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Primitive extensions of rank 4 of multiply transitive permutation groups

(Part I. The case where all the orbits are self-paired)

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Introduction.

In [1] the author has determined the permutation groups which are primitive extensions of rank 3 of 4-ply transitive permutation groups. This note is a continuation of [1], and here we consider primitive extensions of rank 4 of multiply (5-ply) transitive permutation groups. Here we say that a permutation group $(\mathfrak{G}, \Omega)$ is a primitive extension of rank r of a (transitive) permutation group (G, Δ) if the following conditions are satisfied: (i) $\mathfrak{G}$ is primitive and of rank r on the set Ω, and (ii) there exists an orbit $\Delta(a)$ of the stabilizer $\mathfrak{G}_a$ $(a \in \Omega)$ such that the action of $\mathfrak{G}_a$ on $\Delta(a)$ is faithful and that $(\mathfrak{G}_a, \Delta(a))$ and (G, Δ) are isomorphic as permutation groups.

In this note we will prove the following theorem:

THEOREM 1. *Let (G, Δ) be a 5-ply transitive permutation group. If (G, Δ) has a primitive extension of rank 4 $(\mathfrak{G}, \Omega)$ such that the orbits of $\mathfrak{G}_a$ $(a \in \Omega)$ on Ω are all self-paired, then* (i) $|\Delta|=7$ *and* $G=S_7$ *or* A_7 *(symmetric and alternating groups on 7 letters, respectively)*[1], *or* (ii) $|\Delta|=379, 1379, 3404, 6671, 18529$ *or* 166754 *and* $G \neq S_{|\Delta|}, A_{|\Delta|}$.

In the present note we devote ourselves to the case where all orbits are self-paired. The remaining case where there exists non-self-paired orbit will be treated in a subsequent paper. There it will be shown that any 4-ply transitive permutation group (G, Δ) has no primitive extension of rank 4 $(\mathfrak{G}, \Omega)$ such that there exist non-self-paired orbits. Thus the determination of primitive extensions of rank 4 of 5-ply transitive permutation group is almost completed.

Our main idea of the proof of Theorem 1 is indebted to the concept of intersection matrices due to D. G. Higman [3], and is also indebted to some results of W. A. Manning (cf. P. J. Cameron [2]).

Just before this work has been done, S. Iwasaki has determined the pri-

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1) In these cases (G, Δ) have indeed primitive extensions of rank 4 $(\mathfrak{G}, \Omega)$ with regular normal subgroup of order 64.

mitive extensions of rank 4 of the alternating groups (A_n, Δ), $|\Delta|=n$ (cf. [**5**], [**6**]). Our result obtained here is a sort of generalization of that of [**6**]. The author thanks Mr. S. Iwasaki for kindly making the preprint of [**5**] available before publication and giving him the valuable remarks by reading the manuscript. Although, we assume no familiarity with [**5** and **6**] to the reader of the present note.

§ 1. Notations and preliminary results.

A) *Notations.*

We fix the following notation throughout this note. $(\mathfrak{G}, \Omega)$ is a primitive extension of rank 4 of a 5-ply transitive permutation group (G, Δ). That is to say, $\mathfrak{G}$ is a primitive permutation group on a set Ω such that the orbits of the stabilizer $\mathfrak{G}_a$ $(a \in \Omega)$ are $\{a\}$, $\Delta(a)$, $\Gamma(a)$ and $\Lambda(a)$ with subdegrees 1, k, l and m respectively, and moreover $\mathfrak{G}_a$ is faithful on $\Delta(a)$ and the permutation group $(\mathfrak{G}_a, \Delta(a))$ is identified with (G, Δ). Henceforth we assume that $\Delta(a)$, $\Gamma(a)$ and $\Lambda(a)$ are all self-paired. We choose the orbits so that $\Delta(a)^g = \Delta(a^g)$, $\Gamma(a)^g = \Gamma(a^g)$ and $\Lambda(a)^g = \Lambda(a^g)$ hold for any $a \in \Omega$ and $g \in \mathfrak{G}$. Let us set $\Gamma_0(a)=\{a\}$, $\Gamma_1(a)=\Delta(a)$, $\Gamma_2(a)=\Gamma(a)$, $\Gamma_3(a)=\Lambda(a)$ and $\mu_{ij}^{(\alpha)}=|\Gamma_\alpha(b) \cap \Gamma_i(a)|$ for $b \in \Gamma_j(a)$. For the fundamental properties and the relations among them, see D. G. Higman [3], (4.1) and (4.2). Also see [2] and [3] for the geometric (graph theoretical) interpretation of the intersection numbers $\mu_{ij}^{(\alpha)}$.

We use the conventional notation $\mu_{ij}=\mu_{ij}^{(1)}$ and

$$M=M_1=(\mu_{ij})=\begin{bmatrix} 0 & 1 & 0 & 0 \\ k & \mu_{11} & \mu_{12} & \mu_{13} \\ 0 & \mu_{21} & \mu_{22} & \mu_{23} \\ 0 & \mu_{31} & \mu_{32} & \mu_{33} \end{bmatrix}.$$

Sometimes we write $\mu=\mu_{12}$, $\nu=\mu_{13}$.

Especially the following relations hold among the intersection numbers of M $(=M_1)$ (these are all contained in (4.1) and (4.2) in [3]).

$$(1.1) \qquad \begin{cases} 1+\mu_{11}+\mu_{21}+\mu_{31}=\mu_{12}+\mu_{22}+\mu_{32}=\mu_{13}+\mu_{23}+\mu_{33}=k\,, \\ k\mu_{21}=l\mu_{12}\,, \qquad k\mu_{31}=m\mu_{13}\,, \qquad l\mu_{32}=m\mu_{23}\,. \end{cases}$$

B) *Preliminary results.*

LEMMA 1. (i) *If the intersection matrix M is tridiagonal (i.e., of maximal diameter), then M has 4 distinct eigen values, say $\alpha_0=k$, α_1, α_2 and α_3 (all of which are real).*

(ii) *Moreover, let f_1, f_2, f_3 be the degree of the non-identity irreducible characters of $\mathfrak{G}$ appearing in the permutation character on Ω. Then the following relations hold:*

$$f_1+f_2+f_3=k+l+m\ ,$$

$$\alpha_1 f_1+\alpha_2 f_2+\alpha_3 f_3=-k\ ,$$

$$\alpha_1^2 f_1+\alpha_2^2 f_2+\alpha_3^2 f_3=k(l+m+1)\ .$$

(iii) *Moreover, if k, l, m are not all equal, then at least one of the eigen values α_1, α_2, α_3 is an integer.*

PROOF. The assertion (i) is already stated in D. G. Higman [3], page 35, and so we omit the proof since it is easily done and well known.

