\beta J)\sum_{l=0}^{m-1}\frac{t^{l}}{l\,!}(\mathrm{ad}\,(JE)-(\alpha+\beta J))^{l}P\\
&=e^{\alpha t}\{\cos\beta t\sum_{l=0}^{m-1}\frac{t^{l}}{l\,!}(\mathrm{ad}\,(JE)-(\alpha+\beta J))^{l}P\\
&\qquad+\sin\beta t\sum_{l=0}^{m-1}\frac{t^{l}}{l\,!}(\mathrm{ad}\,(JE)-(\alpha+\beta J))^{l}JP\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\rho(JE,\ \exp t\,\mathrm{ad}\,(JE)P)\\
&\quad=e^{\alpha t}\{\cos\beta t\sum_{l=0}^{m-1}\frac{1}{l\,!}\rho(JE,\ (\mathrm{ad}\,(JE)-(\alpha+\beta J))^{l}P)t^{l}\\
&\qquad+\sin\beta t\sum_{l=0}^{m-1}\frac{1}{l\,!}\rho(JE,\ (\mathrm{ad}\,(JE)-(\alpha+\beta J))^{l}JP)t^{l}\}\,.
\end{aligned}$$

We put

$$h(t)=\sum_{l=0}^{m-1}\frac{1}{l!}\rho(JE, (\mathrm{ad}\,(JE)-(\alpha+\beta J))^l P)t^l\,,$$

$$k(t)=\sum_{l=0}^{m-1}\frac{1}{l!}\rho(JE, (\mathrm{ad}\,(JE)-(\alpha+\beta J))^l JP)t^l\,.$$

Then, $h(t)$ and $k(t)$ are polynomials whose degrees are $m-1$ at most. We have then

$$h(t)\cos\beta t+k(t)\sin\beta t=ae^{-(1+\alpha)t}\,,$$

$$\left|\frac{h(t)}{t^m}\cos\beta t+\frac{k(t)}{t^m}\sin\beta t\right|=\left|a\frac{e^{-(1+\alpha)t}}{t^m}\right|. \tag{4.17}$$

We assume that $a\neq 0$. Since $1+\alpha>0$ and since $h(t)$ and $k(t)$ are polynomials of degree $\leqq m-1$, the left side of the above formula (4.17) approaches to 0 and the right side to ∞, when $t\to-\infty$. This is a contradiction, and we must have $a=0$. This implies that

$$\rho(JE, \exp t\,\mathrm{ad}\,(JE)P)=0\,,\qquad \text{for } P\in\mathfrak{p}_{(\alpha+i\beta)}\,.$$

Hence we have

$$\rho(JE, \exp t\,\mathrm{ad}\,(JE)P)=0\,,\qquad \text{for } P\in\mathfrak{p}\,.$$

Therefore

$$e^{2\alpha t}\rho(JP, P)=be^t+c\,.$$

This implies that $\alpha=0$ or $1/2$, which proves Lemma 4.

Now, put

$$\mathfrak{p}_{[\alpha]}=\sum_{\beta}\mathfrak{p}_{(\alpha+i\beta)}\,.$$

Then, we have

$$J\mathfrak{p}_{[\alpha]}\subset\mathfrak{p}_{[\alpha]},\ \mathrm{ad}\,(JE)\mathfrak{p}_{[\alpha]}\subset\mathfrak{p}_{[\alpha]}\,,$$

$$\mathfrak{p}=\mathfrak{p}_{[0]}+\mathfrak{p}_{[1]}\,.$$

Moreover the real parts of the eigenvalues of $\mathrm{ad}\,(JE)$ on $\mathfrak{p}_{[\alpha]}$ are equal to α. Hence we get the following decomposition;

$$\mathfrak{g}^n=\mathfrak{g}_{[0]}+\mathfrak{g}_{[1/2]}+\mathfrak{g}_{[1]}\,, \tag{4.18}$$

where $\mathfrak{g}_{[0]}=\{JE\}+\mathfrak{p}_{[0]}$, $\mathfrak{g}_{[1/2]}=\mathfrak{p}_{[1/2]}$ and $\mathfrak{g}_{[1]}=\{E\}$. Moreover, $\mathrm{ad}\,(JE)\mathfrak{g}_{[\alpha]}\subset\mathfrak{g}_{[\alpha]}$ and the real parts of the eigenvalues of $\mathrm{ad}\,(JE)$ on $\mathfrak{g}_{[\alpha]}$ are equal to α. We put

