J. Math. Soc. Japan
Vol. 24, No. 4, 1972

Object logic and morphism logic

By Nobuyoshi MOTOHASHI

(Received Jan. 31, 1972)

Introduction.

In studying model theory by using proof theoretic techniques, the author noticed the utility of distinguishing two kinds of logics, which will be called here "object logic" and "morphism logic". The former are those to which algebraic structures are related, while the morphisms between algebraic structures are related to the latter. The exact definitions of these logics will be given below in § 1 and § 2; however we explain here briefly how to construct a morphism logic from a family of object logics $\{L_\lambda\}_{\lambda\in\Lambda}$. Let PC be a set of predicate constants which do not appear in any logics in $\{L_\lambda\}_{\lambda\in\Lambda}$; every predicate constant in PC is intended to denote a morphism. Then the morphism logic $L=L(L_\lambda)_{\lambda\in\Lambda}$ for the family of object logics $\{L_\lambda\}_{\lambda\in\Lambda}$ is defined as the logic with the usual Gentzen type inference rules whose set of formulas are obtained by applying the usual first order operations $\neg$, $\wedge$, $\vee$, $\forall$, $\exists$ ($\wedge$ and $\vee$ are used to mean countable conjunction and countable disjunction) to the formulas of the following two types:

(i) The formulas of the form $P(x_1, \cdots, x_n)$, where $P\in PC$, and $x_1, \cdots, x_n$ are free variables;

(ii) The formulas in one of L_λ, $\lambda\in\Lambda$.

It will be shown in this paper that a kind of cut-elimination theorem, called "normal derivation theorem", holds in the morphism logic L. This theorem will be stated explicitly in § 3 and proved in § 4 below. Roughly speaking, this theorem asserts that cut rules whose cut formulas contain predicate constants denoting morphisms can be eliminated in L. Cut rules whose cut formulas are included in an object logic can not always be eliminated, but the above assertion is powerful enough to simplify the proofs and to generalize interpolation theorems and preservation theorems in mathematical logic. This shows the significance of making distinction between object logics and morphism logics.

The purpose of this paper is to prove the "normal derivation theorem." The applications will be appeared in sequels to this paper, [2], [3], [4].

Finally the author would like to thank Professor Shoji Maehara for his valuable advices during his research.

§ 1. Object logics.

Consider a family of logics $\{L_\lambda\}_{\lambda\in\Lambda}$. Let FM_λ be the set of formulas in L_λ and PFM_λ be the set of provable formulas in L_λ for each $\lambda\in\Lambda$. If $\vec{x}=\langle x_1,\cdots,x_n\rangle$ is a finite sequence of distinct free individual variables in L_λ, $\vec{y}=\langle y_1,\cdots,y_n\rangle$ is a finite sequence of individual variables in L_λ and $\theta\in FM_\lambda$, then by $\theta\binom{\vec{x}}{\vec{y}}$ we shall denote the expression obtained from θ by replacing $x_1,\cdots,x_n$ by $y_1,\cdots,y_n$ respectively. We may write θ for $\theta(\vec{x})$ and $\theta\binom{\vec{x}}{\vec{y}}$ for $\theta(\vec{y})$ as far as no confusion is likely to occur.

Now we impose the following requirements on $\{L_\lambda\}_{\lambda\in\Lambda}$. When these requirements are satisfied, each L_λ is called an *object logic.*

1.1. REQUIREMENTS ON $\{L_\lambda\}_{\lambda\in\Lambda}$.

① Each L_λ has the common countable infinite set of free individual variables (this set will be denoted by FV), and the common uncountable set of bound individual variables (this set will be denoted by BV).

② $\{FM_\lambda\}_{\lambda\in\Lambda}$ are mutually disjoint sets.

③ Every formula in L_λ has only finitely many free individual variables.

④ For any sequences of free individual variables $\vec{x},\vec{y}$ of the same length such that all the variables in $\vec{x}$ are distinct, $\theta(\vec{x})\in FM_\lambda$ implies $\theta(\vec{y})\in FM_\lambda$, and $\theta(\vec{x})\in PFM_\lambda$ implies $\theta(\vec{y})\in PFM_\lambda$.

⑤ If $\theta\in FM_\lambda$ then $\neg\theta\in FM_\lambda$.

⑥ If Φ is a non-empty countable set of formulas in FM_λ which has only finitely many free individual variables, then $\wedge\Phi$, $\vee\Phi\in FM_\lambda$.

⑦ If $\theta(x)\in FM_\lambda$, $x\in FV$ and $v\in BV$ does not occur in $\theta(x)$, then $(\forall v)\theta(v)$, $(\exists v)\theta(v)\in FM_\lambda$.

By x, y, z (with or without suffixes) we shall denote elements in FV and by u, v, w (with or without suffixes) we shall denote elements in BV. By θ, φ, ψ (with or without suffixes) we shall denote elements in $\bigcup_{\lambda\in\Lambda} FM_\lambda$.

§ 2. Morphism logic $L=L(L_\lambda)_{\lambda\in\Lambda}$.

Roughly speaking, the morphism logic $L=L(L_\lambda)_{\lambda\in\Lambda}$ for a family of object logics $\{L_\lambda\}_{\lambda\in\Lambda}$ is a logic obtained from $\{L_\lambda\}_{\lambda\in\Lambda}$ by applying first order operations. Now we give the explicit definition of L. Let PC be a set of predicate constants which are not contained in L_λ for any $\lambda\in\Lambda$.

Then the set of formulas in L (denoted by FM) is defined recursively by the following rules:

① If $P\in PC$ is an n-ary predicate constant and $x_1,\cdots,x_n\in FV$ then $P(x_1,\cdots,x_n)$ is a formula in L (called an *atomic m-formula*).

② If $\theta \in FM_\lambda$ then θ is a formula in L (called a *λ-formula*) for each $\lambda \in \Lambda$.

③ If F is a formula in L, then $\urcorner F$ is a formula in L.

④ If K is a non-empty countable set of formulas in L such that only finitely many free individual variables occur in K, then $\wedge K$, $\vee K$ are formulas in L.

⑤ If $F(x)$ is a formula in L and $v \in BV$ does not occur in $F(x)$, then $(\forall v)F(v)$, $(\exists v)F(v)$ are formulas in L.

⑥ All the formulas in L are obtained from ①—⑤.

A formula F in L is said to be *m-formula* if F is not a λ-formula for any $\lambda \in \Lambda$. By F, G (with or without suffixes), we shall denote formulas in L. A sequent in L is a configuration of the form $\Gamma \to \Theta$ where Γ and Θ are countable (possibly empty) sets of formulas in L such that only finitely many free individual variables occur in $\Gamma \cup \Theta$.

