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On deformations of holomorphic maps I

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Notation.

$\boldsymbol{C}$: the field of complex numbers.
X, Y, Z: compact complex manifolds.
$\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$, M, M', N: (connected) complex manifolds.
If $f: X \to Y$ is a holomorphic map,
$\Theta_{X/Y}$: the sheaf of germs of relative vector fields,
$\Theta_X = \Theta_{X/C}$: the sheaf of germs of holomorphic vector fields on X.
If E is a vector bundle (or a locally free sheaf) on X,
$\mathcal{A}^{0,q}(E)$: the sheaf of germs of differentiable $(0, q)$-forms with coefficients in E,
$A^{0,q}(E) = \Gamma(X, \mathcal{A}^{0,q}(E))$.
If $p: \mathcal{X} \to M$ is a family of compact complex manifolds,
X_t: the fibre over $t \in M$.
If $q: \mathcal{Y} \to N$ is another family of compact complex manifolds and if
$(\Phi, s): (\mathcal{X}, p, M) \to (\mathcal{Y}, q, N)$ is a morphism of families (i.e., $\Phi: \mathcal{X} \to \mathcal{Y}$, $s: M \to N$, $q \circ \Phi = s \circ p$),
$\Phi_t: X_t \to Y_{s(t)}$: the holomorphic map induced by Φ.
If $\{U_i\}$ is an open covering of X
$U_{ij\cdots k} = U_i \cap U_j \cap \cdots \cap U_k$.
For any vector $t = (t_1, t_2, \cdots, t_r)$,
$|t| = \max_{\lambda} |t_\lambda|$.
We denote by ν the multi-index $(\nu_1, \cdots, \nu_r)$, and
$t^\nu = t_1^{\nu_1} t_2^{\nu_2} \cdots t_r^{\nu_r}$,
$|\nu| = \nu_1 + \nu_2 + \cdots + \nu_r$.

Introduction.

The modern deformation theory has begun with the splendid work of Kodaira-Spencer [5] followed by [6], [7]. Moreover Kodaira has investigated families of submanifolds of a fixed complex manifold in [8]. The next natural problem is to investigate "deformations of holomorphic maps." First we define the notion of families of holomorphic maps. There are several aspects.

i) A family of holomorphic maps is a collection $\{f_t: X \to Y | t \in M\}$ parametrized by M (X, Y being fixed).

If we allow the complex manifold X to vary, we obtain the second definition:

ii) A family of holomorphic maps consists of a family $\{X_t | t \in M\}$ plus a collection $\{f_t: X_t \to Y | t \in M\}$ (Y being fixed).

Moreover, we may also allow the complex manifold Y to vary. But we fix once for all a family $\mathcal{Y} \to S$, e. g., a complete family of deformations of some given Y_0, in which Y should vary. From this aspect we obtain the third definition:

iii) A family of holomorphic maps consists of a family $\{X_t | t \in M\}$, a holomorphic map $s: M \to S$ and a collection $\{f_t: X_t \to Y_{s(t)} | t \in M\}$ ($\mathcal{Y} \to S$ being fixed).

This paper is the first part of a study of "germs" of deformations of holomorphic maps. The main results of this study were announced in a short note [4].

In this paper, we take the second definition, and define a characteristic map τ, then prove two fundamental theorems which correspond to the results of [6], [7], under the assumption that holomorphic maps in consideration are non-degenerate, i. e., the rank of the Jacobian matrix at some point $x \in X$ is equal to $\dim X$.

In the Appendix, we give an elementary proof of the existence of effectively parametrized complete family as a formal analytic space. The author does not know whether one can prove convergences from this approach.

§ 1. Infinitesimal deformations.

In this section we define a characteristic map for deformations of non-degenerate holomorphic maps, which plays a fundamental role.

Let Y be a fixed compact complex manifold.

DEFINITION 1.1. By *a family of holomorphic maps into* Y, we mean a quadruplet $(\mathcal{X}, \Phi, p, M)$ of complex manifolds $\mathcal{X}$, M and holomorphic maps $\Phi: \mathcal{X} \to \mathcal{Y} = Y \times M$, $p: \mathcal{X} \to M$ with following properties:

i) p is a surjective smooth proper holomorphic map,

ii) $q \circ \Phi = p$ where $q: \mathcal{Y} \to M$ is the projection onto the second factor.

Two families $(\mathcal{X}, \Phi, p, M)$ and $(\mathcal{X}', \Phi', p', M')$ of holomorphic maps into Y are said to be *equivalent* if there exist analytic isomorphisms $\Psi: \mathcal{X} \to \mathcal{X}'$ and $\psi: M \to M'$ such that the following diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\Psi} & \mathcal{X}' \\ \Phi \downarrow & & \downarrow \Phi' \\ Y\times M & \xrightarrow{id\times\phi} & Y\times M' \end{array}$$

is commutative.

If $(\mathcal{X}, \Phi, p, M)$ is a family of holomorphic maps into Y, and if $h: N\rightarrow M$ is a holomorphic map, we can define *the family* $(\mathcal{X}', \Phi', p', N)$ *induced by* h as follows:

i) $\mathcal{X}'=\mathcal{X}\times_M N$,

ii) $\Phi'=\Phi\times id: \mathcal{X}'\rightarrow(Y\times M)\times_M N=Y\times N$,

iii) $p'=p_N: \mathcal{X}'\rightarrow N$

(for the notation, see [2]).

In particular, if N is a submanifold of M and if h is the natural injection, we call $(\mathcal{X}', \Phi', p', N)$ the *restriction* on N and denote it by $(\mathcal{X}|_N, \Phi|_N, p|_N, N)$.

DEFINITION 1.2. A family $(\mathcal{X}, \Phi, p, M)$ of holomorphic maps into Y is *complete at* $o\in M$ if, for any family $(\mathcal{X}', \Phi', p', N)$ such that $\Phi'_o: X'_o\rightarrow Y$ is equivalent to $\Phi_o: X_o\rightarrow Y$ for a point $o'\in N$, there exists a holomorphic map h of a neighborhood U of o' in N into M with $h(o')=o$ such that the restriction of $(\mathcal{X}', \Phi', p', N)$ on U is equivalent to the family induced by h from $(\mathcal{X}, \Phi, p, M)$.

Now we define a characteristic map. Let $(\mathcal{X}, \Phi, p, M)$ be a family of holomorphic maps into Y, $o\in M$, $X=X_o$ and $f=\Phi_o: X\rightarrow Y$. Then we have an exact sequence

$$0\longrightarrow\Theta_{X/Y}\longrightarrow\Theta_X\xrightarrow{F}f^*\Theta_Y$$

of coherent sheaves on X ([2], VII). Let $\mathcal{T}=\mathcal{T}_{X/Y}$ be the cokernel of the canonical homomorphism F, then we have an exact sequence

$$0\longrightarrow\Theta_{X/Y}\longrightarrow\Theta_X\xrightarrow{F}f^*\Theta_Y\xrightarrow{P}\mathcal{T}\longrightarrow 0. \tag{1.1}$$

Restricting M on a neighborhood of o if necessary, we may assume the following:

i) M is an open set in C^r with coordinates $t=(t_1, \cdots, t_r)$ and $o=(0, \cdots, 0)$.

ii) $\mathcal{X}$ is covered by a finite number of Stein coordinate neighborhoods $\mathcal{U}_i$. Each $\mathcal{U}_i$ is covered by a system of coordinates (z_i, t) such that $p(z_i, t)=t$ (we indicate by (z_i, t) a set of $n+r$ complex numbers $z_i^1, \cdots, z_i^n, t_1, \cdots, t_r$, and the point on $\mathcal{U}_i$ with the coordinates $(z_i^1, \cdots, z_i^n, t_1, \cdots, t_r)$).

iii) Y is covered by a finite number of Stein coordinate neighborhoods V_i with a system of coordinates $w_i=(w_i^1, \cdots, w_i^m)$, $\Phi(\mathcal{U}_i)\subset\mathcal{V}_i=V_i\times M$, and in terms of these coordinates Φ is given by

$$w_i=\Phi_i(z_i, t),$$

and let $f_i(z_i)=\Phi_i(z_i, 0)$.

iv) (z_i, t) coincides with (z_j, t) if and only if

$$z_i=\phi_{ij}(z_j, t).$$

v) (w_i) coincides with (w_j) if and only if

$$w_i=\psi_{ij}(w_j).$$

Then we have

$$\Phi_i(\phi_{ij}(z_j, t), t)=\psi_{ij}(\Phi_j(z_j, t)). \tag{1.2}$$

Let $T_0(M)$ denote the tangent space of M at 0. For any $\frac{\partial}{\partial t}\in T_0(M)$, let

$$\tau_i=\sum_\rho \frac{\partial\Phi_i^\rho}{\partial t}\Big|_{t=0}\frac{\partial}{\partial w_i^\rho} \tag{1.3}$$

$\left(\text{where } \frac{\partial\Phi}{\partial t}=\sum v_\lambda\frac{\partial\Phi}{\partial t_\lambda} \text{ for } \frac{\partial}{\partial t}=\sum v_\lambda\frac{\partial}{\partial t_\lambda}\right)$ which is regarded as an element of $\Gamma(U_i, f^*\Theta_Y)$ $(U_i=X\cap\mathcal{U}_i)$. Then from the equality (1.2) we infer that

$$\tau_j-\tau_i=F\Big(\sum_\sigma \frac{\partial\phi_{ij}^\sigma}{\partial t}\Big|_{t=0}\frac{\partial}{\partial z_i^\sigma}\Big). \tag{1.4}$$

Therefore the collection $\{P\tau_i\}$ defines an element of $H^0(X, \mathfrak{T})$. Thus we define a linear map

$$\tau: T_0(M)\longrightarrow H^0(X, \mathfrak{T})$$

which we call *the* (partial) *characteristic map* of the family of holomorphic maps into Y at o.