(ii) The first equation is clear, and the second and the third equation is immediately obtained by calculating the trace of the incidence matrix B_1 and B_1^2 respectively, where $B_1=(\beta_{ab}^{(1)})$, $\beta_{ab}=1$ if $a\in\Delta(b)$ and $\beta_{ab}=0$ otherwise (cf. [3], (4.13), or see [5], Lemma 2.3).

The assertion (iii) is already proved in [5], Lemma 2.3. Here we look (iii) from a slightly more general view point, and we have the following assertion:

(1.2) PROPOSITION. *Let $\mathfrak{G}$ be a transitive permutation group of any rank on a set Ω, and let M be an intersection matrix for some orbit of length k of $\mathfrak{G}_a$ ($a\in\Omega$). If the eigenvalues of M are all distinct and the characteristic polynomial $m(x)=\det(M-Ix)$ is a product of $(x-k)$ and the irreducible polynomial over the rational field, then the orbits ($\neq\{a\}$) of $\mathfrak{G}_a$ are all of equal length.*

Lemma 1 (iii) is immediately obtained from Proposition (1.2). Proposition (1.2) is proved as follows: since the polynomial $m(x)/(x-k)$ is irreducible, all eigenvalues $\neq k$ of M are mutually conjugate by the action of the Galois group over the rational field. By Theorem (5.5) in [3] (or by (4.13) in [3]), the degree of irreducible characters corresponding to conjugate eigenvalues are equal. Thus we immediately have the assertion from Theorem 30.2 in [7], q. e. d.

LEMMA 2. (i) *Let G be a 2-ply transitive permutation group on a set Σ, $|\Sigma|=n$, and let H be a subgroup of G of index n. If H is transitive on Σ, then H is 2-ply transitive on Σ.*

(ii) *Let G be a 3-ply transitive permutation group on a set Σ, $|\Sigma|=n$, and let H be a subgroup of G of index n. If H is transitive on Σ, then H is (5/2)-ply transitive.*

(iii) *Let G be a 4-ply transitive permutation group on a set Σ, $|\Sigma|=n$, and let H be a subgroup of G of index n. If H is transitive on Σ, then H is 3-ply transitive except the only one case $|\Sigma|=6$ and $G\cong A_6$.*

(iv) *Let G be a 5-ply transitive permutation group on a set Σ, $|\Sigma|=n$, and let H be a subgroup of G of index n. If H is transitive on Σ, then H is 3-ply transitive.*

PROOF. (i) Let $a\in\Sigma$. Then $G_a\cap H$ is a subgroup of index n of the transitive permutation group $(G_a, \Sigma-\{a\})$ of degree $n-1$. Therefore $G_a \cap H$ is transitive on $\Sigma-\{a\}$ by Theorem 17.3 in Wielandt [7]. Thus (i) is proved.

(ii) By (i), H is 2-ply transitive. Let $a, b \in \Sigma$, $a \neq b$. Then $G_{a,b} \cap H$ is a subgroup of index n of the transitive permutation group $(G_{a,b}, \Sigma-\{a, b\})$ of degree $n-2$. Therefore either $G_{a,b} \cap H$ is transitive on $\Sigma-\{a, b\}$ (thus H is 3-ply transitive), or $\Sigma-\{a, b\}$ is decomposed into two orbits of $G_{a,b} \cap H$ of equal length, by Lemma 17.1 in Wielandt [7]. Thus (ii) is proved.

(iii) By (ii), we may assume that the orbits on $\Sigma-\{a, b\}$ by the action of $G_{a,b} \cap H$ is two and both are of length $(n-2)/2$. Let $a, b, c \in \Sigma$, $a \neq b \neq c \neq a$. $G_{a,b,c} \cap H$ is a subgroup of index $n/2$ of the transitive permutation group $(G_{a,b,c}, \Sigma-\{a, b, c\})$ of degree $n-3$. Therefore, since the greatest common divisor of $n/2$ and $n-3$ divides 3, by Lemma 17.1 in Wielandt [7], all orbits of $G_{a,b,c} \cap H$ are of length $(i/3)(n-3)$, $i=1, 2, 3$. While $G_{a,b,c} \cap H$ has a union of orbits whose total length is $(n-2)/2$. Thus we have a contradiction unless $n=6$, and we immediately have the assertion.

(iv) is obvious from (iii), q. e. d.

§ 2. Proof of Theorem 1.

(Step 1. Determination of the possible intersection matrix M.)

From now on we assume that there exists $(\mathfrak{G}, \Omega)$ which satisfies the assumptions of Theorem 1. Henceforth we always assume that $k \geqq 5$.

The main object of this section is to show that the intersection matrix M of $(\mathfrak{G}, \Omega)$ is of very restricted structure. That is, M must be one of those listed in Main Proposition (2.12) given below.

(2.1) PROPOSITION. *Either* μ_{12} $(=\mu)$ *or* μ_{13} *is different from* 0 *and we may assume that* $\mu_{12} \neq 0$.

PROOF. Because otherwise $(\mathfrak{G}, \Omega)$ is imprimitive (cf. [3], (4.8)). If $\mu_{13} \neq 0$, we have only to interchange the role of $\Gamma(a)$ and $\Lambda(a)$, q. e. d.

(2.2) PROPOSITION. $\mu_{11}=0$, $\mu_{31}=0$ *and* $\mu_{21}=k-1$, *and* $1 \leqq \mu_{12}(=k(k-1)/l) < k-1$. *Moreover* $\mu_{13}=0$.

Proof is immediate from the doubly (5-ply) transitivity of $(\mathfrak{G}_a, \Delta(a))$ $(\cong (G, \Delta))$ and Theorem 1 of [2] together with the parameter relations (1.1), q. e. d.

(2.3) PROPOSITION. μ_{12} $(=\mu)=1$ *or* 2.

PROOF. Since $\Delta(a)$ is self-paired, there exists an element $x \in \mathfrak{G}$ which interchanges a and b, where $b \in \Delta(a)$. We denote by σ the automorphism of $\mathfrak{G}_{a,b}$ induced by the conjugation by x. Then there exist a point $c \in \Delta(a)-\{b\}$ and a point $d \in \Gamma(a)$ such that $(\mathfrak{G}_{a,b,c})^{\sigma} \leqq \mathfrak{G}_{a,d}$, since there exist $c \in \Delta(a)$ and $d \in \Gamma(a)$ such that $c^x=d$ because of μ_{21} and $\mu_{12} \neq 0$. While $(\mathfrak{G}_{a,b,c})^{\sigma}$ is a subgroup of index $|\Delta|-1$ of the 3-ply (4-ply) transitive permutation group $(\mathfrak{G}_{a,b}, \Delta(a)-\{b\})$. Thus by Satz 3 of N. Ito [4], either

(1) $(\mathfrak{G}_{a,b,c})^{\sigma}$ is transitive on $\Delta(a)-\{b\}$, or

(2) $(\mathfrak{G}_{a,b,c})^{\sigma}=\mathfrak{G}_{a,b,e}$ for some $e\in\Delta(a)-\{b\}$.

