J. Math. Soc. Japan
Vol. 25, No. 1, 1973

Minimal 2-regular digraphs with given girth

By Mehdi BEHZAD

(Received July 13, 1971)
(Revised Nov. 12, 1971)

§ 1. Abstract.

A digraph D is r-regular if degree $v=r$, $r\geqq 1$, for every vertex v of D. The girth n, $n\geqq 2$, of D containing directed cycles is the length of the smallest cycle in D. The minimum number of vertices of r-regular digraphs having girth n is denoted by $g(r, n)$. In this note we prove that $g(2, n)=2n-1$.

§ 2. Introduction and definitions.*

The smallest number of vertices that a regular graph of degree r, $r\geqq 1$, and girth n, $n\geqq 2$, may possess is denoted by $f(r, n)$. The determination of the value of $f(r, n)$ has been the subject of many investigations in recent years. (See, for example, [3], [4], and [5].) Yet, with few exceptions, the numbers $f(r, n)$ are unknown for $r\geqq 3$ and $n\geqq 5$. In [2] the analogous problem for digraphs (directed graphs) was considered.

A digraph D is r-regular, $r\geqq 1$, if $\mathrm{id}\, v=\mathrm{od}\, v=r$ for every vertex v of D, where $\mathrm{id}\, v$ is the in-degree of v, while $\mathrm{od}\, v$ is the out-degree of the vertex v of D. For positive integers $n\geqq 2$ and $r\geqq 1$ the number $g(r, n)$ is defined to be the minimum number of vertices r-regular digraphs having girth n(the length of the smallest cycle in the digraph) may possess. The upper bound $r(n-1)+1$ for $g(r, n)$ was obtained in [2] and it was conjectured that $g(r, n)=r(n-1)+1$. Moreover, the values of $g(r, n)$ for the elements of the subset S of the set of all lattice points of the $r-n$ plane were obtained where:

$$S=\{(r, n): n=2, 3\}\cup\{(r, n): r=1\}\cup\{(2, 4), (3, 4), (4, 4), (3, 5)\}.$$

In this article we propose to prove that the conjecture is true for the case $r=2$ as well.

---

* Definitions not given here can be found in [1].

§ 3. The function $g(2, n)$.

First we show that $g(2, n)$ is an increasing function of n.

LEMMA 1. *Let $n \geqq 2$. Then $g(2, n+1) > g(2, n)$.*

PROOF. We use induction on n. For $n=2$, and 3, the lemma is obviously true. Assume D is a 2-regular digraph of order $f(2, n+1)$ whose girth is $n+1$, $n \geqq 3$. Then D contains a cycle $C: v_1, v_2, \cdots, v_{n+1}, v_1$ of length $n+1$. Each vertex v_i of C is adjacent to and adjacent from an element of $V(D)-V(C)$, say u_j and u_k, $j \neq k$, respectively, where $V(D)$ denotes the vertex set of D. There exists an integer i, $1 \leqq i \leqq n+1$, such that the edge $\overrightarrow{u_k u_j}$ is not in D, for otherwise D contains at least $4n+4$ edges, while $g(2, n+1) \leqq 2n+1$ and the regularity of D show that D has at most $4n+2$ edges. Now, we remove the vertex v_i together with its incident edges and add two new edges $\overrightarrow{v_{i-1} v_{i+1}}$ and $\overrightarrow{u_k u_j}$—if $i=1$, then $i-1$ is replaced by $n+1$ and if $i=n+1$, then $i+1$ is replaced by 1—to obtain a new 2-regular digraph of order $g(2, n+1)-1$ and girth n. Hence, $g(2, n) < g(2, n+1)$ as was required to prove.

We say a vertex v of a digraph D having girth n, $n \geqq 3$, is *adjacent with* a vertex u of D if either v is adjacent to or is adjacent from the vertex u. From now on the subscripts are computed in terms of the integers modulo n.

LEMMA 2. *Assume there exists a 2-regular digraph D of order $g(2, n) = 2n-2$ having girth n, $n \geqq 4$. If $C: v_1, v_2, \cdots, v_n, v_1$ is a cycle of length n of D, then every vertex of $V(D)-V(C)$ is adjacent with either 2 or 3 vertices of C.*

PROOF. Let u be an element of the nonempty set $V(D)-V(C)$. Suppose u is adjacent to v_i. Then u can be adjacent from no vertices of C other than v_{i-1} and v_{i-2}. Now it is clear that the vertex u can be adjacent to no other vertices of C. This proves that u is adjacent with at most 3 vertices of C.