PROPOSITION 1.3. *The linear map τ, defined above, is independent of the choice of coverings and systems of coordinates.*

PROOF. Clearly τ is invariant under a refinement of coverings. Hence it suffices to consider the case of fixed coverings. Let (z_i', t) and (w_i') be other systems of coordinates on $\mathcal{U}_i$ and on V_i, respectively, then

i) (z_i', t) coincides with (z_i, t) if and only if

$$z_i'=h_i(z_i, t),$$

ii) (w_i') coincides with (w_i) if and only if

$$w_i=g_i(w_i'),$$

iii) Φ is given in terms of new coordinates by

$$w_i'=\Phi_i'(z_i', t).$$

We must have the equality:

$$\Phi_i(z_i, t)=g_i(\Phi_i'(h_i(z_i, t), t)). \tag{1.5}$$

Let τ'_i be the element of $\Gamma(U_i, f^*\Theta_Y)$ defined by the formula (1.3) with the aid of Φ'_i, then from (1.5) it follows that

$$\tau_i - \tau'_i = F\Big(\sum \frac{\partial h_i^\sigma}{\partial t}\Big|_{t=0} \frac{\partial}{\partial z_i^\sigma}\Big).$$

This proves the assertion.

A holomorphic map $f: X \to Y$ is called *non-degenerate*, if $\operatorname{rank}_z df = \dim X$ for some point $z \in X$. If f is non-degenerate, the set of $z \in X$ such that $\operatorname{rank}_z df < \dim X$ forms a proper analytic subset of X, and the exact sequence (1.1) reduces to

$$(1.1)' \qquad 0 \longrightarrow \Theta_X \overset{F}{\longrightarrow} f^*\Theta_Y \overset{P}{\longrightarrow} \mathfrak{T} \longrightarrow 0.$$

A family $(\mathcal{X}, \Phi, p, M)$ of holomorphic maps into Y is called a *family of non-degenerate holomorphic maps into* Y if $\Phi_t: X_t \to Y_t$ is non-degenerate for any point $t \in M$.

In this paper, we restrict ourselves to families of non-degenerate holomorphic maps (general case will be discussed in a future).

PROPOSITION 1.4. *Let $(\mathcal{X}, \Phi, p, M)$ be a family of non-degenerate holomorphic maps into Y, $o \in M$ and $X = X_0$, then the diagram*

$$\begin{array}{ccc} T_0(M) & \overset{\tau}{\longrightarrow} & H^0(X, \mathfrak{T}) \\ & \rho \searrow & \downarrow \delta \\ & & H^1(X, \Theta_X) \end{array}$$

is commutative, where ρ is the infinitesimal deformation map of the family $(\mathcal{X}, p, M)$ at o [Kodaira-Spencer **5**, chap. II, 5] *and δ is the coboundary map of cohomology groups.*

PROOF. By definition, $\rho\Big(\frac{\partial}{\partial t}\Big)$ is the cohomology class of the 1-cocycle $\sum \frac{\partial \phi_{ij}^\sigma}{\partial t}\Big|_{t=0} \frac{\partial}{\partial z_i^\sigma}$. Our assertion follows from the equality (1.4).

REMARK 1.5. Proposition 1.4 assures that in the case of a family of non-degenerate holomorphic maps into Y, the characteristic map τ completely describes the infinitesimal deformations. Or more explicitly, given a non-degenerate holomorphic map $f: X \to Y$, the set of classes of "families" of holomorphic maps into Y "with base space Spec$(C[t]/t^2)$" (cf. Grothendieck [**2**]) extending f, has a natural structure of principal homogeneous space under the group $H^0(X, \mathfrak{T})$.

§2. A theorem of completeness.

In this section we prove the following theorem analogous to [**6**].

THEOREM 2.1. *Let $(\mathcal{X}, \Phi, p, M)$ be a family of non-degenerate holomorphic* *maps into Y, $o\in M$, $X=X_o$ and $f=\Phi_o$: $X\to Y$. If the characteristic map τ: $T_o(M)\to H^0(X,\mathcal{T})$ at o is surjective, then the family is complete at o (in the sense of Definition* 1.2).

To prove the theorem, it suffices to prove that, for any family $(\mathcal{X}', \Phi', p', M')$ of holomorphic maps into Y, such that $\Phi'_{o'}$: $X'_{o'}\to Y$ is equivalent to f: $X\to Y$ for some $o'\in M'$, there exist a holomorphic map t: $M'\to M$ with $t(o')=o$ (restricting M' on a neighborhood of o' if necessary) and a holomorphic map g: $\mathcal{X}'\to\mathcal{X}$ over t which maps $X'_{t'}$ biregularly onto $X_{t(t')}$.

We employ the notation of § 1 ("$'$" indicates something on $\mathcal{X}'$). Moreover we assume that M, M', $\mathcal{U}_i$, $\mathcal{U}'_i$ and V_i are open balls in some affine spaces. Since $\Phi'_{o'}$ is equivalent to Φ_o, we may assume that $U'_i=U_i$, $z'_i=z_i$ on U_i, $\Phi'_i(z_i, 0)=\Phi_i(z_i, 0)=f_i(z_i)$, $\phi'_{ij}(z_j, 0)=\phi_{ij}(z_j, 0)=b_{ij}(z_j)$. Let $M'_\varepsilon=\{t'\in M': |t'|<\varepsilon\}$ with $\varepsilon>0$ sufficiently small. We construct holomorphic maps t: $M'_\varepsilon\to M$ and $g_i: U_i\times M'_\varepsilon\to \boldsymbol{C}^n$ which satisfy the following conditions:

$$g_i(z_i, 0)=z_i\,,\qquad t(0)=0\,, \tag{2.0}$$

$$g_i(\phi'_{ij}, t')=\phi_{ij}(g_j, t(t'))\,, \tag{2.1}$$

$$\Phi_i(g_i, t(t'))=\Phi'_i\,. \tag{2.2}$$

(We do not indicate the domain where the equality should hold, if no confusion arises.)

I) Existence of formal solutions. We first prove the existence of formal power series g_i and t' satisfying (2.0)—(2.2). For this purpose, we introduce some notations. Let $P(s)=P(s_1, \cdots, s_r)$, $Q(s)$ be power series in s with coefficients in some module. We write

$$P(s)=P_0(s)+P_1(s)+\cdots+P_\mu(s)+\cdots$$

where $P_\mu(s)$ is a homogeneous polynomial in s of degree μ. We indicate by $P^\mu(s)$ the polynomial

$$P^\mu(s)=P_0(s)+P_1(s)+\cdots+P_\mu(s)\,.$$

Moreover we write $P(s)\underset{\mu}{\equiv}0$ if $P^\mu(s)=0$, and $P(s)\underset{\mu}{\equiv}Q(s)$ if $P(s)-Q(s)\underset{\mu}{\equiv}0$. We identify a holomorphic function with its power series expansion.

With these notations, (2.1) and (2.2) are equivalent to the following systems of congruences:

$$g_i^\mu(\phi'_{ij}, t')\underset{\mu}{\equiv}\phi_{ij}(g_j^\mu, t^\mu)\qquad \mu=0, 1, \cdots, \tag{2.3$_\mu$}$$

$$\Phi_i(g_i^\mu, t^\mu)\underset{\mu}{\equiv}\Phi'_i\qquad \mu=0, 1, \cdots. \tag{2.4$_\mu$}$$

We construct g_i^μ and t^μ satisfying $(2.3)_\mu$ and $(2.4)_\mu$ by induction on μ. Suppose therefore that $t^{\mu-1}$ and $g_i^{\mu-1}$ satisfying $(2.3)_{\mu-1}$ and $(2.4)_{\mu-1}$ are already determined.

We define $\Gamma_{ij|\mu}\in\Gamma(U_{ij},\Theta_X)$ (resp. $\gamma_{i|\mu}\in\Gamma(U_i,f^*\Theta_Y)$) (by this we mean that $\Gamma_{ij|\mu}$ is a homogeneous polynomial of degree μ with coefficients in $\Gamma(U_{ij},\Theta_X)$ and so on, by abuse of notation), by the congruence

$$\Gamma_{ij|\mu}\underset{\mu}{\equiv}(g_i^{\mu-1}(\phi'_{ij},t')-\phi_{ij}(g_j^{\mu-1},t^{\mu-1}))\cdot\frac{\partial}{\partial z_i} \tag{2.5}$$

(resp.

$$\gamma_{i|\mu}\underset{\mu}{\equiv}(\Phi'_i-\Phi_i(g_i^{\mu-1},t^{\mu-1}))\cdot\frac{\partial}{\partial w_i}\Big), \tag{2.6}$$

(where we use the following notation: $h\cdot\frac{\partial}{\partial z_i}=\sum h^\sigma\frac{\partial}{\partial z_i^\sigma}$ for any n-vector $h=(h^\sigma)$).

Now we prove that we have the following equalities:

$$\Gamma_{jk|\mu}-\Gamma_{ik|\mu}+\Gamma_{ij|\mu}=0\,, \tag{2.7}$$

$$F\Gamma_{ij|\mu}=\gamma_{j|\mu}-\gamma_{i|\mu}\,. \tag{2.8}$$

PROOF. The equality (2.7) is proved in [6, Lemma 2]. The equality (2.8) can be proved in a similar way as follows (we omit the indices $\mu-1$, if no confusion is possible).

$$\begin{aligned}\Phi'_i(\phi'_{ij},t')&=\phi_{ij}(\Phi'_j)\\&\underset{\mu}{\equiv}\phi_{ij}(\Phi_j(g_j,t)+\gamma_{j|\mu})\\&\underset{\mu}{\equiv}\phi_{ij}(\Phi_j(g_j,t))+C_{ij}\gamma_{j|\mu}\qquad\text{where } C_{ij}=\Big(\frac{\partial\phi_{ij}}{\partial w_j}\Big).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\phi_{ij}(\Phi_j(g_j,t))&\underset{\mu}{\equiv}\Phi_i(\phi_{ij}(g_j,t))\\&\underset{\mu}{\equiv}\Phi_i(g_i(\phi'_{ij},t')-\Gamma_{ij|\mu})\\&\underset{\mu}{\equiv}\Phi_i(g_i(\phi'_{ij},t'))-F_i\Gamma_{ij|\mu}\qquad\text{where } F_i=\Big(\frac{\partial f_i}{\partial z_i}\Big).\end{aligned}$$

It follows that

$$\gamma_{i|\mu}\underset{\mu}{\equiv}\gamma_{i|\mu}(\phi'_{ij},t')\underset{\mu}{\equiv}C_{ij}\gamma_{j|\mu}-F_i\Gamma_{ij|\mu}\,.$$ q. e. d.