$$\mathfrak{p}_n=\mathfrak{p}_{[1/2]}\,, \tag{4.19}$$

$$\mathfrak{g}_n=\{JE\}+\{E\}+\mathfrak{p}_{[1/2]}\,, \tag{4.20}$$

$$\mathfrak{g}^{n+1}=\mathfrak{p}_{[0]}\,. \tag{4.21}$$

Then, $\mathfrak{g}_n$ is clearly a J-invariant subalgebra. We prove that $\mathfrak{g}^{n+1}$ is also a J-invariant subalgebra. First we have $J\mathfrak{g}^{n+1}\subset\mathfrak{g}^{n+1}$. Now, since $[\mathfrak{g}_{[0]}, \mathfrak{g}_{[0]}]\subset\mathfrak{g}_{[0]}$, if $P, Q \in \mathfrak{g}^{n+1}$, we have

$$[P, Q]=\lambda JE+P',$$

where $\lambda \in \boldsymbol{R}$ and $P' \in \mathfrak{p}_{[0]}$, and therefore

$$[[P, Q], E]=\lambda E.$$

On the other hand, we have

$$[[P, Q], E]=[[P, E], Q]+[P, [Q, E]]=0.$$

This implies that $\lambda=0$. Thus we have $[P, Q] \in \mathfrak{g}^{n+1}$, which shows that $\mathfrak{g}^{n+1}$ is a subalgebra.

Now, we shall show that the factors of the decomposition

$$\mathfrak{g}^{n}=\{JE\}+\{E\}+\mathfrak{p}_{[1/2]}+\mathfrak{p}_{[0]}$$

are mutually orthogonal with respect to the non-degenerate form η. Indeed, for $P \in \mathfrak{p}_{[1/2]}$, $Q \in \mathfrak{p}_{[0]}$, put $P'=[JP, Q]$. Then $P' \in \mathfrak{p}_{[1/2]}$. Since the real parts of the eigenvalues of $\mathrm{ad}(JE)$ on $\mathfrak{p}_{[1/2]}$ are equal to 1/2, $\mathrm{ad}(JE)$ is non-singular on $\mathfrak{p}_{[1/2]}$ and hence there exists an element $P'' \in \mathfrak{p}_{[1/2]}$ such that $[JE, P'']=P'$. We have then $2\eta(P, Q)=\phi([JP, Q])=\phi([JE, P''])=-\phi([E, JP''])=0$. This shows that $\mathfrak{p}_{[1/2]}$ and $\mathfrak{p}_{[0]}$ are orthogonal with respect to η. It is clear that the other pairs of factors are mutually orthogonal with respect to η.

LEMMA 5.

$$\phi(E)>0.$$

PROOF. Recall that $(\mathfrak{p}_k, J, \sigma_k)$ is a symplectic space where σ_k is defined in (2.13) and that f_k is a symplectic representation of $\mathfrak{g}^n$ in $\mathfrak{p}_k$ defined by (2.14). Since

$$[f_k(JE), f_k(E)]=f_k(E),$$

$$[J, f_k(JE)-(1/2)[J, f_k(E)]]=0,$$

we have by [10]

1) $\mathfrak{p}_k=\mathfrak{p}_k^{+}+\mathfrak{p}_k^{-}+\mathfrak{p}_k^{0}$ direct sum;

2) $\mathfrak{p}_k^{+}$, $\mathfrak{p}_k^{-}$ and $\mathfrak{p}_k^{0}$ are invariant by $f_k(JE)$;

3) the real parts of the eigenvalues of $f_k(JE)$ on $\mathfrak{p}_k^{+}$, $\mathfrak{p}_k^{-}$ and $\mathfrak{p}_k^{0}$ are 1/2, $-1/2$ and 0 respectively;

4) $J\mathfrak{p}_k^{-}=\mathfrak{p}_k^{+}$, $J\mathfrak{p}_k^{0}=\mathfrak{p}_k^{0}$, in particular $\dim \mathfrak{p}_k^{-}=\dim \mathfrak{p}_k^{+}$, $\mathrm{Tr}_{\mathfrak{p}_k} f_k(JE)=0$;

5) $$f_k(E)=\begin{cases} J & \text{on } \mathfrak{p}_k^{-}, \\ 0 & \text{on } \mathfrak{p}_k^{+}+\mathfrak{p}_k^{0}. \end{cases}$$

