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Bordism groups of dihedral groups

By Masayoshi KAMATA and Haruo MINAMI

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Let G be a finite group. By a G-manifold we mean a closed oriented manifold together with an orientation preserving action of G without fixed points or a closed weakly complex manifold together with a weakly complex structure preserving action of G without fixed points. We denote a G-manifold by a pair (M, f) where M is a G-manifold and f a free action of G on $M: G\times M\rightarrow M$ and its bordism class by $[M, f]$. Moreover we denote by $\tilde{\Omega}_m^{SO}(G)$ the oriented reduced bordism group of G of dimension m and by $\tilde{\Omega}_m^U(G)$ the weakly complex reduced bordism group of G of dimension m.

Let D_n be the dihedral group of order $2n$. In this paper the authors show a mapping splitting theorem for $\tilde{\Omega}_m^{SO}(D_n)$ and $\tilde{\Omega}_m^U(D_n)$ when n is odd and determine the additive structure of $\tilde{\Omega}_m^U(D_p)$, p an odd prime.

In the following sections we denote $\tilde{\Omega}_m^{SO}(G)$ or $\tilde{\Omega}_m^U(G)$ by $\tilde{\Omega}_m^L(G)$.

§ 1. A mapping splitting theorem for $\tilde{\Omega}_m^L(G)$.

Let G be a finite group and BG a classifying space of G. Let (M, f) be a G-manifold of dimension m. Then $\pi: M\rightarrow M/G$ is a principal G-bundle and there exists a classifying map $g: M/G\rightarrow BG$. The correspondence $[M, f]\mapsto[M/G, g]$ is well-defined homomorphism of $\tilde{\Omega}_m^L(G)$ into $\tilde{\Omega}_m^L(BG)$ and we have the following known result.

THEOREM 1.1 (Conner-Floyd [1]). *The above defined homomorphism* ρ_*: $\tilde{\Omega}_*^L(G)\rightarrow\tilde{\Omega}_*^L(BG)$ *is an isomorphism of degree* 0 *as an* Ω_*^L*-module homomorphism.*

Let $\alpha: H\rightarrow G$ be a homomorphism of finite groups and $B\alpha: BH\rightarrow BG$ a map induced by α. We denote by $\alpha_*: \tilde{\Omega}_*^L(BH)\rightarrow\tilde{\Omega}_*^L(BG)$ the homomorphism induced by $B\alpha$ and we also denote $\rho_*^{-1}\alpha_*\rho_*: \tilde{\Omega}_*^L(H)\rightarrow\tilde{\Omega}_*^L(G)$ by α_*. Then we have

$$(1.1)\qquad \alpha_*([M, f])=[G\times_H M, f_G], \qquad [M, f]\in\tilde{\Omega}_*^L(H)$$

where $G\times_H M=G\times M/(g, x)\sim(g\alpha(h)^{-1}, f(h, x))$, $g\in G$, $h\in H$ *and* $x\in M$ *on which* G *acts by the rule*

$$f_G(g, g'\times_H x)=gg'\times_H x, \qquad g, g'\in G,\ x\in M.$$

Let H be a normal subgroup of G, and put $\Gamma = G/H$. Let $i: H \to G$ be the inclusion of H and $\pi: G \to \Gamma$ the projection. Then we have

$$\pi_* i_* = 0 \tag{1.2}$$

because $(B\pi)(Bi) \simeq 0$.

For an H-manifold (M, f) and $g \in G$, (M, f^g) denotes an H-manifold consisting of the manifold M and the action f^g defined by

$$f^g(h, x) = f(g^{-1}hg, x), \quad x \in M,\ h \in H \text{ and } g \in G.$$

Then, since (M, f^g) and (M, f^{gh}) are diffeomorphic as H-manifolds for any $h \in H$ we can define an action of Γ on $\tilde{\Omega}^L_m(H)$ by

$$[M, f]^\gamma = [M, f^g] \quad \textit{for } [M, f] \in \tilde{\Omega}^L_m(H) \textit{ and } \gamma = gH \in \Gamma. \tag{1.3}$$

By $\tilde{\Omega}^L_m(H)^\Gamma$ we denote a subgroup of $\tilde{\Omega}^L_m(H)$ consisting of invariant elements under the action of Γ.

