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Model theory on a positive second order logic with countable conjunctions and disjunctions

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Introduction.

This paper is a sequel to our paper [8], in Chapter V of which we developed a general theory of so-called "preservation theorem" without using any model theoretic notions, so that we can apply it to different kinds of logics. In this paper, we shall show some applications of it to the model theory on a positive second order logic $\mathfrak{L}$ (in the sense of [17]) with countable conjunctions and disjunctions. (Cf. Theorem 4.1 and Theorem 4.2 in §4.)

Suppose $(\exists\xi)\varphi(\xi)$, $(\forall\eta)\psi(\eta)$, φ_1, ψ_1 are sentences in the second order logic such that $\varphi(\xi)$, $\psi(\eta)$, φ_1, ψ_1 have no second order quantifiers and ξ, η are second order variables. Notice that the sentence $(\exists\xi)\varphi(\xi)\supset(\forall\eta)\psi(\eta)$ is a sentence in the positive second order logic $\mathfrak{L}$. Hence, Craig's interpolation theorem can be expressed in the following form:

(1) If $\vdash(\exists\xi)\varphi(\xi)\supset(\forall\eta)\psi(\eta)$, then $\vdash(\exists\xi)\varphi(\xi)\supset\theta$ and $\vdash\theta\supset(\forall\eta)\psi(\eta)$ for some *first order* sentence θ.

Also, Łos-Tarski's theorem on extension can be expressed in the following form:

(2) If every extension of a model of φ_1 is a model of ψ_1 then $\vdash\varphi_1\supset\theta$ and $\vdash\theta\supset\psi_1$ for some *existential* sentence θ.

Combining (1) and (2), we can get

(3) If every extension of a model of $(\exists\xi)\varphi(\xi)$ is a model of $(\forall\eta)\psi(\eta)$, then $\vdash(\exists\xi)\varphi(\xi)\supset\theta$ and $\vdash\theta\supset(\forall\eta)\psi(\eta)$ for some *first order existential* sentence θ.

This is an example of preservation theorems in the positive second order logic $\mathfrak{L}$. Our Theorem 4.1 is a generalization of the preservation theorems of the form (3) to $L_{\omega_1\omega}$.

On the other hand, Tarski's theorem on PC_δ-class can be expressed in the following form:

(4) The class of substructures of models of $(\exists\xi)\varphi(\xi)$ is an EC_δ-class.

Our Theorem 4.2 is a generalization of infinitary analogues of (4) to $L_{\omega_1\omega}$. After some preparations in §1, 2 and 3, we shall prove these theorems in

§4. In §5, we shall see many instances of our main theorems with respect to different morphisms between models e. g. identity relation, homomorphism, strong homomorphism, extension, substructure, U-extension, end-extension, super end-extension, endomorphism, retract, direct root of direct power, direct factor, k-isomorphic union and ω-isomorphic union.

§1. Logic $\mathfrak{L}$.

Let $\mathfrak{L}$ be a fixed second order logic with equality $\doteq$. We do not consider $\doteq$ as a logical symbol, but as a designated binary predicate constant. For simplicity, we assume that $\mathfrak{L}$ has neither function symbols nor individual constant symbols. Let $PF(\mathfrak{L})$, $PB(\mathfrak{L})$, $PC(\mathfrak{L})$, $FV(\mathfrak{L})$, $BV(\mathfrak{L})$ and $FM(\mathfrak{L})$ be the set of free predicate variables (denoted by $\alpha, \beta, \cdots$), the set of bound predicate variables (denoted by $\xi, \eta, \cdots$), the set of predicate constants (denoted by $P, \cdots$), the set of free individual variables (denoted by $x, y, w, \cdots$), the set of bound individual variables (denoted by $u, v, \cdots$) and the set of formulas (denoted by $\theta, \varphi, \psi, \cdots$) in $\mathfrak{L}$ respectively. We assume that $\doteq \in PC(\mathfrak{L})$, $PF(\mathfrak{L})$ and $FV(\mathfrak{L})$ are countably infinite, $PB(\mathfrak{L})$ and $BV(\mathfrak{L})$ are uncountably infinite, $PF(\mathfrak{L})$ and $PB(\mathfrak{L})$ have sufficiently many k-ary predicate variables for each $k<\omega$. As logical symbols, we shall use $\neg$(negation), $\wedge$(countable conjunction), $\vee$(countable disjunction), $\forall$(universal quantification) and $\exists$(existential quantification). Moreover, $\mathfrak{L}$ has two propositional constants $\top$(truth) and $\perp$(false). We shall use $\supset$(implication), $\equiv$(equivalence) as abbreviations as usual. For any $\theta \in FM(\mathfrak{L})$, let $PF(\theta)$, $FV(\theta)$ and $V(\theta)$ be the set of free predicate variables in θ, the set of free individual variables in θ and the set of free variables in θ. We always assume that $V(\theta)$ is finite for each $\theta \in FM(\mathfrak{L})$ throughout this paper. A sentence θ is a formula such that $V(\theta)=\phi$.

Let t, s be two variables (free or bound, predicate or individual). We say that "t and s are of the same type" if both t and s are individual variables or both t and s are predicate variables with the same number of argument places. Let $\vec{t}$ and $\vec{s}$ are two finite sequences of variables. $\vec{t}$ and $\vec{s}$ are called of the same type if the lengths of $\vec{t}$ and $\vec{s}$ are the same and the i-th variable of $\vec{t}$ and the i-th variable of $\vec{s}$ are of the same type for each i.

Let θ be a formula, $\vec{t}$ and $\vec{s}$ be two sequences of variables of the same type such that all variables in $\vec{t}$ are free variables and distinct. By $\theta\binom{\vec{t}}{\vec{s}}$, we mean the expression (not always formula) obtained from θ by substituting $t_1, \cdots, t_n$ by $s_1, \cdots, s_n$ respectively, where n is the length of $\vec{t}$ and $\vec{s}$. When no confusion is to be feared we shall write θ in this situation by $\theta(\vec{t})$ and $\theta\binom{\vec{t}}{\vec{s}}$ by $\theta(\vec{s})$.