Firstly, let us assume that the case (1) holds: then the orbits on $\Delta(a)$ by the action of $\mathfrak{G}_{a,d}$ $(\geqq(\mathfrak{G}_{a,b,c})^{\sigma})$ are either $\Delta(a)$ itself, or $\{b\}$ and $\Delta(a)-\{b\}$. Therefore either $\mu=1$, $\mu=k-1$, $\mu=k$ or $\mu=0$. Thus we have $\mu=1$ by Proposition (2.2). Secondly let us assume that the case (2) holds: then there exists an orbit by the action of $\mathfrak{G}_{a,d}$ $(\geqq(\mathfrak{G}_{a,b,c})^{\sigma})$ on $\Delta(a)$ which contains the subset $\Delta-\{b, e\}$. Therefore either $\mu=1$, $\mu=2$, $\mu=k-2$, $\mu=k-1$, $\mu=k$ or $\mu=0$. The last three cases are impossible by Proposition (2.2), and $\mu=k-2$ is also impossible, otherwise by the relation (1.1) $l=k(k-1)/(k-2)$ is not an integer for $k\geqq 5$, a contradiction, q. e. d.

(2.4) PROPOSITION. *One of the following cases* (I), (II) *and* (III) *holds* (*where* $b, c\in\Delta(a)$, $b\neq c$):

(I) $\mu=1$ *and* $(\mathfrak{G}_{a,b,c})^{\sigma}$ *is transitive on* $\Delta(a)-\{b\}$, *moreover* $\mathfrak{G}_{a,d}$ $(d\in\Gamma(a))$ $=(\mathfrak{G}_{a,b,c})^{\sigma}$.

(II) $\mu=1$ *and* $(\mathfrak{G}_{a,b,c})^{\sigma}=\mathfrak{G}_{a,b,e}$ *for some* $e\in\Delta(a)-\{b\}$, *moreover* $\mathfrak{G}_{a,d}$ $(d\in\Gamma(a))=\mathfrak{G}_{a,b,e}$.

(III) $\mu=2$ *and* $(\mathfrak{G}_{a,b,c})^{\sigma}=\mathfrak{G}_{a,b,e}$ *for some* $e\in\Delta(a)-\{b\}$, *moreover* $\mathfrak{G}_{a,d}$ $(d\in\Gamma(a))=\mathfrak{G}_{a,\{b,e\}}$.

Proof of this proposition is already contained in the proof of Proposition (2.3), q. e. d.

(2.5) PROPOSITION. $\mu_{23}\neq 0$, *and* $\mu_{32}\neq 0$.

PROOF. Because otherwise $(\mathfrak{G}, \Omega)$ is imprimitive (cf. [3], (4.8)), since μ_{13} and μ_{31} are zero, q. e. d.

(2.6) PROPOSITION. *If the case* (I) (*in Proposition* (2.4)) *holds, then* $\mu_{22}=0$, *and so* $\mu_{32}=k-1$.

PROOF. Since $\mathfrak{G}_{a,d}$ is transitive on $\Delta(a)-\{b\}$, we have either $\mu_{22}=0$ or $k-1$. However $\mu_{22}=k-1$ is impossible, because otherwise $\mu_{32}=0$ and it contradicts to Proposition (2.5), q. e. d.

(2.7) PROPOSITION. *If the case* (I) *holds, then* $\mu_{23}=1, 2, k-2, k-1$ *or* k.

PROOF. Since $\Gamma(a)$ is self-paired, there exists an element $y\in\mathfrak{G}$ which interchanges a and d, where $d\in\Gamma(a)$. We denote by τ the automorphism of $\mathfrak{G}_{a,d}$ induced by the conjugation by y. Then there exist a point $b\in\Delta(a)\cap\Delta(d)$ $(d\in\Gamma(a))$, a point $c\in\Delta(a)$ and a point $f\in\Lambda(a)$ such that $(\mathfrak{G}_{a,d,c})^{\tau}\leqq\mathfrak{G}_{a,f}$, since $\mu_{32}\neq 0$. Let $c=b$. Then the orbits on $\Delta(a)$ by the action of $(\mathfrak{G}_{a,d,c})^{\tau}$ is $\{b\}$ and $\Delta(a)-\{b\}$. Therefore $\mu_{23}=0, 1, k-1$ or k. Let $c\neq b$. Then $(\mathfrak{G}_{a,d,c})$ is a subgroup of index $k-1$ of the group $\mathfrak{G}_{a,d}$, and the group $\mathfrak{G}_{a,d}$ is 3-ply transitive on the set $\Delta(a)-\{b\}$, unless $(\mathfrak{G}_{a,b}, \Delta(a)-\{b\})\cong(A_6$, on 6 letters).

Because $\mathfrak{G}_{a,d}$ is a subgroup of index $k-1$ of the 4-ply transitive group $(\mathfrak{G}_{a,b}, \Delta(a)-\{b\})$ of degree $k-1$, the assertion is true by Lemma 2 (iii). Thus if $(\mathfrak{G}_{a,d}, \Delta(a)-\{b\})$ is 3-ply transitive then by Satz 3 of N. Ito [4] and even if $(\mathfrak{G}_{a,b}, \Delta(a)-\{b\})\cong(A_6$, on 6 letters) then by the direct consideration, we have either (1) $(\mathfrak{G}_{a,d,c})^{\tau}$ is transitive on $\Delta(a)-\{b\}$, or (2) $(\mathfrak{G}_{a,d,c})^{\tau}$ fixes two points, say b and e, and transitive on remaining points $\Delta(a)-\{b, e\}$. Thus in any case, $(\mathfrak{G}_{a,d,c})^{\tau}$ has an orbit of length $\geqq k-2$. Therefore we have either $\mu_{23}=0, 1, k-2, k-1$ or k. However $\mu_{23}\neq 0$, by Proposition 2.5, q. e. d.

(2.8) PROPOSITION. *Let the case* (II) *hold. Then* $\mu_{22}=0$, 1 *or* $k-2$.