A sequent of the forms $\{F\} \to \{F\}$ (where F is an m-atomic formula or a λ-formula) or $\to \{F\}$ (where $F \in PFM_\lambda$) is called an *axiom sequent*. An arrangement of sequents in tree form (possibly infinite) is called a *derivation* of the lowest sequent, provided the following conditions are satisfied:

(a) all uppermost sequents are axiom sequents,

(b) consecutive sequents in any branch of the tree are connected by one of the following inference rules;

Structural inference rules

$$(W, \Gamma_1, \Theta_1)\frac{\Gamma \to \Theta}{\Gamma_1 \cup \Gamma \to \Theta \cup \Theta_1}$$

$$(C, F)\frac{\Gamma_1 \to \Theta_1 \qquad \Gamma_2 \to \Theta_2}{\Gamma_1 \cup \Gamma_2 - \{F\} \to \Theta_1 - \{F\} \cup \Theta_2},$$

where, of course, $\Gamma_1 \cup \Gamma_2 - \{F\}$ means $\Gamma_1 \cup (\Gamma_2 - \{F\})$ and $\Theta_1 - \{F\} \cup \Theta_2$ means $(\Theta_1 - \{F\}) \cup \Theta_2$.

Logical inference rules

$$(\urcorner\to, F, \urcorner F)\ \frac{\Gamma \to \Theta \cup \{F\}}{\{\urcorner F\} \cup \Gamma \to \Theta} \qquad (\to\urcorner, F, \urcorner F)\ \frac{\{F\} \cup \Gamma \to \Theta}{\Gamma \to \Theta \cup \{\urcorner F\}}$$

$$(\wedge\to, F, \wedge K)\ \frac{\{F\} \cup \Gamma \to \Theta}{\{\wedge K\} \cup \Gamma \to \Theta}(F \in K) \qquad (\to\wedge, K, \wedge K)\frac{\Gamma \to \Theta \cup \{F\}\ \ (F \in K)}{\Gamma \to \Theta \cup \{\wedge K\}}$$

$$(\vee\to, K, \vee K)\ \frac{\{F\} \cup \Gamma \to \Theta\ \ (F \in K)}{\{\vee K\} \cup \Gamma \to \Theta} \qquad (\to\vee, F, \vee K)\frac{\Gamma \to \Theta \cup \{F\}}{\Gamma \to \Theta \cup \{\vee K\}}\ (F \in K)$$

$$\left(\forall\to, F\binom{x}{y}, (\forall v)F(v)\right)\frac{\left\{F\binom{x}{y}\right\} \cup \Gamma \to \Theta}{\{(\forall v)F(v)\} \cup \Gamma \to \Theta} \qquad (\to\forall, F(x), (\forall v)F(v))\frac{\Gamma \to \Theta \cup \{F(x)\}}{\Gamma \to \Theta \cup \{(\forall v)F(v)\}}$$

$$(\exists\rightarrow,\ F(x),\ (\exists v)F(v))\frac{\{F(x)\}\cup\Gamma\rightarrow\Theta}{\{(\exists v)F(v)\}\cup\Gamma\rightarrow\Theta}\quad \left(\rightarrow\exists,\ F\binom{x}{y},\ (\exists v)F(v)\right)\frac{\Gamma\rightarrow\Theta\cup\left\{F\binom{x}{y}\right\}}{\Gamma\rightarrow\Theta\cup\{(\exists v)F(v)\}}$$

with the restrictions on variables; the individual variable x in $(\rightarrow\forall, F(x), (\forall v)F(v))$, $(\exists\rightarrow, F(x), (\exists v)F(v))$ is called the *eigenvariable* of these inference rules and must not occur in the lower sequent.

$(\neg\rightarrow)$ rules are rules of the form $(\neg\rightarrow, F, \neg F)$ for some F. Similarly the notions "$(\rightarrow\neg)$, $(\wedge\rightarrow)$, $(\rightarrow\wedge)$, $(\vee\rightarrow)$, $(\rightarrow\vee)$, $(\forall\rightarrow)$, $(\rightarrow\forall)$, $(\exists\rightarrow)$, $(\rightarrow\exists)$ rules" will be used. By *left rules*, we shall mean the rules $(\neg\rightarrow)$, $(\wedge\rightarrow)$, $(\vee\rightarrow)$, $(\forall\rightarrow)$, $(\exists\rightarrow)$ and by *right rules*, $(\rightarrow\neg)$, $(\rightarrow\wedge)$, $(\rightarrow\vee)$, $(\rightarrow\forall)$, $(\rightarrow\exists)$. Notice that every logical inference rule R has the form $R=(r_0, r_1, r_2)$, where r_1 or every its element is called a *side formula* of R and r_2 called the *principal formula* of R. The formula F in (C, F) is called the *cut formula* of this inference rule and if F is an m-formula, (C, F) is called an *m-cut rule.* A sequent $\Gamma\rightarrow\Theta$ is provable in L (expressed by $\underset{L}{\vdash}\Gamma\rightarrow\Theta$) if there is a derivation $\mathfrak{D}$ whose lowest sequent is $\Gamma\rightarrow\Theta$, when we say that $\mathfrak{D}$ is a derivation of $\Gamma\rightarrow\Theta$. A formula F is provable in L (expressed by $\underset{L}{\vdash}F$) if $\underset{L}{\vdash}\rightarrow\{F\}$. If f is a mapping from FV to FV, $F\in FM$ and $\mathfrak{D}$ is a derivation, then by $f(F)$ and $f(\mathfrak{D})$, we shall denote the expressions obtained from F and $\mathfrak{D}$ by replacing every $x\in FV$ by $f(x)$, respectively. Obviously $f(F)$ is a formula in L (cf. 1.1 ③ and ④) but $f(\mathfrak{D})$ is not always a derivation. A part of the form $R\ \dfrac{\overset{\downarrow}{\Gamma_h\rightarrow\Theta_h}(h\in H)}{\underset{\downarrow}{\Gamma\rightarrow\Theta}}$ in a derivation $\mathfrak{D}$ is called an *instance* of R in $\mathfrak{D}$.

§ 3. Normal derivation.

In this section, we shall state the main theorem in this paper, "normal derivation theorem", which means roughly the following: Every provable sequent has a derivation in which any m-cut rule does not appear, the rules $(\wedge\rightarrow)$, $(\rightarrow\vee)$, $(\forall\rightarrow)$, $(\rightarrow\exists)$ are applied as early as possible and the rules $(\vee\rightarrow)$, $(\rightarrow\wedge)$, $(\exists\rightarrow)$, $(\rightarrow\forall)$ are applied as late as possible. We divide the set of inference rules into two types. The set of inference rules of *type* 1 consists of the following inference rules: $(\wedge\rightarrow)$, $(\rightarrow\vee)$, $(\forall\rightarrow)$, $(\rightarrow\exists)$ rules whose side formulas are *m-formulas but not atomic m-formulas*, $(\exists\rightarrow)$ rules whose side formulas are *m-formulas and have the forms* $\vee K$ *and* $(\exists v)G(v)$, $(\rightarrow\forall)$ rules whose side formulas are *m-formulas and have the forms* $\wedge K$ *and* $(\forall u)G(u)$. The set of inference rules of *type* 2 consists of the rules which are not of type 1.