Our purpose is to determine

$$t^\mu(t')=t^{\mu-1}(t')+t_\mu(t')\,,$$

and

$$g_i^\mu(z_i,t')=g_i^{\mu-1}(t')+g_{i|\mu}(t')\,,$$

which satisfy $(2.3)_\mu$ and $(2.4)_\mu$.

We prove that $(2.3)_\mu$ and $(2.4)_\mu$ are equivalent to

$$\Gamma_{ij|\mu} = g_{j|\mu} - g_{i|\mu} + \sum t_{\lambda|\mu}\rho_{ij\lambda}, \tag{2.9}$$

$$\gamma_{i|\mu} = F g_{i|\mu} + \sum t_{\lambda|\mu}\tau_{i\lambda}, \tag{2.10}$$

where we denote by the same letter g the section $\sum g^\rho \frac{\partial}{\partial w^\rho}$ of the sheaf $f^*\Theta_Y$, etc., and

$$\rho_{ij\lambda} = \frac{\partial\phi_{ij}}{\partial t_\lambda}\Big|_{t=0}, \quad \tau_{i\lambda} = \frac{\partial\Phi_i}{\partial t_\lambda}\Big|_{t=0}.$$

PROOF. The first equivalence is proved in [6, p. 290]. The second equivalence follows from the congruence

$$\begin{aligned}\Phi_i(g_i^\mu, t^\mu) &= \Phi_i(g_i^{\mu-1} + g_{i|\mu}, t^{\mu-1} + t_\mu) \\ &\underset{\mu}{\equiv} \Phi_i(g_i^{\mu-1}, t^{\mu-1}) + F_i g_{i|\mu} + \sum t_{\lambda|\mu}\tau_{i\lambda}.\end{aligned}$$

Now we prove the existence of t_μ and $g_{i|\mu}$.

LEMMA 2.2. *We can find t_μ and $g_{i|\mu}$ which satisfy* (2.9) *and* (2.10).

PROOF. First note that the equality (2.10) implies (2.9), for F is injective. The equality (2.8) shows that the collection $\{P\gamma_{i|\mu}\}$ determines a homogeneous polynomial of degree μ with coefficients in $H^0(X, \mathfrak{T})$. Since the characteristic map τ is assumed to be surjective, we can find $t_{\lambda|\mu}$ such that

$$P\gamma_{i|\mu} = \sum t_{\lambda|\mu} P\tau_{i\lambda}.$$

This proves the lemma.

This lemma completes our inductive construction of $t^\mu(t')$ and $g_i^\mu(z_i, t')$.

II) Proof of convergence. Now we prove that, if we choose solutions $t_\mu(t')$ and $g_{i|\mu}(z_i, t')$ of the equations (2.9) and (2.10) properly in each step of the above construction, the power series

$$t(t') = t_1(t') + t_2(t') + \cdots + t_\mu(t') + \cdots$$

$$g_i(z_i, t') = z_i + g_{i|1}(z_i, t') + \cdots + g_{i|\mu}(z_i, t') + \cdots$$

converge absolutely and uniformly for $|t'| < \varepsilon$ provided that $\varepsilon > 0$ is sufficiently small.

Consider a power series

$$g(s) = \sum g_{\nu_1\nu_2\cdots\nu_r} s_1^{\nu_1} s_2^{\nu_2} \cdots s_r^{\nu_r}$$

whose coefficients $g_{\nu_1\nu_2\cdots\nu_r}$ are vectors, and a power series

$$a(s) = \sum a_{\nu_1\nu_2\cdots\nu_r} s_1^{\nu_1} s_2^{\nu_2} \cdots s_r^{\nu_r}$$

with non-negative coefficients. We indicate by writing $g(s) \ll a(s)$ that $|g_{\nu_1\nu_2\cdots\nu_r}| \leqq a_{\nu_1\nu_2\cdots\nu_r}$.

Let

$$A(s)=\frac{b}{16c}\sum_{\mu=1}^{\infty}\frac{1}{\mu^2}c^{\mu}(s_1+s_2+\cdots+s_r)^{\mu}.$$

We remark that

$$A(s)^{\nu}\ll\left(\frac{b}{c}\right)^{\nu-1}A(s)\qquad\text{for }\nu=2,3,4,\cdots.$$

For our purpose it suffices to derive the estimates

$$(2.11)_{\mu}\qquad t^{\mu}(t')\ll A(t'),\qquad g_i^{\mu}(z_i,t')-z_i\ll A(t')$$

by induction on μ provided that the coefficients b, c are chosen properly. For $\mu=1$ the estimates $(2.11)_1$ are obvious if b is sufficiently large. Assume therefore that the estimates $(2.11)_{\mu-1}$ are established for some μ. Then we have

$$(2.12)\qquad \Gamma_{ij|\mu}(z_i,t')\ll\left(\frac{K_1}{b}+\frac{K_2}{c}+\frac{K_3b}{c}\right)A(t'),$$

$$(2.13)\qquad \gamma_{i|\mu}(z_i,t')\ll\left(\frac{K_4}{b}+\frac{K_5b}{c}\right)A(t')$$

where $K_1, K_2, \cdots, K_5$ are constants independent of μ.

PROOF. The estimate (2.12) is proved in [6, pp. 292-294]. So we prove only the estimate (2.13). We expand $\Phi_i(z_i+y,t)$ into power series in $n+r$ variables $y_1,\cdots,y_n$, $t_1,\cdots,t_r$ and let

$$L_i(z_i,y,t)=[\Phi_i(z_i+y,t)]_1$$

be the linear term of the power series. Then we may assume that the power series expansion of $\Phi_i(z_i+y,t)$ in (y,t) satisfies

$$\begin{aligned}&\Phi_i(z_i+y,t)-f_i(z_i)-L_i(z_i,y,t)\\&\qquad\ll\frac{b_0}{c_0}\sum_{\mu=2}^{\infty}c_0^{\mu}(y_1+\cdots+y_n+t_1+\cdots+t_r)^{\mu}.\end{aligned}$$

Letting $y=g_i^{\mu-1}(z_i,t')-z_i$, $t=t^{\mu-1}(t')$ and using our inductive assumption $(2.11)_{\mu-1}$ we obtain the estimate

$$[\Phi_i(g_i^{\mu-1}(z_i,t'),t^{\mu-1}(t'))]_{\mu}\ll\frac{b_0}{c_0}\sum_{\mu=2}^{\infty}c_0^{\mu}(n+r)^{\mu}A(t')^{\mu}.$$

Assume that

$$(2.14)\qquad \frac{(n+r)bc_0}{c}<\frac{1}{2}.$$

Then we have

$$\begin{aligned}\sum_{\mu=2}^{\infty}c_0^{\mu}(n+r)^{\mu}A(t')^{\mu}&\ll\sum_{\mu=2}^{\infty}c_0^{\mu}(n+r)^{\mu}\left(\frac{b}{c}\right)^{\mu-1}A(t')\\&\ll\frac{2(n+r)^2bc_0^2}{c}A(t')\end{aligned}$$

and therefore

$$[\Phi_i(g_i^{\mu-1}(z_i, t'), t^{\mu-1}(t'))]_\mu \ll \frac{2(n+r)^2 bb_0 c_0}{c} A(t') .$$

While we may assume

$$\Phi_i'(z_i, t') - f_i(z_i) \ll \frac{b_0}{c_0} \sum_{\mu=1}^{\infty} c_0^\mu (t_1' + \cdots + t_{r'}')^\mu .$$

Assume that

$$(2.15) \qquad b > b_0 , \qquad c > c_0 ,$$

then we obtain the estimate (2.13). q. e. d.

We may assume that ρ_{ij}, τ_i, F_i, C_{ij} and $B_{ij} = \dfrac{\partial b_{ij}}{\partial z_j}$ are uniformly bounded.

For any pair $\sigma = (\Gamma, \gamma)$ of a 1-cocycle $\{\Gamma_{ij}\}$ and a 0-cochain $\{\gamma_i\}$ satisfying

$$(2.8)' \qquad F\Gamma_{ij} = \gamma_j - \gamma_i ,$$

we define the norm $\|\sigma\|$ by

$$\|\sigma\| = \|\Gamma\| + \|\gamma\| ,$$

where

$$\|\Gamma\| = \max_{i,j} \sup_{z_i} |\Gamma_{ij}(z_i)| , \qquad \|\gamma\| = \max_i \sup_{z_i} |\gamma_i(z_i)| .$$

LEMMA 2.3. *For any pair $\sigma = (\Gamma, \gamma)$ of a 1-cocycle Γ and a 0-cochain γ satisfying* (2.8)′, *we can find $g_i(z_i)$ and t_λ satisfying*

$$(2.9)' \qquad \Gamma_{ij} = g_j - g_i + \sum t_\lambda \rho_{ij\lambda} ,$$

$$(2.10)' \qquad \gamma_i = Fg_i + \sum t_\lambda \tau_{i\lambda} ,$$

$$|g_i(z_i)| \leqq K_6 \|\sigma\| , \qquad |t| \leqq K_6 \|\sigma\| ,$$

where K_6 is a constant independent of σ.

