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The nullity spaces of the conformal curvature tensor

By Masami SEKIZAWA

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§ 1. Introduction.

A. Gray [2] has studied the nullity space of the Riemannian tensor which is a tensor field of type (1, 3) on a Riemannian manifold having the same formal properties as the curvature tensor field, and unified the studies of the nullity spaces of several tensor fields. But the Weyl conformal curvature tensor C on a Riemannian manifold is not a Riemannian tensor. It is invariant under a conformal change of the metric and vanishes identically on 3-dimensional Riemannian manifold. The invariant tensor on 3-dimensional Riemannian manifold is the tensor field c defined by (2.7) in § 2.

We shall define the nullity space C_p of the conformal curvature tensor as the subspace of the tangent space $T_p(M)$ at $p \in M$ spanned by $X \in T_p(M)$ such that $C_{XY}=0$ and $c(X, Y)=0$ for any $Y \in T_p(M)$, and prove that a maximal integral manifold of the distribution $p \to C_p$ is totally umbilic and conformally flat.

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§ 2. Conformal curvature tensor.

Throughout this paper, we denote by M an n-dimensional differentiable Riemannian manifold of class C^∞ $(n>2)$, by $T_p(M)$ the tangent space of M at $p \in M$. Let $\mathfrak{F}(M)$ be the algebra of differentiable real-valued functions on M, $\mathfrak{X}(M)$ the Lie algebra of differentiable vector fields on M. The metric tensor field will be denoted by $\langle\,,\rangle$, the Riemannian connection by ∇_X $(X \in \mathfrak{X}(M))$, and the curvature operator by $R_{XY}=[\nabla_X, \nabla_Y]-\nabla_{[X,Y]}$ $(X, Y \in \mathfrak{X}(M))$. The tensors on each tangent space determined by the tensor fields will be denoted by the same symbols. The Weyl conformal curvature tensor on M is the tensor field C of type (1, 3) defined by

$$(2.1)\qquad C_{XY}Z=R_{XY}Z+(1/(n-2))\{S(X, Z)Y-S(Y, Z)X+\langle X, Z\rangle QY-\langle Y, Z\rangle QX\} -(K/(n-1)(n-2))\{\langle X, Z\rangle Y-\langle Y, Z\rangle X\}$$

for any $X, Y, Z \in \mathfrak{X}(M)$, where we denote by S, Q and K the Ricci tensor, the Ricci operator defined by $\langle X, QY\rangle = S(X, Y)$ and the scalar curvature respectively.

Now we prepare the identities of the conformal curvature tensor, which are obtained by straightforward calculations.

LEMMA 1. *The Weyl conformal curvature tensor satisfies the following equations:*

$$C_{XY} = -C_{YX}, \tag{2.2}$$

$$\langle C_{XY}Z, W\rangle = -\langle C_{XY}W, Z\rangle, \tag{2.3}$$

$$\underset{XYZ}{\mathfrak{S}} C_{XY}Z = 0, \tag{2.4}$$

$$\operatorname{trace}(Z \longrightarrow C_{ZX}Y) = 0, \tag{2.5}$$

$$\underset{XYZ}{\mathfrak{S}} (\nabla_X C)_{YZ} W = (1/(n-2)) \underset{XYZ}{\mathfrak{S}} \{\langle c(X, Y), W\rangle Z - \langle Z, W\rangle c(X, Y)\}, \tag{2.6}$$

where the tensor field c of type (1, 2) *is defined by*

$$c(X, Y) = (\nabla_X Q)Y - (\nabla_Y Q)X - (1/2(n-1))\{(XK)Y - (YK)X\}. \tag{2.7}$$

Because of the equation (2.6), the Weyl conformal curvature tensor C is not a Riemannian tensor on M, (if M has the parallel Ricci tensor then C is a Riemannian tensor). The following lemma is also proved by direct calculations.

LEMMA 2. *The tensor field c satisfies the following equations:*

$$c(X, Y) = -c(Y, X), \tag{2.8}$$

$$\underset{XYZ}{\mathfrak{S}} \langle c(X, Y), Z\rangle = 0, \tag{2.9}$$

$$\underset{XYZ}{\mathfrak{S}} (\nabla_X c)(Y, Z) = \underset{XYZ}{\mathfrak{S}} R_{XY}QZ = \underset{XYZ}{\mathfrak{S}} C_{XY}QZ, \tag{2.10}$$

$$\operatorname{trace}(W \longrightarrow (\nabla_W C)_{XY}Z) = ((n-3)/(n-2))\langle c(X, Y), Z\rangle. \tag{2.11}$$

A Riemannian manifold on which $C \equiv 0$ for $n > 3$ and $c \equiv 0$ for $n = 3$ is said to be conformally flat.