A derivation $\mathfrak{D}$ is said to be *m-cut free* if $\mathfrak{D}$ has no instances of m-cut rules and satisfies the following eigenvariable condition $(*)$:

(*) If a free variable x is used as an eigenvariable in an instance of a rule R in $\mathfrak{D}$, then x does not occur in $\mathfrak{D}$ except in sequents *above* the lower sequent of this instance.

3.1. DEFINITION OF NORMAL DERIVATION. A derivation $\mathfrak{D}$ is said to be *normal* if $\mathfrak{D}$ is m-cut free and satisfies (**) below.

(**) If an inference rule R of type 1 is applied next to an instance of a rule R_1, then the principal formula of R_1 is the side formula R and R, R_1 are both left rules or both right rules.

3.2. EXAMPLES.

EXAMPLE 1. Let $\mathfrak{D}_1$ be

$$R_1 \frac{R_2 \dfrac{\{P(x)\} \to \{P(x)\}}{\{P(x)\} \to \{(\exists v)P(v)\}}}{\{(\forall v)P(v)\} \to \{(\exists v)P(v)\}},$$

where $R_1 = (\to\exists,\ P(x),\ (\exists v)P(v))$

$R_2 = (\forall\to,\ P(x),\ (\forall v)P(v))$.

Then $\mathfrak{D}_1$ is normal because R_1, R_2 are of type 2.
(Notice that the side formula of R_1 and R_2 is an atomic m-formula.)

EXAMPLE 2. Let $\mathfrak{D}_2$ be

$$\begin{array}{l} R_3\ \underline{\{(\exists v)\theta(x, v)\} \to \{(\exists v)\theta(x, v)\}} \\ R_4\ \underline{\{(\exists v)\theta(x, v)\} \to \{(\exists u)(\exists v)\theta(u, v)\}} \\ \quad \{(\exists u)(\exists v)\theta(u, v)\} \to \{(\exists u)(\exists v)\theta(u, v)\} \end{array},$$

where $R_3 = (\to\exists,\ (\exists v)\theta(x, v),\ (\exists u)(\exists v)\theta(u, v))$

$R_4 = (\exists\to,\ (\exists v)\theta(x, v),\ (\exists u)(\exists v)\theta(u, v))$.

Then $\mathfrak{D}_2$ is normal because R_3, R_4 are of type 2.
(Notice that the side formula of R_3 and R_4 is not an m-formula.)

EXAMPLE 3. Let $\mathfrak{D}_3$ be

$$\begin{array}{l} R_5\ \underline{\{P_1(x_1, y_1)\} \to \{P_1(x_1, y_1)\}} \\ R_6\ \underline{\{P_1(x_1, y_1)\} \to \{(\exists v_1)P_1(x_1, v_1)\}} \\ \quad \{P_1(x_1, y_1)\} \to \{(\exists v_1)P_1(x_1, v_1), (\exists v_2)P_2(x_2, v_2)\} \end{array},$$

$\mathfrak{D}_4$ be

$$\begin{array}{l} R_7\ \underline{\{P_2(x_2, y_2)\} \to \{P_2(x_2, y_2)\}} \\ R_8\ \underline{\{P_2(x_2, y_2)\} \to \{(\exists v_2)P_2(x_2, v_2)\}} \\ \quad \{P_2(x_2, y_2)\} \to \{(\exists v_1)P_1(x_1, v_1), (\exists v_2)P_2(x_2, v_2)\} \end{array},$$

$\mathfrak{D}_5$ be

$$\begin{array}{l} \qquad\qquad \downarrow \mathfrak{D}_3 \qquad\qquad\qquad\qquad \downarrow \mathfrak{D}_4 \\ R_9\ \underline{\{P_1(x_1, y_1)\} \to \Theta \qquad\qquad \{P_2(x_2, y_2)\} \to \Theta} \\ R_{10}\ \underline{\{P_1(x_1, y_1) \vee P_2(x_2, y_2)\} \to \Theta} \\ R_{11}\ \underline{\{(\exists v_2)(P_1(x_1, y_1) \vee P_2(x_2, v_2))\} \to \Theta} \\ R_{12}\ \underline{\{(\exists v_1)(\exists v_2)(P_1(x_1, v_1) \vee P_2(x_2, v_2))\} \to \Theta} \\ R_{13}\ \underline{\{(\forall u_2)(\exists v_1)(\exists v_2)(P_1(x_1, v_1) \vee P_2(u_2, v_2))\} \to \Theta} \\ \quad \{(\forall u_1)(\forall u_2)(\exists v_1)(\exists v_2)(P_1(u_1, v_1) \vee P_2(u_2, v_2))\} \to \Theta \end{array},$$

where $\Theta=\{(\exists v_1)P_1(x_1, v_1), (\exists v_2)P_2(x_2, v_2)\}$

$\mathfrak{D}_6$ be

$$\begin{array}{ll} & \downarrow \mathfrak{D}_5 \\ & \Gamma\to\{(\exists v_1)P_1(x_1, v_1), (\exists v_2)P_2(x_2, v_2)\} \\ R_{14} & \overline{\Gamma\to\{(\forall u_1)(\exists v_1)P_1(u_1, v_1), (\exists v_2)P_2(x_2, v_2)\}} \\ R_{15} & \overline{\Gamma\to\{(\forall u_1)(\exists v_1)P_1(u_1, v_1)\vee(\forall u_2)(\exists v_2)P_2(u_2, v_2), (\exists v_2)P_2(x_2, v_2)\}} \\ R_{16} & \overline{\Gamma\to\{(\forall u_1)(\exists v_1)P_1(u_1, v_1)\vee(\forall u_2)(\exists v_2)P_2(u_2, v_2), (\forall u_2)(\exists v_2)P_2(u_2, v_2)\}} \\ R_{17} & \overline{\Gamma\to\{(\forall u_1)(\exists v_1)P_1(u_1, v_1)\vee(\forall u_2)(\exists v_2)P_2(u_2, v_2)\}} \end{array},$$

where $\Gamma=\{(\forall u_1)(\forall u_2)(\exists v_1)(\exists v_2)(P_1(u_1, v_1)\vee P_2(u_2, v_2))\}$

and

$R_5=(\to\exists, P_1(x_1, y_1), (\exists v_1)P_1(x_1, v_1))$,

$R_6=(W, \phi, \{(\exists v_2)P_2(x_2, v_2)\})$,

$R_7=(\to\exists, P_2(x_2, y_2), (\exists v_2)P_2(x_2, v_2))$,

$R_8=(W, \phi, \{(\exists v_1)P_1(x_1, v_1)\})$,

$R_9=(\vee\to, \{P_1(x_1, y_1), P_2(x_2, y_2)\}, P_1(x_1, y_1)\vee P_2(x_2, y_2))$,