These show that $\mathrm{Tr}_{\mathfrak{p}_k} Jf_k(E)=-\dim \mathfrak{p}_k^{-}$. On the other hand $\mathrm{Tr}_{\mathfrak{g}^n}(\mathrm{ad}(JE)-J\,\mathrm{ad}(E))>0$. Indeed, $\mathrm{Tr}_{\mathfrak{g}^n} J\,\mathrm{ad}(E)=-1$ because $J\,\mathrm{ad}(E)\mathfrak{g}^n \subset \{JE\}$ and $J\,\mathrm{ad}(E)JE=-JE$. Moreover, the real parts of the eigenvalues of $\mathrm{ad}(JE)$ on $\mathfrak{g}^n$ are equal to 0, 1/2 or 1, and $\mathrm{ad}(JE)E=E$. Therefore we have $\mathrm{Tr}_{\mathfrak{g}^n}\mathrm{ad}(JE)>0$.

These imply that $\mathrm{Tr}_{\mathfrak{g}^n}(\mathrm{ad}\,(JE)-J\,\mathrm{ad}\,(E))>0$. Therefore we have

$$\begin{aligned}\psi(E)&=-\sum_{k=1}^{n-1}\mathrm{Tr}_{\mathfrak{p}_k}Jf_k(E)+\mathrm{Tr}_{\mathfrak{g}^n}(\mathrm{ad}\,(JE)-J\,\mathrm{ad}\,(E))\\&=\sum_{k=1}^{n-1}\dim\mathfrak{p}_k^-+\mathrm{Tr}_{\mathfrak{g}^n}(\mathrm{ad}\,(JE)-J\,\mathrm{ad}\,(E))>0\,.\end{aligned}$$

Q. E. D.

LEMMA 6. *η is positive definite on $\mathfrak{p}_{[1/2]}$.*

PROOF. We shall first prove that the decomposition $\mathfrak{p}_{[1/2]}=\sum_\beta \mathfrak{p}_{(1/2+i\beta)}$ is an orthogonal decomposition with respect to η. Let P and Q be non-zero elements in $\mathfrak{p}_{(1/2+i\beta)}$, $\mathfrak{p}_{(1/2+i\beta')}$ respectively and assume $\beta\neq\beta'$. Then there exist positive integers m, n such that

$$(\mathrm{ad}\,(JE)-(1/2+\beta J))^m P=0\,,$$

$$(\mathrm{ad}\,(JE)-(1/2+\beta' J))^n Q=0\,.$$

Hence we have

$$\exp t\,\mathrm{ad}\,(JE)P=\exp t(1/2+\beta J)\sum_{l=0}^{m-1}\frac{t^l}{l\,!}(\mathrm{ad}\,(JE)-(1/2+\beta J))^l P\,,$$

$$\exp t\,\mathrm{ad}\,(JE)Q=\exp t(1/2+\beta' J)\sum_{l=0}^{n-1}\frac{t^l}{l\,!}(\mathrm{ad}\,(JE)-(1/2+\beta' J))^l Q\,.$$

Since $[\mathfrak{p}_{[1/2]},\mathfrak{p}_{[1/2]}]\subset\{E\}$, it becomes that $[JP,Q]=\lambda E$, where $\lambda\in\boldsymbol{R}$. By Lemma 1, we have

$$\begin{aligned}&\frac{d}{dt}\rho(\exp t\,\mathrm{ad}\,(JE)JP,\ \exp t\,\mathrm{ad}\,(JE)Q)\\&\qquad=\rho(JE,\ \exp t\,\mathrm{ad}\,(JE)[JP,Q])\,.\end{aligned}\tag{4.22}$$

The left side of this equation is equal to

$$\begin{aligned}&\frac{d}{dt}\rho(J\exp t\,\mathrm{ad}\,(JE)P,\ \exp t\,\mathrm{ad}\,(JE)Q)\\&=\frac{d}{dt}\rho(J\exp t(1/2+\beta J)\sum_{l=0}^{m-1}\frac{t^l}{l\,!}(\mathrm{ad}\,(JE)-(1/2+\beta J))^l P\,,\\&\qquad\exp t(1/2+\beta' J)\sum_{l=0}^{n-1}\frac{t^l}{l\,!}(\mathrm{ad}\,(JE)-(1/2+\beta' J))^l Q)\\&=\frac{d}{dt}e^t\rho(\exp\beta tJ\sum_{l=0}^{m-1}\frac{t^l}{l\,!}(\mathrm{ad}\,(JE)-(1/2+\beta J))^l JP\,,\\&\qquad\exp\beta' tJ\sum_{l=0}^{n-1}\frac{t^l}{l\,!}(\mathrm{ad}\,(JE)-(1/2+\beta' J))^l Q)\\&=\frac{d}{dt}e^t\rho(\{\cos\beta t+(\sin\beta t)J\}u(t),\ \{\cos\beta' t+(\sin\beta' t)J\}v(t))\end{aligned}$$