PROOF. We define

$$\iota(\sigma) = \inf \max \{\sup_{z_i} |g_i(z_i)| , |t|\}$$

where the "inf" is taken with respect to all solutions $g_i(z_i)$, t_λ of the equations (2.9)′ and (2.10)′. It suffices to prove the existence of a constant K_6 such that $\iota(\sigma) \leqq K_6 \|\sigma\|$. Suppose that such a constant K_6 does not exist. Then we can find a sequence $\sigma^{(1)}, \sigma^{(2)}, \cdots, \sigma^{(\nu)}, \cdots$ of pairs $\sigma^{(\nu)} = (\Gamma^{(\nu)}, \gamma^{(\nu)})$ satisfying (2.8)′ such that $\iota(\sigma^{(\nu)}) = 1$, $\|\sigma^{(\nu)}\| < 1/\nu$. $\iota(\sigma^{(\nu)}) = 1$ implies that there exist $g^{(\nu)}$ and $t^{(\nu)}$ such that

$$(2.16) \qquad \Gamma_{ij}^{(\nu)} = g_j^{(\nu)} - g_i^{(\nu)} + \sum t_\lambda^{(\nu)} \rho_{ij\lambda} ,$$

$$(2.17) \qquad \gamma_i^{(\nu)} = Fg_i^{(\nu)} + \sum t_\lambda^{(\nu)} \tau_{i\lambda} ,$$

$$|g_i^{(\nu)}(z_i)| < 2 , \qquad |t^{(\nu)}| < 2 .$$

Hence replacing $\sigma^{(1)}, \sigma^{(2)}, \cdots$ by a suitable subsequence if necessary, we may suppose that $g_i(z_i)=\lim g_i^{(\nu)}(z_i)$ and $t_\lambda=\lim t_\lambda^{(\nu)}$ exist, where the convergence $g_i^{(\nu)}(z_i)\to g_i(z_i)$ is uniform on each compact subset of U_i and $g_i(z_i)$ is holomorphic on U_i. Since

$$|\Gamma_{ij}^{(\nu)}(z_i)| \longrightarrow 0, \quad |\gamma_i^{(\nu)}(z_i)| \longrightarrow 0 \quad (\nu \longrightarrow \infty) \tag{2.18}$$

we obtain

$$0=g_j-g_i+\sum t_\lambda \rho_{ij\lambda}, \tag{2.19}$$

$$0=Fg_i+\sum t_\lambda \tau_{i\lambda}. \tag{2.20}$$

Let $\{U_i^*\}$ be a covering of X such that U_i^* is a relatively compact subset of U_i. For each point $z_i\in U_i$ there exists at least one U_j^* such that $z_i\in U_i\cap U_j^*$. Hence we infer from (2.16) and (2.18) that $g_i^{(\nu)}(z_i)$ converges to $g_i(z_i)$ uniformly on the whole of U_i. Letting $g_i'(z_i)=g_i^{(\nu)}(z_i)-g_i(z_i)$ and $t_\lambda'=t_\lambda^{(\nu)}-t_\lambda$ for a sufficiently large integer ν, we have therefore

$$|g_i'(z_i)|<1/2, \qquad |t_\lambda'|<1/2,$$

while we infer from (2.16), (2.17), (2.19) and (2.20) that

$$\Gamma_{ij}^{(\nu)}=g_j'-g_i'+\sum t_\lambda' \rho_{ij\lambda},$$

$$\gamma_i^{(\nu)}=Fg_i'+\sum t_\lambda' \tau_{i\lambda}.$$

This contradicts $\iota(\sigma^{(\nu)})=1$. q. e. d.

Consequently we can choose solutions $g_{i|\mu}(z_i, t')$ and $t_\mu(t')$ of the equations (2.9) and (2.10) such that

$$g_{i|\mu}(z_i, t')\ll K_6K^*A(t') \quad \text{and} \quad t_\mu(t')\ll K_6K^*A(t')$$

where

$$K^*=\frac{K_1+K_4}{b}+\frac{K_2}{c}+\frac{(K_3+K_5)b}{c}.$$

On the other hand, by a proper choice of the constants b and c satisfying (2.14) and (2.15), we obtain

$$K_6K^*<1,$$

and we infer that $g_{i|\mu}(z_i, t')\ll A(t')$ and $t_\mu(t')\ll A(t')$. This proves $(2.11)_\mu$.

q. e. d.

§ 3. A theorem of existence.

The purpose of this section is to prove the following theorem.

THEOREM 3.1. *Let $f: X\to Y$ be a non-degenerate holomorphic map. If $H^1(X, \mathcal{T}_{X/Y})=0$, then there exist a family $(\mathcal{X}, \Phi, p, M)$ of non-degenerate holomorphic maps into Y and a point $o\in M$ such that*

i) *$\Phi_o: X_o\to Y$ is equivalent to $f: X\to Y$,*

ii) *$\tau: T_o(M)\to H^0(X, \mathcal{T}_{X/Y})$ is bijective.*

First we define the Poisson bracket of differentiable vector (0, q)-forms. We denote by $A^{0,q}(\Theta_X)$ the linear space of differentiable vector (0, q)-forms

$$\psi=(\psi^1, \cdots, \psi^\alpha, \cdots, \psi^n)$$

$$\psi^\alpha=\frac{1}{q!}\sum\psi^\alpha_{\bar\mu_1\cdots\bar\mu_q}d\bar z^{\mu_1}\wedge\cdots\wedge d\bar z^{\mu_q}.$$

The exterior derivative $\bar\partial\psi$ of ψ is defined by

$$\bar\partial\psi=(\bar\partial\psi^1, \cdots, \bar\partial\psi^\alpha, \cdots, \bar\partial\psi^n).$$

We define the Poisson bracket

$$[\phi, \psi]=([\phi, \psi]^1, \cdots, [\phi, \psi]^\alpha, \cdots, [\phi, \psi]^n)$$

of $\phi\in A^{0,p}(\Theta_X)$ and $\psi\in A^{0,q}(\Theta_X)$ by

$$[\phi, \psi]^\alpha=\sum_{\mu=1}^n(\phi^\mu\wedge\partial_\mu\psi^\alpha+(-1)^{pq+1}\psi^\mu\wedge\partial_\mu\phi^\alpha)$$

where

$$\partial_\mu\psi^\alpha=\frac{1}{q!}\sum\frac{\partial\psi^\alpha_{\bar\mu_1\cdots\bar\mu_q}}{\partial z^\mu}d\bar z^{\mu_1}\wedge\cdots\wedge d\bar z^{\mu_q}.$$

$[\phi, \psi]$ is a vector form in $A^{0,p+q}(\Theta_X)$.

For any locally free sheaf E (of finite rank), we denote by $\mathcal{A}^{0,q}(E)$ the sheaf of germs of differentiable vector (0, q)-forms with coefficients in E, and let $A^{0,q}(E)=\Gamma(X, \mathcal{A}^{0,q}(E))$. Then we have the Dolbeault isomorphisms

$$H^q_{\bar\partial}(A^{0,*}(E))\cong H^q(X, E).$$

The canonical homomorphism $F\colon \Theta_X\to f^*\Theta_Y$ can be extended to a homomorphism $\mathcal{A}^{0,q}(\Theta_X)\to\mathcal{A}^{0,q}(f^*\Theta_Y)$ which we denote by the same letter F. Since f is assumed to be non-degenerate, each F is injective. Let $\mathcal{A}^{0,q}(\mathcal{T})$ be the cokernel of $F\colon \mathcal{A}^{0,q}(\Theta_X)\to\mathcal{A}^{0,q}(f^*\Theta_Y)$, $P\colon \mathcal{A}^{0,q}(f^*\Theta_Y)\to\mathcal{A}^{0,q}(\mathcal{T})$ the natural projection. Then the exterior derivative induces $\bar\partial\colon \mathcal{A}^{0,q}(\mathcal{T})\to\mathcal{A}^{0,q+1}(\mathcal{T})$ and we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Theta_X & \longrightarrow & \mathcal{A}^{0,0}(\Theta_X) & \longrightarrow & \mathcal{A}^{0,1}(\Theta_X) & \longrightarrow & \mathcal{A}^{0,2}(\Theta_X) & \longrightarrow \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & f^*\Theta_Y & \longrightarrow & \mathcal{A}^{0,0}(f^*\Theta_Y) & \longrightarrow & \mathcal{A}^{0,1}(f^*\Theta_Y) & \longrightarrow & \mathcal{A}^{0,2}(f^*\Theta_Y) & \longrightarrow \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{T} & \longrightarrow & \mathcal{A}^{0,0}(\mathcal{T}) & \longrightarrow & \mathcal{A}^{0,1}(\mathcal{T}) & \longrightarrow & \mathcal{A}^{0,2}(\mathcal{T}) & \longrightarrow \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0 & & 0
\end{array}$$

where all horizontal and vertical lines are exact.

Since $\mathcal{A}^{0,q}(\Theta_X)$ and $\mathcal{A}^{0,q}(f^*\Theta_Y)$ are fine sheaves, we have

$$H^p(X, \mathcal{A}^{0,q}(\mathcal{T}))=0 \qquad \text{for } p>0\,.$$

Hence the spectral sequence

$$H^p_{\bar\partial}(H^q(X, \mathcal{A}^{0,*}(\mathcal{T}))) \underset{p}{\Longrightarrow} H^n(X, \mathcal{T}) \qquad [3, \text{Ch. II}, 4.5]$$

degenerates, and we have an isomorphism

$$H^p_{\bar\partial}(A^{0,*}(\mathcal{T})) \cong H^p(X, \mathcal{T})\,.$$

Note that this isomorphism is compatible with the Dolbeault isomorphisms for Θ_X and for $f^*\Theta_Y$.

With these preparations we prove Theorem 3.1 following the idea of Kodaira-Nirenberg-Spencer [7].

We may assume the following:

i) X is covered by a finite number of coordinate neighborhoods U_i with a system of coordinates $(z_i^1, \cdots, z_i^n)$ and

$$U_i=\{(z_i)\in \boldsymbol{C}^n \mid |z_i|<1\}\,.$$

ii) Y is covered by a finite number of coordinate neighborhoods V_i with a system of coordinates $(w_i^1, \cdots, w_i^n)$ and

$$V_i=\{(w_i)\in \boldsymbol{C}^m \mid |w_i|<1\}\,.$$

iii) $f(U_i)\subset V_i$, and in terms of above coordinates f is given by

$$w_i=f_i(z_i)\,.$$

iv) $z_i\in U_i$ coincides with $z_j\in U_j$ if and only if

$$z_i=b_{ij}(z_j)\,.$$

v) $w_i\in V_i$ coincides with $w_j\in V_j$ if and only if

$$w_i=g_{ij}(w_j)\,.$$

Let $r=\dim H^0(X, \mathcal{T})$ and $M=\{t\in \boldsymbol{C}^r \mid |t|<\varepsilon\}$ with $\varepsilon>0$ sufficiently small.

