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Products of two semi-algebraic groups

By Morikuni GOTO[1)]

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§ 1. Introduction.

Let $\boldsymbol{R}$ and $\boldsymbol{C}$ denote the field of real numbers and complex numbers, respectively. For a field Φ, we denote by $GL(n, \Phi)$ the group of all n by n non-singular matrices over Φ. A subgroup A of $GL(n, \Phi)$ is called *algebraic* if there exists a family of polynomials of n^2 variables over Φ which defines A. In this paper, we are mainly interested in subgroups of $GL(n, \boldsymbol{R})$. A subgroup P of $GL(n, \boldsymbol{R})$ is said to be *pre-algebraic*[2)] if there exists an algebraic subgroup A of $GL(n, \boldsymbol{R})$ which contains P as a subgroup of finite index. A pre-algebraic group is closed, and a closed subgroup G of $GL(n, \boldsymbol{R})$ is pre-algebraic if and only if the Lie algebra $\mathfrak{g}$ of G is algebraic and G has only finitely many connected components. For any subgroup G of $GL(n, \boldsymbol{R})$, we can find the smallest pre-algebraic group $\mathcal{A}(G)$ containing G. Let us call $\mathcal{A}(G)$ the *pre-algebraic hull* of G. For a topological group G, we adopt the notation G_e for the identity component group of G. The identity component of $\mathcal{A}(G)$ will be denoted by $\mathcal{A}_e(G)$, i.e. $\mathcal{A}_e(G)=(\mathcal{A}(G))_e$.

Let G be a connected Lie subgroup of $GL(n, \boldsymbol{R})$. Then the pre-algebraic hull $\mathcal{A}(G)$ is connected, and the commutator subgroup of $\mathcal{A}(G)$ is closed and is contained in G. In Goto [4], the author defined G to be semi-algebraic if G contains a maximal compact subgroup of $\mathcal{A}(G)$. If G is semi-algebraic, then G is a closed normal subgroup of $\mathcal{A}(G)$ such that the factor group $\mathcal{A}(G)/G$ is isomorphic with a vector group $\boldsymbol{R}^k$ of a certain dimension k, and vice versa. Let us extend the definition of semi-algebraic groups to the non-connected case.

DEFINITION. A closed subgroup S of $GL(n, \boldsymbol{R})$ is said to be *semi-algebraic* if S is a normal subgroup of its pre-algebraic hull $\mathcal{A}(S)$ and the factor group $\mathcal{A}(S)/S$ is a vector group.

It is obvious that this generalizes the definition of the connected case and that a pre-algebraic group is semi-algebraic.

Let S be a semi-algebraic group. Then $S\mathcal{A}_e(S)$ is of finite index in $\mathcal{A}(S)$,

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2) About pre-algebraic groups, see Goto-Wang [5].

and so is $S\mathcal{A}_e(S)/S$ in $\mathcal{A}(S)/S$. Since $\mathcal{A}(S)/S$ is connected, we have that $\mathcal{A}(S)=S\mathcal{A}_e(S)$, and accordingly $\mathcal{A}(S)/S\cong\mathcal{A}_e(S)/\mathcal{A}_e(S)\cap S$. Because $\mathcal{A}(S)/S$ is simply connected, $\mathcal{A}_e(S)\cap S$ is connected and coincides with S_e. Hence we have $\mathcal{A}(S)/\mathcal{A}_e(S)\cong S/S_e$ and $\mathcal{A}(S)/S\cong\mathcal{A}_e(S)/S_e$. Hence, in particular, S_e is of finite index in S, and $\mathcal{A}_e(S)/S_e$ is a vector group. Also the finiteness of S/S_e implies that $S\mathcal{A}(S_e)=\mathcal{A}(S)$, and so $\mathcal{A}(S_e)=\mathcal{A}_e(S)$. Therefore S_e is semi-algebraic. Thus we have the following proposition.

PROPOSITION. *If S is semi-algebraic, then S_e is semi-algebraic and is of finite index in S, and conversely.*

The purpose of this paper is to prove the following theorem.

THEOREM. *Let A and B be semi-algebraic subgroups of $GL(n, \boldsymbol{R})$. Then the set $AB=\{ab\,;\,a\in A,\, b\in B\}$ is locally compact.*

L. Pukanszky kindly informed the author that related or similar results have been announced or published by C. Chevalley, J. Dixmier, L. Pukanszky and M. Rosenlicht (cf. [2], [6] and [7]).