Next we define a homomorphism $t: \tilde{\Omega}^L_m(G) \to \tilde{\Omega}^L_m(H)$, which is called the transfer: Regard a G-manifold (M, f) as an H-manifold with the restriction f_H of f to H and put $t([M, f]) = [M, f_H]$.

When we denote the elements of Γ by $\gamma_1, \gamma_2, \cdots, \gamma_k$, we have the following

THEOREM 1.2 (Conner-Floyd [1]). *Let H be a normal subgroup of G, then*

$$ti_*([M, f]) = \sum_{j=1}^{k} [M, f]^{\gamma_j}$$

for any $[M, f] \in \tilde{\Omega}^L_m(H)$ and in particular, if $[M, f] \in \tilde{\Omega}^L_m(H)^\Gamma$, then $ti_*([M, f]) = k[M, f]$ for every $m \geqq 0$ and $L = SO$ or U.

THEOREM 1.3. *Let H be a normal and abelian subgroup of G, $k = [\Gamma : 1]$, $l = [H : 1]$ and assume that k and l are relatively prime. Then there exists a homomorphism*

$$\Phi^L_m : \tilde{\Omega}^L_m(H)^\Gamma \oplus \tilde{\Omega}^L_m(\Gamma) \longrightarrow \tilde{\Omega}^L_m(G)$$

and it is injective for every $m \geqq 0$ and $L = SO$ or U.

PROOF. First we define the homomorphism Φ^L_m. From Theorem (7.5) of Curtis and Reiner [3], we see that G is isomorphic to a semi-direct product $H \cdot \Gamma$ of H and Γ. Namely there exists a homomorphism $j: \Gamma \to G$ such that $\pi j = 1$. Then,

$$\pi_* j_* = 1 \tag{1.4}$$

because $(B\pi)(Bj) \simeq 1$.

Let $\tilde{i}_*: \tilde{\Omega}^L_m(H)^\Gamma \to \tilde{\Omega}^L_m(G)$ be the restriction of i_* to $\tilde{\Omega}^L_m(H)^\Gamma$. Then we define Φ^L_m by $\tilde{i}_* + j_*$.

Next we prove that Φ^L_m is injective. Suppose that $\Phi^L_m(\alpha, \beta) = 0$ for $(\alpha, \beta) \in \tilde{\Omega}^L_m(H)^\Gamma \oplus \tilde{\Omega}^L_m(\Gamma)$, then $\tilde{i}_*(\alpha) = -j_*(\beta)$. It follows from (1.2) and (1.4), that

$$\beta=\pi_* j_*(\beta)=-\pi_* i_*(\alpha)=0 \quad \text{and so} \quad i_*(\alpha)=\tilde{i}_*(\alpha)=0\,.$$

From Theorem 1.2 and $i_*(\alpha)=0$ we have

$$k\alpha=ti_*(\alpha)=0\,.$$

Since H is a finite abelian group of order l and $(l, k)=1$, the elements of $\tilde{H}_m(BH: Z)$ are divisible by k. So we see that the elements of $\tilde{\Omega}^L_m(H)$ are divisible by k using the bordism spectral sequence. Therefore, $\alpha=0$. Consequently, $(\alpha, \beta)=0$. This shows that Φ^L_m is injective. q. e. d.

§ 2. $\tilde{H}_*(D_n: Z)$ and $\tilde{\Omega}^L_*(D_n)$.