A *quasi* $\mathfrak{L}$-*structure* $\mathfrak{A}$ is a mapping from $PC(\mathfrak{L})$ to the set of all finitary relations on a non empty set $|\mathfrak{A}|$ (called the universe of $\mathfrak{A}$) such that $\mathfrak{A}(P) \subset |\mathfrak{A}|^k$ for each k-ary $P \in PC(\mathfrak{L})$ and $\mathfrak{A}(\doteq)$ is a congruence relation with respect to $\mathfrak{A}(P)$ for each $P \in PC(\mathfrak{L})$; $\mathfrak{A}(\doteq)$ is an equivalence relation on $|\mathfrak{A}|$ and for any k-ary predicate constant $P \in PC(\mathfrak{L})$, any k-ary sequences $\langle a_1, \cdots, a_k\rangle$, $\langle b_1, \cdots, b_k\rangle$ of elements in $|\mathfrak{A}|$ such that $\langle a_1, b_1\rangle \in \mathfrak{A}(\doteq), \cdots, \langle a_k, b_k\rangle \in \mathfrak{A}(\doteq)$, $\langle a_1, \cdots, a_k\rangle \in \mathfrak{A}(P)$ implies $\langle b_1, \cdots, b_k\rangle \in \mathfrak{A}(P)$. $\mathfrak{A}$ is said to be *countable* if $|\mathfrak{A}|$ is countable. If $\mathfrak{A}(\doteq)$ is the identity relation on $|\mathfrak{A}|$, $\mathfrak{A}$ is said to be an $\mathfrak{L}$-structure. For any quasi $\mathfrak{L}$-structure $\mathfrak{A}$, we shall associate an $\mathfrak{L}$-structure $\mathfrak{A}^*$ by $|\mathfrak{A}^*| = |\mathfrak{A}|/\mathfrak{A}(\doteq)$, $\mathfrak{A}^*(P) = \mathfrak{A}(P)/\mathfrak{A}(\doteq)$, $P \in PC(\mathfrak{L})$ (usual factorization through the equivalence relation $\mathfrak{A}(\doteq)$). For each $a \in |\mathfrak{A}|$, by a^* we shall denote the equivalence class of a. By an *assignment* τ in $\mathfrak{A}$, we mean a mapping from $PF(\mathfrak{L}) \cup FV(\mathfrak{L})$ to the union set of $|\mathfrak{A}|$ and the set of all finitary relations on $|\mathfrak{A}|$, compatible with $\mathfrak{A}(\doteq)$, such that $\tau(x) \in |\mathfrak{A}|$ and $\tau(\alpha) \subseteq |\mathfrak{A}|^k$ for any $x \in FV(\mathfrak{L})$ and any k-ary $\alpha \in PF(\mathfrak{L})$. For any assignment τ in $\mathfrak{A}$, τ^* is an assignment in $\mathfrak{A}^*$ naturally defined by τ and $\mathfrak{A}$. If $\mathfrak{A}$ is a quasi-$\mathfrak{L}$-structure, τ is an assignment in $\mathfrak{A}$ and θ is a formula, then the notion "θ is satisfied in $\mathfrak{A}$ by τ" (denoted by $\mathfrak{A} \models \theta[\tau]$) can be defined as usual. Obviously $\mathfrak{A} \models \theta[\tau]$ if and only if $\mathfrak{A}^* \models \theta[\tau^*]$. If $FV(\theta) = \{x_1, \cdots, x_n\}$, $PF(\theta) = \{\alpha_1, \cdots, \alpha_m\}$, $\tau(x_1) = a_1, \cdots, \tau(x_n) = a_n$, $\tau(\alpha_1) = Q_1, \cdots, \tau(\alpha_m) = Q_m$, we shall write $\mathfrak{A} \models \theta\left[{}^{x_1, \cdots, x_n, \alpha_1, \cdots, \alpha_m}_{a_1, \cdots, a_n, Q_1, \cdots, Q_m}\right]$ instead of $\mathfrak{A} \models \theta[\tau]$. If in particular θ is a sentence and $\mathfrak{A}$ is an $\mathfrak{L}$-structure, we shall write $\mathfrak{A} \models \theta$ and say that $\mathfrak{A}$ is a *model* of θ. For any set Ψ of sentences in $\mathfrak{L}$, $\mathfrak{A}$ is a model of Ψ if $\mathfrak{A}$ is a model of every θ in Ψ. θ is said to be valid (written by $\models \theta$) if for any $\mathfrak{L}$-*structure* $\mathfrak{A}$ and any assignment τ in $\mathfrak{A}$, '$\mathfrak{A} \models \theta[\tau]$' holds (of course this is equivalent to "for any *quasi-*$\mathfrak{L}$-*structure* $\mathfrak{A}$ and any assignment τ in $\mathfrak{A}$, '$\mathfrak{A} \models \theta[\tau]$' holds"). Let $PFM(\mathfrak{L}) = \{\theta ; \models \theta\}$. Then obviously $PFM(\mathfrak{L})$ satisfies the requirements mentioned in § 1 of Chapter I of Motohashi [8]. If $\theta \in PFM(\mathfrak{L})$, we write $\vdash_{\mathfrak{L}} \theta$. Hence $\models \theta$ is equivalent to $\vdash_{\mathfrak{L}} \theta$.

Let $\mathfrak{A}$, $\mathfrak{B}$ be two $\mathfrak{L}$-structures and f be a mapping from $|\mathfrak{A}|$ to $|\mathfrak{B}|$. Then $f(\mathfrak{A})$ is an $\mathfrak{L}$-structure defined by $|f(\mathfrak{A})| = \{f(a) : a \in |\mathfrak{A}|\}$ (denoted by $f(|\mathfrak{A}|)$) and $(f(\mathfrak{A}))(P) = \{\langle f(a_1) \cdots f(a_k)\rangle : \langle a_1 \cdots a_k\rangle \in \mathfrak{A}(P)\}$ (denoted by $f(\mathfrak{A}(P))$). If f is a bijection and $f(\mathfrak{A}) = \mathfrak{B}$, we say that f is an *isomorphism* of $\mathfrak{A}$ to $\mathfrak{B}$. If there is an isomorphism of $\mathfrak{A}$ to $\mathfrak{B}$, we say that $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic and write $\mathfrak{A} \cong \mathfrak{B}$. If f is a surjection and $f(\mathfrak{A}(P)) \subseteq \mathfrak{B}(P)$ for each $P \in PC(\mathfrak{L})$, we say that f is a homomorphism of $\mathfrak{A}$ to $\mathfrak{B}$. If there is a homomorphism of $\mathfrak{A}$ to $\mathfrak{B}$, we say that $\mathfrak{A}$ is homomorphic to $\mathfrak{B}$ and write $\mathfrak{A} \precsim \mathfrak{B}$. If f is an injection and $f(\mathfrak{A}(P)) = \mathfrak{B}(P) \cap (f(|\mathfrak{A}|))^k$ for each k-ary $P \in PC(\mathfrak{L})$, we say that f is an embedding of $\mathfrak{A}$ to $\mathfrak{B}$. If $|\mathfrak{A}| \subseteq |\mathfrak{B}|$ and the inclusion mapping is an embedding, we say that $\mathfrak{A}$ is a *substructure* of $\mathfrak{B}$ or $\mathfrak{B}$ is an *extension* of $\mathfrak{A}$ and write $\mathfrak{A} \subseteq \mathfrak{B}$. The *product* of $\mathfrak{A}$ and $\mathfrak{B}$ (denoted by $\mathfrak{A} \times \mathfrak{B}$) is an $\mathfrak{L}$-structure $\mathfrak{C}$ defined by $|\mathfrak{C}|=|\mathfrak{A}|\times|\mathfrak{B}|$, $\mathfrak{C}(P)=\{\langle\langle a_1, b_1\rangle, \cdots, \langle a_k, b_k\rangle\rangle ; \langle a_1, \cdots, a_k\rangle\in\mathfrak{A}(P), \langle b_1, \cdots, b_k\rangle \in\mathfrak{B}(P)\}$ for any k-ary $P\in PC(\mathfrak{L})$. Let S be a set of $\mathfrak{L}$-structures. Then the union of S is the $\mathfrak{L}$-structure $\mathfrak{A}$ (denoted by $\cup S$) such that $|\mathfrak{A}|=\bigcup_{\mathfrak{B}\in S}|\mathfrak{B}|$, $\mathfrak{A}(P)=\bigcup_{\mathfrak{B}\in S}\mathfrak{B}(P)$ for each $P\in PC(\mathfrak{L})$. S is called a k-family if for any $a_1, \cdots, a_k\in |\cup S|$, there is a $\mathfrak{A}\in S$ such that $a_1, \cdots, a_k\in|\mathfrak{A}|$. S is called an ω-family if S is a k-family for any $k<\omega$. If S is a k-family, $\cup S$ is called the *k-union* of S. If S is an ω-family, $\cup S$ is called the *ω-union* of S.

A formula is called *first order* if it does not contain any second order quantifiers.

Now, we define recursively as follows (following [**17**]) the notion "a formula is positive or negative with respect to the second order quantifiers", which will be expressed by "a formula is positive or negative" for brevity.

(1) Every first order formula is positive and negative.

(2) If $\theta(\alpha)$ is positive, then $(\forall\xi)\theta(\xi)$ is positive.
If $\theta(\alpha)$ is negative, then $(\exists\xi)\theta(\xi)$ is negative.

(3) $(\forall v)\theta(v)$, $(\exists v)\theta(v)$ are positive or negative according as θ is positive or negative.

(4) $\neg\theta$ is positive or negative according as θ is negative or positive.