Next, assume that u is an element of $V(D)-V(C)$ which is adjacent with at most one vertex of C. Suppose that u is adjacent from the vertices u_1 and u_3 and is adjacent to the vertices u_2 and u_4 of D. (In case u is adjacent with one vertex of C, then exactly one of the elements of the set $\{u_1, u_2, u_3, u_4\}$ is a vertex of C.) Now remove the edges of C from D and denote the resulting digraph by D^*. We show that D^* contains a cycle C_2 of length n by considering the following cases.

CASE 1. *At least one of the two edges $\overrightarrow{u_1 u_2}$ and $\overrightarrow{u_3 u_4}$ is an edge of D.* Then the edges $\overrightarrow{u_3 u_2}$ and $\overrightarrow{u_1 u_4}$ are not in D. If D^* has no cycle of length n, then we remove the vertex u together with its incident edges from the digraph D and add the new edges $\overrightarrow{u_3 u_2}$ and $\overrightarrow{u_1 u_4}$ to the resulting digraph to obtain a 2-regular digraph of order $g(2, n)-1$ having girth n. But this contradicts the minimality of $g(2, n)$.

CASE 2. *Neither* $\overrightarrow{u_1u_2}$ *nor* $\overrightarrow{u_3u_4}$ *is an edge of* D. In this case, too, following the above argument and replacing $\overrightarrow{u_3u_2}$ and $\overrightarrow{u_1u_4}$ by $\overrightarrow{u_1u_2}$ and $\overrightarrow{u_3u_4}$, we reach the conclusion that D^* contains a cycle of length n.

Now remove the edges of C_2 from D^* and denote the resulting digraph by D^{**}. Since D contains $4n-4$ edges, D^{**} contains $2n-4$ edges. Starting from a nonisolated vertex of D^{**} and traversing along the directed edges of D^{**} we obtain a cycle C_3 of length $\mu=2n-4$. Clearly $\mu \geqq n$. In case $\mu<2n-4$ then D^{**} would necessarily contain a cycle of length less than n which is impossible. Thus, D is the sum of three edge-disjoint cycles C_1, C_2 and C_3 such that the length of C_i, $i=1, 2$, is n and the length of C_3 is $2n-4$. The vertex set of D consists of the $2n-4$ vertices of C_3 and two additional vertices w_1 and w_2. Both cycles C_1 and C_2 contain both vertices w_1 and w_2; moreover, the two cycles C_1 and C_2 have no other vertices in common. Since D has girth n the length of the directed path w_1-w_2 (resp. w_2-w_1) in C_1 is the same as the length of the directed path w_1-w_2 (resp. w_2-w_1) in C_2. The length of each of these 4 paths is greater than one, and no vertex of each of the directed paths w_1-w_2 can be adjacent with either a vertex of the path w_2-w_1 in C_1 or a vertex of the path w_2-w_1 in C_2. (See Figure 1.)

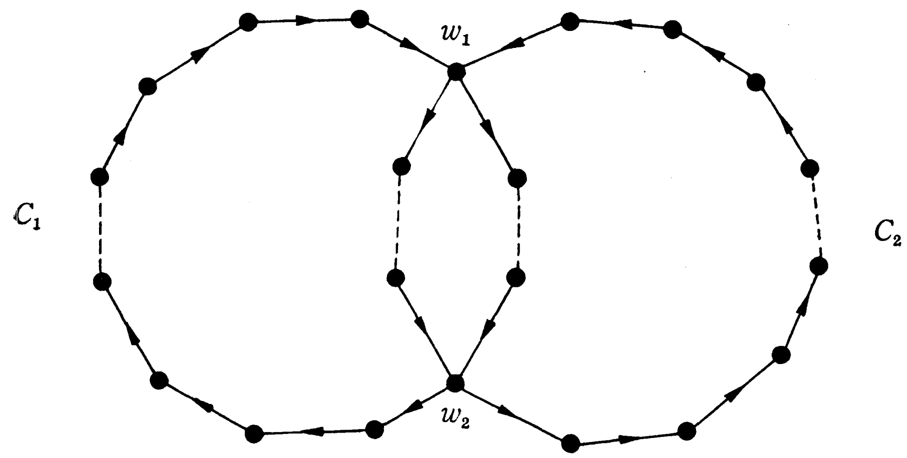

Fig. 1.

Hence D can contain no cycle of length $2n-4$ which does not pass through w_1 and w_2. This contradiction completes the proof of the lemma.

Our main result is:

For any integer $n \geqq 2$, $g(2, n)=2n-1$.