$R_{10}=(\exists\to, P_1(x_1, y_1)\vee P_2(x_2, y_2), (\exists v_2)(P_1(x_1, y_1)\vee P_2(x_2, v_2)))$,

$R_{11}=(\exists\to, (\exists v_2)(P_1(x_1, y_1)\vee P_2(x_2, v_2)), (\exists v_1)(\exists v_2)(P_1(x_1, v_1)\vee P_2(x_2, v_2)))$,

$R_{12}=(\forall\to, (\exists v_1)(\exists v_2)(P_1(x_1, v_1)\vee P_2(x_2, v_2)),$
$(\forall u_2)(\exists v_1)(\exists v_2)(P_1(x_1, v_1)\vee P_2(u_2, v_2)))$,

$R_{13}=(\forall\to, (\forall u_2)(\exists v_1)(\exists v_2)(P_1(x_1, v_1)\vee P_2(u_2, v_2)),$
$(\forall u_1)(\forall u_2)(\exists v_1)(\exists v_2)(P_1(u_1, v_1)\vee P_2(u_2, v_2)))$,

$R_{14}=(\to\forall, (\exists v_1)P_1(x_1, v_1), (\forall u_1)(\exists v_1)P_1(u_1, v_1))$,

$R_{15}=(\to\vee, (\forall u_1)(\exists v_1)P_1(u_1, v_1), (\forall u_1)(\exists v_1)P_1(u_1, v_1)\vee(\forall u_2)(\exists v_2)P_2(u_2, v_2))$,

$R_{16}=(\to\forall, (\exists v_2)P_2(x_2, v_2), (\forall u_2)(\exists v_2)P_2(u_2, v_2))$,

$R_{17}=(\to\vee, (\forall u_2)(\exists v_2)P_2(u_2, v_2), (\forall u_1)(\exists v_1)P_1(u_1, v_1)\vee(\forall u_2)(\exists v_2)P_2(u_2, v_2))$.

Then $\mathfrak{D}_3$, $\mathfrak{D}_4$, $\mathfrak{D}_5$ and $\mathfrak{D}_6$ are normal, R_{10}, R_{11}, R_{12}, R_{13}, R_{15}, R_{17}, are inference rules of type 1.

Let $\mathfrak{D}_7$ be

$$\begin{array}{ll} & \downarrow \mathfrak{D}_5 \\ & \Gamma\to\{(\exists v_1)P_1(x_1, v_1), (\exists v_2)P_2(x_2, v_2)\} \\ R_{14} & \overline{\Gamma\to\{(\forall u_1)(\exists v_1)P_1(u_1, v_1), (\exists v_2)P_2(x_2, v_2)\}} \\ R_{16} & \overline{\Gamma\to\{(\forall u_1)(\exists v_1)P_1(u_1, v_1), (\forall u_2)(\exists v_2)P_2(u_2, v_2)\}} \\ R_{15} & \overline{\Gamma\to\{(\forall u_1)(\exists v_1)P_1(u_1, v_1)\vee(\forall u_2)(\exists v_2)P_2(u_2, v_2), (\forall u_2)(\exists v_2)P_2(u_2, v_2)\}} \\ R_{17} & \overline{\Gamma\to\{(\forall u_1)(\exists v_1)P_1(u_1, v_1)\vee(\forall u_2)(\exists v_2)P_2(u_2, v_2)\}} \end{array}.$$

Then $\mathfrak{D}_7$ is *not* normal because R_{16}, R_{15} and R_{15}, R_{17} do not satisfy (**).

3.3. PROPOSITION. *If $\{\mathfrak{D}_h\}_{h\in H}$ are normal derivations, R is an inference rule of type 2 and $\mathfrak{D}$ is an m-cut free derivation of the form* $R\,\dfrac{\mathfrak{D}_h(h\in H)}{\Gamma\to\Theta}$, *then* $\mathfrak{D}$ *is* normal.

3.4. THEOREM (Normal derivation theorem). *For any derivation* $\mathfrak{D}$, *there is a normal derivation of the lowest sequent of* $\mathfrak{D}$.

3.5. REMARK. We can not hope that "general cut elimination theorem" holds in every morphism logic, because $\neg\varphi \vee \psi \in PFM_\lambda$ implies $\vdash_L \varphi \to \psi$ but obviously we can not prove this fact without using cut rules in $L = L(L_\lambda)_{\lambda \in \Lambda}$, for some $\{L_\lambda\}_{\lambda \in \Lambda}$.

§ 4. A proof of the normal derivation theorem.

4.1. LEMMA. *Suppose* $\mathfrak{D}$ *is an m-cut free derivation of a sequent* $\Gamma \to \Theta$ *and f is a mapping from FV to FV satisfying the following conditions: Let* $x, y \in FV$; *if* $f(x) = f(y)$ *and x is used as an eigenvariable in* $\mathfrak{D}$, *then* $x = y$. *Then* $f(\mathfrak{D})$ *is an m-cut free derivation.*

PROOF. Obvious from the definition of derivations.

4.2. LEMMA. *For any m-cut free derivations* $\mathfrak{D}_1$ *of* $\Gamma_1 \to \Theta_1$, $\mathfrak{D}_2$ *of* $\Gamma_2 \to \Theta_2$ *and any formula F, there is an m-cut free derivation of*

$$\Gamma_1 \cup \Gamma_2 - \{F\} \to \Theta_1 - \{F\} \cup \Theta_2 .$$

PROOF. First we should remark that Lemma 4.1 permits us to assume, without loss of generality, that the derivation $(C, F)\dfrac{\overset{\downarrow \mathfrak{D}_1}{\Gamma_1 \to \Theta_1} \qquad \overset{\downarrow \mathfrak{D}_2}{\Gamma_2 \to \Theta_2}}{\Gamma_1 \cup \Gamma_2 - \{F\} \to \Theta_1 - \{F\} \cup \Theta_2}$ satisfies the eigenvariable condition.

Formally we have to prove this lemma by induction on F using, in each induction step, induction on $(\mathfrak{D}_1, \mathfrak{D}_2)$. However in dividing the cases as follows, one easily obtains a proof of this lemma which can be immediately rewritten in formal language. (Cf. Feferman [1].)