The dihedral group D_n, $n\geqq 3$, is a subgroup of the symmetric group S_n generated by the permutations $g=(1, 2, \cdots, n)$ and $t=\begin{pmatrix}1, & 2, \cdots, n\\ n, & n-1, \cdots, 1\end{pmatrix}$ with the relations $g^n=t^2=1$ and $tgt=g^{-1}$. In particular, $D_3=S_3$. Let Z_n and Z_2 be the cyclic subgroups of D_n generated by g and t respectively. Then Z_n is normal in D_n, the quotient group D_n/Z_n is isomorphic to Z_2 and $D_n=Z_n\cdot Z_2$.

From now we shall construct a classifying space of D_n. Let S^{2l+1} denote the unit $(2l+1)$-dimensional sphere in C^{l+1} with the coordinate $(z_0, z_1, \cdots, z_l)$ and let S^m denote the unit m-dimensional sphere in R^{m+1} with the coordinate $(x_0, x_1, \cdots, x_m)$. Consider the product space $S^{2l+1}\times S^m$ and define an action ϕ of D_n on $S^{2l+1}\times S^m$ by the rule

$$\phi((g^i, t^j), (z, x))=(\rho^i c^j(z), (-1)^j x)\,, \qquad z\in S^{2l+1},\ x\in S^m$$

where $\rho=\exp((2\pi\sqrt{-1})/n)$, $c(z)$ denotes the conjugate point of z and $-x$ the antipodal point of x and we define $c^{j+1}(z)$ and $(-1)^{j+1}x$ inductively by setting

$$c^{j+1}(z)=c(c^j(z)) \quad \text{and} \quad (-1)^{j+1}x=-((-1)^j x) \qquad \text{for } j\geqq 1\,.$$

Then we see that this action of D_n on $S^{2l+1}\times S^m$ is a free action. Denote by $D(l, m)$ the quotient space $(S^{2l+1}\times S^m)/D_n$. Then the direct limit space of $D(m, m)$ with respect to the natural inclusions $D(m, m)\subset D(m+1, m+1)$ becomes a classifying space of D_n, that is, $BD_n=\lim_m D(m, m)$.

Consider the product space of $L^l(n)\times S^m$ and define a homeomorphism

$$T: L^l(n)\times S^m \longrightarrow L^l(n)\times S^m$$

by $T([z], x)=([c(z)], -x)$, where $L^l(n)$ denotes the standard $(2l+1)$-dimensional lens space and $[z]$ the point in the quotient space corresponding to $z\in S^{2l+1}$. Let $D'(l, m)$ be the quotient space obtained from $L^l(n)\times S^m$ by identifying $([z], x)$ with $T([z], x)$. Then clearly we have

LEMMA 2.1.

$$D'(l, m) \approx D(l, m) \qquad \textit{for } l,\ m \geqq 1 .$$

Let $P^m(R)$ be the real m-dimensional projective space. When we define BZ_n, $n>2$, and BZ_2 by the direct limit spaces of $L^m(n)$ and $P^m(R)$ with respect to the natural inclusions $L^m(n) \subset L^{m+1}(n)$ and $P^m(R) \subset P^{m+1}(R)$ respectively, we have the maps

$$i:\ BZ_n \longrightarrow BD_n \quad \text{and} \quad j:\ BZ_2 \longrightarrow BD_n$$

by the definition of BD_n and moreover the inclusion maps

$$i_1:\ P^m(R) \longrightarrow BZ_2 \quad \text{and} \quad i_2:\ L^m(n) \longrightarrow BZ_n .$$

Let X be an oriented manifold. By $[X]$ we denote the fundamental class of X.

THEOREM 2.2. *If n is odd, we have*

$$\tilde{H}_{2q}(BD_n\,;\, Z) = 0\,, \qquad \tilde{H}_{4k-1}(BD_n\,;\, Z) = Z_2 \oplus Z_n$$

where Z_2 is generated by $(ji_1)_*([P^{4k-1}(R)])$ and Z_n by $(ii_2)_*([L^{2k-1}(n)])$, and

$$\tilde{H}_{4k-3}(BD_n\,;\, Z) = Z_2$$

where Z_2 is generated by $(ji_1)_*([P^{4k-3}(R)])$ for every $q \geqq 0$ and $k \geqq 1$.