PROOF. We use induction on n. It is known that the theorem is true for $n=2, 3, 4$ and 5. Assume that the theorem is true for $n-1$ and consider a 2-regular digraph D having girth n, $n \geqq 6$ and order $g(2, n)$. Then $g(2, n) \leqq 2n-1$ and by the induction hypothesis $g(2, n-1)=2n-3$. These and Lemma

1 imply that $g(2, n)$ is either $2n-2$ or $2n-1$. Assume $g(2, n)=2n-2$ and let $C: v_1, v_2, \cdots, v_n, v_1$ be a cycle of length n of D. By Lemma 2 each element of $V(D)-V(C)$ is adjacent with 2 or 3 vertices of C. In fact, exactly 4 elements of $V(D)-V(C)$, say u_1, u_2, u_3 and u_4 are adjacent with 3 vertices of C and each of the remaining $n-6$ elements of $V(D)-V(C)$, say $u_5, u_6, \cdots, u_{n-2}$, are adjacent with two vertices of C. To see this, we observe that the only partition of the even integer $2n$ with $n-2$ summands belonging to the set $\{2, 3\}$ is 3, 3, 3, 3, 2, 2, $\cdots$, 2. Next we show that such a situation is impossible.

CASE 1. *Assume that two of the elements of the set* $\{u_1, u_2, u_3, u_4\}$ *are adjacent.* Without loss of generality, we may suppose that u_1 is adjacent to the vertex u_2. Then u_1 is adjacent to a vertex of C, say v_1, and is adjacent from 2 vertices of C. These two vertices are necessarily v_n and v_{n-1}. Then the only vertex of C to which the vertex u_2 can be adjacent is v_2. But this produces a contradiction because the vertex u_2 must be adjacent to two vertices of C. For an illustration, see Figure 2.

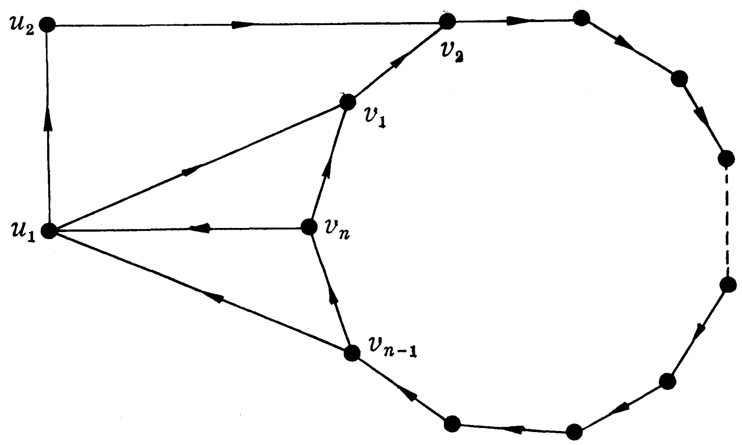

Fig. 2.

CASE 2. *The only alternative is that* $n\geqq 8$ *and that two elements of the set* $\{u_1, u_2, u_3, u_4\}$, *say* u_1 *and* u_2, *are joined by a semipath* P *of length* t, $2\leqq t \leqq n-6$, *all of whose vertices belong to* $V(D)-V(C)$. Let $P: u_1, u_5, u_6, \cdots, u_k, u_2$, where $5\leqq k\leqq n-3$. We denote u_1 by w_1, u_5 by w_2, u_6 by $w_3, \cdots$, u_k by w_{k-3}, and u_2 by w_{k-2}. Then $P: w_1, w_2, \cdots, w_{k-2}$.

Now we have two cases to consider.

i) *The vertex* w_1 *is adjacent to the vertex* w_2. without loss of generality, we assume that w_1 is adjacent to v_1. Then vertices v_n and v_{n-1} must be adjacent to w_1. The vertex w_2 is adjacent to at least one vertex of C and that must be v_2. Hence, the vertex w_2 must be adjacent to w_3 as well. Continuing this process, we observe that the vertex w_i can be adjacent to only one vertex of C, namely v_i, for $1\leqq i\leqq k-2$; therefore the vertex w_i must be adjacent to w_{i+1}, for $1 \leqq i \leqq k-3$. But then the adjacency of $w_{k-2}=u_2$ to two of the vertices of C is impossible. (Note that the semipath P turns out to be a (directed) path from u_1 to u_2.)