PROOF. Since $\mathfrak{G}_{a,d}=\mathfrak{G}_{a,b,e}$ $(d\in\Gamma(a))$ for some b, $e\in\Delta(a)$, $b\neq e$, and $\mathfrak{G}_{a,d}$ has an orbit of length $\geqq k-2$ on $\Delta(a)$, we have $\mu_{22}=0, 1, k-2$ or $k-1$. However $\mu_{22}\neq k-1$, since otherwise we have a contradiction to Proposition (2.3) and (2.5), q. e. d.

(2.9) PROPOSITION. *Let the case* (II) *hold. Then* $\mu_{23}=1, 2, 3, k-3, k-2, k-1$ *or* k.

PROOF. We may assume that $\mathfrak{G}_{a,f}\geqq(\mathfrak{G}_{a,d,h})^{\tau}$, since $\mu_{32}\neq 0$, where $d\in\Gamma(a)$, $\mathfrak{G}_{a,d}=\mathfrak{G}_{a,b,e}$ for some $b, e\in\Delta(a)$, $h\in\Delta(a)$. $(\mathfrak{G}_{a,d,h})^{\tau}$ is a subgroup of index no more than $k-2$ of the 3-ply transitive permutation group $\mathfrak{G}_{a,b,e}$ of degree $k-2$. Therefore by Satz 3 of N. Ito [4], $(\mathfrak{G}_{a,d,h})^{\tau}$ has on $\Delta(a)$ an orbit of length $\geqq k-3$. Thus we have either $\mu_{23}=0, 1, 2, 3, k-3, k-2, k-1$ or k. However $\mu_{23}\neq 0$ (cf. [3], (4.8)), q. e. d.

(2.10) PROPOSITION. *Let the case* (III) *hold. Then* $\mu_{22}=0$.

Proof is quite the same as that of Proposition (2.8), q. e. d.

(2.11) PROPOSITION. *Let the case* (III) *hold. Then* $\mu_{23}=1, 2, 3, k-3, k-2, k-1$ *or* k.

Proof is quite the same as that of Proposition (2.9), q. e. d.

Combining Propositions (2.1)~(2.11) so far obtained, we have the following main proposition of this section.

(2.12) MAIN PROPOSITION. *As for the permutation group* $(\mathfrak{G}, \Omega)$ *and the intersection matrix M, one of the following cases holds (where* $b, c\in\Delta(a), b\neq c$*):*

(I) $\mu=1$ *and* $(\mathfrak{G}_{a,b,c})^{\sigma}$ *is transitive on* $\Delta(a)-\{b\}$. *Moreover the intersection matrix M is as follows:*

$$M=\begin{bmatrix} 0 & 1 & 0 & 0 \\ k & 0 & 1 & 0 \\ 0 & k-1 & 0 & \alpha \\ 0 & 0 & k-1 & k-\alpha \end{bmatrix}, \qquad \alpha\in\{1, 2, k-2, k-1, k\},$$

and the characteristic (=minimal) polynomial $m(x)$ *of M is given by*

$$m(x)=(x-k)\{x^3+\alpha x^2+(\alpha-2k+1)x-\alpha(k-1)\}.$$

(II) $\mu=1$ *and* $(\mathfrak{G}_{a,b,c})^{\sigma}=\mathfrak{G}_{a,b,e}$ *for some* $e\in\Delta(a)-\{b\}$. *Moreover the intersection matrix M is one of the following:*

(II, A)

$$M=\begin{bmatrix} 0 & 1 & 0 & 0 \\ k & 0 & 1 & 0 \\ 0 & k-1 & 0 & \alpha \\ 0 & 0 & k-1 & k-\alpha \end{bmatrix}, \qquad \alpha\in\{1,\ 2,\ 3,\ k-3,\ k-2,\ k-1,\ k\}\ ,$$

and $m(x)=(x-k)\{x^3+\alpha x^2+(\alpha-2k+1)x-\alpha(k-1)\}$;

(II, B)

$$M=\begin{bmatrix} 0 & 1 & 0 & 0 \\ k & 0 & 1 & 0 \\ 0 & k-1 & 1 & \alpha \\ 0 & 0 & k-2 & k-\alpha \end{bmatrix}, \qquad \alpha\in\{1,\ 2,\ 3,\ k-3,\ k-2,\ k-1,\ k\}\ ,$$

and $m(x)=(x-k)\{x^3+(\alpha-1)x^2+(1+\alpha-2k)x+(-\alpha k+\alpha+k)\}$;

(II, C)

$$M=\begin{bmatrix} 0 & 1 & 0 & 0 \\ k & 0 & 1 & 0 \\ 0 & k-1 & k-2 & k-\beta \\ 0 & 0 & 1 & \beta \end{bmatrix}, \qquad \beta\in\{0,\ 1,\ 2,\ 3,\ k-3,\ k-2,\ k-1\}\ ,$$

and $m(x)=(x-k)\{x^3+(2-\beta)x^2+(-k+1-\beta)x+(\beta k-k-\beta)\}$.

(III) $\mu=2$ *and* $(\mathfrak{G}_{a,b,c})^{\sigma}=\mathfrak{G}_{a,b,e}$ *for some* $e\in\Delta(a)-\{b\}$. *Moreover the intersection matrix M is given by*

$$M=\begin{bmatrix} 0 & 1 & 0 & 0 \\ k & 0 & 2 & 0 \\ 0 & k-1 & 0 & \alpha \\ 0 & 0 & k-2 & k-\alpha \end{bmatrix}, \qquad \alpha\in\{1,\ 2,\ 3,\ k-3,\ k-2,\ k-1,\ k\}\ ,$$

and $m(x)=(x-k)\{x^3+\alpha x^2+(2\alpha-3k+2)x-\alpha(k-2)\}$.

§ 3. Proof of Theorem 1 (continued).

(Step 2. Completion of the proof.)

The main purpose of this section is to show the non-existence of the group $(\mathfrak{G},\Omega)$ whose intersection matrix is one of those listed in Proposition (2.12) by leading a contradiction for each case. But, we can get no contradiction from the intersection matrix $M_1=\begin{pmatrix} 0 & 1 & 0 & 0 \\ k & 0 & 2 & 0 \\ 0 & k-1 & 0 & 3 \\ 0 & 0 & k-2 & k-3 \end{pmatrix}$ with $k=7$ (in this case, there really exist such $(\mathfrak{G}, \Omega)$'s), and $M_1=\begin{pmatrix} 0 & 1 & 0 & 0 \\ k & 0 & 1 & 0 \\ 0 & k-1 & 0 & 2 \\ 0 & 0 & k-1 & k-2 \end{pmatrix}$ with k, $(8k-7)\,|\,11^2\cdot 7^2\cdot 3^2\cdot 5^2$. However from the assumption of 5-ply transitivity of (G, Δ), we can exclude some of the latter cases. But there still remain 6 cases, i.e., $k=379$, 1379, 3404, 6671, 18529, 166754. (Probably there exist no such groups.)