(5) Let $\wedge\Phi$, $\vee\Phi$ be formulas in $\mathfrak{L}$.
If all formulas in Φ are positive, then $\wedge\Phi$, $\vee\Phi$ are positive.
If all formulas in Φ are negative, then $\wedge\Phi$, $\vee\Phi$ are negative.

(6) All positive or negative formulas are obtained by (1)-(5).

Now we state a well-known theorem due to D. Scott in the terminology defined above. This theorem will be used in § 4.

(I) Scott's isomorphism theorem: *For any countable $\mathfrak{L}$-structure $\mathfrak{A}$, there is a first order sentence φ (called a Scott sentence of $\mathfrak{A}$) such that for any countable $\mathfrak{L}$-structure $\mathfrak{B}$,*

$$\mathfrak{B}\models\varphi \text{ if and only if } \mathfrak{A}\cong\mathfrak{B}.$$

(Cf. Chang [**3**], Keisler [**7**], Scott [**15**]).

§ 2. Logic $\mathfrak{L}^I$.

From $\mathfrak{L}$, we construct two object logics $\mathfrak{L}^1$ and $\mathfrak{L}^2$ by the method in [**20**]. Then for each $\theta\in FM(\mathfrak{L})$, θ^1 is the formula of $\mathfrak{L}^1$ obtained from θ by replacing every α, ξ, P, $\top$, $\perp$ in θ by α^1, ξ^1, P^1, $\top^1$, $\perp^1$ respectively. Similarly θ^2 is obtained.

Let $\{I_i\}_{i<\omega}$ be a set of binary predicate constants which do not appear in $\mathfrak{L}$, $\mathfrak{L}^1$, $\mathfrak{L}^2$. Then the formation rules of formulas in L^I can be expressed as follows;

(1) $I_i(x, y)$ is a formula in L^I for each $i<\omega$, x, y.

(2) θ^1, θ^2 are formulas in L^I for each $\theta \in FM(\mathfrak{L})$.

(3) If F is a formula in L^I, then $\neg F$ is a formula in L^I.

(4) If K is a non empty countable set of formulas in L^I such that the set of free variables in K is finite, then $\wedge K$, $\vee K$ are formulas in L^I.

(5) If $F(x)$ is a formula in L^I and v does not occur in $F(x)$, then $(\forall v)F(v)$, $(\exists v)F(v)$ are formulas in L^I.

(6) All the formulas in L^I are obtained from (1)-(5).

By F, G (with or without suffixes) we shall denote formulas in L^I and $FM(L^I)$ the set of formulas in L^I. Every notion above mentioned with respect to formulas in $\mathfrak{L}$ is also used to formulas in L^I (e.g. $FV(F)$, $V(F)$, $\cdots$). A formula F in L^I is said to be a 1-formula if $F=\theta^1$ for some $\theta \in FM(\mathfrak{L})$ and a 2-formula if $F=\theta^2$ for some $\theta \in FM(\mathfrak{L})$. If F is neither 1-formula nor 2-formula, F is called an *I-formula.* As for formulas in L^I, we can define a notion "a formula is positive or negative (with respect to the second order quantifiers)" by the same method as in the case of formulas in $\mathfrak{L}$.

An L^I-structure M is a mapping from $\{I_i\}_{i<\omega} \cup \{P^1, P^2; P \in PC(\mathfrak{L})\}$ to the set of all finitary relations on a non-empty set $|M|$ (called the universe of M), such that M_1 and M_2 are quasi-$\mathfrak{L}$-structures, where $|M_1|=|M_2|=|M|$, $M_1(P)=M(P^1)$, $M_2(P)=M(P^2)$, for each $P \in PC(\mathfrak{L})$ and $M(I_i) \subseteq |M|^2$ for each I_i. For each M, each $i<\omega$, let $M(I_i)^* = \{\langle a^*, b^*\rangle ; \langle a, b\rangle \in M(I_i),\ a^* \in |M_1^*|,\ b^* \in |M_2^*|\}$. Then $M(I)^*$ is a relation between M_1^* and M_2^*. M is countable if $|M|$ is countable. Then the notions "an assignment ν in M", "$M \models F$", "M is a model of F" "$\models F$" are defined similarly as in §1.

As for axioms, inference rules and derivations of L^I, we use those described in [20]. Then by the straight-forward generalization of the method in Lopez-Escobar [14], we can get the following completeness theorem for positive sentences in L^I and Löwenheim-Skolem's theorem for positive sentences in L^I.

(II) Completeness theorem for positive sentences in L^I: $\models F \Leftrightarrow \vdash_{L^I} F$ *for any positive sentence* F *in* L^I.

(III) Löwenheim-Skolem's theorem for positive sentences in L^I (c.f. Vaught [18]): *For any positive sentence F, if $\neg F$ has a model, then $\neg F$ has a countable model.*

§3. Interpolation theorem and characterization theorem.

In this section, we shall quote some results in the Chapter V of the paper [8], which will be used in §4. For simplicity, we shall omit (I-) from every notion, i.e. we shall write "Interpolation theorem" instead of "I-interpolation theorem" etc. Divide $FV(\mathfrak{L})$ into mutually disjoint, infinite sets $\{V_i\}_{i<\omega}$. By

w^i (with or without suffixes) we shall denote an element in V_i. Let Ψ be a set of sentences in L^I.

Then the *invariant set* $In(\Psi)$ of Ψ is the set of all formulas $\theta(w^i)$ such that $\Psi \underset{LI}{\vdash} (\forall\vec{u})(\forall\vec{v})(I_i(\vec{u}, \vec{v}) \wedge \theta^1(\vec{u}) \supset \theta^2(\vec{v}))$, where w^i is a repetition-free enumeration of free individual variables in θ and $I_i(\vec{u}, \vec{v})$ is $I_{i_1}(u_1, v_1) \wedge \cdots \wedge I_{i_n}(u_n, v_n)$.

A set $\varDelta \subseteq In(\Psi)$ is a *characterization set* of Ψ if for any formula $\theta \in In(\Psi)$, there is a formula $\theta' \in \varDelta$ such that $\underset{\mathfrak{L}}{\vdash} \theta \equiv \theta'$ and every free variable in θ' occurs in θ.

3.1. DEFINITION. Ψ is *interpolatable* if for any finite sequence of distinct free variables w^i, any formulas $\varphi(w^i)$, $\phi(w^i)$, $\Psi \underset{LI}{\vdash} (\forall\vec{u})(\forall\vec{v})(I_i(\vec{u}, \vec{v}) \wedge \varphi^1(\vec{u}) \supset \phi^2(\vec{v}))$ implies $\underset{\mathfrak{L}}{\vdash} \varphi \supset \theta$ and $\underset{\mathfrak{L}}{\vdash} \theta \supset \phi$ for some $\theta \in In(\Psi)$ whose free individual variables are among w^i.

3.2. DEFINITION OF PRIMITIVE SENTENCES.

(1) $(\forall\vec{u})(\forall\vec{v})(I_i(\vec{u}, \vec{v}) \wedge \phi^1(\vec{u}) \supset \phi^2(\vec{v}))$ is the primitive sentence of type 1 defined by $\phi(w^i)$, where w^i is a repetition-free enumeration of free variables in ϕ.

(2) $(\forall\vec{u})(\exists\vec{v})I_i(\vec{u}, \vec{v})$ is the primitive sentence of type 2 defined by $\vec{i}$.

(3) $(\forall\vec{v})(\exists\vec{u})I_i(\vec{u}, \vec{v})$ is the primitive sentence of type 3 defined by $\vec{i}$.

(4) $(\forall\vec{u})(\exists\vec{v}) \bigvee_{i_0 \leqq i} I_i(\vec{u}, \vec{v})$ is the primitive sentence of type 4 defined by (i_0, n), where n is the length of $\vec{u}$.