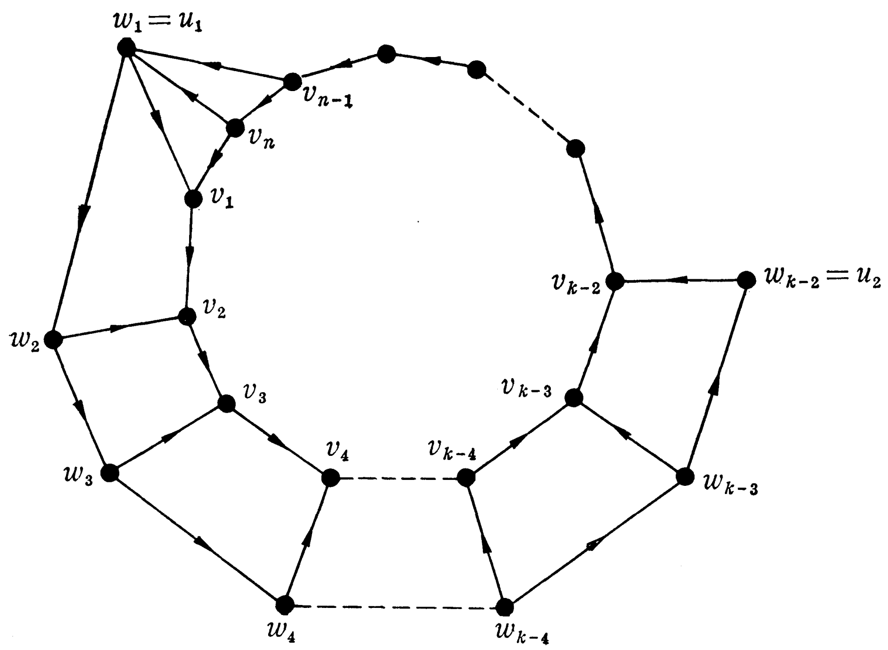

Fig. 3.

Hence, the assumption $g(2, n)=2n-2$ leads to a contradiction.

ii) *The vertex* w_1 *is adjacent from the vertex* w_2. We may assume that the vertex v_{n-1} of C is also adjacent to w_1. Therefore, the two vertices of C to which the vertex w_1 is adjacent are v_1 and v_n. Next, at least one vertex of C must be adjacent to w_2 and that without any other choice is v_{n-2}. Hence, the vertex w_3 is adjacent to the vertex w_2. Continuing this process, we conclude that the only vertex of C adjacent to w_i is v_{n-i} for $i=1, 2, \cdots, k-2$. Hence, the vertex w_i is also adjacent from the vertex w_{i+1}, for $1 \leqq i \leqq k-3$. But this contradicts the fact that the vertex $w_{k-2}=u_2$ is adjacent from two of the vertices of C. (In this case the semipath P is a directed path from u_2 to u_1.) This contradicts the assumption that $g(2, n)=2n-2$. For an illustration, see Figure 4. Hence, in any case $g(2, n)=2n-1$ as was required to prove.

We conclude this article by mentioning that with some modifications, this method seems to work for the determination of the value of the function $g(3, n)$ and this result may appear elsewhere.

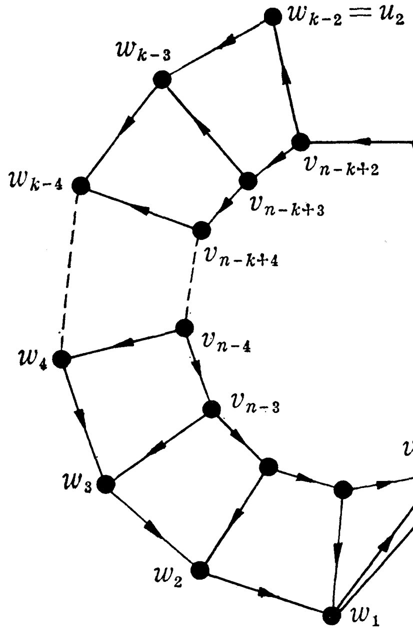

Fig. 4.

Department of Mathematics
Arya-Mehr University of Technology
P. O. Box 3406, Tehran, Iran

References

[1] M. Behzad and G. Chartrand, An Introduction to Theory of Graphs, Allyn and Bacon Inc., 1971.

[2] M. Behzad, G. Chartrand and C. E. Wall, On Minimal Regular Digraphs with Given Girth, Fund. Math., **69** (1970), 227–231.

[3] P. Erdös and H. Sachs, Regular Graphen Gegebener Tailleneites mit Minimaler Konetenzahl, Wiss. Z. Univ. Hulle. Math.-Nat., **12**, No. 3 (1963), 251–258.

[4] A. J. Hoffman and R. R. Singleton, On Moore Graph with Diameters two and three, I. B. M. J. Res. Develop., **4** (1960), 497–504.

[5] W. T. Tutte, The Connectivity of Graphs, Toronto Univ. Press, Toronto, 1967.