The method of the proof is as follows:

(1) the polynomial $m(x)/(x-k)$ is irreducible over the rational field (i.e., $m(x)/(x-k)=0$ has an integral solution x) except for some special cases;

(2) even though the polynomial $m(x)/(x-k)$ is not irreducible, the degree f_1 (or f_2, f_3) of the irreducible character is not an integer except for some finite number of possibilities for k;

(3) the elimination of the remaining finite number of possibilities by ad hoc consideration, that is, by considering the number of elements of $\mathfrak{G}$ which are conjugate to a fixed element, or by exploiting the known classification theorems about multiply transitive permutation groups.

(3.1) PROPOSITION. *The case* (I) *in Proposition* (2.12) *does not hold.*

PROOF. Since $k>l$, by Lemma 1 the equation $m(x)/(x-k)=x^3+\alpha x^2+(\alpha-2k+1)x-\alpha(k-1)=0$ must have an integral solution $x=s$.

(i) If $\alpha=1$, then $s^3+s^2-2(k-1)s-(k-1)=0$ and so $s^3+s^2=(2s+1)(k-1)$, $2s+1\neq 0$. Thus $k-1=(s^3+s^2)/(2s+1)=(1/8)(4s^2+2s-1+1/(2s+1))$. Thus in order that $k-1$ is an integer, $s=-1$ or 0. However $s=-1$ or 0 implies $k=1<5$, and this is a contradiction.

(ii) If $\alpha=2$, then $x^3+2x^2+(3-2k)x-2(k-1)=(x+1)(x^2+x+2-2k)$. Thus $\alpha_1=-1$, $\alpha_2=(-1+\sqrt{8k-7})/2$, $\alpha_3=(-1-\sqrt{8k-7})/2$. From the equations of Lemma 1, (ii), we have

$$f_1+f_2+f_3=k+l+m\,,$$

$$-f_1+\alpha_2 f_2+\alpha_3 f_3=-k\,,$$

$$f_1+\alpha_2^2 f_2+\alpha_3^2 f_3=k(l+m+1)\,,$$

and moreover

$$f_2=\frac{(k-\alpha_3)}{(1+\alpha_2)(\alpha_2-\alpha_3)}(l+m)$$

$$=\frac{(\sqrt{8k-7}-1)(k+(1+\sqrt{8k-7})/2)}{4\sqrt{8k-7}}\cdot\frac{k(k+1)}{2}\,,$$

and moreover

$$f_3=\frac{(\sqrt{8k-7}+1)(k+(1-\sqrt{8k-7})/2)}{4\sqrt{8k-7}}\cdot\frac{k(k+1)}{2},$$

$$f_1=\frac{k(k^2-k+2)}{4}.$$

In order that f_2 is an integer, $(2k+1)k(k+1)/\sqrt{8k-7}$ must be an integer, and moreover $(2k+1)^2k^2(k+1)^2/(8k-7)$ must be an integer. While, $(2k+1, 8k-7)$ (i.e., the greatest common divisor of $2k+1$ and $8k-7$) divides 11, $(k, 8k-7)$ divides 7 and $(k+1, 8k-7)$ divides 15. Thus $8k-7$ divides $11^2\cdot 7^2\cdot 3^2\cdot 5^2$ and $8k-7$ must be a square. Thus we have $k=7$, 16, 29, 56, 137, 154, 379, 742, 1379, 3404, 6671, 18529 or 166754 (since $k\geqq 5$). If $(G, \Delta)\cong(S_k, \Delta)$ or (A_k, Δ), $|\Delta|=k$, then we have a contradiction unless $k=7$, since G_b $(b\in\Delta(a))$ $(\cong S_{k-1}$ or $A_{k-1})$ has no subgroup of index $k-1$ which is transitive on $\Delta(a)-\{b\}$. It is proved by the argument given in pp. 39~40 of Wielandt [7] that there exists no non-trivial 5-ply transitive permutation group of degree $16=13+3$, $29=2\cdot 13+3$, $56=53+3$, $137=2\cdot 67+3$, $154=151+3$ and $742=739+3$. Thus we may assume that k is one of 7, 379, 1379, 3404, 6671, 18529 and 166754. For the elimination of the case $k=7$, see Iwasaki [6]. Unfortunately, we can say nothing any more in these remaining 6 cases at present.

(iii) If $\alpha=k-2$, then $m=k(k-1)^2/(k-2)$. Thus $k<5$, a contradiction.

(iv) If $\alpha=k-1$, then $s^3+(k-1)s^2-ks-(k-1)^2=0$. Setting $h=k-1$, we have $s^3+hs^2-(h+1)s-h^2=0$ and so $-h^2+(s^2-s)h+s^3-s=0$. Thus the discriminant $D=(s^2-s)^2+4(s^3-s)=s^4+2s^3+s^2-4s$ must be a square. Now $(s^2+s-1)^2<D<(s^2+s+1)^2$ for $s\geqq 2$ and $s\leqq -3$. If $s=-2$ then D is not a square, and if $-1\leqq s\leqq 1$ we have $k<5$, a contradiction.

(vi) If $\alpha=k$, then $m(x)/(x-k)=x^3+kx^2+(-k+1)x-k(k-1)=(x+k)(-k+x^2+1)=0$. Thus the eigenvalues of M are k, $\alpha_1=-k$, $\alpha_2=\sqrt{k-1}$, $\alpha_3=-\sqrt{k-1}$. Moreover $l=k(k-1)$ and $m=(k-1)^2$. Now the two relations $f_1+f_2+f_3=k+l+m$ and $k^2+\alpha_1^2f_1+\alpha_2^2f_2+\alpha_3^2f_3=k(1+k+l+m)$ (=trace of the matrix B_1^2) lead $f_1=1$. Thus $(\mathfrak{G}, \Omega)$ is not primitive. Because, if N denotes the normal subgroup of $\mathfrak{G}$ which is the kernel of the irreducible character of degree f_1, then NG is a proper subgroup of $\mathfrak{G}$, but since the normal subgroup N is transitive on Ω, $NG\neq G$, and we have the assertion.

Thus we have completed the proof of Proposition (3.1).

(3.2) PROPOSITION. *The case* (II, A) *in Proposition* (2.12) *does not hold.*

PROOF. The case $\alpha=1$, $k-2$, $k-1$ and k have been treated in Proposition (3.1). Thus we may assume that $\alpha=2$, 3 or $k-3$.