(5) $(\forall\vec{v})(\exists\vec{u}) \bigvee_{i_0 \leqq i} I_i(\vec{u}, \vec{v})$ is the primitive sentence of type 5 defined by (i_0, n), where n is the length of $\vec{v}$.

(6) $(\forall\vec{u}_2)(\forall\vec{v}_2)(\forall\vec{u}_1)(\exists\vec{v}_1)(\varphi^1(\vec{u}_1, \vec{u}_2) \wedge I_{\vec{j}}(\vec{u}_2, \vec{v}_2) \supset I_i(\vec{u}_1, \vec{v}_1))$ is the primitive sentence of type 6 defined by (φ, w^i) where $w^i \frown w_1^{\vec{j}}$ is a repetition-free enumeration of free variables in $\varphi(w^i, w_1^{\vec{j}})$.

(7) $(\forall\vec{v}_2)(\forall\vec{u}_2)(\forall\vec{v}_1)(\exists\vec{u}_1)(\varphi^2(\vec{v}_1, \vec{v}_2) \wedge I_{\vec{j}}(\vec{u}_2, \vec{v}_2) \supset I_i(\vec{u}_1, \vec{v}_1))$ is the primitive sentence of type 7 defined by (φ, w^i), where $w^i \frown w_1^{\vec{j}}$ is a repetition-free enumeration of free variables in $\varphi(w^i, w_1^{\vec{j}})$.

(8) $(\forall\vec{u})(\exists\vec{v})(\phi^1(\vec{u}) \supset I_i(\vec{u}, \vec{v}) \wedge \phi^2(\vec{v}))$ is the primitive sentence of type 8 defined by $\phi(w^i)$, where w^i is a repetition-free enumeration of free variables in ϕ.

(9) $(\forall\vec{v})(\exists\vec{u})(\phi^2(\vec{v}) \supset I_i(\vec{u}, \vec{v}) \wedge \phi^1(\vec{u}))$ is the primitive sentence of type 9 defined by $\phi(w^i)$, where w^i is a repetition-free enumeration of free variables in ϕ.

3.3. DEFINITION OF PRIMITIVE SETS. A set Ψ of sentences in L^I is primitive if

(1) every sentence in Ψ is primitive,

(2) if Ψ has the primitive sentence of type 6 (or 7) defined by (φ, w^i),

then Ψ has the primitive sentence of type 1 defined by φ,

(3) if Ψ has the primitive sentence of type 4 (or 5) defined by (i_0, n) then

(a) Ψ has the primitive sentence of type 4 (or 5) defined by (i_0+1, n),

(b) if $F \in \Psi$ is a primitive sentence whose type is neither 4 nor 5, and $i \geqq i_0$ then $F\binom{i_0}{i} \in \Psi$.

A primitive set Ψ is called *of first order* if *every sentence in* Ψ *is of first order and* Ψ *is countable.*

3.4. DEFINITION OF $\Delta(\Psi)$

(1) If Ψ has the primitive sentence of type 1 defined by ϕ, then $\phi \in \Delta(\Psi)$.

(2) If Ψ has the primitive sentence of type 2 defined by $\vec{i}$ and $\theta(w^i) \in \Delta(\Psi)$, then $(\exists \vec{u})\theta(\vec{u}) \in \Delta(\Psi)$.

(3) If Ψ has the primitive sentence of type 3 defined by $\vec{i}$ and $\theta(w^i) \in \Delta(\Psi)$, then $(\forall \vec{u})\theta(\vec{u}) \in \Delta(\Psi)$.

(4) If Ψ has the primitive sentence of type 4 defined by (i_0, n) and $\theta(\vec{w}^{i_0}) \in \Delta(\Psi)$, where the length of $\vec{w}^{i_0}$ is n and every free variable in θ except those in $\vec{w}^{i_0}$ belongs to $\bigcup_{j<i_0} V_i$, then $(\exists \vec{u})\theta(\vec{u}) \in \Delta(\Psi)$.

(5) If Ψ has the primitive sentence of type 5 defined by (i_0, n) and $\theta(\vec{w}^{i_0}) \in \Delta(\Psi)$, where the length of $\vec{w}^{i_0}$ is n and every free variable in θ except those in $\vec{w}^{i_0}$ belongs to $\bigcup_{i<i_0} V_i$, then $(\forall \vec{u})\theta(\vec{u}) \in \Delta(\Psi)$.

(6) If Ψ has the primitive sentence of type 6 defined by $(\varphi(w^i), w^i)$ and $\theta(w^i) \in \Delta(\Psi)$, then $(\exists \vec{u})(\varphi(\vec{u}) \wedge \theta(\vec{u})) \in \Delta(\Psi)$.

(7) If Ψ has the primitive sentence of type 7 defined by $(\varphi(w^i), w^i)$ and $\theta(w^i) \in \Delta(\Psi)$, then $(\forall \vec{u})(\varphi(\vec{u}) \supset \theta(\vec{u})) \in \Delta(\Psi)$.

(8) If Ψ has the primitive sentence of type 8 defined by $\phi(w^i)$ and $\theta(w^i) \in \Delta(\Psi)$, then $(\exists \vec{u})(\phi(\vec{u}) \wedge \theta(\vec{u})) \in \Delta(\Psi)$.

(9) If Ψ has the primitive sentence of type 9 defined by $\phi(w^i)$ and $\theta(w^i) \in \Delta(\Psi)$, then $(\forall \vec{u})(\phi(\vec{u}) \supset \theta(\vec{u})) \in \Delta(\Psi)$.

(10) $\top$, $\bot \in \Delta(\Psi)$.

(11) If Φ is a non-empty, countable set of formulas in $\Delta(\Psi)$ such that only finitely many free variables occur in Φ, then $\wedge\Phi$, $\vee\Phi \in \Delta(\Psi)$.

(12) If $\theta(w^i) \in \Delta(\Psi)$, then $\theta(w^i_1) \in \Delta(\Psi)$, where w^i is a finite sequence of distinct free variables and $w^i = \langle w_1^{i_1}, \cdots, w_n^{i_n}\rangle$, $w^i_1 = \langle w_{11}^{i_1}, \cdots, w_{1n}^{i_n}\rangle$.

(13) All the formulas in $\Delta(\Psi)$ are obtained from (1)-(12).

When we have the following theorems.

(IV) Interpolation theorem on primitive sets: *Every primitive set is interpolatable.*

(V) Characterization theorem on primitive sets: *$\Delta(\Psi)$ is a characterization set of Ψ for each primitive set Ψ.*

§ 4. Preservation theorems.

Let R be an arbitrary relation between $\mathfrak{L}$-structures (we write $\mathfrak{A}R\mathfrak{B}$ instead of $\langle\mathfrak{A}, \mathfrak{B}\rangle \in R$) and Ψ be a countable set of sentences in L^I. R is expressible in L^I by Ψ if the following two conditions are satisfied:

(*) For any countable $\mathfrak{L}$-structures $\mathfrak{A}$, $\mathfrak{B}$,

$\mathfrak{A}R\mathfrak{B}$ implies $\mathfrak{A} \cong M_1^*$ and $\mathfrak{B} \cong M_2^*$ for some $\mathfrak{M} \models \wedge\Psi$.

(**) For any countable L^I-structure M,

$M \models \wedge\Psi$ implies $\mathfrak{A} \cong M_1^*$ and $\mathfrak{B} \cong M_2^*$ for some $\mathfrak{A}$, $\mathfrak{B}$ such that $\mathfrak{A}R\mathfrak{B}$.