$$=\frac{d}{dt}e^t(\cos\beta t\cos\beta' t+\sin\beta t\sin\beta' t)\rho(u(t),\ v(t))$$

$$+(\sin\beta t\cos\beta' t-\cos\beta t\sin\beta' t)\rho(Ju(t),\ v(t))$$

$$=\frac{d}{dt}e^t\{h(t)\cos(\beta-\beta')t+k(t)\sin(\beta-\beta')t\}$$

$$=e^t\{a(t)\cos(\beta-\beta')t+b(t)\sin(\beta-\beta')t\}\ ,$$

where

$$u(t)=\sum_{l=0}^{m-1}\frac{t^l}{l!}(\mathrm{ad}\,(JE)-(1/2+\beta J))^l JP\,,$$

$$v(t)=\sum_{l=0}^{n-1}\frac{t^l}{l!}(\mathrm{ad}\,(JE)-(1/2+\beta' J))^l Q\,,$$

$$h(t)=\rho(u(t),\ v(t))\,,$$

$$k(t)=\rho(Ju(t),\ v(t))\,,$$

$$a(t)=h(t)+h'(t)+(\beta-\beta')k(t)\,,$$

$$b(t)=k(t)+k'(t)-(\beta-\beta')h(t)\,.$$

Hence $a(t)$ and $b(t)$ are polynomials. On the other hand, the right side of the equation (4.22) is equal to

$$\rho(JE,\ \exp t\,\mathrm{ad}\,(JE)[JP,\ Q])=\rho(JE,\ \lambda e^t E)$$
$$=e^t\lambda\rho(JE,\ E)\,.$$

Therefore we have

$$a(t)\cos(\beta-\beta')t+b(t)\sin(\beta-\beta')t=\lambda\rho(JE,\ E)\,.$$

Since $a(t)-\lambda\rho(JE, E)$ is a polynomial and since $a(t_n)-\lambda\rho(JE, E)=0$ for $t_n=2n\pi/(\beta-\beta')$, where n integer, it follows that $a(t)$ is a constant a. Similarly $b(t)$ is a constant b. Hence we have

$$a\cos(\beta-\beta')t+b\sin(\beta-\beta')t=\lambda\rho(JE,\ E)\,.$$

By this formula, we have $(\beta-\beta')^2\lambda\rho(JE, E)=0$. Since $\beta-\beta'\neq 0$ and $\rho(JE, E)>0$, λ must be 0. Thus we have

$$\eta(P,\ Q)=\phi([JP,\ Q])=\lambda\phi(E)=0\,.$$

This implies that $\mathfrak{p}_{(1/2+i\beta)}$ and $\mathfrak{p}_{(1/2+i\beta')}$ are mutually orthogonal with respect to η.

Now, let P be a non-zero element in $\mathfrak{p}_{(1/2+i\beta)}$. Then, there exists a positive integer m such that

$$(\mathrm{ad}\,(JE)-(1/2+\beta J))^m P=0\,,$$

and hence

$$\exp t\,\mathrm{ad}\,(JE)P = \exp t\,(1/2+\beta J)u(t)\,,$$

where $u(t)=\sum_{l=0}^{m-1}\frac{t^l}{l!}(\mathrm{ad}\,(JE)-(1/2+\beta J))^l P$. On the other hand, we have by Lemma 1

$$(4.23)\qquad \frac{d}{dt}\rho(\exp t\,\mathrm{ad}\,(JE)JP,\ \exp t\,\mathrm{ad}\,(JE)P) = \rho(JE,\ \exp t\,\mathrm{ad}\,(JE)[JP,\ P])\,.$$