CASE 1. $\mathfrak{D}_1$ is an axiom sequent. If $F \notin \Theta_1$,

$$(W, \Gamma_2 - \{F\}, \Theta_2)\dfrac{\Gamma_1 \to \Theta_1}{\Gamma_1 \cup \Gamma_2 - \{F\} \to \Theta_1 - \{F\} \cup \Theta_2}$$

is an m-cut free derivation. So, we assume that $F \in \Theta_1$. Then $\Gamma_1 \to \Theta_1$ is $\{F\} \to \{F\}$ or $\to \{F\}$. If $\Gamma_1 \to \Theta_1$ is $\{F\} \to \{F\}$ then $\Gamma_1 \cup \Gamma_2 - \{F\} \to \Theta_1 - \{F\} \cup \Theta_2$ is $\{F\} \cup \Gamma_2 \to \Theta_2$. Hence $(W, \{F\}, \emptyset)\dfrac{\overset{\downarrow \mathfrak{D}_2}{\Gamma_2 \to \Theta_2}}{\{F\} \cup \Gamma_2 \to \Theta_2}$ is an m-cut free derivation of

$$\Gamma_1 \cup \Gamma_2 - \{F\} \to \Theta_1 - \{F\} \cup \Theta_2 .$$

If $\Gamma_1 \to \Theta_1$ is $\to \{F\}$ then F is a λ-formula for some $\lambda \in \Lambda$. Hence

$$(C, F)\dfrac{\to \{F\} \qquad \overset{\downarrow \mathfrak{D}_2}{\Gamma_2 \to \Theta_2}}{\Gamma_2 - \{F\} \to \Theta_2}$$

is an m-cut free derivation.

CASE 2. $\mathfrak{D}_2$ is an axiom sequent. Similar to case 1.

Let R_1 be the last rule of $\mathfrak{D}_1$ and R_2 be the last rule of $\mathfrak{D}_2$.

CASE 3. R_1 is $(W, \Gamma_1', \Theta_1')$, i.e. $\mathfrak{D}_1$ is $R_1 \dfrac{\overset{\downarrow}{\Gamma_1'' \to \Theta_1''}}{\Gamma_1' \cup \Gamma_1'' \to \Theta_1'' \cup \Theta_1'}$. By the hypothesis of induction, there is an m-cut free derivation $\mathfrak{D}'$ of

$$\Gamma_1'' \cup \Gamma_2 - \{F\} \to \Theta_1'' - \{F\} \cup \Theta_2 .$$

Then

$$(W, \Gamma_1', \Theta_1' - \{F\}) \dfrac{\overset{\downarrow \mathfrak{D}'}{\Gamma_1'' \cup \Gamma_2 - \{F\} \to \Theta_1'' - \{F\} \cup \Theta_2}}{\Gamma_1 \cup \Gamma_2 - \{F\} \to \Theta_1 - \{F\} \cup \Theta_2}$$

is an m-cut free derivation.

CASE 4. R_2 is $(W, \Gamma_2', \Theta_2')$. Similar to case 3.

CASE 5. R_1 is (C, G), i.e. $\mathfrak{D}_1$ is

$$R_1 \dfrac{\overset{\downarrow \mathfrak{D}_{11}}{\Gamma_{11} \to \Theta_{11}} \qquad \overset{\downarrow \mathfrak{D}_{12}}{\Gamma_{12} \to \Theta_{12}}}{\Gamma_{11} \cup \Gamma_{12} - \{G\} \to \Theta_{11} - \{G\} \cup \Theta_{12}},$$

where

$$\Gamma_1 = \Gamma_{11} \cup \Gamma_{12} - \{G\}$$

$$\Theta_1 = \Theta_{11} - \{G\} \cup \Theta_{12} .$$

Then by the hypotheses of induction, there are two m-cut free derivations $\mathfrak{D}_{11}'$ of $\Gamma_{11} \cup \Gamma_2 - \{F\} \to \Theta_{11} - \{F\} \cup \Theta_2$ and $\mathfrak{D}_{12}'$ of

$$\Gamma_{12} \cup \Gamma_2 - \{F\} \to \Theta_{12} - \{F\} \cup \Theta_2 .$$

Since G is not an m-formula,

$$(C, G) \dfrac{\overset{\downarrow \mathfrak{D}_{11}'}{\Gamma_{11} \cup \Gamma_2 - \{F\} \to \Theta_{11} - \{F\} \cup \Theta_2} \quad \overset{\downarrow \mathfrak{D}_{12}'}{\Gamma_{12} \cup \Gamma_2 - \{F\} \to \Theta_{12} - \{F\} \cup \Theta_2}}{\Gamma_{11} \cup \Gamma_{12} - \{G\} \cup \Gamma_2 - \{F\} \to \Theta_{11} - \{F, G\} \cup \Theta_{12} - \{F\} \cup \Theta_2}$$

is an m-cut free derivation of $\Gamma_1 \cup \Gamma_2 - \{F\} \to \Theta_1 - \{F\} \cup \Theta_2$.

CASE 6. R_2 is (C, F). Similar to case 5.

CASE 7. R_1 is a logical inference rule but not a right inference rule whose principal formula is F.

SUBCASE 7.1. R_1 is $(\neg\to, F_1, \neg F_1)$, i.e. $\mathfrak{D}_1$ is

$$R_1 \dfrac{\overset{\downarrow \mathfrak{D}_1'}{\Gamma_1' \to \Theta_1 \cup \{F_1\}}}{\{\neg F_1\} \cup \Gamma_1' \to \Theta_1},$$

where

$$\Gamma_1 = \{\neg F_1\} \cup \Gamma_1' .$$

If $F_1 = F$, by the hypothesis of induction, there is an m-cut free derivation $\mathfrak{D}_1''$ of $\Gamma_1' \cup \Gamma_2 - \{F\} \to \Theta_1 - \{F\} \cup \Theta_2$. By applying $(W, \{\neg F_1\}, \emptyset)$, we get an

m-cut free derivation of $\Gamma_1 \cup \Gamma_2 - \{F\} \to \Theta_1 - \{F\} \cup \Theta_2$. Suppose $F_1 \neq F$. Then by the hypothesis of induction, there is an m-cut free derivation $\mathfrak{D}_1''$ of $\Gamma_1' \cup \Gamma_2 - \{F\} \to (\Theta_1 \cup \{F_1\}) - \{F\} \cup \Theta_2$. By applying R_1, we get an m-cut free derivation of $\Gamma_1 \cup \Gamma_2 - \{F\} \to \Theta_1 - \{F\} \cup \Theta_2$.

SUBCASE 7.2. R_1 is $(\to \neg)$ or $(\wedge\to)$ or $(\to\wedge)$ or $(\vee\to)$ or $(\to\vee)$ or $(\forall\to)$ or $(\to\forall)$ or $(\exists\to)$ or $(\to\exists)$. Similar to subcase 7.1.

CASE 8. R_2 is a logical inference rule but not a left inference rule whose principal formula is F. Similar to case 7.

CASE 9. R_1 is a right inference rule whose principal formula is F and R_2 is a left inference rule whose principal formula is F.