(i) If $\alpha=2$, then $k(k-1)^2/2$ must divide $k(k-1)(k-2)$, since $\mathfrak{G}_a\geqq\mathfrak{G}_{a,f}\geqq(\mathfrak{G}_{a,d,h})^{\sim}\geqq(\mathfrak{G}_{a,b,e,h})^{\sim}$ $(h\in\Delta(a)$, $b, e\in\Delta(a)$, $b\neq e\neq h\neq b)$ (cf. proof of Proposition

(2.9)). Thus $k<5$, a contradiction.

(ii) If $\alpha=3$, then $m=k(k-1)2/3$ must divide $k(k-1)(k-2)$.

(iii) If $\alpha=k-3$, then $m=k(k-1)^2/(k-3)$ must divide $k(k-1)(k-2)$. Thus $k<5$, a contradiction.

Thus we have completed the proof of Proposition (3.2).

(3.3) PROPOSITION. *The case* (II, B) *in Proposition* (2.12) *does not hold.*

PROOF. Since $k<l$, by Lemma 1 the equation $m(x)/(x-k)=x^3+(\alpha-1)x^2+(1+\alpha-2k)x+(-\alpha k+\alpha+k)=0$ must have an integral solution s.

(i) If $\alpha=1$, then $s^3+(2-2k)s+1=0$, $s\neq 0$. Thus $k=(s^3+2s+1)/2s$, and k is not an integer $\geqq 5$, a contradiction.

(ii) If $\alpha=2$, then $s^3+s^2+(3-2k)s+(-k+2)=0$, and so $(-2s-1)k+s^3+s^2+3s+2=0$, $2s+1\neq 0$. Thus $k=(s^3+s^2+3s+2)/(2s+1)=(1/8)(4s^2+2s+11+5/(2s+1))$. Thus $s=-3$, -1, 0 or 2. Thus $k=5$ $(s=-3)$, since $k\geqq 5$. Thus $G=S_5$, $|\Omega|=56$ and $|\mathfrak{G}|=2^6\cdot 3\cdot 5\cdot 7$. A minimal normal subgroup $\mathfrak{G}'$ of $\mathfrak{G}$ is simple and of order $2^5\cdot 3\cdot 5\cdot 7$ or $2^6\cdot 3\cdot 5\cdot 7$. Moreover a Sylow 5 subgroup of $\mathfrak{G}'$ is self-centralizing, and we can easily show that there exists no such simple group of order $2^5\cdot 3\cdot 5\cdot 7$ or $2^6\cdot 3\cdot 5\cdot 7$ by using the theorems of Sylow and Burnside together with the standard consideration about principal 5 and 7 blocks of $\mathfrak{G}'$.

(iii) If $\alpha=3$, $m(x)/(x-k)=x^3+2x^2+(4-2k)x+(3-2k)=(x+1)(x^2+x-2k+3)$. Thus $\alpha_1=-1$, $\alpha_2=(-1+\sqrt{8k-11})/2$, $\alpha_3=(-1-\sqrt{8k-11})/2$. As in the proof of Proposition (3.1), (ii), we have

$$f_2=\frac{(k-\alpha_3)}{(1+\alpha_2)(\alpha_2-\alpha_3)}\cdot(l+m)$$
$$=\frac{(\sqrt{8k-11}-1)(k+(1+\sqrt{8k-11})/2)}{2(2k-3)\sqrt{8k-11}}\cdot\frac{1}{3}k(k-1)(k+1).$$

In order that f_2 is an integer, $(2k+1)k(k-1)(k+1)/\sqrt{8k-11}$ must be an integer. While the G. C. D. $(2k+1, 8k-11)$ divides 15, $(k, 8k-11)$ divides 11, $(k-1, 8k-11)$ divides 3 and $(k+1, 8k-11)$ divides 19. Thus $8k-11$ divides $3^4\cdot 5^2\cdot 11^2\cdot 19^2$ and $8k-11$ must be a square. However this is impossible, because $3^2\equiv 5^2\equiv 11^2\equiv 19^2\equiv 1 \pmod 8$ and so $8k-11\equiv 1 \pmod 8$, hence k is not an integer and this is a contradiction.

(iv) If $\alpha=k-3$, then $m=k(k-1)(k-2)/(k-3)$. Thus $k=9$, 6 or 5 (since $k\geqq 5$). If $k=9$, then $|\Omega|=1+9+72+84=166=2\cdot 83$ where 83 is a prime. Thus this contradicts to Theorem 31.1 of Wielandt [7]. If $k=6$, 5 then $\alpha=3$, 2 respectively and these are impossible as were already proved in (ii) and (iii).

(v) If $\alpha=k-2$, then $s^3+(k-3)s^2+(-k-1)s+(-k^2+4k-2)=0$, and so $-k^2+(s^2-s+4)k+s^3-3s^2-s-2=0$. The discriminant $D=s^4+2s^3-3s^2-12s+8$.

While $(s^2+s-1)^2 > D > (s^2+s-3)$ for $s \geqq 4$ and $s \leqq -6$. $(s^2+s-2)^2 = D$ implies $s=1/2$, a contradiction. If $-5 \leqq s \leqq 3$, then D is not a square unless $s=2$ or -1. If $s=2$, then $k=2$ or $4<5$, a contradiction. If $s=-1$, then $k=5$ and $\alpha=3$, and this is already excluded by (iii).

(vi) If $\alpha=k-1$, then $s^3+(k-2)s^2+(-k)s+(-k^2+3k-1)=0$, and so $-k^2+(s^2-s+3)k+(s^3-2s^2-1)=0$. The discriminant $D=s^4+2s^3-s^2-6s+5$. While $(s^2+s)^2 > D > (s^2+s-2)^2$ for $s \geqq 1$ and $s \leqq -4$. $(s^2+s-1)^2 = D$ implies $s=1$. If $s=0, -2, -3$, then D is not a square, a contradiction. If $s=-1$ or 1, then D is a square, but we have $k<5$, a contradiction.

(vii) If $\alpha=k$, then $s^3+(k-1)s^2+(1-k)s+(-k^2+2k)=0$, and so $-k^2+(s^2-s+2)k+(s^3-s^2+s)=0$. The discriminant $D=s^4+2s^3+s^2+4$. While $(s^2+s+1)^2 > D > (s^2+s)^2$ for $s \leqq -2$ and $s \geqq 1$. If $s=-1$ or 0, then $k<5$, a contradiction.

Thus we have completed the proof of Proposition (3.3).

(3.4) PROPOSITION. *The case* (II, C) *in Proposition* (2.12) *does not hold.*

PROOF. Since $k<l$, and so by Lemma 1, $m(x)/(x-k)=x^3+(2-\beta)x^2+(-k+1-\beta)x+(\beta k-k-\beta)=0$ must have an integral solution s.