§2. Algebraic group case.

Let A be an algebraic subgroup of $GL(n, \boldsymbol{R})$. The set of polynomials over $\boldsymbol{R}$ defining A in $GL(n, \boldsymbol{R})$ also defines an algebraic subgroup of $GL(n, \boldsymbol{C})$, which will be called the *complexification* of A. Let B also be an algebraic subgroup of $GL(n, \boldsymbol{R})$, and let $\tilde{A}$ and $\tilde{B}$ denote the complexifications of A and B, respectively. The product group $\tilde{A}\times\tilde{B}$ acts on $GL(n, \boldsymbol{C})$ as a group of rational transformations: for $\alpha\in\tilde{A}$, $\beta\in\tilde{B}$ and $g\in GL(n, \boldsymbol{C})$, let $g(\alpha, \beta)=\alpha^{-1}g\beta$. $\tilde{A}\tilde{B}$ is an orbit of the transformation group, and is Zariski open in the Zariski closure of $\tilde{A}\tilde{B}$, by a theorem of Chevalley [1]. Hence, in particular, $\tilde{A}\tilde{B}$ is a locally compact set.

Let π denote the map $\tilde{A}\times\tilde{B}\ni(\alpha, \beta)\mapsto\pi(\alpha, \beta)=\alpha^{-1}\beta\in\tilde{A}\tilde{B}$. Because $A\times\tilde{B}$ is a locally compact group which is a union of countably many compact sets and $\tilde{A}\tilde{B}$ is a locally compact Hausdorff space, the map π is open and induces a homeomorphism between the coset space $D\backslash(\tilde{A}\times\tilde{B})=\{D(\alpha, \beta)\,;\,\alpha\in\tilde{A},\, \beta\in\tilde{B}\}$ and $\tilde{A}\tilde{B}$, where $\tilde{C}=\tilde{A}\cap\tilde{B}$ and $D=\{(\gamma, \gamma)\,;\,\gamma\in\tilde{C}\}$.

We put $R=\tilde{A}\tilde{B}\cap GL(n, \boldsymbol{R})$ and $S=\pi^{-1}(R)$. Then R is a locally compact set. In order to prove that AB is open in R, it suffices to show that $\pi^{-1}(AB)=D(A\times B)$ is open in S. On the other hand, the product group $\tilde{C}\times A\times B$ is acting as a transformation group on S: for $\gamma\in\tilde{C}$, $a\in A$, $b\in B$ and $(\alpha, \beta)\in S$, let $(\alpha, \beta)(\gamma, a, b)=(\gamma^{-1}\alpha a, \gamma^{-1}\beta b)$. Then $\pi^{-1}(AB)$ is an orbit of the transformation group passing through (e, e), where e denotes the identity matrix. Thus, in order to prove the theorem for algebraic A and B, it is enough to find an open set U of S with $D(A\times B)\supset U\ni(e, e)$.

For a matrix ξ over $\boldsymbol{C}$, let us denote by $\bar{\xi}$ the matrix whose entries are the complex conjugate of the corresponding entries of ξ. An element ξ of $GL(n, \boldsymbol{C})$, sufficiently close to e and satisfying $\xi\bar{\xi}=e$, can be written as $\xi=\exp X$, $X+\bar{X}=0$. Hence, putting $\eta=\exp\frac{1}{2}X$ we have that $\xi=\eta^2$ and $\eta\bar{\eta}=e$. Therefore we can find a neighborhood V of e in $\tilde{C}$ such that if $V\ni\gamma$ and $\gamma\bar{\gamma}=e$ then there exists $\delta\in V$ with $\delta^2=\gamma$ and $\delta\bar{\delta}=e$. For this V, let us take a neighborhood W of e in $\tilde{A}$ such that $W\ni\alpha$ and $\bar{\alpha}\alpha^{-1}\in\tilde{C}$ implies that $\bar{\alpha}\alpha^{-1}\in V$.