We regard $X\times M$ as a differentiable manifold and prove the existence of a vector $(0, 1)$-form

$$\phi(t)=\sum \phi_i^\nu(z_i, t)\frac{\partial}{\partial z_i^\nu}=\sum \phi_{i\bar\alpha}^\nu(z_i, t)d\bar z_i^\alpha \frac{\partial}{\partial z_i^\nu}$$

depending holomorphically on t and a vector valued differentiable functions $\Phi_i(z_i, t)$ on $U_i\times M$ depending holomorphically on t which satisfy the following equalities:

$$\phi(0)=0\,, \tag{3.1}$$

(3.2) $$\bar{\partial}\phi-(1/2)[\phi, \phi]=0\,,$$

(3.3) $$\Phi_i(z_i, 0)=f_i(z_i)\,,$$

(3.4) $$\bar{\partial}\Phi_i-\phi\cdot\Phi_i=0 \qquad \text{where } \phi\cdot\Phi=\sum\phi_i^\nu\frac{\partial\Phi}{\partial z_i^\nu}\,,$$

(3.5) $$\Phi_i(b_{ij}(z_j), t)=g_{ij}(\Phi_j(z_j, t))\,.$$

I) Existence of formal solutions. Using the notation of § 2, let

$$\phi(t)=\sum\phi_\mu(t)\,, \qquad \Phi_i(z_i, t)=\sum\Phi_{i|\mu}(z_i, t)$$

where $\phi_\mu(t)$ and $\Phi_{i|\mu}(z_i, t)$ are homogeneous in t of degree μ, and let

$$\phi^\mu(t)=\phi_0(t)+\phi_1(t)+\cdots+\phi_\mu(t)$$

$$\Phi_i^\mu(z_i, t)=\Phi_{i|0}(z_i, t)+\Phi_{i|1}(z_i, t)+\cdots+\Phi_{i|\mu}(z_i, t)\,.$$

In view of (3.1) and (3.3), we set

(3.6) $$\phi_0=0\,, \qquad \Phi_{i|0}(z_i, t)=f_i(z_i)\,.$$

Clearly (3.2), (3.4) and (3.5) are equivalent to the following systems of congruences:

$(3.7)_\mu$ $$\bar{\partial}\phi^\mu-(1/2)[\phi^\mu, \phi^\mu]\underset{\mu}{\equiv}0\,,$$

$(3.8)_\mu$ $$\bar{\partial}\Phi_i^\mu-\phi^\mu\cdot\Phi_i^\mu\underset{\mu}{\equiv}0\,,$$

$(3.9)_\mu$ $$\Phi_i^\mu(b_{ij}(z_j), t)\underset{\mu}{\equiv}g_{ij}(\Phi_j^\mu(z_j, t))\,,$$

for $\mu=1, 2, 3, \cdots$ (we do not indicate domains on which the equations should hold, if no confusion is possible).

We construct solutions of (3.1)—(3.5) by induction on μ. We suppose that $\phi^{\mu-1}$ and $\Phi_i^{\mu-1}$ satisfying $(3.7)_{\mu-1}$, $(3.8)_{\mu-1}$ and $(3.9)_{\mu-1}$ are already determined.

We define homogeneous polynomials $\xi_\mu\in A^{0,2}(\Theta_X)$, $\Xi_{i|\mu}\in\Gamma(U_i, \mathcal{A}^{0,1}(f^*\Theta_Y))$ and $\Gamma_{ij|\mu}\in\Gamma(U_{ij}, \mathcal{A}^{0,0}(f^*\Theta_Y))$ (for the convention of notation, see § 2) by the following congruences:

(3.10) $$\xi_\mu\underset{\mu}{\equiv}\bar{\partial}\phi^{\mu-1}-(1/2)[\phi^{\mu-1}, \phi^{\mu-1}]\,,$$

(3.11) $$-\Xi_{i|\mu}\underset{\mu}{\equiv}(\bar{\partial}\Phi_i^{\mu-1}-\phi^{\mu-1}\cdot\Phi_i^{\mu-1})\cdot\frac{\partial}{\partial w_i}\,,$$

(3.12) $$\Gamma_{ij|\mu}\underset{\mu}{\equiv}(\Phi_i^{\mu-1}-g_{ij}(\Phi_j^{\mu-1}))\cdot\frac{\partial}{\partial w_i}\,.$$

Then we have the following equalities:

(3.13) $$\bar{\partial}\xi_\mu=0 \qquad \text{in } \Gamma(X, \mathcal{A}^{0,3}(\Theta_X))\,,$$

(3.14) $$\bar{\partial}\Xi_{i|\mu}=F\xi_\mu \qquad \text{in } \Gamma(U_i, \mathcal{A}^{0,2}(f^*\Theta_Y))\,,$$

(3.15) $$\Xi_{j|\mu}-\Xi_{i|\mu}=\bar{\partial}\Gamma_{ij|\mu} \quad \text{in } \Gamma(U_{ij}, \mathcal{A}^{0,1}(f^*\Theta_Y)),$$

(3.16) $$\Gamma_{jk|\mu}-\Gamma_{ik|\mu}+\Gamma_{ij|\mu}=0 \quad \text{in } \Gamma(U_{ijk}, \mathcal{A}^{0,0}(f^*\Theta_Y)).$$

PROOF OF (3.13). This equality is proved in [7, p. 454].

PROOF OF (3.14). We suppress the indices $\mu-1$, if no confusion is possible. Also we suppress the subscript i. With these conventions, we have

$$\bar{\partial}\Xi_\mu \underset{\mu}{\equiv} \sum \bar{\partial}\phi^\sigma\partial_\sigma\Phi-\sum\phi^\nu\wedge\partial_\nu\bar{\partial}\Phi \underset{\mu}{\equiv} \sum\bar{\partial}\phi^\sigma\partial_\sigma\Phi-\sum\phi^\nu\wedge\partial_\nu\phi^\sigma\partial_\sigma\Phi$$
$$\underset{\mu}{\equiv} \sum\xi_\mu^\sigma\partial_\sigma\Phi \underset{\mu}{\equiv} F\xi_\mu.$$

PROOF OF (3.15). We suppress the indices $\mu-1$. Let $G_{ij\lambda}^\rho=\dfrac{\partial g_{ij}^\rho}{\partial w_j^\lambda}$ as in § 2. Then we have

$$\Xi_{i|\mu}^\rho=[\sum\phi_i^\sigma\partial_{i\sigma}\Phi_i^\rho]_\mu=[\sum\phi_i^\sigma\partial_{i\sigma}(g_{ij}^\rho(\Phi_j))]_\mu$$
$$=[\sum\phi_j^\sigma G_{ij\lambda}^\rho(\Phi_j)\partial_{j\sigma}\Phi_j^\lambda]_\mu$$
$$-\bar{\partial}\Gamma_{ij\mu}^\rho=[\bar{\partial}g_{ij}^\rho(\Phi_j)]_\mu=[\sum G_{ij\lambda}^\rho(\Phi_j)\bar{\partial}\Phi_j^\lambda]_\mu$$
$$=-\sum G_{ij\lambda}^\rho\Xi_{j|\mu}+[\sum G_{ij\lambda}^\rho(\Phi_j)\phi_j^\sigma\partial_{j\sigma}\Phi_j^\lambda]_\mu. \qquad \text{q. e. d.}$$

PROOF OF (3.16). Since $g_{ij}(g_{jk}(w_k))=g_{ik}(w_k)$, we have

$$g_{ik}(\Phi_k)=g_{ij}(g_{jk}(\Phi_k)) \underset{\mu}{\equiv} g_{ij}(\Phi_j-\Gamma_{jk|\mu})$$
$$\underset{\mu}{\equiv} g_{ij}(\Phi_j)-G_{ij}(\Phi_j)\Gamma_{jk|\mu}$$
$$\underset{\mu}{\equiv} \Phi_i-\Gamma_{ij|\mu}-G_{ij}(f_j)\Gamma_{jk|\mu}. \qquad \text{q. e. d.}$$

Our purpose is to determine

$$\phi^\mu=\phi^{\mu-1}+\phi_\mu, \qquad \Phi_i^\mu=\Phi_i^{\mu-1}+\Phi_{i|\mu}$$

which satisfy $(3.7)_\mu$, $(3.8)_\mu$ and $(3.9)_\mu$.

We prove that $(3.7)_\mu$, $(3.8)_\mu$ and $(3.9)_\mu$ are equivalent to the following equalities:

(3.17) $$\bar{\partial}\phi_\mu=-\xi_\mu,$$

(3.18) $$\Xi_{i|\mu}=\bar{\partial}\Phi_{i|\mu}-F\phi_\mu,$$

(3.19) $$\Gamma_{ij|\mu}=\Phi_{j|\mu}-\Phi_{i|\mu},$$

where as in § 2, we denote by the same letter Φ_i the section $\sum\Phi_i^\rho\dfrac{\partial}{\partial w_i^\rho}$ of the sheaf $f^*\Theta_Y$, etc.

PROOF. The first equivalence is easy to prove, so we omit it. We have a congruence

$$\phi^{\mu}\cdot\Phi^{\rho,\mu}=(\phi^{\mu-1}+\phi_{\mu})(\Phi^{\rho,\mu-1}+\Phi^{\rho}_{\mu})$$
$$\underset{\mu}{\equiv}\phi^{\mu-1}\cdot\Phi^{\rho,\mu-1}+\phi_{\mu}\cdot f^{\rho}.$$

It follows that $(3.8)_{\mu}$ is equivalent to

$$0\underset{\mu}{\equiv}(\bar{\partial}\Phi^{\mu-1}-\phi^{\mu-1}\cdot\Phi^{\mu-1})+\bar{\partial}\Phi_{\mu}-F\phi_{\mu}.$$

This proves the assertion.