THEOREM 4.1 (Preservation theorem). *Suppose R is expressible in L^I by a first order primitive set Ψ. Then the following two conditions are equivalent for any positive sentence $\varphi \supset \phi$.*

(i) *For any countable $\mathfrak{L}$-structures $\mathfrak{A}$, $\mathfrak{B}$, if $\mathfrak{A}R\mathfrak{B}$ and $\mathfrak{A} \models \varphi$ then $\mathfrak{B} \models \phi$.*

(ii) *There is a sentence $\theta \in \Delta(\Psi)$ such that $\models \varphi \supset \theta$ and $\models \theta \supset \phi$.*

PROOF. (i) $\Leftrightarrow \models \varphi^1 \wedge \wedge\Psi \supset \phi^2$ (By Assumption and (III))

$\Leftrightarrow \Psi \underset{L^I}{\vdash} \varphi^1 \supset \phi^2$ (By (II))

$\Leftrightarrow \underset{\mathfrak{L}}{\vdash} \varphi \supset \theta$ and $\underset{\mathfrak{L}}{\vdash} \theta \supset \phi$ for some sentence $\theta \in \Delta(\Psi)$ (By (IV) (V))

$\Leftrightarrow \models \varphi \supset \theta$ and $\models \theta \supset \phi$ for some sentence $\theta \in \Delta(\Psi)$

$\Leftrightarrow$ (ii). Q. E. D.

THEOREM 4.2. *Suppose R is expressible in L^I by a first order primitive set Ψ. Let φ be a negative sentence in $\mathfrak{L}$. Then the class of all countable $\mathfrak{L}$-structures which are isomorphic to some $\mathfrak{B}$ such that $\mathfrak{A}R\mathfrak{B}$ for some countable $\mathfrak{A} \models \varphi$ is identical to the class of all countable models of the sentences $\theta \in \Delta(\Psi)$ such that $\models \varphi \supset \theta$.*

PROOF. Let S_1 be the class of all countable $\mathfrak{L}$-structures which are isomorphic to some $\mathfrak{B}$ such that $\mathfrak{A}R\mathfrak{B}$ for some countable $\mathfrak{A} \models \varphi$ and S_2 be the class of all countable models of the sentences $\theta \in \Delta(\Psi)$ such that $\models \varphi \supset \theta$. Then clearly $S_1 \subseteqq S_2$ by the assumptions.

Assume $\mathfrak{B} \in S_2 - S_1$. Then $\mathfrak{B} \models \theta$ for any sentence $\theta \in \Delta(\Psi)$ such that $\models \varphi \supset \theta$.

Let ϕ be a Scott sentence of $\mathfrak{B}$ (by (I)).

Now we want to show $\models \varphi^1 \wedge \wedge\Psi \supset \neg\phi^2$.

Assume "not $\models \varphi^1 \wedge \wedge\Psi \supset \neg\phi^2$". Since $\varphi^1 \wedge \wedge\Psi \supset \neg\phi^2$ is positive, we can get a countable L^I-structure M such that $M \models \varphi^1 \wedge \wedge\Psi \wedge \phi^2$ by (III). Hence $M \models \wedge\Psi$, $M_1^* \models \varphi$ and $M_2^* \models \phi$. Since R is expressible in L^I by Ψ, there are $\mathfrak{A}_1$, $\mathfrak{B}_1$ such that $\mathfrak{A}_1 R \mathfrak{B}_1$ and $\mathfrak{A}_1 \cong M_1^*$, $\mathfrak{B}_1 \cong M_2^*$. Hence $\mathfrak{A}_1 \models \varphi$ and $\mathfrak{B}_1 \models \phi$. Since ϕ is a Scott sentence of $\mathfrak{B}$, $\mathfrak{B} \cong \mathfrak{B}_1$. This implies $\mathfrak{B} \in S_1$. Contradiction!

Hence we have $\models \varphi^1 \wedge \wedge\Psi \supset \neg\phi^2$.

By (II), $\Psi \underset{L^I}{\vdash} \varphi^1 \supset \neg\phi^2$.

By (IV), (V), $\models\varphi\supset\theta$ and $\models\theta\supset\neg\psi$ for some sentence $\theta\in\Delta(\Psi)$. This means $\mathfrak{B}\models\theta$ and $\mathfrak{B}\models\neg\theta$. Contradiction! Therefore $S_1=S_2$. Q. E. D.

4.3. REMARK. Assume that $\mathfrak{L}$, L^I are finitary logics. Then the class of all $\mathfrak{L}$-structures which are isomorphic to substructures of some $\mathfrak{B}$ such that $\mathfrak{A}R\mathfrak{B}$ for some $\mathfrak{A}\models\varphi$, is identical to the class of all models of the universal sentences ψ such that $\models\varphi\supset\theta$ and $\models\theta\supset\psi$ for some $\theta\in\Delta(\Psi)$,

i. e. of $S_1'=\{\mathfrak{C}\,;\ \mathfrak{C}\cong{}^{\exists}\mathfrak{C}'\subseteqq{}^{\exists}\mathfrak{B},\ {}^{\exists}\mathfrak{A}R\mathfrak{B},\ \mathfrak{A}\models\varphi\}$,

$S_2'=\{\mathfrak{C}\,;\ \mathfrak{C}\models\psi$, ψ is universal sentence such that

$\models\varphi\supset\theta$ and $\models\theta\supset\psi$ for some $\theta\in\Delta(\Psi)\}$,

then $S_1'=S_2'$.

This is a direct generalization of Tarski's theorem on PC_δ-class (cf. [**16**]).

4.4. PROPOSITION (Local preservation theorem, cf. Reyes [**11**]). *Suppose R is expressible in L^I by a first order primitive set Ψ. For any countable $\mathfrak{L}$-structures $\mathfrak{A}$, $\mathfrak{B}$, there are $\mathfrak{A}_1\cong\mathfrak{A}$, $\mathfrak{B}_1\cong\mathfrak{B}$ such that $\mathfrak{A}_1R\mathfrak{B}_1$ if and only if $\mathfrak{A}\models\theta$ implies $\mathfrak{B}\models\theta$ for any sentence $\theta\in\Delta(\Psi)$.*

PROOF. "Only if" part is obvious. Assume "$\mathfrak{A}\models\theta$ implies $\mathfrak{B}\models\theta$" for any sentence $\theta\in\Delta(\Psi)$.

Let φ, ψ be Scott sentences of $\mathfrak{A}$ and $\mathfrak{B}$ respectively.

Assume not $\mathfrak{A}_1R\mathfrak{B}_1$ for any $\mathfrak{A}_1\cong\mathfrak{A}$, $\mathfrak{B}_1\cong\mathfrak{B}$. Then $\models\varphi^1\wedge\bigwedge\Psi\supset\neg\psi^2$. Hence $\models\psi\supset\theta$ and $\models\theta\supset\neg\psi$ for some sentence $\theta\in\Delta(\Psi)$. Since $\mathfrak{A}\models\varphi$, we have $\mathfrak{A}\models\theta$. Hence $\mathfrak{B}\models\theta$. On the other hand $\mathfrak{B}\models\psi$ and $\models\psi\supset\neg\theta$. Hence $\mathfrak{B}\models\neg\theta$. Contradiction! Therefore for some $\mathfrak{A}_1\cong\mathfrak{A}$, $\mathfrak{B}_1\cong\mathfrak{B}$, $\mathfrak{A}_1R\mathfrak{B}_1$. Q. E. D.

§ 5. Examples of relations expressible in L^I by a first order primitive set.

(5.1) *Identity relation*

Let $R_i=\{\langle\mathfrak{A},\mathfrak{A}\rangle\,;\ \mathfrak{A}$ is an $\mathfrak{L}$-structure$\}$

and
$$\begin{aligned}\Psi_i=\{&(\forall u)(\exists v)I_0(u,v),\ (\forall v)(\exists u)I_0(u,v),\\ &(\forall\vec{u})(\forall\vec{v})(I_0(\vec{u},\vec{v})\wedge P^1(\vec{u})\supset P^2(\vec{v})),\\ &(\forall\vec{u})(\forall\vec{v})(I_0(\vec{u},\vec{v})\wedge\neg P^1(\vec{u})\supset\neg P^2(\vec{v}))\,;\ P\in PC(\mathfrak{L})\}.\end{aligned}$$

Then Ψ_i is a first order primitive set and R_i is expressible in L^I by Ψ_i. $\Delta(\Psi_i)$ is essentially the set of first order formulas in negation normal forms in $\mathfrak{L}$.