PROOF. Let $e^{2k+\varepsilon}$, $\varepsilon=0$ or 1, denote an open $(2k+\varepsilon)$-cell of $S^{2l+1} \subset C^{l+1}$ defined by

$$e_j^{2k+1} = \{(z_0, \cdots, z_k, 0, \cdots, 0) \in S^{2l+1} \mid\ z_k \neq 0 \text{ and } 2\pi j/n < \arg z_k < 2\pi(j+1)/n\}$$

and

$$e_j^{2k} = \{(z_0, \cdots, z_k, 0, \cdots, 0) \in S^{2l+1} \mid\ z_k \neq 0,\ \arg z_k = 2\pi j/n\}$$

for $0 \leqq j \leqq n-1$ and $0 \leqq k \leqq l$. Let $\phi_1 : S^{2l+1} \to L^l(n)$ denote the projection and put $C_r = \phi_1(e_0^r)$, $0 \leqq r \leqq 2l+1$, then $C_r = \phi_1(e_0^r) = \phi_1(e_1^r) = \cdots = \phi_1(e_{n-1}^r)$.

Let D_j^+ (D_j^-) be an open j-cell of $S^m \subset R^{m+1}$ defined by $x_{j+1} = x_{j+2} = \cdots = x_m = 0$, $x_j > 0$ $(x_j < 0)$. Then $\{C_i \times D_j^{\pm} \mid i=0, 1, \cdots, l\,;\ j=0, 1, \cdots, m\}$ forms an oriented cellular decomposition of $L^l(n) \times S^m$ whose boundary relations are given by

$$\partial(C_{2i+1} \times D_j^{\pm}) = (-1)^{j+1} C_{2i+1} \times (D_{j-1}^+ - D_{j-1}^-)\,, \qquad 0 \leqq i \leqq 1,\ 1 \leqq j \leqq m\,,$$

$$\partial(C_{2i} \times D_j^{\pm}) = nC_{2i-1} \times D_j^{\pm} + (-1)^j C_{2i} \times (D_{j-1}^+ - D_{j-1}^-)\,, \qquad 1 \leqq i \leqq l,\ 1 \leqq j \leqq m\,,$$

$$\partial(C_{2i+1} \times D_0^{\pm}) = 0\,, \qquad 0 \leqq i \leqq l\,,$$

$$\partial(C_{2i} \times D_0^{\pm}) = nC_{2i-1} \times D_0^{\pm}\,, \qquad 1 \leqq i \leqq l\,,$$

$$\partial(C_0 \times D_j^{\pm}) = (-1)^j C_0 \times (D_{j-1}^+ - D_{j-1}^-)\,, \qquad 1 \leqq j \leqq m\,.$$

The homeomorphism T is a cellular map with respect to the above cellular decomposition and satisfies

$$T(C_{2i+\varepsilon}\times D_j^{\pm})=(-1)^{i+j+\varepsilon}C_{2i+\varepsilon}\times D_j^{\mp}$$

for $\varepsilon=0, 1$, $0\leqq i\leqq l$ and $0\leqq j\leqq m$.

Let ϕ_2: $L^l(n)\times S^m\to D(l, m)$ denote the composition of the projection $L^l(n)\times S^m\to D'(l, m)$ and the homeomorphism $D'(l, m)\to D(l, m)$ in Lemma 2.1 and write $(C_i, D_j)=\phi_2(C_j, D_j^+)$. Then $\{(C_i, D_j)\mid i=0, 1, \cdots, l;\ j=0, 1, \cdots, m\}$ is a cellular decomposition of $D(l, m)$ whose boundary relations are given by

$$\partial(C_{2i+1}, D_j)=((-1)^i+(-1)^{j+1})(C_{2i+1}, D_{j-1}), \quad 0\leqq i\leqq l,\ 1\leqq j\leqq m,$$
$$\partial(C_{2i}, D_j)=n(C_{2i-1}, D_j)+((-1)^i+(-1)^j)(C_{2i}, D_{j-1}), \quad 1\leqq i\leqq l,\ 1\leqq j\leqq m,$$
$$\partial(C_{2i+1}, D_0)=0, \quad 0\leqq i\leqq l,$$
$$\partial(C_{2i}, D_0)=n(C_{2i-1}, D_0), \quad 1\leqq i\leqq l,$$
$$\partial(C_0, D_j)=(1+(-1)^j)(C_0, D_{j-1}), \quad 1\leqq j\leqq m.$$