(i) If $\beta=0$, then the intersection matrix M_3 with respect to the orbit $\Lambda(a)$ must have the form

$$M_3=\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & m & m-1 & 0 \\ m & 0 & 0 & m-1 \end{bmatrix} \qquad \text{(where } m=k-1)$$

since $\mu_{23}^{(3)} \leqq m-1$ (cf. the parameter relations [3], (4.1) and (4.2)). This implies that $(\mathfrak{G}, \Omega)$ is not primitive (cf. [3], (4.8). Change the second and the fourth rows and columns of M_3!) and this is a contradiction.

(ii) If $\beta=1$, then $s^3+s^2+(-s)k-1=0$, $s \neq 0$ and so $k=s^2+s-1/s$. Thus k is not an integer $\geqq 5$, a contradiction.

(iii) If $\beta=2$, then $m=k(k-1)/(k-2)$ must be an integer, and this implies $k<5$, a contradiction.

(iv) If $\beta=3$, then $m=k(k-1)/(k-3)$ must be an integer, and this implies $k=5$, 6 or 9, since $k \geqq 5$. If $k=5$ then $|\Omega|=1+5+20+10=36$, and this is excluded similarly as in the case of Proposition 3.3 (ii) $k=5$. If $k=6$, then $|\Omega|=1+6+30+10=47$, a prime, and this is impossible. If $k=9$, then $|\Omega|=1+9+72+12=94=2\cdot 47$ with 47 a prime, and this contradicts to Theorem 31.1 of Wielandt [7].

(v) If $\beta=k-3$ then $s^3+(5-k)s^2+(-2k+4)s+(k^2-5k+3)=0$ and so $k^2+(-s^2-2s-5)k+s^3+5s^2+4s+3=0$. The discriminant $D=s^4-6s^2+4s+13$. While $(s^2-2)^2 > D > (s^2-4)^4$ for $s \leqq -3$ and $s \geqq 4$, and $(s^2-3)^2 = D$ implies $s=-1$. If

$-2 \leqq s \leqq 3$, then $s=-1$ (since D must be a square), and so $k<5$, a contradiction.

(vi) If $\beta=k-2$, then $s^3+(4-k)s^2+(-2k+3)s+(k^2-4k+2)=0$ and so $k^2+(-s^2-2s-4)k+s^3+4s^2+3s+2=0$. The discriminant $D=s^4-4s^2+4s+8$. While $(s^2-1)^2>D>(s^2-3)^2$ for $s\leqq -3$ and $s\geqq 4$, and $(s^2-2)^2=D$ implies $s=-1$. If $-2\leqq s\leqq 3$, then $s=-2$, -1, 1 or 2 (since D must be a square), and $k<5$ except for the two cases $k=5$ $(s=1)$ and $k=8$ $(s=2)$. If $k=5$ then $\beta=3$ and this is impossible as was already proved in (iv). If $k=8$, then $G=S_8$ or A_8. It is immediately shown that $(G, \Gamma(a))\cong G$ acting on the set of ordered pairs of $\Delta(a)$, and also that $(G, \Lambda(a))\cong G$ acting on the set of unordered pairs of $\Delta(a)$. Now $|\Omega|=1+8+56+28=93$, and an element $\tau\in G$ $(=S_8$ or $A_8)$ consisting of one 3-cycle fixes $1+5+20+10=36$ points of Ω, and other elements of G of order 3 and not consisting of one 3-cycle fix less than 36 points. Thus the number of elements of $\mathfrak{G}$ which are conjugate to τ is given by $|G: C_G(\tau)|\cdot(|\Omega|/36)=8\cdot 7\cdot 2\cdot 93/36$. However this is not an integer, and this is a contradiction.

(vii) If $\beta=k-1$, then $s^3+(3-k)s^2+(-2k+2)s+(k^2-3k+1)=0$ and so $k^2+(-s^2-2s-3)k+s^3+3s^2+2s+1$. The discriminant $D=s^4-2s^2+4s+5$. While $s^4>D>(s^2-2)^2$ for $s\leqq -2$ and $s\geqq 3$, and $D=(s^2-1)^2$ implies $s=-1$. If $-1\leqq s\leqq 2$, then $s=-1$ (since D must be a square), and so $k<5$, a contradiction.

Thus we have completed the proof of Proposition (3.4).

(3.5) PROPOSITION. *The case* (III) *in Proposition* (2.12) *does not hold, unless* $\alpha=3$ *and* $k=7$.

PROOF. Since $k<l$, by Lemma 1 $m(x)/(x-k)=(x-k)\cdot\{x^3+\alpha x^2+(2\alpha-3k+2)x-\alpha(k-2)\}$ has an integral solution s.

(i) If $\alpha=1$, then $s^3+s^2+(4-3k)s-(k-2)=0$ and so $s^3+s^2+4s+2=(3s+1)k$, $3s+1\neq 0$. Thus $k=(1/27)(9s^2+6s+34+20/(3s+1))$. In order that k is an integer $\geqq 5$, $k=5$ $(s=3)$ or $k=16$ $(s=-7)$. If $k=5$ then the eigenvalues of M are 5, $\alpha_1=3$, $\alpha_2=-2+\sqrt{3}$, $\alpha_3=-2-\sqrt{3}$. Thus $f_2=f_3$ (cf. Proof of Lemma 1). Moreover, from the first and the second relations in Lemma 1, (ii), we easily have a contradiction. But if $k=16$ then $|\Omega|=1+16+8\cdot 15+8\cdot 15\cdot 14=1817=23\cdot 79$. But this case is also excluded by the same method as in the case of $k=5$. i.e., $\alpha_1=-7$, $\alpha_2=3+\sqrt{11}$, $\alpha_3=3-\sqrt{11}$, $f_2=f_3=16\cdot 6\cdot 299/(2\cdot 41)$ is not an integer.

(ii) If $\alpha=2$, then $s^3+2s^2+(6-3k)s-2(k-2)=0$ and so $s^3+2s^2+6s+4=(3s+2)k$, $3s+2\neq 0$. Thus $k=(s^3+2s^2+6s+4)/(3s+2)=(1/27)(9s^2+12s+46+16/(3s+2))$. It is easily shown that there exists no integral solution $k\geqq 5$, unless $s=-6$. If $s=-6$, then $k=11$, $m=k(k-1)(k-2)/2\alpha=11\cdot 10\cdot 9/(2\cdot 2)$ is not an integer, a contradiction.