The statement obtained from Theorem 4.1 by replacing R, Ψ by R_i, Ψ_i is essentially Craig's interpolation theorem extended by Maehara & Takeuti (Theorem 1 in [**17**]).

(5.2) *Homomorphism*

Let $R_h = \{\langle \mathfrak{A}, \mathfrak{B}\rangle ; \mathfrak{A} \precsim \mathfrak{B}\}$,

$$\Psi_h = \{(\forall u)(\exists v) I_0(u, v),\ (\forall v)(\exists u) I_0(u, v),\ (\forall \vec{u})(\forall \vec{v})(I_0(\vec{u}, \vec{v}) \wedge P^1(\vec{u}) \supset P^2(\vec{v}))\ ;\ P \in PC(\mathfrak{L})\} .$$

Then Ψ_h is a first order primitive set and R_h is expressible in L^I by Ψ_h. $\Delta(\Psi_h)$ is essentially the set of first order formulas having no negation symbols in $\mathfrak{L}$.

The statement obtained from Theorem 4.1 by replacing R, Ψ by R_h, Ψ_h is an extension of Lyndon's theorem [12] in the model theory on finitary first order logic and Lopez-Escobar's theorem [14] in the model theory of $L_{\omega_1\omega}$.

The statement obtained from Theorem 4.2 by replacing R, Ψ by R_h, Ψ_h is an extension of Makkai's theorem [9] in the model theory of $L_{\omega_1\omega}$.

(5.3) *Strong homomorphism*

A homomorphism f of $\mathfrak{A}$ to $\mathfrak{B}$ is a *strong homomorphism* if for any $P \in PC(\mathfrak{L})$, $f(\mathfrak{A}(P)) = \mathfrak{B}(P)$.

Let $R_{sh} = \{\langle \mathfrak{A}, \mathfrak{B}\rangle$; there is a strong homomorphism of $\mathfrak{A}$ to $\mathfrak{B}\}$,

$$\Psi_{sh} = \Psi_h \cup \{(\forall \vec{v})(\exists \vec{u})(P^2(\vec{v}) \supset I_0(\vec{u}, \vec{v}) \wedge P^1(\vec{u}))\ ;\ P \in PC(\mathfrak{L})\} .$$

Then Ψ_{sh} is a first order primitive set and R_{sh} is expressible in L^I by Ψ_{sh}. $\Delta(\Psi_{sh})$ is defined by the least set Δ satisfying the following (1)-(4):

(1) Every atomic formula whose free variables are among V_0 belongs to Δ.

(2) Δ is closed under countable conjunctions and disjunctions.

(3) Δ is closed under the first order quantifications.

(4) If $\theta(\vec{w}^0) \in \Delta$ then $(\forall \vec{u})(P(\vec{u}) \supset \theta(\vec{u})) \in \Delta$.

The statement obtained from Theorem 4.1 by replacing R, Ψ by R_{sh} and Ψ_{sh} is an extension of Keisler's theorem [6] in the model theory of $L_{\omega\omega}$ and Makkai's theorem [9] in the model theory of $L_{\omega_1\omega}$.

The statement obtained from Theorem 4.2 by replacing R, Ψ by R_{sh} and Ψ_{sh} is an extension of Makkai's theorem [9].

(5.4) *Extension*

Let $R_e = \{\langle \mathfrak{A}, \mathfrak{B}\rangle ; \mathfrak{A} \subseteqq \mathfrak{B}\}$,

$$\Psi_e = \Psi_i - \{(\forall v)(\exists u) I_0(u, v)\} .$$

Then Ψ_e is a first order primitive set and R_e is expressible in L^I by Ψ_e. $\Delta(\Psi_e)$ is defined by the least set Δ satisfying the following (1)-(3):

(1), (2) in (5.3).

(3) Δ is closed under first order existential quantifications.

The statement obtained from Theorem 4.1 by replacing R, Ψ by R_e, Ψ_e

is an extension of Łos-Tarski theorem [13], [16] in $L_{\omega\omega}$ and Malitz's theorem [10] in $L_{\omega_1\omega}$.

The statement obtained from Theorem 4.2 by replacing R, Ψ by R_e, Ψ_e is an extension of Bairwise theorem [1] and Makkai's theorem [9] in $L_{\omega_1\omega}$.

(5.5) *Substructure*

Let $R_s = \{\langle\mathfrak{A}, \mathfrak{B}\rangle ; \mathfrak{B} \subseteq \mathfrak{A}\}$,

$$\Psi_s = \Psi_i - \{(\forall u)(\exists v) I_0(u, v)\} .$$

Then Ψ_s is a first order primitive set and R_s is expressible in L^I by Ψ_s.

$\Delta(\Psi_s)$ is defined by the least set Δ satisfying the following (1)-(3):

(1), (2) in (5.3).

(3) Δ is closed under first order universal quantifications.

Above remarks stated in the last paragraph of 5.4 are true of R_s, Ψ_s.

(5.6) *U-extension*

Let U be a fixed unary predicate constant in $\mathfrak{L}$.

$\mathfrak{B}$ is an U-extension of $\mathfrak{A}$ if $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{A}(U) = \mathfrak{B}(U)$.

Let $R_{Ue} = \{\langle\mathfrak{A}, \mathfrak{B}\rangle ; \mathfrak{B}$ is an U-extension of $\mathfrak{A}\}$,

$$\Psi_{Ue} = \Psi_e \cup \{(\forall v)(\exists u)(U^2(v) \supset I_0(u, v) \wedge U^1(u))\} .$$

Then Ψ_{Ue} is a first order primitive set and R_{Ue} is expressible in L^I by Ψ_{Ue}.

$\Delta(\Psi_{Ue})$ is defined by the least set Δ satisfying the following (1)-(4):

(1), (2), (3) in (5.4).

(4) If $\theta(w^0) \in \Delta$ then $(\forall u)(U(u) \supset \theta(u)) \in \Delta$.

(5.7) *End-extension*

Let $<$ be a fixed binary predicate constant in $\mathfrak{L}$.

$\mathfrak{B}$ is an end-extension of $\mathfrak{A}$ if $\mathfrak{A} \subseteq \mathfrak{B}$ and for any $a \in |\mathfrak{A}|$, $b \in |\mathfrak{B}|$,

$$\langle b, a\rangle \in \mathfrak{B}(<) \text{ implies } b \in |\mathfrak{A}| .$$

Let $R_{ee} = \{\langle\mathfrak{A}, \mathfrak{B}\rangle ; \mathfrak{B}$ is an end-extension of $\mathfrak{A}\}$,

$$\Psi_{ee} = \Psi_e \cup \{(\forall v_1)(\forall u_1)(\forall v)(\exists u)(v <^2 v_1 \wedge I_0(u_1, v_1) \supset I_0(u, v))\} .$$

Then Ψ_{ee} is a first order primitive set and R_{ee} is expressible in L^I by Ψ_{ee}.

$\Delta(\Psi_{ee})$ is defined by the least set Δ satisfying the following (1)-(4):

(1), (2), (3) in (5.4).

(4) If $\theta(w_1^0) \in \Delta$, then $(\forall u)(u < w_2^0 \supset \theta(u)) \in \Delta$.

The statement obtained from Theorem 4.1 by replacing R, Ψ by R_{ee}, Ψ_{ee} is an extension of Feferman-Kreisel's theorem [4] in $L_{\omega_1\omega}$.

The statement obtained from Theorem 4.2 by replacing R, Ψ by R_{ee}, Ψ_{ee} is an extension of Makkai's theorem [9] in $L_{\omega_1\omega}$.