As to the last equivalence, we have a congruence (Φ should be regarded as a vector $(\Phi^{1},\cdots,\Phi^{m})$)

$$g_{ij}(\Phi^{\mu}_{j})=g_{ij}(\Phi^{\mu-1}_{j}+\Phi_{j|\mu})$$
$$\underset{\mu}{\equiv}g_{ij}(\Phi^{\mu-1}_{j})+G_{ij}(f_{j})\Phi_{j|\mu}$$
$$\underset{\mu}{\equiv}\Phi^{\mu-1}_{i}-\Gamma_{ij|\mu}+G_{ij}(f_{j})\Phi_{j|\mu}.$$

It follows that $(3.9)_{\mu}$ is equivalent to

$$0\underset{\mu}{\equiv}\Phi_{i|\mu}+\Gamma_{ij|\mu}-G_{ij}(f_{j})\Phi_{j|\mu}.$$

This completes the proof.

The final step of I) is to prove the following lemma:

LEMMA 3.2. *Under the hypothesis of Theorem* 3.1 *we can find*

$$\phi_{\mu}\in A^{0,1}(\Theta_{X})\quad\textit{and}\quad\Phi_{i|\mu}\in\Gamma(U_{i},\mathcal{A}^{0,0}(f^{*}\Theta_{Y}))$$

satisfying (3.17), (3.18) *and* (3.19).

PROOF. In virtue of the equality (3.16), we can find $\Gamma_{i|\mu}\in\Gamma(U_{i},\mathcal{A}^{0,0}(f^{*}\Theta_{Y}))$ such that

$$\Gamma_{ij|\mu}=\Gamma_{j|\mu}-\Gamma_{i|\mu}.\tag{3.20}$$

From the equalities (3.15) and (3.20), we infer that

$$\Xi'_{\mu}=\Xi_{i|\mu}-\bar{\partial}\Gamma_{i|\mu}\tag{3.21}$$

determines a global section $\Xi'_{\mu}\in A^{0,1}(f^{*}\Theta_{Y})$, and from (3.14) it follows that $\bar{\partial}\Xi'_{\mu}=F\xi_{\mu}$; consequently $P\Xi'_{\mu}\in A^{0,1}(\mathfrak{T})$ is $\bar{\partial}$-closed.

By hypothesis $H^{1}(X,\mathfrak{T})=0$, $P\Xi'_{\mu}$ is $\bar{\partial}$-exact; this implies that we can find $\phi'_{\mu}\in A^{0,1}(\Theta_{X})$ and $\Phi''_{\mu}\in A^{0,0}(f^{*}\Theta_{Y})$ such that

$$\bar{\partial}\Phi''_{\mu}=\Xi'_{\mu}+F\phi'_{\mu}.$$

Then it follows that $F\bar{\partial}\phi'_{\mu}=-F\xi_{\mu}$. Since F is injective, we obtain

$$\bar{\partial}\phi'_{\mu}=-\xi_{\mu}.\tag{3.22}$$

Reversing the process, take any $\phi'\in A^{0,1}(\Theta_{X})$ satisfying the equality (3.22).

Then it follows that

$$\bar{\partial}(\Xi'_\mu + F\phi') = 0 . \tag{3.23}$$

In the exact sequence

$$H^1(X, \Theta_X) \xrightarrow{F} H^1(X, f^*\Theta_Y) \xrightarrow{P} H^1(X, \mathfrak{T})$$

the cohomology class (in $H^1(X, f^*\Theta_Y)$) corresponding to the $\bar{\partial}$-closed form $\Xi'_\mu + F\phi'_\mu$ (by the Dolbeault isomorphism) is in Ker P. It follows that we can find $\chi_\mu \in A^{0,1}(\Theta_X)$ and $\Phi'_\mu \in A^{0,0}(f^*\Theta_Y)$ such that

$$\bar{\partial}\chi_\mu = 0 , \tag{3.24}$$

$$\Xi'_\mu + F\phi'_\mu = F\chi_\mu + \bar{\partial}\Phi'_\mu . \tag{3.25}$$

Let

$$\phi_\mu = \phi'_\mu - \chi_\mu , \tag{3.26}$$

$$\Phi_{i|\mu} = \Phi'_\mu + \Gamma_{i|\mu} . \tag{3.27}$$

Then from (3.26), (3.24) and (3.22) it follows that

$$\bar{\partial}\phi_\mu = \bar{\partial}\phi'_\mu = -\xi_\mu .$$

From (3.27), (3.25), (3.26) and (3.21) it follows that

$$\bar{\partial}\Phi_{i|\mu} = \Xi'_\mu + F\phi_\mu + \bar{\partial}\Gamma_{i|\mu} = \Xi_{i|\mu} + F\phi_\mu .$$

From (3.27) and (3.20) it follows that

$$\Phi_{j|\mu} - \Phi_{i|\mu} = \Gamma_{j|\mu} - \Gamma_{i|\mu} = \Gamma_{ij|\mu} .$$

This proves the lemma.

For $\mu = 1$, we determine ϕ_1 and Φ_1 as follows: Take $\Phi_{1\lambda} \in A^{0,0}(f^*\Theta_Y)$ such that $\{P\Phi_{1\lambda}\}$ $(\lambda = 1, 2, \cdots, r)$ forms a basis of the linear space $H^0(X, \mathfrak{T})$. Then we can find $\phi_{1\lambda} \in A^{0,1}(\Theta_X)$ such that $\bar{\partial}\Phi_{1\lambda} = F\phi_{1\lambda}$, and let $\phi_1 = \sum \phi_{1\lambda} t_\lambda$ and $\Phi_1 = \sum \Phi_{1\lambda} t_\lambda$. It is clear that ϕ_1 and Φ_1 satisfy the congruences $(3.7)_1$, $(3.8)_1$ and $(3.9)_1$.

Once we determine ϕ_1 and Φ_1, we can extend them to formal power series in t satisfying (3.1)—(3.5), as we have already seen.

II) Proof of convergence. Let E be a locally free sheaf of rank r on X, such that $E|_{U_i}$ is trivial. We define a norm $|\ |_{k+\alpha}$ (k: an integer $\geqq 2$, $0 < \alpha < 1$) for sections of $\mathcal{A}^{0,q}(E)$ as follows: Let $\psi \in \Gamma(U_i, \mathcal{A}^{0,q}(E))$ and we write ψ explicitly in the form

$$\psi = (\psi^1, \cdots, \psi^\beta, \cdots, \psi^r) \qquad \psi^\beta = \frac{1}{q\,!} \sum \psi^\beta_{i\bar{\mu}_1 \cdots \bar{\mu}_q}(z_i) d\bar{z}_i^{\mu_1} \wedge \cdots \wedge d\bar{z}_i^{\mu_q}$$

in terms of local coordinates $(z_i^1, \cdots, z_i^n)$ and let

(3.28)
$$|\phi|^{U_i}_{k+\alpha}=\sum_{h=0}^{k}\sup|D_i^h\phi^\beta_{i\bar\mu_1\cdots\bar\mu_q}(z_i)| + \sup\frac{|D_i^k\phi^\beta_{i\bar\mu_1\cdots\bar\mu_q}(z_i)-D_i^k\phi^\beta_{i\bar\mu_1\cdots\bar\mu_q}(y_i)|}{|z_i-y_i|^\alpha}$$

where the "sup" is extended over all points z_i, $y_i\in U_i$, all indices β, $\bar\mu_1,\cdots,\bar\mu_q$ and all partial derivatives D_i^h, D_i^k of order h, k with respect to $z_i^1,\cdots,z_i^n$, $\bar z_i^1,\cdots,\bar z_i^n$. For $\phi\in A^{0,q}(E)$ we define

$$|\phi|_{k+\alpha}=\max_i|\phi|^{U_i}_{k+\alpha}.$$

For $\phi\in\Gamma(U_{ij},\mathcal{A}^{0,q}(E))$ we define a norm $|\phi|^{U_{ij}}_{k+\alpha}$ by the formula (3.28) with additional restriction z_i, $y_i\in U_{ij}$. We do not indicate explicitly the domain, if no confusion is possible.

We introduce a harmonic theory on the sheaf Θ_X, denote by ϑ the adjoint operator of $\bar\partial$, and let $\square=\vartheta\bar\partial+\bar\partial\vartheta$ be the complex Laplace-Beltrami operator and G the Green's operator.

Consider a formal power series

$$\phi=\phi(t)=\sum\phi_{\nu_1\cdots\nu_r}t_1^{\nu_1}\cdots t_r^{\nu_r}$$

with coefficients in $A^{0,q}(E)$ (or in $\Gamma(U_i,\mathcal{A}^{0,q}(E))$ or in $\Gamma(U_{ij},\mathcal{A}^{0,q}(E))$) and a power series

$$a(t)=\sum a_{\nu_1\cdots\nu_r}t_1^{\nu_1}\cdots t_r^{\nu_r}\qquad a_{\nu_1\cdots\nu_r}\geqq 0.$$

We indicate by $|\phi|_{k+\alpha}\ll a(t)$ that

$$|\phi_{\nu_1\cdots\nu_r}|_{k+\alpha}\leqq a_{\nu_1\cdots\nu_r}.$$

Let

$$A(t)=\frac{b}{16c}\sum_{\mu=1}^{\infty}\frac{1}{\mu^2}c^\mu(t_1+\cdots+t_r)^\mu.$$

Now we show that for a fixed integer $k\geqq 2$ and α, $0<\alpha<1$, the construction of ϕ and Φ_i can be carried out in such a way that

(3.29) $$|\phi|_{k+\alpha}\ll A(t),$$

(3.30) $$|\Phi_i-f_i|_{k+\alpha}\ll A(t).$$

For this purpose it suffices to prove

$(3.29)_\mu$ $$|\phi^\mu|_{k+\alpha}\ll A(t),$$

$(3.30)_\mu$ $$|\Phi_i^\mu-f_i|_{k+\alpha}\ll A(t),$$

for $\mu=1,2,\cdots$.

The estimates $(3.29)_1$ and $(3.30)_1$ hold for sufficiently large b. Therefore we may assume that $\phi^{\mu-1}$ and $\Phi_i^{\mu-1}$ are already determined in such a way that $(3.29)_{\mu-1}$ and $(3.30)_{\mu-1}$ hold.