LEMMA 3. *For* $X, Y, Z \in \mathfrak{X}(M)$ *we have*

$$\underset{XYZ}{\mathfrak{S}} \{[\nabla_X, C_{YZ}] - C_{[X,Y]Z}\} W = (1/(n-2)) \underset{XYZ}{\mathfrak{S}} \{\langle c(X, Y), W\rangle Z - \langle Z, W\rangle c(X, Y)\}, \tag{2.12}$$

$$\underset{XYZ}{\mathfrak{S}} \{\nabla_X(c(Y, Z)) - c([X, Y], Z)\} = \underset{XYZ}{\mathfrak{S}} C_{XY}QZ. \tag{2.13}$$

PROOF. We have $\nabla_X Y - \nabla_Y X = [X, Y]$ for $X, Y \in \mathfrak{X}(M)$, and so by (2.6)

$$(1/(n-2))\underset{XYZ}{\mathfrak{S}}\{\langle c(X, Y), W\rangle Z-\langle Z, W\rangle c(X, Y)\}$$

$$=\underset{XYZ}{\mathfrak{S}}\{[\nabla_X, C_{YZ}]-C_{\nabla_X Y Z}-C_{Y\nabla_X Z}\}W$$

$$=\underset{XYZ}{\mathfrak{S}}\{[\nabla_X, C_{YZ}]-C_{[X,Y]Z}\}W.$$

Similarly we have the equation (2.13). q. e. d.

§ 3. Nullity space of the conformal curvature tensor.

We shall define the nullity space $\mathcal{C}_p$ of the conformal curvature tensor and study the differentiability and the integrability of the distribution $p\rightarrow\mathcal{C}_p$.

DEFINITION. Let $p\in M$. We define

$$\mathcal{C}_p=\{X\in T_p(M)\,|\,C_{XY}=0 \text{ and } c(X, Y)=0 \text{ for any } Y\in T_p(M)\}$$

and we denote by $\mathcal{C}$ the distribution $p\rightarrow\mathcal{C}_p$. We call the subspace $\mathcal{C}_p$ of $T_p(M)$ *the nullity space of the conformal curvature tensor at* p, and $\mu_C(p)=\dim\mathcal{C}_p$ *the index of nullity of the conformal curvature tensor at* p. We call the orthogonal complement of $\mathcal{C}_p$ in $T_p(M)$ *the conullity space of the conformal curvature tensor at* p, and denote it by $\mathcal{C}_p^{\perp}$.

The function μ_C is upper semicontinuous, and the set on which μ_C assumes its minimum value is open in M.

LEMMA 4. *For each point* $p\in M$, *either* $\mu_C(p)=n$ *or* $\mu_C(p)\leqq n-2$.

PROOF. If we assume $\mu_C(p)\leqq n-1$, then we can choose a non-zero vector $X\notin\mathcal{C}_p$. It follows that there is a vector $Y\in T_p(M)$ such that $C_{XY}\neq 0$ or a vector $Z\in T_p(M)$ such that $c(X, Z)\neq 0$. In both cases, Y (resp. Z) does not belong to $\mathcal{C}_p$ and it is linearly independent of X because of (2.2) (resp. (2.8)). Hence $\dim\mathcal{C}_p^{\perp}\geqq 2$. q. e. d.

THEOREM 1. *In a region* U *of* M *where* $\mu_C(p)$ *is positive and constant for any point* $p\in U$, *the distribution* $\mathcal{C}$ *is differentiable.*

PROOF. For $p\in U$, let $\mathcal{A}_p$ be the linear subspace of $T_p(M)$ spanned by vectors of the form $C_{XY}Z$ and $A(X, Y)$, where $X, Y, Z\in T_p(M)$ and $\langle A(X, Y), Z\rangle=\langle c(Z, Y), X\rangle$. Then, if $W\in\mathcal{C}_p$, the relations

$$\langle C_{XY}Z, W\rangle=\langle C_{ZW}X, Y\rangle=0,$$

$$\langle A(X, Y), W\rangle=\langle c(W, Y), X\rangle=0$$

hold, which shows that $\mathcal{A}_p\subset\mathcal{C}_p^{\perp}$. If $\mathcal{A}_p\neq\mathcal{C}_p^{\perp}$, there is a non-zero vector $W\in\mathcal{C}_p$ such that $\langle W, \mathcal{A}_p\rangle=0$. But then, for any $X, Y, Z\in T_p(M)$, we have

$$\langle C_{WX}Y, Z\rangle=\langle C_{YZ}W, X\rangle=-\langle C_{YZ}X, W\rangle=0,$$

$$\langle c(W, X), Y\rangle=\langle A(Y, X), W\rangle=0,$$

so that $W \in \mathcal{C}_p$. Hence, it follows that $W=0$ and hence $\mathcal{A}_p = \mathcal{C}_p^{\perp}$.