From this formulas we obtain

$$\tilde{H}_{2q}(BD_n; Z)=0 \quad\text{and}\quad \tilde{H}_{4k-1}(BD_n; Z)=Z_2\oplus Z_n$$

where Z_2 is generated by (C_0, D_{4k-1}) and Z_n by (C_{4k-1}, D_0) and

$$\tilde{H}_{4k-3}(BD_n; Z)=Z_2$$

where Z_2 is generated by (C_0, D_{4k-3}). Furthermore (C_0, D_{2k-1}) and (C_{4k-1}, D_0) are $(ji_1)_*([P^{2k-1}(R)])$ and $(ii_2)_*([L^{2k-1}(n)])$ respectively. This completes the proof.

Denote by

$$\mu^L:\ \tilde{\Omega}_*^L(G)\longrightarrow H_*(BG; Z)$$

the Thom homomorphism for $L=SO$ or U defined by

$$\mu^L([M, f])=g_*([M/G])$$

where $[M/G]$ denotes the fundamental class of the quotient manifold M/G and g a classifying map of the principal G-bundle $M\to M/G$. Define an action of $Z_n\subset D_n$ on $S^{2m-1}\subset C^m$ and an action of $Z_2\subset D_n$ on $S^{l-1}\subset R^l$ by

$$T_n(g, (z_0, z_1, \cdots, z_{m-1}))=(\rho z_0, \rho z_1, \cdots, \rho z_{m-1}), \quad \rho=\exp(2\pi\sqrt{-1}/n)$$

and $T_2(t, (x_0, x_1, \cdots, x_{l-1}))=(-x_0, -x_1, \cdots, -x_{l-1})$ respectively where g and t are the generators of D_n. Then $S^{2m-1}/Z_n=L^{m-1}(n)$ and $S^{l-1}/Z_2=P^{l-1}(R)$. We recall the following

THEOREM 2.3 (Conner-Floyd [1], Conner-Smith [2]).

(i) *$\mu^L:\ \tilde{\Omega}_*^L(Z_n)\to\tilde{H}_*(BZ_n; Z)$ is onto for every $n\geqq 2$ and*

$$\mu^L([S^{2i-1}, T_2])=i_{1*}([P^{2i-1}(R)]) \quad \textit{for } n=2$$

and

$$\mu^L([S^{2i-1}, T_n])=i_{2*}([L^{i-1}(n)]) \quad \textit{for } n>2.$$

(ii) $\{[S^{2i-1}, T_n]; i\geqq 1\}$ *forms a generating set for* $\tilde{\Omega}_*^U(Z_n)$ *as an* $\tilde{\Omega}_*^U$*-module for* $n\geqq 2$.

Then we see that

PROPOSITION 2.4. *The Thom homomorphism*

$$\mu^L: \tilde{\Omega}_*^L(D_n) \longrightarrow \tilde{H}_*(BD_n; Z)$$

is onto when n is odd for $L=SO$ *or* U.

PROOF. Using the notation of Theorem 2.3 $[S^{4i-1}, T_n]+[S^{4i-1}, T_n]^t$ is contained in $\tilde{\Omega}_{4i-1}^L(Z_n)^{Z_2}$ for each $i\geqq 1$ since $t^2=1$ where t is the generator of Z_2 and moreover it is divisible by 2 as n is odd. Consider the images of $j_*([S^{2i-1}, T_2])$ and $i_*((1/2)([S^{4i-1}, T_n]+[S^{4i-1}, T_n]^t))$ by μ^L. Then, by Theorem 2.3 and the naturality of μ^L

$$\mu^L j_*([S^{2i-1}, T_2])=(ji_1)_*([P^{2i-1}(R)])$$

and

$$\mu^L i_*((1/2)([S^{4i-1}, T_n]+[S^{4i-1}, T_n]^t))=(ii_2)_*([L^{2i-1}(n)])$$

for $i\geqq 1$. Therefore, from Theorem 2.2 we get Proposition 2.4.