(iii) If $\alpha=3$, then $m(x)/(x-k)=x^3+3x^2+(8-3k)x-3(k-2)$, and so the eigenvalues of M are k, $\alpha_1=-1$, $\alpha_2=-1+\sqrt{3k-5}$, $\alpha_3=-1-\sqrt{3k-5}$. From the equations of Lemma 1, (ii), we have

$$f_2=\frac{(k-\alpha_3)}{(1+\alpha_2)(\alpha_2-\alpha_3)}(l+m)$$
$$=\frac{(k+1+\sqrt{3k-5})}{\sqrt{3k-5}\cdot 2\sqrt{3k-5}}\cdot\frac{k(k-1)(k+1)}{6}.$$

Clearly $f_2\neq f_3=((k-\alpha_2)/(1+\alpha_3)(\alpha_3-\alpha_2))(l+m)$, and so $\sqrt{3k-5}$ must be an integer. Thus in order that f_2 is an integer, $(k+1)^2k(k-1)/\sqrt{3k-5}$ must be an integer, and moreover $(k+1)^4k^2(k-1)^2/(3k-5)$ must be an integer. While $(k+1, 3k-5)$ divides 8, $(k, 3k-5)$ divides 5, $(k-1, 3k-5)$ divides 2. Therefore $3k-5$ divides $8^4\cdot 2^2\cdot 5^2$ and $3k-5$ must be a square. However, if 5^2 divides $3k-5$ then the denominator of the above formula giving f_2 is divisible by 5^2, while the numerator is not divisible by 5^2, a contradiction. Thus $3k-5$ divides 2^{14}. Moreover we can easily show that f_2 is never an integer unless $k=7$ or 23 (here $k\geqq 5$). If $k=23$ then $G=S_{23}$ or A_{23}, $|\Omega|=1+23+253+1771=2048=2^{11}$. It is quickly shown that $(G, \Gamma(a))\cong G$ acting on the set of unordered pairs of $\Delta(a)$, and that $(G, \Lambda(a))\cong G$ acting on the set of unordered triples of $\Delta(a)$. Now an element $\tau\in G$ $(=S_{23}$ or $A_{23})$ consisting of one 3-cycle fixes $1+20+190+1141=1352=2^3\cdot 13^2$ points of Ω, and other elements of G of order 3 and not consisting of one 3-cycle fix less than $2^3\cdot 13^2$ points. Thus the number of elements of $\mathfrak{G}$ which are conjugate to τ is given by $|G: C_G(\tau)|\cdot 2^{11}/(2^3\cdot 13^2)$ and this is not an integer, a contradiction.

For $k=7$, there really exist primitive extensions of rank 4 of (A_7, Δ) and (S_7, Δ), $|\Delta|=7$, and these extensions have regular normal subgroups of order 64. (For the detailed exposition on these extensions, see S. Iwasaki [**6**].)

(iv) If $\alpha=k-3\neq 2$, $\neq 3$, then $m=k(k-1)(k-2)/2(k-3)$ must be an integer, and this implies $k=9$. If $k=9$, then $m=42$ and $G=S_9$ or A_9. However it is quickly shown that G has no subgroup of index 42, a contradiction.

(v) If $\alpha=k-2\neq 3$, then $s^3+(k-2)s^2+(-k-2)s+(-k^2+4k-4)=0$ and so $-k^2+(s^2-s+4)k+s^3-2s^2-2s-4=0$. The discriminant $D=s^4+2s^3+s^2-16s$. While $(s^2+s+1)^2>D>(s^2+s-1)^2$ for $s\leqq -9$ and $s\geqq 8$, and $D=(s^2+s)^2$ implies $s=0$. If $-8\leqq s\leqq 7$, then $s=-2, -1, 0, 2$ (since D must be a square), and $k<6$ except for the case $k=8$ and $s=-2$. If $k=8$, then the eigenvalues of M are 8, $\alpha_1=-2$, $\alpha_2=-2+\sqrt{22}$ and $\alpha_3=-2-\sqrt{22}$. Quite the same argument as in the proof of (i) for $k=5$ shows a contradiction.

(vi) If $\alpha=k-1$, then $s^3+(k-1)s^2+(-k)s-k^2+3k-2=0$ and so $-k^2+(s^2-s+3)k+s^3-s^2-2=0$. The discriminant $D=s^4+2s^3+3s^2-6s+1$. While $(s^2+s+2)^2>D>(s^2+s)^2$ for $s\leqq -5$ and $s\geqq 3$, and $D=(s^2+s+1)^2$ implies $s=0$. If $-4\leqq s\leqq 2$, then $s=-2, -1, 0$ or 1 (since D must be a square), and $k<5$

except for the case $k=7$ and $s=-2$. If $k=7$, then $|\Omega|=1+7+21+18=47$, a prime, and this is a contradiction.

(vii) If $\alpha=k$, then $m(x)=(x-k)(x+k)(x^2-k+2)$. Thus the eigenvalues of M are k, $\alpha_1=-k$, $\alpha_2=\sqrt{k-2}$, $\alpha_3=-\sqrt{k-2}$. Moreover $l=k(k-1)/2$ and $m=(k-1)(k-2)/2$. Quite the same argument as in the proof of Proposition (3.1), (vi) shows $f_1=1$, and this is a contradiction.

Thus we have completed the proof of Proposition (3.5).

Thus Theorem 1 is completely proved.

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References

[1] E. Bannai, On rank 3 groups with a multiply transitive constituent, J. Math. Soc. Japan, **24** (1972), 252–254.
[2] P. J. Cameron, Proofs of some theorems of W. A. Manning, Bull. London Math. Soc., **1** (1969), 349–352.
[3] D. G. Higman, Intersection matrices for finite permutation groups, J. Algebra, **6** (1967), 22–42.
[4] N. Ito, Über die Gruppen $PSL_n(q)$, die eine Untergruppe von Primzahlindex enthalten, Acta Sci. Math. Szeged, **21** (1960), 206–217.
[5] S. Iwasaki, On finite permutation groups of rank 4 (to appear).
[6] S. Iwasaki, A note on primitive extensions of rank 4 of the alternating groups, Proc. Japan Acad., **48** (1972), 5–8.
[7] H. Wielandt, Finite permutation groups, Academic Press, New York and London, 1964.

Added in Proof: Part II of this paper has appeared in J. Fac. Sci. Univ. Tokyo, **19** (1972), 151–154. Statement of Theorem 2 in Part II needs a slight correction, according to the fact that Theorem 1 of this paper (Part I) was slightly corrected from that of the original version. That is,

THEOREM 2. *Let* (G, Δ) *be a* 5-*ply transitive permutation group. If* (G, Δ) *has a primitive extension of rank* 4, *then either* (i) $|\Delta|=7$ *and* $G=S_7$ *or* A_7, *or* (ii) $|\Delta|=379$, 1379, 3404, 6671, 18529 *or* 166754 *and* $G\neq S_{|\Delta|}$, $A_{|\Delta|}$.