For any fixed point $q \in U$, let $F=(F_1, \cdots, F_n)$ be a frame field defined on a neighborhood U_0 of q in U, and let the vector fields C_{abc} and A_{ab} be defined by formulas $C_{abc} = C_{F_a F_b} F_c$ and $A_{ab} = A(F_a, F_b)$. These vector fields are differentiable in U_0. Since $\mathcal{A}_p = \mathcal{C}_p^{\perp}$, we see that the vectors $C_{abc}(p)$ and $A_{de}(p)$ span $\mathcal{C}_p^{\perp}$ for each $p \in U_0$. So, let us suppose that the vectors $\{C_{abc}(q), A_{de}(q)\}_{(abcde) \in I}$, where I is an index set, are a basis for $\mathcal{C}_q^{\perp}$. Then the vector fields C_{abc} and A_{de} $((abcde) \in I)$ are differentiable vector fields defined on U_0; they are linearly independent in some (possibly smaller) neighborhood V of q, and they span $\mathcal{C}_p^{\perp}$ for each $p \in V$, because of the fact that the index μ_C is constant on V. Since the distributions $\mathcal{C}$ and $\mathcal{C}^{\perp}$ are orthogonal, it follows that the distribution $\mathcal{C}$ is differentiable. q. e. d.

THEOREM 2. *Let U be a region of M on which the index μ_C is positive and constant. Then the distribution $\mathcal{C}$ is integrable on U.*

PROOF. Let X and Y be vector fields in $\mathcal{C}$. From Lemma 3 it follows that $[X, Y]$ is in $\mathcal{C}$. q. e. d.

§ 4. Local properties of the integral manifolds.

Let L be a Riemannian manifold isometrically immbedded into another Riemannian manifold M. Let $\bar{\mathfrak{X}}(L)$ be the restriction of vector fields on M to L, then we write

$$\bar{\mathfrak{X}}(L) = \mathfrak{X}(L) \oplus \mathfrak{X}(L)^{\perp}$$

where $\mathfrak{X}(L)^{\perp}$ is the collection of vector fields normal to L. Let P denote the orthogonal projection of $\bar{\mathfrak{X}}(L)$ to $\mathfrak{X}(L)$. For $X \in \mathfrak{X}(L)$ we denote the Riemannian connection on L by $\bar{\nabla}_X$. It is known that $\bar{\nabla}_X Y = P \nabla_X Y$ holds for X, $Y \in \mathfrak{X}(L)$.

The *configuration tensor* (cf. [1], [3]) of L in M is an $\mathfrak{F}(M)$-linear map $T: \mathfrak{X}(L) \times \bar{\mathfrak{X}}(L) \to \bar{\mathfrak{X}}(L)$ defined by

$$T_X Y = \nabla_X Y - \bar{\nabla}_X Y \quad \text{for } X,\ Y \in \mathfrak{X}(L),$$

$$T_X Z = P \nabla_X Z \quad \text{for } X \in \mathfrak{X}(L),\ Z \in \mathfrak{X}(L)^{\perp}.$$

We now list some well-known properties of this operator. (For the proof see [1] for example.)

LEMMA 5. *The configuration tensor T has the following properties:*

(4.1) $$T_X Y = T_Y X \quad \text{for } X,\ Y \in \mathfrak{X}(L),$$

(4.2) $$\langle T_X Y, Z \rangle = -\langle Y, T_X Z \rangle \quad \text{for } X \in \mathfrak{X}(L),\ Y,\ Z \in \bar{\mathfrak{X}}(L),$$

(4.3) $$T_X(\mathfrak{X}(L)) \subset \mathfrak{X}(L)^{\perp} \text{ and } T_X(\mathfrak{X}(L)^{\perp}) \subset \mathfrak{X}(L) \quad \text{for } X \in \mathfrak{X}(L).$$

We note that the configuration tensor is determined by its effect on $\mathfrak{X}(L)$ or on $\mathfrak{X}(L)^{\perp}$, and it has the same informations as the second fundamental form.

Next we shall prove that a maximal integral manifold L of the distribution $\mathcal{C}$ is totally umbilic. First we state a lemma, the proof of which is given in a way similar to that used in the first step of the proof of Theorem 1.