Let us put $U=(W\times\tilde{B})\cap S$. U is an open subset of S. For $(\alpha, \beta)\in U$ we have that $\alpha^{-1}\beta=\overline{\alpha^{-1}\beta}=\bar{\alpha}^{-1}\bar{\beta}$. Since $\alpha\in\tilde{A}$ implies $\bar{\alpha}\in\tilde{A}$, we have that $\gamma=\bar{\alpha}\alpha^{-1}=\bar{\beta}\beta^{-1}\in\tilde{C}$. Hence $\gamma\in V$. Because $\gamma\bar{\gamma}=e$ we can find $\delta\in\tilde{C}$ with $\delta^2=\gamma$ and $\delta\bar{\delta}=e$. Then $\overline{\delta\alpha}=\bar{\delta}\bar{\alpha}=\delta^{-1}\gamma\alpha=\delta\alpha$, and $\delta\beta=\overline{\delta\beta}$. Hence $(\alpha, \beta)=(\delta^{-1}, \delta^{-1})(\delta\alpha, \delta\beta)\in D(A\times B)$.

Thus the theorem is proved for algebraic A and B.

REMARK. In Goto [**3**], the author mistakenly mentioned that for algebraic groups A and B in $GL(n, \boldsymbol{R})$, AB is Zariski open in its Zariski closure. He still does not know if it is true, although he has an impression it will not be. However for the purposes of the paper [**3**], the result of this section is sufficient.

§3. Proof of Theorem.

Let A_1 and B_1 be pre-algebraic subgroups of $GL(n, \boldsymbol{R})$. Let A_2 and B_2 be the smallest algebraic groups containing A_1 and B_1, respectively. We set

$$A_0=(A_1)_e=(A_2)_e\,,\qquad B_0=(B_1)_e=(B_2)_e\,,$$

$$C_2=A_2\cap B_2\quad\text{and}\quad D_2=\{(c, c)\,;\, c\in C_2\}\,.$$

Then obviously $D_2(A_1\times B_1)$ is a union of cosets of $A_0\times B_0$ in the direct product group $A_2\times B_2$, and is open and closed in $A_2\times B_2$. Hence A_1B_1 is open and closed in A_2B_2.

Next, let A and B be semi-algebraic subgroups of $GL(n, \boldsymbol{R})$. We suppose that $\mathcal{A}(A)=A_1$ and $\mathcal{A}(B)=B_1$. Then $A\times B$ is a closed normal subgroup of $A_1\times B_1$ and the factor group $(A_1\times B_1)/(A\times B)$ is a vector group. We set $C_1=A_1\cap B_1$ and $D_1=\{(c, c)\,;\, c\in C_1\}$. Then C_1 is pre-algebraic, and has only finitely many connected components. Hence $D_0=(D_1)_e$ is of finite index in D_1. Because any connected Lie subgroup of a vector group is closed, $D_0(A\times B)/(A\times B)$ is a closed subgroup of $(A_1\times B_1)/(A\times B)$, and so $D_0(A\times B)$ is closed in $A_1\times B_1$. On the other hand, $(A_1\times B_1)/D_0(A\times B)$ is a vector group and contains no element of finite order except the identity, and $D_0(A\times B)$ is of finite index in $D_1(A\times B)$. Hence $D_1(A\times B)=D_0(A\times B)$. Since $D_1(A\times B)$ is closed in $A_1\times B_1$, AB is closed in A_1B_1. This completes the proof of the theorem.

Department of Mathematics
University of Pennsylvania
Philadelphia, Pa. 19104
U. S. A.

References

[1] C. Chevalley, Théorie des groupes de Lie, Tome III, Paris, 1955.
[2] J. Dixmier, Représentations induites holomorphes des groupes résolubles algébraiques, Bull. Soc. Math. France, **94** (1966), 181–206.
[3] M. Goto, Orbits of one-parameter groups II (Linear group case), J. Math. Soc. Japan, **22** (1970), 123–133.
[4] M. Goto, Orbits of one-parameter groups III (Lie group case), J. Math. Soc. Japan, **23** (1971), 95–102.
[5] M. Goto and H. C. Wang, Non-discrete uniform subgroups of semi-simple Lie groups, Math. Ann., **198** (1972), 259–286.
[6] B. L. Pukanszky, Unitary representations of solvable Lie groups, Ann. Sci. École Norm. Sup., **4** (1971), 457–608.
[7] M. Rosenlicht, A remark on quotient spaces, An. Acad. Brasil Ci., **35** (1963), 487–489.