(5.8) *Super end-extension* ($\tilde{\Sigma}_1$-extension in Takahashi [**21**])

Let $<$ be a fixed binary predicate constant in $\mathfrak{L}$. $\mathfrak{B}$ is a super end-extension of $\mathfrak{A}$ if $\mathfrak{B}$ is an end-extension of $\mathfrak{A}$ and for any $a \in |\mathfrak{A}|$, $b \in |\mathfrak{B}|$

$$\{c\,;\ \langle c, b\rangle \in \mathfrak{B}(<),\ c \in |\mathfrak{B}|\} \subseteqq \{c\,;\ \langle c, a\rangle \in \mathfrak{B}(<),\ c \in |\mathfrak{B}|\}$$

implies $b \in |\mathfrak{A}|$, i.e.

$$\mathfrak{B} \models (\forall u)(u < x \supset u < y)[b, a] \text{ implies } b \in |\mathfrak{A}|.$$

Let $R_{se} = \{\langle \mathfrak{A}, \mathfrak{B}\rangle\,;\ \mathfrak{B}$ is a super end-extension of $\mathfrak{A}\}$,

$$\Psi_{se} = \Psi_{ee} \cup \{(\forall v_1)(\forall u_1)(\forall v_2)(\exists u_2)((\forall v)(v <^2 v_2 \supset v <^2 v_1)_\wedge I_0(u_1, v_1) \supset I_0(u_2, v_2))\}.$$

Then Ψ_{se} is a first order primitive set and R_{se} is expressible in L^I by Ψ_{se}. $\Delta(\Psi_{se})$ is defined by the least set Δ satisfying the following (1)-(5):

(1)-(4) in (5.7).

(5) If $\theta(w_1^0) \in \Delta$, then $(\forall u)((\forall v)(v < u \supset v < w_2^0) \supset \theta(u)) \in \Delta$.

(5.9) *Endomorphism*

Let $R_{ed} = \{\langle \mathfrak{A}, \mathfrak{B}\rangle\,;\ \mathfrak{A} \precsim \mathfrak{B}$ and $\mathfrak{B} \subseteqq \mathfrak{A}\}$,

$$\Psi_{ed} = \Psi_h \cup \{(\forall v)(\exists u) I_1(u, v),\ (\forall \vec{u})(\forall \vec{v})(I_1(\vec{u}, \vec{v})_\wedge P^1(\vec{u}) \supset P^2(\vec{v}))$$
$$(\forall \vec{u})(\forall \vec{v})(I_1(\vec{u}, \vec{v})_\wedge \neg P^1(\vec{u}) \supset \neg P^2(\vec{v}))\,;\ P \in PC(\mathfrak{L})\}.$$

Then Ψ_{ed} is a first order primitive set and R_{ed} is expressible in L^I by Ψ_{ed}. $\Delta(\Psi_{ed})$ is defined by the least set Δ satisfying the following (1)-(5):

(1) Every atomic formula whose free variables are among V_0 belongs to Δ.

(2) Every atomic formula and its negation whose free variables are among V_1 belong to Δ.

(3) Δ is closed under countable conjunctions and disjunctions.

(4) If $\theta(w^0) \in \Delta$ then $(\forall v)\theta(v)$, $(\exists v)\theta(v) \in \Delta$.

(5) If $\theta(w^1) \in \Delta$ then $(\forall v)\theta(v) \in \Delta$.

The statement obtained from Theorem 4.1 by replacing R, Ψ by R_{ed}, Ψ_{ed} is an extension of Makkai's theorem [**9**] in $L_{\omega_1\omega}$ and the statement obtained from Theorem 4.2 by replacing R, Ψ by R_{ed}, Ψ_{ed} is an extension of Makkai's theorem [**9**] in $L_{\omega_1\omega}$.

(5.10) *Retract*

Let $R_r = \{\langle \mathfrak{A}, \mathfrak{B}\rangle\,;\ \mathfrak{B} \subseteqq \mathfrak{A}$ and there is a homomorphism f of $\mathfrak{A}$ to $\mathfrak{B}$ such that $f(b) = b$ for any $b \in |\mathfrak{B}|\}$,

$$\Psi_r = \Psi_{ed} \cup \{(\forall v_1)(\forall u_1)(\forall v_2)(\forall u_2)(I_0(u_1, v_1)_\wedge I_1(u_2, v_2)_\wedge u_1 \doteq^1 u_2 \supset v_1 \doteq^2 v_2)\}.$$

Then Ψ_r is a first order primitive set and R_r is expressible in L^I by Ψ_r. $\Delta(\Psi_r)$ is defined by the least set Δ satisfying the following (1)-(6).

(1)-(5) in (5.8).

(6) $w^0 \doteq w^1 \in \Delta$ for any w^0, w^1.

The statement obtained from Theorem 4.1 by replacing R, Ψ by R_r, Ψ_r is an extension of Keisler's theorem [6] in $L_{\omega\omega}$ and Makkai's theorem [9] in $L_{\omega_1\omega}$ and the statement obtained from Theorem 4.2 by the same method above is an extension of Makkai's theorem [9] in $L_{\omega_1\omega}$.

(5.11) *Direct root of direct power.*

Let $R_{dp} = \{\langle\mathfrak{A}, \mathfrak{B}\rangle\,;\ \mathfrak{A}\times\mathfrak{A} \cong \mathfrak{B}\times\mathfrak{B}\}$,

$$\begin{aligned}\Psi_{dp} = \{&(\forall u_1)(\forall u_2)(\exists v_1)(\exists v_2)(I_1(u_1, v_1)\wedge I_2(u_2, v_2))\\ &(\forall v_1)(\forall v_2)(\exists u_1)(\exists u_2)(I_1(u_1, v_1)\wedge I_2(u_2, v_2))\\ &(\forall \vec{u}_1)(\forall \vec{u}_2)(\forall \vec{v}_1)(\forall \vec{v}_2)(I_1(\vec{u}_1, \vec{v}_1)\wedge I_2(\vec{u}_2, \vec{v}_2)\wedge P^1(\vec{u}_1)\wedge P^1(\vec{u}_2)\\ &\qquad\qquad \supset P^2(\vec{v}_1)\wedge P^2(\vec{v}_2))\\ &(\forall \vec{u}_1)(\forall \vec{u}_2)(\forall \vec{v}_1)(\forall \vec{v}_2)(I_1(\vec{u}_1, \vec{v}_1)\wedge I_2(\vec{u}_2, \vec{v}_2)\wedge \neg(P^1(\vec{u}_1)\wedge P^1(\vec{u}_2))\\ &\qquad\qquad \supset \neg(P^2(\vec{v}_1)\wedge P^2(\vec{v}_2)))\,;\ P\in PC(\mathfrak{L})\}\,.\end{aligned}$$

Then Ψ_{dp} is a first order primitive set and R_{dp} is expressible in L^I by Ψ_{dp}.

$\Delta(\Psi_{dp})$ is defined by the least set Δ satisfying the following (1)-(3):

(1) $\top$, $\bot$, $P(\vec{w}^1)\wedge P(\vec{w}^2)$, $\neg(P(\vec{w}^1)\wedge P(\vec{w}^2)) \in \Delta$ for any $P\in PC(\mathfrak{L})$.

(2) Δ is closed under countable conjunctions and disjunctions.

(3) If $\theta(w^1, w^2)\in\Delta$, then $(\forall u_1)(\forall u_2)\theta(u_1, u_2)$, $(\exists v_1)(\exists v_2)\theta(v_1, v_2)\in\Delta$.

The statement obtained from Theorem 4.1 by replacing R, Ψ by R_{dp}, Ψ_{dp} is an extension of Keisler's theorem [6] in $L_{\omega\omega}$ and Makkai's theorem [9] in $L_{\omega_1\omega}$ and the statement obtained from Theorem 4.2 by the same method above is an extension of Makkai's theorem [9] in $L_{\omega_1\omega}$.