According to Proposition 2.4, we obtain the following

COROLLARY 2.5. *When n is odd,* $\{j_*([S^{2i-1}, T_2]), i_*((1/2)([S^{4i-1}, T_n]+[S^{4i-1}, T_n]^t)); i\geqq 1\}$ *forms a generating set for* $\tilde{\Omega}_*^U(D_n)$ *as an* $\tilde{\Omega}_*^U$*-module.*

From Theorem 1.3 and Corollary 2.5 we obtain immediately the following

THEOREM 2.6. *If n is odd, then*

$$\Phi_m^U: \tilde{\Omega}_m^U(Z_n)^{Z_2}\oplus\tilde{\Omega}_m^U(Z_2) \longrightarrow \tilde{\Omega}_m^U(D_n)$$

is an isomorphism for every $m\geqq 0$.

§ 3. The structure of $\tilde{\Omega}_*^U(Z_p)^{Z_2}$, p an odd prime.

In this section we suppose that p is an odd prime. Consider an element

$$L^k=[S^{2k+1}, T_p]+[S^{2k+1}, T_p]^t$$

where t is the generator of Z_2. The element L^k belongs to $\tilde{\Omega}_{2k+1}^U(Z_p)^{Z_2}$.

Denote by $\Gamma_*(p)$ the polynomial subring in $\Omega_*^U=Z[x_1, x_2, \cdots]$ which is generated by x_i $(i\neq p-1)$.

PROPOSITION 3.1. *Suppose that*

$$\sum_{k=0}^{n}\alpha_{2t+4n-4k}L^{2k+1}=0, \qquad \alpha_{2t+4n-4k}\in\Gamma_{2t+4n-4k}(p).$$

Then,

$$\alpha_{2t+4n-4k}\in p^{\left[\frac{2k+1}{p-1}\right]+1}\Gamma_{2t+4n-4k}(p)$$

where [ ] *is the Gaussian symbol.*

PROOF. Consider the Thom homomorphism

$$\mu^U : \tilde{\Omega}_*^U(Z_p) \longrightarrow \tilde{H}_*(BZ_p\,;\,Z)$$

where BZ_p is the classifying space of Z_p. It is easy to see that $\mu^U(L^m)=\{1+(-1)^{m+1}\}g$, where g is the generator of $\tilde{H}_{2m+1}(BZ_p\,;\,Z)$. According to Kamata [4], we have the following representation

$$L^{2k+1}=\sum_{j=0}^{2k+1} a_j[S^{2j+1},\,T_p]\,, \quad a_j\in\Gamma_{4k-2j+2}(p)\cdots(1)\,.$$

Applying the homomorphism μ^U to the above equation, we have

$$a_{2k+1}\equiv 2 \qquad (\mathrm{mod}\, p)\cdots(2)\,.$$

Using (1), we describe the equation $\sum_{k=0}^{n}\alpha_{2t+4n-4k}L^{2k+1}=0$ as follows:

$$a_{2n+1}\alpha_{2t}[S^{4n+3},\,T]+\sum_{j=0}^{2n} b_j[S^{2j+1},\,T_p]=0$$

where the coefficient b_j belongs to $\Gamma_*(p)$.

Therefore it follows from Kamata [4] and (2) that $\alpha_{2t}\in p^{[\frac{2n+1}{p-1}]+1}\Gamma_*(p)$. Since L^k has order of $p^{[\frac{k}{p-1}]+1}$, by induction, Proposition 3.1 follows.