In (3.22), we may assume

$$\phi'_\mu = -\vartheta G \xi_\mu , \tag{3.31}$$

and we infer from the results of Douglis-Nirenberg [1], that

$$|\phi'_\mu|_{k+\alpha} \ll K_1 |\xi_\mu|_{k-1+\alpha} \tag{3.32}$$

where K_1 is a constant which is independent of ξ_μ and μ.

Moreover, we may amplify the condition (3.24) by

$$\Box \chi_\mu = 0 . \tag{3.24)*$$

Now we prove the following key lemma:

LEMMA 3.3. *Suppose that* $\phi \in A^{0,1}(\Theta_X)$ *and* $\Xi \in A^{0,1}(f^*\Theta_Y)$ *satisfying* $\bar\partial(\Xi + F\phi) = 0$ *are given. Then we can find* $\chi \in A^{0,1}(\Theta_X)$ *and* $\Phi \in A^{0,0}(f^*\Theta_Y)$ *in such a way that*

$$\Box \chi = 0 , \tag{3.24)*$$

$$\Xi + F\phi = F\chi + \bar\partial \Phi , \tag{3.25}$$

$$|\chi|_{k+\alpha} \ll K_2(|\phi|_{k+\alpha} + |\Xi|_{k-1+\alpha}) , \tag{3.33}$$

$$|\Phi|_{k+\alpha} \ll K_2(|\phi|_{k+\alpha} + |\Xi|_{k-1+\alpha}) , \tag{3.34}$$

where K_2 *is a constant which is independent of* ϕ *and* Ξ.

PROOF. For any pair $\sigma = (\phi, \Xi)$ as above, let

$$\|\sigma\| = |\phi|_{k+\alpha} + |\Xi|_{k-1+\alpha} ,$$

$$\iota(\sigma) = \inf |\chi|_{k+\alpha} ,$$

where the "inf" is taken with respect to all solutions (χ, Φ) of the equalities (3.24)* and (3.25).

We introduce a harmonic theory on the sheaf $f^*\Theta_Y$, and denote by ϑ' and G', respectively the adjoint operator of $\bar\partial$ and Green's operator. It suffices to prove the existence of a constant K_2 such that

$$\iota(\sigma) \leqq K_2 \|\sigma\| \qquad \text{for all pairs } \sigma .$$

In fact, if the assertion is valid, we can find χ and Φ, satisfying (3.24)*, (3.25) and (3.33) (replacing K_2 by a larger constant if necessary). Then we can replace Φ by $\vartheta' G'(\Xi + F\phi - F\chi)$, and we obtain (3.34) from (3.33) (replacing K_2 by a larger constant, if necessary).

Now we prove the existence of such a constant K_2. Assume that there is no such constant. Then we can find a sequence $\sigma^{(1)}, \sigma^{(2)}, \cdots, \sigma^{(\nu)}, \cdots$ of pairs $\sigma^{(\nu)} = (\phi^{(\nu)}, \Xi^{(\nu)})$ such that

$$\iota(\sigma^{(\nu)}) = 1 \quad \text{and} \quad \|\sigma^{(\nu)}\| < 1/\nu .$$

The first equality implies the existence of $\chi^{(\nu)} \in A^{0,1}(\Theta_X)$ and $\Phi^{(\nu)} \in A^{0,0}(f^*\Theta_Y)$ such that

$$\square\chi^{(\nu)}=0\,, \tag{3.35}$$

$$\Xi^{(\nu)}+F\phi^{(\nu)}=F\chi^{(\nu)}+\bar{\partial}\Phi^{(\nu)}\,, \tag{3.36}$$

$$|\chi^{(\nu)}|_{k+\alpha}<2\,. \tag{3.37}$$

Moreover we may assume that $\Phi^{(\nu)}=\vartheta' G'(\Xi^{(\nu)}+F\phi^{(\nu)}-F\chi^{(\nu)})$. From (3.37), it follows that, replacing $\sigma^{(1)}, \sigma^{(2)}, \cdots$ by a suitable subsequence if necessary, we may assume that

$$\chi=\lim \chi^{(\nu)}$$

exists in the norm $|\ |_k$. A priori, χ is of class C^k. But by virtue of [1] Theorem 5, χ is in fact of class C^∞, for χ satisfies an elliptic partial differential equation $\square\chi=0$. Moreover, from [1] Theorem 4 it follows that

$$|\chi^{(\nu)}-\chi|_{k+\alpha}\leqq \text{const.}|\chi^{(\nu)}-\chi|_0\,.$$

Hence $\chi^{(\nu)}$ converges to χ in the norm $|\ |_{k+\alpha}$. Moreover by the construction of $\Phi^{(\nu)}$, $\Phi^{(\nu)}$ converges to a C^∞ section Φ. From (3.36) it follows that

$$0=F\chi+\bar{\partial}\Phi\,. \tag{3.38}$$

Consequently we infer from (3.36) and (3.38) that

$$\Xi^{(\nu)}+F\phi^{(\nu)}=F(\chi^{(\nu)}-\chi)+\bar{\partial}(\Phi^{(\nu)}-\Phi)\,.$$

On the other hand, we have

$$|\chi^{(\nu)}-\chi|_{k+\alpha}<1/2$$

for sufficiently large integer ν; this contradicts $\iota(\sigma)=1$. q. e. d.

Now we prove the following inequalities:

$$|\xi_\mu|_{k-1+\alpha}\ll \frac{K_3 b}{c}A(t) \tag{3.39}$$

$$|\Xi_{i|\mu}|_{k-1+\alpha}\ll \frac{K_4 b}{c}A(t) \tag{3.40}$$

$$|\Gamma_{ij|\mu}|_{k+\alpha}\ll \frac{K_5 b}{c}A(t) \tag{3.41}$$

where K_3, K_4 and K_5 are constants which are independent of μ.

PROOF. The inequalities (3.39) and (3.40) can be easily deduced from induction hypotheses $(3.29)_{\mu-1}$ and $(3.30)_{\mu-1}$.

Now we prove the inequality (3.41). Let $w_j+u=(w_j^1+u_1, \cdots, w_j^m+u_m)$. We expand $g_{ij}(w_j+u)$ into a power series in m variables $u_1, \cdots, u_m$, and let $L_{ij}(w_j, u)=[g_{ij}(w_j+u)]_1$ be the linear term of the power series. We may assume that

$$|g_{ij}(w_j+u)-g_{ij}(w_j)-L_{ij}(w_j, u)|_{k+\alpha}$$
$$\ll \frac{b_0}{c_0}\sum_{\mu=2}^{\infty} c_0^{\mu}(u_1+\cdots+u_m)^{\mu}.$$

Let $u(z_j, t)=\Phi_j^{\mu-1}(z_j, t)-f_j(z_j)$. Then, by inductive hypothesis, $|u|_{k+\alpha}\ll A(t)$. Let K_0 be a constant such that

$$|\phi\psi|_{k+\alpha}<K_0|\phi|_{k+\alpha}|\psi|_{k+\alpha}$$

for any ϕ and ψ. Then it follows that

$$|[g_{ij}(\Phi_j^{\mu-1}(z_j, t))]_{\mu}|_{k+\alpha}\ll \frac{b_0}{c_0}\sum_{\mu=2}^{\infty} K_0 c_0^{\mu}|(u_1+\cdots+u_m)^{\mu}|_{k+\alpha}$$
$$\ll \frac{b_0}{c_0}\sum_{\mu=2}^{\infty} c_0^{\mu} m^{\mu} K_0^{\mu} A(t)^{\mu}.$$

Assume that

(3.42) $$\frac{mbc_0K_0}{c}<\frac{1}{2}.$$

Then we have

$$\sum_{\mu=2}^{\infty} c_0^{\mu} m^{\mu} K_0^{\mu} A(t)^{\mu} \ll \sum_{\mu=2}^{\infty} c_0^{\mu} m^{\mu} K_0^{\mu} (b/c)^{\mu-1} A(t) \ll \frac{2m^2bc_0^2K_0^2}{c} A(t).$$

This proves the inequality (3.41).

Let $\{p_i(z)\}$ be a partition of unity subordinate to the covering $\{U_i\}$. As a solution $\Gamma_{i|\mu}$ of the equations (3.20), we take

$$\Gamma_{i|\mu}=\sum_j p_j(z)\Gamma_{ji|\mu} \qquad \text{defined on } U_i.$$

Then there exists a constant K_6 (which is independent of μ) such that

(3.43) $$|\Gamma_{i|\mu}|_{k+\alpha}\ll K_6|\Gamma_{ij|\mu}|_{k+\alpha}.$$

Consequently, from (3.32) and (3.39), it follows that

(3.44) $$|\phi'_{\mu}|_{k+\alpha}\ll \frac{K_3K_1b}{c}A(t).$$

From (3.21), (3.40), (3.41) and (3.43), it follows that

$$|\Xi'_{\mu}|_{k-1+\alpha}\ll \left(\frac{K_4b}{c}+\frac{K_5K_6b}{c}\right)A(t).$$

Combining these with Lemma 3.3, we can find χ_{μ} and Φ'_{μ} satisfying (3.24)* and (3.25) in such a way that

(3.45) $$|\chi_{\mu}|_{k+\alpha}\ll K_2K^*A(t),$$

(3.46) $$|\Phi'_{\mu}|_{k+\alpha}\ll K_2K^*A(t),$$

where

$$K^*=\frac{K_3K_1b}{c}+\frac{K_4b}{c}+\frac{K_5K_6b}{c}.$$

It follows from (3.26), (3.44) and (3.46) that

$$|\phi_\mu|_{k+\alpha}\ll\Big(\frac{K_3K_1b}{c}+K_2K^*\Big)A(t).$$

From (3.27), (3.41), (3.43) and (3.46) we get

$$|\Phi_{i|\mu}|_{k+\alpha}\ll\Big(\frac{K_5K_6b}{c}+2K_2K^*\Big)A(t).$$

On the other hand, we can choose b and c satisfying (3.42) such that

$$\frac{K_3K_1b}{c}+2K_2K^*<1,\qquad \frac{K_5K_6b}{c}+2K_2K^*<1.$$

Consequently we obtain

$$|\phi_\mu|_{k+\alpha}\ll A(t),\qquad |\Phi_{i|\mu}|_{k+\alpha}\ll A(t).$$

This proves $(3.29)_\mu$ and $(3.30)_\mu$.