SUBCASE 9.1. $F = \neg G$, i.e. $R_1 = (\to \neg, G, F)$ and $R_2 = (\neg\to, G, F)$. Then

$$\mathfrak{D}_1 \text{ is } R_1 \frac{\{G\} \cup \Gamma_1 \to \Theta_1'}{\Gamma_1 \to \Theta_1' \cup \{F\}} \quad \text{and} \quad \mathfrak{D}_2 \text{ is } R_2 \frac{\Gamma_2' \to \Theta_2 \cup \{G\}}{\{F\} \cup \Gamma_2' \to \Theta_2},$$

where

$$\Theta_1 = \Theta_1' \cup \{F\} \quad \text{and} \quad \Gamma_2 = \{F\} \cup \Gamma_2'.$$

By the hypotheses of induction, there are two m-cut free derivations $\mathfrak{D}_1'$ of

$$\{G\} \cup \Gamma_1 \cup \Gamma_2' - \{F\} \to \Theta_1' - \{F\} \cup \Theta_2$$

and $\mathfrak{D}_2'$ of

$$\Gamma_1 \cup \Gamma_2' - \{F\} \to \Theta_1' - \{F\} \cup \Theta_2 \cup \{G\}.$$

Since F is more complex than G, again by the hypothesis of induction, there is an m-cut free derivation of

$$\Gamma_1 \cup \Gamma_2' - \{F\} \cup \Gamma_1 - \{G\} \cup \Gamma_2' - \{F, G\} \to \Theta_1' - \{F, G\} \cup \Theta_2 - \{G\} \cup \Theta_1' - \{F\} \cup \Theta_2,$$

i.e.

$$\Gamma_1 \cup \Gamma_2 - \{F\} \to \Theta_1 - \{F\} \cup \Theta_2.$$

SUBCASE 9.2. $F = \wedge K$, i.e. R_1 is $(\to\wedge, K, \wedge K)$ and R_2 is $(\wedge\to, G_0, \wedge K)$ for some $G_0 \in K$. Then

$$\mathfrak{D}_1 \text{ is } R_1 \frac{\Gamma_1 \to \Theta_1' \cup \{G\} \ (G \in K)}{\Gamma_1 \to \Theta_1' \cup \{F\}} \quad \text{and } \mathfrak{D}_2 \text{ is } R_2 \frac{\{G_0\} \cup \Gamma_2' \to \Theta_2}{\{F\} \cup \Gamma_2' \to \Theta_2},$$

where

$$\Theta_1 = \Theta_1' \cup \{F\} \quad \text{and} \quad \Gamma_2 = \{F\} \cup \Gamma_2'.$$

By the hypotheses of induction, there are two m-cut free derivations of

$$\Gamma_1 \cup \Gamma_2' - \{F\} \to \Theta_1' - \{F\} \cup \{G_0\} \cup \Theta_2$$

and

$$\Gamma_1 \cup \{G_0\} \cup \Gamma_2' - \{F\} \to \Theta_1' - \{F\} \cup \Theta_2.$$

Again by the hypothesis of induction, there is an m-cut free derivation of

$$\Gamma_1 \cup \Gamma_2' - \{F\} \to \Theta_1' - \{F\} \cup \Theta_2\,, \qquad \text{i.e.}\quad \Gamma_1 \cup \Gamma_2 - \{F\} \to \Theta_1 - \{F\} \cup \Theta_2\,.$$

SUBCASE 9.3. $F = \vee K$. Similar to subcase 9.2.

SUBCASE 9.4. $F = (\forall u)G(u)$, i.e. R_1 is $(\to\forall, G(x), (\forall u)G(u))$ and R_2 is $\left(\forall\to, G\binom{x}{y}, (\forall u)G(u)\right)$. Then

$$\mathfrak{D}_1 \text{ is } R_1 \frac{\begin{array}{c}\downarrow \mathfrak{D}_{11}\\ \Gamma_1 \to \Theta_1' \cup \{G(x)\}\end{array}}{\Gamma_1 \to \Theta_1' \cup \{F\}} \quad \text{and } \mathfrak{D}_2 \text{ is } R_2 \frac{\begin{array}{c}\downarrow\\ \left\{G\binom{x}{y}\right\} \cup \Gamma_2' \to \Theta_2\end{array}}{\{F\} \cup \Gamma_2' \to \Theta_2}\,,$$

where

$$\Theta_1 = \Theta_1' \cup \{F\} \qquad \text{and} \qquad \Gamma_2 = \{F\} \cup \Gamma_2'\,.$$

Define f: FV to FV by $f(z) = z$ if $z \neq x$ and $f(x) = y$. (Without loss of generality, we can assume that y is not used in $\mathfrak{D}_{11}$ as an eigenvariable.) Then $\mathfrak{D}_{11}$ and f satisfy the hypotheses of Lemma 4.1. Hence $f(\mathfrak{D}_{11})$ is an m-cut free derivation of $\Gamma_1 \to \Theta_1' \cup \left\{G\binom{x}{y}\right\}$. By the hypotheses of induction, there are two m-cut free derivations of

$$\Gamma_1 \cup \left\{G\binom{x}{y}\right\} \cup \Gamma_2' - \{F\} \to \Theta_1' - \{F\} \cup \Theta_2$$

and

$$\Gamma_1 \cup \Gamma_2' - \{F\} \to \Theta_1' - \{F\} \cup \left\{G\binom{x}{y}\right\} \cup \Theta_2\,.$$

Again by the hypothesis of induction, there is an m-cut free derivation of

$$\Gamma_1 \cup \Gamma_2 - \{F\} \to \Theta_1 - \{F\} \cup \Theta_2\,.$$

SUBCASE 9.5. $F = (\exists u)G(u)$. Similar to subcase 9.4. (Q. E. D.)

4.3. LEMMA. *For any sequent $\Gamma \to \Theta$, if $\vdash_L \Gamma \to \Theta$, there is an m-cut free derivation of $\Gamma \to \Theta$.*

PROOF. By the induction on a derivation $\mathfrak{D}$ of $\Gamma \to \Theta$. This is obvious from Lemmas 4.1 and 4.2.

4.4. NORMAL SEQUENCE. A sequence $\vec{R} = \langle R_1, \cdots, R_n \rangle$ of *logical inference rules* is said to be *normal* if

① R_1 is of type 2,

② $R_2, \cdots, R_n$ are of type 1,

③ the principal formula of R_i is the side formula of R_{i+1} and R_i, R_{i+1} are both left rules or both right rules for $i = 1, \cdots, n-1$.

4.5. DEFINITION OF $n(\mathfrak{D})$ AND $R(\mathfrak{D})$. For each normal derivation $\mathfrak{D}$, we shall associate a normal derivation $n(\mathfrak{D})$ and a normal sequence $R(\mathfrak{D})$ by the following rules:

CASE 1. $\mathfrak{D}$ is an axiom sequent. Let $n(\mathfrak{D}) = \mathfrak{D}$ and $R(\mathfrak{D}) = 0$ (the empty sequence). Let R be the last rule of $\mathfrak{D}$.