The left side of this equation is equal to

$$\begin{aligned}
&\frac{d}{dt}\rho(J\exp t\,\mathrm{ad}\,(JE)P,\ \exp t\,\mathrm{ad}\,(JE)P)\\
&=\frac{d}{dt}\rho(J\exp t(1/2+\beta J)u(t),\ \exp t(1/2+\beta J)u(t))\\
&=\frac{d}{dt}\rho(\exp t(1/2+\beta J)Ju(t),\ \exp t(1/2+\beta J)u(t))\\
&=\frac{d}{dt}e^{(1/2+i\beta)t}\overline{e^{(1/2+i\beta)t}}\rho(Ju(t),\ u(t))\\
&=\frac{d}{dt}e^{t}\rho(Ju(t),\ u(t))\\
&=e^t(h'(t)+h(t))\,,
\end{aligned}$$

where $h(t)=\rho(Ju(t),\ u(t))$, and $h(t)$ is a polynomial of degree $\leqq 2m-2$. Because $[JP,\ P]=\lambda E$ where $\lambda\in\boldsymbol{R}$, the right side of the equation (4.23) is equal to

$$\rho(JE,\ \lambda e^t E)=e^t\lambda\rho(JE,\ E)\,.$$

Hence we have

$$h'(t)+h(t)=\lambda\rho(JE,\ E)\,.$$

The solution of this equation is $h(t)=ce^{-t}+\lambda\rho(JE,\ E)$, where c is an arbitrary constant. However, $h(t)$ is a polynomial, and so c must be 0. Hence we have

$$h(t)=\lambda\rho(JE,\ E)\,,$$

and hence it follows that

$$(4.24)\qquad \lambda=\frac{h(t)}{\rho(JE,\ E)}=\frac{h(0)}{\rho(JE,\ E)}=\frac{\rho(JP,\ P)}{\rho(JE,\ E)}>0\,.$$

Therefore we have by Lemma 5 and (4.24)

$$\eta(P,\ P)=\psi([JP,\ P])=\lambda\psi(E)>0\,.$$

This shows that η is positive definite on $\mathfrak{p}_{(1/2+i\beta)}$, and hence on $\mathfrak{p}_{[1/2]}=\sum_\beta \mathfrak{p}_{(1/2+i\beta)}$.

Q. E. D.

This completes the proof of Proposition 2.

As explained in § 2 our theorem follows then by induction on n, applying Propositions 1 and 2 successively.

§ 5. Proof of corollaries

PROOF OF COROLLARY 1. Let $\hat{K}$ be the isotropy subgroup of $\hat{G}$ at the point o. Since $\hat{G}=I^0(M)$, $\hat{K}$ is a compact subgroup of $\hat{G}$. By a theorem of Iwasawa, $\hat{G}/\hat{K}=G/K$ is homeomorphic to a Euclidean space. Hence $\hat{K}$ is a maximal compact subgroup of $\hat{G}$. Let $\hat{\mathfrak{k}}$ be the Lie subalgebra of $\hat{\mathfrak{g}}$ corresponding to $\hat{K}$. By a theorem of Hano (Introduction (vi)), the adjoint group $\tilde{G}=\mathrm{Ad}_{\hat{\mathfrak{g}}}(\hat{G})$ of $\hat{G}$ is the identity component of a real algebraic group in $GL(\hat{\mathfrak{g}}, \boldsymbol{R})$. Since the center of $\hat{G}$ is finite and since $\tilde{K}=\mathrm{Ad}_{\hat{\mathfrak{g}}}(\hat{K})$ is a compact subgroup of $\tilde{G}$, $\tilde{K}$ is a maximal compact subgroup of $\tilde{G}$. Hence there exists a connected triangular subgroup $\tilde{T}$ of $\tilde{G}$ such that $\tilde{G}=\tilde{T}\tilde{K}$, where $\tilde{T}\cap\tilde{K}$ consists of the identity only [12]. Thus we have $\hat{G}=\hat{T}\hat{K}$, where $\hat{T}=\mathrm{Ad}_{\hat{\mathfrak{g}}}^{-1}(\tilde{T})$, and $\hat{T}$ is a solvable Lie group which acts transitively on M. The Kähler algebra corresponding to $\hat{T}$ is normal, solvable and the canonical hermitian form is non-degenerate. By our theorem, M is holomorphically isomorphic to a homogeneous bounded domain.

PROOF OF COROLLARY 2. Since M is complete, simply connected, and since o has no conjugate point, the exponential map $\exp_0 : T_0(M)\rightarrow M$ is a homeomorphism. Therefore the isotropy subgroup of $I^0(M)$ at o is a maximal compact subgroup of $I^0(M)$. Therefore, by Corollary 1 the proof is completed.

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