LEMMA 6. *Let L be a maximal integral manifold of $\mathcal{C}$. If X, Y, Z belong to $\mathfrak{X}(L)^{\perp}$, then $C_{XY}Z$ and $c(X, Y)$ also belong to $\mathfrak{X}(L)^{\perp}$.*

THEOREM 3. *Let L be a maximal integral manifold of $\mathcal{C}$; then L is totally umbilic.*

PROOF. Let $X\in\mathfrak{X}(L)$ and $Y, Z, U\in\mathfrak{X}(L)^{\perp}$. Since $C_{YZ}U\in\mathfrak{X}(L)^{\perp}$ we have

$$\begin{aligned} P\underset{XYZ}{\mathfrak{S}}\nabla_X(C_{YZ}U) &= P\{\nabla_X(C_{YZ}U)+\nabla_Y(C_{ZX}U)+\nabla_Z(C_{XY}U)\} \\ &= T_X C_{YZ}U . \end{aligned}$$

On the other hand, we get by (2.12) and Lemma 6

$$\begin{aligned} &(n-2)P\underset{XYZ}{\mathfrak{S}}\nabla_X(C_{YZ}U) \\ &= P\underset{XYZ}{\mathfrak{S}}\{\langle c(X, Y), U\rangle Z-\langle Z, U\rangle c(X, Y)+(n-2)(C_{YZ}\nabla_X U+C_{[X,Y]Z}U)\} \\ &= P\{\langle c(Y, Z), U\rangle X+(n-2)(C_{YZ}P\nabla_X U+C_{P[X,Y]Z}U+C_{P[Z,X]Y}U)\} \\ &= \langle c(Y, Z), U\rangle X . \end{aligned}$$

Therefore we see that

$$T_X C_{YZ}U=(1/(n-2))\langle c(Y, Z), U\rangle X . \tag{4.4}$$

Next let $W\in\mathfrak{X}(L)$. Then $T_X W\in\mathfrak{X}(L)^{\perp}$ and so by Lemma 6 $C_{YZ}T_X W\in\mathfrak{X}(L)$. From (4.4) and Lemma 5, we have

$$(n-2)\langle C_{YZ}T_X W, U\rangle=(n-2)\langle T_X C_{YZ}U, W\rangle=\langle X, W\rangle\langle c(Y, Z), U\rangle .$$

Hence

$$\langle (n-2)C_{YZ}T_X W-\langle X, W\rangle c(Y, Z), U\rangle=0$$

and so

$$(n-2)C_{YZ}T_X W-\langle X, W\rangle c(Y, Z)\in\mathfrak{X}(L)\cap\mathfrak{X}(L)^{\perp} .$$

This implies

$$(n-2)C_{YZ}T_X W-\langle X, W\rangle c(Y, Z)=0 . \tag{4.5}$$

Especially setting $X=W$ and $\langle X, X\rangle=1$, we have

$$C_{YZ}T_X X-(1/(n-2))c(Y, Z)=0 . \tag{4.6}$$

Let us put

$$\tau_X W=T_X W-\langle X, W\rangle T_X X$$

for any unit $X\in\mathfrak{X}(L)$. Then by (4.5) and (4.6), we have

$$C_{YZ}\tau_X W=0 \tag{4.7}$$

for any unit $X\in\mathfrak{X}(L)$ and any $W\in\mathfrak{X}(L)$, $Y, Z\in\mathfrak{X}(L)^{\perp}$; however, the equation (4.7) holds also for any $Y, Z\in\mathfrak{X}(L)$, and so by (2.3) and (4.6), we see that

$$C_{\tau_X WY}=0 \quad \text{and} \quad c(\tau_X W, Y)=0$$

for any unit $X\in\mathfrak{X}(L)$ and any $W\in\mathfrak{X}(L)$, $Y\in\bar{\mathfrak{X}}(L)$. Hence $\tau_X W\in\mathfrak{X}(L)$ and so $\tau_X W=0$. This implies

$$T_X W=\langle X, W\rangle T_X X$$

for any unit vector field X on L and any vector field W on L. Taking account of (4.1), we see that $T_X X$ is independent of the choice of a unit vector field X on L. This proves that L is totally umbilic. q. e. d.

We remark that a maximal integral manifold L of $\mathcal{C}$ is conformally flat provided $\dim L>3$ by the Gauss equation for the Weyl conformal curvature tensor (cf. [7]).

On a Riemannian manifold, we know another interesting curvature tensor B defined by

$$B_{XY}Z=(p-1)R_{XY}Z+(1/2)\{S(X, Z)Y-S(Y, Z)X+\langle X, Z\rangle QY-\langle Y, Z\rangle QX\}$$
$$-((n-p)K/n(n-1))\{\langle X, Z\rangle Y-\langle Y, Z\rangle X\}, \qquad p=2, 3, \cdots, n-1,$$

which has appeared in Tachibana [5] and Tomonaga [6]. Our method can be applied to studies of nullity spaces of this curvature tensor. Precisely, we define the tensor field b by

$$b(X, Y)=(\nabla_X Q)Y-(\nabla_Y Q)X-((n-p)/n(n-1))\{(XK)Y-(YK)X\},$$

then we have the same results as in Theorems 1, 2 and 3 for B and b.

Department of Mathematics
Science University of Tokyo
Wakamiya Shinjuku-ku
Tokyo, Japan

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