CASE 2. R is a structural inference rule. Let $n(\mathfrak{D})=\mathfrak{D}$ and $R(\mathfrak{D})=0$.

CASE 3. R is a logical inference rule of type 2. Let $n(\mathfrak{D})=\mathfrak{D}$ and $R(\mathfrak{D})=\langle R\rangle$.

CASE 4. R is of type 1. Then $\mathfrak{D}$ has the form $R\,\dfrac{\mathfrak{D}_1}{\Gamma\to\Theta}$. Let $n(\mathfrak{D})=n(\mathfrak{D}_1)$ and $R(\mathfrak{D})=\langle R(\mathfrak{D}_1), R\rangle$.

For example, if $\mathfrak{D}_1, \mathfrak{D}_2, \cdots, \mathfrak{D}_6$ are derivations in example 3.2,

$$
\begin{array}{lll}
n(\mathfrak{D}_1)=\mathfrak{D}_1 & \text{and} & R(\mathfrak{D}_1)=\langle R_2\rangle\,, \\
n(\mathfrak{D}_2)=\mathfrak{D}_2 & \text{and} & R(\mathfrak{D}_2)=\langle R_4\rangle\,, \\
n(\mathfrak{D}_3)=\mathfrak{D}_3 & \text{and} & R(\mathfrak{D}_3)=0\,, \\
n(\mathfrak{D}_4)=\mathfrak{D}_4 & \text{and} & R(\mathfrak{D}_4)=0\,, \\
n(\mathfrak{D}_5)=R_9\,\dfrac{\downarrow\qquad\downarrow}{\cdots\to\Theta} & \text{and} & R(\mathfrak{D}_5)=\langle R_9, R_{10}, R_{11}, R_{12}, R_{13}\rangle\,, \\
n(\mathfrak{D}_6)=R_{16}\,\dfrac{\downarrow}{\Gamma\to\cdots} & \text{and} & R(\mathfrak{D}_6)=\langle R_{16}, R_{17}\rangle\,.
\end{array}
$$

4.6. PROPOSITION. ① *Suppose* $\mathfrak{D}$ *is a normal derivation and* R *is the last rule of* $\mathfrak{D}$. *If* R *is of type* 1, *then* $R(\mathfrak{D})=\langle R_1, \cdots, R_n\rangle$, $n\geqq 2$ *and* $\mathfrak{D}$ *has the form*

$$
\left.\begin{array}{l}
\quad\downarrow \\
R_1\ \overline{\Gamma_1\to\Theta_1}
\end{array}\right\}\ n(\mathfrak{D})
\qquad
\begin{array}{l}
R_2\ \overline{\Gamma_2\to\Theta_2} \\
\qquad\vdots \\
R_n\ \overline{\Gamma_n\to\Theta_n}\,.
\end{array}
$$

② *If* $\{\mathfrak{D}_h\}_{h\in H}$ *are normal derivations,* $\vec{R}=\langle R_1, \cdots, R_n\rangle$ *is a normal sequence and*

$$
\begin{array}{l}
\qquad\mathfrak{D}_h\ (h\in H) \\
\qquad\quad\downarrow \\
R_1\ \overline{\Gamma_1\to\Theta_1} \\
R_2\ \overline{\Gamma_2\to\Theta_2} \\
\qquad\vdots \\
R_n\ \overline{\Gamma_n\to\Theta_n}
\end{array}
$$

is an m-cut free derivation, then this derivation is also normal.

4.7. LEMMA. *For any normal derivation* $\mathfrak{D}$ *of a sequent* $\Gamma\to\Theta$ *and any m-formula* F:

(a) *Let* $K\subseteqq FM$ *and suppose* $\wedge K\in FM$; *If* $F\in K$ *and* $F\in\Gamma$ *then there is a normal derivation of* $\{\wedge K\}\cup\Gamma-\{F\}\to\Theta$ (*i.e.* $\{\wedge K\}\cup(\Gamma\to\{F\})\to\Theta$);

(b) *Let* $K\subseteqq FM$ *and suppose* $\vee K\in FM$; *If* $F\in K$ *and* $F\in\Theta$ *then there is a normal derivation of* $\Gamma\to\{\vee K\}\cup\Theta-\{F\}$;

(c) *If* $F=\vee K\in\Gamma$ *and* $G\in K$, *then there is a normal derivation of* $\{G\}\cup\Gamma-\{F\}\rightarrow\Theta$;

(d) *If* $F=\wedge K\in\Theta$ *and* $G\in K$, *then there is a normal derivation of* $\Gamma\rightarrow\{G\}\cup\Theta-\{F\}$;

(e) *If* $F=F\binom{x}{y}\in\Gamma$, *then there is a normal derivation of* $\{(\forall v)F(v)\}\cup\Gamma-\{F\}\rightarrow\Theta$;

(f) *If* $F=F\binom{x}{y}\in\Theta$, *then there is a normal derivation of* $\Gamma\rightarrow\{(\exists v)F(v)\}\cup\Theta-\{F\}$;

(g) *If* $F=(\exists v)G(v)\in\Gamma$ *and* x *does not appear in* $\Gamma\cup\Theta$, *then there is a normal derivation of* $\{G(x)\}\cup\Gamma-\{F\}\rightarrow\Theta$;

(h) *If* $F=(\forall v)G(v)\in\Theta$ *and* x *does not appear in* $\Gamma\cup\Theta$, *then there is a normal derivation of* $\Gamma\rightarrow\{G(x)\}\cup\Theta-\{F\}$.

PROOF. By the induction on $\mathfrak{D}$.

CASE 1. $\mathfrak{D}$ is an axiom sequent. Obvious.

Let R be the last rule of $\mathfrak{D}$.

CASE 2. R is of type 2. Obvious from Proposition 3.3.

CASE 3. R is of type 1. Let $n(\mathfrak{D})=\mathfrak{D}_1$ and $R(\mathfrak{D})=\langle R_1,\cdots,R_n\rangle$. Then by Proposition 4.6.①, $\mathfrak{D}$ has the form

$$\left.\begin{array}{l}\downarrow\\ R_1\,\overline{\Gamma_1\rightarrow\Theta_1}\end{array}\right\}\mathfrak{D}_1\qquad R_2\,\overline{\Gamma_2\rightarrow\Theta_2}\qquad\vdots\qquad R_n\,\overline{\Gamma_n\rightarrow\Theta_n}$$

and $n\geqq 2$, $R=R_n$, $\Gamma_n=\Gamma$, $\Theta_n=\Theta$. Let F_i be the principal formula of R_i, $i=1,\cdots,n$.