(5.12) *Direct factor*

Let $R_{df} = \{\langle\mathfrak{A}, \mathfrak{B}\rangle\,;\ \mathfrak{A}\cong\mathfrak{B}\times\mathfrak{C}$ for some $\mathfrak{C}\}$,

$$\begin{aligned}\Psi_{df} = \{&(\forall u)(\exists v)\bigvee_{i\leq j} I_j(u, v),\ (\forall v)(\exists u)I_i(u, v)\,,\\ &(\forall \vec{u})(\forall \vec{v})(I_i(\vec{u}, \vec{v})\wedge P^1(\vec{u})\supset P^2(\vec{v}))\,,\\ &(\forall \vec{u}_1)(\forall \vec{u}_2)(\forall \vec{v}_1)(\forall \vec{v}_2)(I_i(\vec{u}_1, \vec{v}_1)\wedge I_i(\vec{u}_2, \vec{v}_2)\wedge P^1(\vec{u}_1)\wedge \neg P^1(\vec{u}_2)\\ &\supset P^2(\vec{v}_1)\wedge\neg P^2(\vec{v}_2))\,;\ i<\omega,\ \vec{i},\ P\in PC(\mathfrak{L})\}\,.\end{aligned}$$

Then obviously Ψ_{df} is a first order primitive set but it is not obvious that R_{df} is expressible in L^I by Ψ_{df} although condition (*) is obvious for R_{df} and Ψ_{df}; to show that, we have only to prove that R_{df} and Ψ_{df} satisfy (**). Suppose M is countable and $M\models\wedge\Psi_{df}$.

Let $D_i = \{a^*\,;$ for some b^*, $\langle a^*, b^*\rangle\in M(I_i)^*\}$, $i<\omega$.

Then (a) $M(I_i)^*$ is a mapping f_i from D_i onto $|M_2^*|$.

(b) $\bigcup_{i<\omega} D_i = |M_1^*|$.

(c) for any $P \in PC(\mathfrak{L})$, $\vec{i} = \langle i_1, \cdots, i_k \rangle$, $a_1^* \in D_{i_1}, \cdots, a_k^* \in D_{i_k}$,

$$b_1^* \in D_{i_1}, \cdots, b_k^* \in D_{i_k},$$

$\langle a_1^*, \cdots, a_k^* \rangle \in M_1^*(P)$ implies $\langle f_{i_1}(a_1^*), \cdots, f_{i_k}(a_k^*) \rangle \in M_2^*(P)$,

$\langle a_1^*, \cdots, a_k^* \rangle \in M_1^*(P)$ and $\langle b_1^*, \cdots, b_k^* \rangle \notin M_1^*(P)$ imply

$$\langle f_{i_1}(b_1^*), \cdots, f_{i_k}(b_k^*) \rangle \notin M_2^*(P).$$

By using (a), (b), (c), we can get

(d) $D_i \cap D_j \neq \phi$ implies $f_i = f_j$.

(e) f_i is a bijection.

Let $i \sim j \Leftrightarrow D_i = D_j$ and $|\mathfrak{C}| = \omega/\sim$,

$$\mathfrak{C}(P) = \{\langle \tilde{i}_1, \cdots, \tilde{i}_k \rangle \,;\, \text{for some } a_1^* \in D_{i_1}, \cdots, a_k^* \in D_{i_k}, \langle a_1^*, \cdots, a_k^* \rangle \in M_1(P)\}.$$

Then $M_1^* \cong M_2^* \times \mathfrak{C}$.

Hence we can conclude that R_{df} is expressible in L^I by Ψ_{df}.

$\Delta(\Psi_{df})$ is defined by the least set Δ satisfying the following (1)-(4):

(1) $\top, \bot, P(w_1^i), P(w_1^i) \wedge \neg P(w_2^i) \in \Delta$ for any $P \in PC(\mathfrak{L})$,

$$w_1^i = \langle w_{11}^{i_1}, \cdots, w_{1k}^{i_k} \rangle,\ w_2^i = \langle w_{21}^{i_1}, \cdots, w_{2k}^{i_k} \rangle.$$

(2) Δ is closed under countable conjunctions and disjunctions.

(3) If $\theta(w^i) \in \Delta$ then $(\forall u)\theta(u) \in \Delta$.

(4) If $\theta(w^i) \in \Delta$ and $V((\exists v)\theta(v)) \subset \bigcup_{j<i} V_j$ then $(\exists v)\theta(v) \in \Delta$.

The statement obtained from Theorem 4.1 by replacing R, Ψ by R_{df}, Ψ_{df} is an extension of Keisler's theorem [6] in $L_{\omega\omega}$ and Makkai's theorem [**9**] in $L_{\omega_1\omega}$ and the statement obtained from Theorem 4.2 by the same method above is an extension of Makkai's theorem [**9**].

(5.13) *k-isomorphic union*

Let $R_{ku} = \{\langle \mathfrak{A}, \mathfrak{B} \rangle \,;\, \text{for some } k\text{-family } S \text{ such that every element in } S \text{ is isomorphic to } \mathfrak{A},\ \mathfrak{B} = \bigcup S\}$,

$$\Psi_{ku} = \{(\forall u)(\exists v)I_i(u, v),\ (\forall v_1) \cdots (\forall v_k)(\exists u_1) \cdots (\exists u_k) \bigvee_{i \leq j} I_j(u_1, v_1) \wedge \cdots \wedge I_j(u_k, v_k),$$
$$(\forall \vec{u})(\forall \vec{v})(I_i(\vec{u}, \vec{v}) \wedge (P^1(\vec{u}) \supset P^2(\vec{v}))$$
$$(\forall \vec{u})(\forall \vec{v})(I_i(\vec{u}, \vec{v}) \wedge \neg P^1(\vec{u}) \supset \neg P^2(\vec{v}))\,;\ P \in PC(\mathfrak{L}),\ i < \omega\}.$$

Then Ψ_{ku} is a first order primitive set and R_{ku} is expressible in L^I by Ψ_{ku}.

$\Delta(\Psi_{ku})$ is defined by the least set Δ satisfying the following (1)-(4):

(1) Every atomic formula and its negation whose free variables are all in V_i for some $i < \omega$ belong to Δ.

(2) Δ is closed under countable conjunctions and disjunctions.

(3) Δ is closed under the first order existential quantifications.

(4) If $\theta(w_1^i, \cdots, w_k^i) \in \Delta$ and $V((\forall u_1) \cdots (\forall u_k)\theta(u_1 \cdots u_k)) \subseteq \bigcup_{j<i} V_j$ then

$$(\forall u_1) \cdots (\forall u_k)\theta(u_1 \cdots u_k) \in \Delta.$$

Especially if we consider only finitary formulas then every formula in $\Delta(\Psi_{ku})$ is so-called *universal existential formula* (see Keisler [**5**], Weistein [**19**]). So, the statement obtained from Theorem 4.1 by replacing R, Ψ by R_{ku}, Ψ_{ku} is an extension of Keisler's theorem [**5**] in $L_{\omega\omega}$.

(5.14) *ω-isomorphic union*

Let $R_{\omega u} = \{\langle \mathfrak{A}, \mathfrak{B} \rangle$; for some ω-family S such that every element in S is isomorphic to $\mathfrak{A}$, $\mathfrak{B} = \bigcup S\}$,

$$\Psi_{\omega u} = \bigcup_{k<\omega} \Psi_{ku}.$$

Then $\Psi_{\omega u}$ is a first order primitive set and $R_{\omega u}$ is expressible in L^I by $\Psi_{\omega u}$.

$\Delta(\Psi_{\omega u})$ is the union set of all $\Delta(\Psi_{ku})$, $k < \omega$.

The statement obtained from Theorem 4.1 by replacing R, Ψ by $R_{\omega u}$, $\Psi_{\omega u}$ is an extension of Łos, Chang's theorem [**2**] in $L_{\omega\omega}$.

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