III) Final step. We fix an integer $k\geqq 2$ and α, $0<\alpha<1$. It follows from (3.29) that

$$\phi(t)=\phi_1(t)+\phi_2(t)+\cdots+\phi_\mu(t)+\cdots$$

converges in the norm $|\ |_{k+\alpha}$ for sufficiently small $|t|$. Note that (3.31) and (3.24)* imply that $\vartheta\phi_\mu(t)=0$ for $\mu\geqq 2$. Hence the argument of Kodaira-Nirenberg-Spencer ([8], pp. 458-459) can be applied to prove that there exists a complex analytic family $p: \mathcal{X}\to M$ of deformations of X, where each fibre $p^{-1}(t)$ is endowed with a complex structure determined by $\phi(t)$. By the equalities (3.3), (3.5), the collection $\{\Phi_i(z_i, t)\}$ defines a differentiable map $\Phi: \mathcal{X}\to Y\times M$ of class C^k which coincides with f on X. From (3.4) it follows that Φ is holomorphic.

The characteristic map is given by the formula

$$\tau\Big(\frac{\partial}{\partial t_\lambda}\Big)=P\Phi_{1\lambda}.$$

Hence, by the construction, $\tau: T_o(M)\to H^0(X, \mathcal{T})$ is bijective. q. e. d.

Appendix. Elementary proof of formal existence theorem.

THEOREM. *Let $f: X\to Y$ be a non-degenerate holomorphic map. If $H^1(X, \mathcal{T}_{X/Y})=0$, then there exists a formal family $(\mathcal{X}, \Phi, p, M)$ of non-degenerate holomorphic maps into Y and a point $o\in M$ such that*

i) $\Phi_o: X_o \to Y$ *is equivalent to* $f: X \to Y$,

ii) $\tau: T_o(M) \to H^0(X, \mathcal{T}_{X/Y})$ *is bijective.*

PROOF. We may assume the conditions i)—v) in the proof of Theorem 3.1. We prove the existence of formal power series $\phi_{ij}(z_j, t)$ and $\Phi_i(z_i, t)$ which satisfy

$$\phi_{ij}(z_j, 0) = b_{ij}(z_j)\,, \tag{A.1}$$

$$\phi_{ij}(\phi_{jk}(z_k, t), t) = \phi_{ik}(z_k, t)\,, \tag{A.2}$$

$$\Phi_i(z_i, 0) = f_i(z_i)\,, \tag{A.3}$$

$$\Phi_i(\phi_{ij}(z_j, t), t) = g_{ij}(\Phi_j(z_j, t))\,. \tag{A.4}$$

Clearly (A.2) and (A.4) are equivalent to the following systems of congruences:

$$\phi_{ij}^{\mu}(\phi_{jk}^{\mu}(z_k, t), t) \underset{\mu}{\equiv} \phi_{ik}^{\mu}(z_k, t)\,, \tag{A.5}_\mu$$

$$\Phi_i^{\mu}(\phi_{ij}^{\mu}(z_j, t), t) \underset{\mu}{\equiv} g_{ij}(\Phi_j^{\mu}(z_j, t))\,. \tag{A.6}_\mu$$

Assume that $\phi_{ij}^{\mu-1}(z_j, t)$ and $\Phi_i^{\mu-1}(z_i, t)$ satisfying $(A.5)_{\mu-1}$ and $(A.6)_{\mu-1}$ are already determined. We define homogeneous polynomials in t of degree μ by the following congruences:

$$\gamma_{ijk|\mu}(z_i, t) \underset{\mu}{\equiv} \phi_{ij}^{\mu-1}(\phi_{jk}^{\mu-1}(z_k, t), t) - \phi_{ik}^{\mu-1}(z_k, t)\,, \tag{A.7}$$

$$\Gamma_{ij|\mu}(z_i, t) \underset{\mu}{\equiv} \Phi_i^{\mu-1}(\phi_{ij}^{\mu-1}(z_j, t), t) - g_{ij}(\Phi_j^{\mu-1}(z_j, t))\,, \tag{A.8}$$

where $z_i = b_{ij}(z_j)$.

Then we have the following equalities:

$$G_{ij}(f_j(z_j))\Gamma_{jk|\mu}(z_j, t) + \Gamma_{ik|\mu}(z_i, t) + \Gamma_{ij|\mu}(z_i, t) = F_i(z_i)\gamma_{ijk|\mu}(z_i, t)\,, \tag{A.9}$$

$$B_{ij}\gamma_{jkl|\mu} - \gamma_{ikl|\mu} + \gamma_{ijl|\mu} - \gamma_{ijk|\mu} = 0\,, \quad \begin{pmatrix} F_i,\ G_{ij} \text{ and } B_{ij} \text{ are} \\ \text{the same as in §3} \end{pmatrix}. \tag{A.10}$$

PROOF. (A.10) follows from (A.9) because F is injective. We prove the equality (A.9).

$$\begin{aligned}
\Gamma_{ik|\mu} &\underset{\mu}{\equiv} \Phi_i^{\mu-1}(\phi_{ik}^{\mu-1}, t) - g_{ik}(\Phi_k^{\mu-1}) \\
&\underset{\mu}{\equiv} \Phi_i^{\mu-1}(\phi_{ij}^{\mu-1}(\phi_{jk}^{\mu-1}, t) - \gamma_{ijk|\mu}, t) - g_{ij}(g_{jk}(\Phi_k^{\mu-1})) \\
&\underset{\mu}{\equiv} \Phi_i^{\mu-1}(\phi_{ij}^{\mu-1}(\phi_{jk}^{\mu-1}, t)) - F_i\gamma_{ijk|\mu} - g_{ij}(\Phi_j^{\mu-1}(\phi_{jk}^{\mu-1}, t) - \Gamma_{jk|\mu}) \\
&\underset{\mu}{\equiv} \Phi_i^{\mu-1}(\phi_{ij}^{\mu-1}(\phi_{jk}^{\mu-1}, t), t) - g_{ij}(\Phi_j^{\mu-1}(\phi_{jk}^{\mu-1}, t)) + G_{ij}\Gamma_{jk|\mu} - F_i\gamma_{ijk|\mu} \\
&\underset{\mu}{\equiv} \Gamma_{ij|\mu} + G_{ij}\Gamma_{jk|\mu} - F_i\gamma_{ijk|\mu}\,.
\end{aligned}$$

q. e. d.

We prove that $(A.5)_\mu$ and $(A.6)_\mu$ are equivalent to the following:

$$-\gamma_{ijk|\mu} = B_{ij}\phi_{jk|\mu} - \phi_{ik|\mu} + \phi_{ij|\mu}, \tag{A.11}$$

$$\Gamma_{ij|\mu} = G_{ij}(f)\Phi_{j|\mu} - \Phi_{i|\mu} - F_i\phi_{ij|\mu}. \tag{A.12}$$

PROOF. We have

$$\begin{aligned}\phi^\mu_{ij}(\phi^\mu_{jk}, t) &= \phi^{\mu-1}_{ij}(\phi^{\mu-1}_{jk} + \phi_{jk|\mu}, t) + \phi_{ij|\mu}(\phi^\mu_{jk}, t)\\ &\underset{\mu}{\equiv} \phi^{\mu-1}_{ij}(\phi^{\mu-1}_{jk}, t) + B_{ij}\phi_{jk|\mu} + \phi_{ij|\mu}.\end{aligned}$$

Hence $(A.5)_\mu$ is equivalent to (A.11).

Similarly

$$\begin{aligned}\Phi^\mu_i(\phi^\mu_{ij}, t) &= \Phi^{\mu-1}_i(\phi^{\mu-1}_{ij} + \phi_{ij|\mu}, t) + \Phi_{i|\mu}(\phi^\mu_{ij}, t)\\ &\underset{\mu}{\equiv} \Phi^{\mu-1}_i(\phi^{\mu-1}_{ij}, t) + F_i\phi_{ij|\mu} + \Phi_{i|\mu},\end{aligned}$$

$$g_{ij}(\Phi^\mu_j) \underset{\mu}{\equiv} g_{ij}(\Phi^{\mu-1}_j) + G_{ij}(f)\Phi_{j|\mu}.$$

It follows that $(A.6)_\mu$ is equivalent to (A.12).

LEMMA. (A.12) *implies* (A.11).

This lemma follows from the fact that F is injective.

Take a basis $\tau_\lambda (1 \leqq \lambda \leqq r)$ of $H^0(X, \mathfrak{T})$. Then τ_λ is locally represented by $\sum \Phi^\rho_{\lambda i}\frac{\partial}{\partial w^\rho_i} \in \Gamma(U_i, f^*\Theta_Y)$ and we can find $\phi_{\lambda ij} \in \Gamma(U_{ij}, \Theta_X)$ such that

$$\sum \Phi^\rho_{\lambda j}\frac{\partial}{\partial w^\rho_j} - \sum \Phi^\rho_{\lambda i}\frac{\partial}{\partial w^\rho_i} = F\phi_{\lambda ij}.$$

It follows that $\phi_{ij|1} = \sum \phi_{\lambda ij}t_\lambda$ and $\Phi_{i|1} = \sum \Phi_{\lambda i}t_\lambda$ satisfy (A.12).

Now assume that $\phi^{\mu-1}_{ij}(z_j, t)$ and $\Phi^{\mu-1}_i(z_i, t)$ are already determined. Then, by the equality (A.9), $\{P\Gamma_{ij|\mu}\}$ represents a 1-cocycle with coefficients in $\mathfrak{T}$. Hence, by hypothesis, this is a coboundary. Hence we can find $\phi_{ij|\mu}$ and $\Phi_{i|\mu}$ satisfying (A.12). This proves the existence of a formal family, and by the construction τ is bijective.

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