We put

$$W_\varepsilon^U(n)=\sum_{k=0}^{n}\Gamma_{2\varepsilon+4n-4k}(p)/p^{[\frac{2k+1}{p-1}]+1}\Gamma_{2\varepsilon+4n-4k}(p)$$

where $\Gamma_i(p)=0,\ i<0$.

THEOREM 3.2. *The homomorphism*

$$\Theta:\ W_\varepsilon^U(n)\longrightarrow \tilde{\Omega}_{4n+2\varepsilon+3}^U(Z_p)^{Z_2}$$

given by

$$\Theta(\sum_{k=0}^{n}\alpha_{2\varepsilon+4n-4k})=\sum_{k=0}^{n}\alpha_{2\varepsilon+4n-4k}L^{2k+1}$$

is isomorphic for $\varepsilon=0,\ -1$.

PROOF. It follows from Proposition 3.1 that Θ is injective. We compute the order of $\tilde{\Omega}_{4n+3}^U(Z_p)^{Z_2}$ and $\tilde{\Omega}_{4n+1}^U(Z_p)^{Z_2}$. Consider the spectral sequence of $\tilde{\Omega}_*^U(BD_p)$ $E_{s,t}^r$ with $E_{s,t}^2=\tilde{H}_s(BD_p\,;\,\Omega_t^U)$. From Proposition 2.4, the spectral sequence collapses. Consider the filtration of $\tilde{\Omega}_*^U(BD_p)$ $J_{s,t}$ with $J_{s,t}/J_{s-1,t+1}\cong E_{s,t}^\infty\cong\tilde{H}_s(BD_p\,;\,\Omega_t^U)$. Denote by σ_t the number of partitions of t. Then we have

$$J_{4s+3,2t}/J_{4s+1,2t+2}=\overbrace{Z_p+\cdots+Z_p}^{\sigma_t}+\overbrace{Z_2+\cdots+Z_2}^{\sigma_t}$$

and

$$J_{4s+1,2t}/J_{4s-1,2t+2}=\overbrace{Z_2+\cdots+Z_2}^{\sigma_t}\,.$$

Therefore, the order of $\tilde{\Omega}^U_{4n+3}(BD_p)$ is $2^a p^b$, $a=\sum_{j=0}^{2n+1}\sigma_j$ and $b=\sum_{j=0}^{n}\sigma_{2j}$, and the order of $\tilde{\Omega}^U_{4n+1}(BD_p)$ is $2^c p^d$, $c=\sum_{j=0}^{2n}\sigma_j$ and $d=\sum_{j=0}^{n-1}\sigma_{2j+1}$.

On the other hand the order of $\tilde{\Omega}^U_{2m+1}(BZ_2)$ is 2^f, $f=\sum_{j=0}^{m}\sigma_j$. From Theorem 2.6, it follows that $\tilde{\Omega}^U_{4n+3}(Z_p)^{Z_2}$ and $\tilde{\Omega}_{4n+1}(Z_p)^{Z_2}$ have order of p^b and p^d respectively. Denote by τ_t the number of partitions of t, containing no $p-1$. $W_0(n)$ and $W_{-1}(n)$ have order of p^u and p^v respectively, where

$$u=\sum_{k=0}^{n}\left\{\left[\frac{2k+1}{p-1}\right]+1\right\}\tau_{2(n-k)},$$

$$v=\sum_{k=0}^{n-1}\left\{\left[\frac{2k+1}{p-1}\right]+1\right\}\tau_{2(n-k)-1}.$$

Using the same method as Conner and Floyd [1], p. 97, we have $b=u$ and $d=v$. q. e. d.

From Theorem 2.6 and Theorem 3.2, we can determine the additive structure of $\tilde{\Omega}^U_*(D_p)$, p an odd prime.

Department of Mathematics
Faculty of General Education
Kyushu University
Ropponmatsu, Fukuoka
Japan

Department of Mathematics
Faculty of Science
Osaka City University
Sugimoto-cho, Sumiyoshi-ku
Osaka, Japan

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