SUBCASE 3.1. R_1 is $(\wedge\rightarrow, F_0, F_1)$. Then by the definition of normal sequence, $R_2,\cdots,R_n$ are all left rules. So, $\mathfrak{D}$ has the form

$$\left.\begin{array}{c}\downarrow\\ \{F_0\}\cup\Gamma'_0\rightarrow\Theta\\ R_1\,\overline{\{F_1\}\cup\Gamma'_1\rightarrow\Theta}\end{array}\right\}\mathfrak{D}_1\qquad R_2\,\overline{\{F_2\}\cup\Gamma'_2\rightarrow\Theta}\qquad\vdots\qquad R_n\,\overline{\{F_n\}\cup\Gamma'_n\rightarrow\Theta}$$

and $F_i\notin\Gamma'_i$, $i=0,\cdots,n$. Then $\Gamma'_n\subseteqq\Gamma'_0\cup\{F_0,F_1,\cdots,F_{n-1}\}$ because $\Gamma'_{i+1}\subseteqq\Gamma'_i\cup\{F_i\}$, $i=0,\cdots,n-1$.

Now we prove (a) only because (b)—(h) are similarly proved. Suppose $\wedge K\in FM$, $F\in\Gamma=\{F_n\}\cup\Gamma'_n$, $F\in K$.

If $F=F_n$, then $(\wedge\rightarrow)\dfrac{\downarrow\mathfrak{D}\quad\{F_n\}\cup\Gamma'_n\rightarrow\Theta}{\{\wedge K\}\cup\Gamma'_n\rightarrow\Theta}$ is a normal derivation of $\{\wedge K\}\cup$

$\Gamma-\{F\}\to\Theta$. (Of course it is necessary to prove that this derivation satisfies the eigenvariable condition (∗), but we can assume this without loss of generality by Lemma 4.1.)

If $F=F_i$, $0<i<n$, then

$$\begin{array}{rl} & \downarrow \\ R_1 & \{F_0\}\cup\Gamma'_0\to\Theta \\ \hline R_2 & \{F_1\}\cup\Gamma'_1\to\Theta \\ \hline & \{F_2\}\cup\Gamma'_2\to\Theta \\ & \vdots \\ R_i & \{F_{i-1}\}\cup\Gamma'_{i-1}\to\Theta \\ \hline R_{i+1} & \{F_i\}\cup\Gamma'_i\to\Theta \\ \hline & \{F_{i+1}\}\cup\Gamma''_{i+1}\to\Theta \\ & \vdots \\ R_n & \\ \hline (W,\{\wedge K\},\emptyset) & \{F_n\}\cup\Gamma''_n\to\Theta \\ \hline & \{\wedge K\}\cup\{F_n\}\cup\Gamma''_n\to\Theta \end{array}$$

is a normal derivation of $\{\wedge K\}\cup\Gamma-\{F\}\to\Theta$, where $\Gamma''_{i+1}=\Gamma'_{i+1}-\{F\},\cdots,$ $\Gamma''_n=\Gamma'_n-\{F\}$.

If $F=F_0$, then

$$\begin{array}{rl} & \downarrow\mathfrak{D} \\ (\wedge\to, F, \wedge K) & \{F_0\}\cup\{F_n\}\cup\Gamma''_n\to\Theta \\ \hline & \{\wedge K\}\cup\{F_n\}\cup\Gamma''_n\to\Theta \end{array}$$

is a normal derivation of $\{\wedge K\}\cup\Gamma-\{F\}\to\Theta$, where $\Gamma''_n=\Gamma'_n-\{F\}$ because $(\wedge\to, F_0, \wedge K)$ is of type 2.

If $F\notin\{F_0, F_1,\cdots, F_n\}$ then $F\in\Gamma'_0$. By the hypothesis of induction, there is a normal derivation of $\{\wedge K\}\cup\{F_0\}\cup\Gamma'_0-\{F\}\to\Theta$. By applying $R_1,\cdots,R_n$, we get a normal derivation of $\{\wedge K\}\cup\{F_n\}\cup\Gamma'_n-\{F\}\to\Theta$ by 4.6.②.

SUBCASE 3.2. R_1 is $(\neg\to)$ or $(\to\neg)$ or $(\to\vee)$ or $(\forall\to)$ or $(\to\exists)$ or $(\vee\to)$ or $(\to\wedge)$ or $(\exists\to)$ or $(\to\forall)$. Similar to subcase 3.1.

PROOF OF THEOREM. By Lemma 4.3, we can assume, without loss of generality, that every derivation considered in the following is m-cut free. Suppose $\mathfrak{D}$ is an m-cut free derivation of $\Gamma\to\Theta$. By induction on $\mathfrak{D}$, we shall prove this theorem.

CASE 1. $\mathfrak{D}$ is an axiom sequent. $\mathfrak{D}$ itself is normal.

Let R be the last rule of $\mathfrak{D}$.

CASE 2. R is an inference rule of type 2. Then $\mathfrak{D}$ has the form $R\dfrac{\mathfrak{D}_h\ (h\in H)}{\Gamma\to\Theta}$. By the hypothesis of induction, we can assume that $\mathfrak{D}_h$ is normal for any $h\in H$. Then by 3.3, $\mathfrak{D}$ is also normal.

CASE 3. R is of type 1.

SUBCASE 3.1. R is $(\wedge\rightarrow, F, \wedge K)$. Then $\mathfrak{D}$ has the form $R\dfrac{\overset{\downarrow \mathfrak{D}_1}{\{F\}\cup\Gamma'\rightarrow\Theta}}{\{\wedge K\}\cup\Gamma'\rightarrow\Theta}$, where $\Gamma=\{\wedge K\}\cup\Gamma'$. By the hypothesis of induction, we can assume that $\mathfrak{D}_1$ is normal. Then by (a) in the Lemma 4.7, there is a normal derivation of $\{\wedge K\}\cup\Gamma'-\{F\}\rightarrow\Theta$. Hence there is a normal derivation of $\{\wedge K\}\cup\Gamma'\rightarrow\Theta$. (Notice that Γ' may have F.)

SUBCASE 3.2. R is $(\rightarrow\vee)$ or $(\forall\rightarrow)$ or $(\rightarrow\exists)$. Similar to subcase 3.1.

SUBCASE 3.3. R is $(\exists\rightarrow, F(x), (\exists v)F(v))$ and $F(x)$ is of the forms $\vee K$ or $(\exists u)G(u)$. Suppose $F(x)$ has the form $(\exists u_1)\cdots(\exists u_n)F_1(u_1, \cdots, u_n, x)$ and $F_1(u_1, \cdots, u_n, x)=(\vee K)(u_1, \cdots, u_n, x)$ is an m-formula. Then $\mathfrak{D}$ has the form

$$R\,\frac{\overset{\downarrow \mathfrak{D}_1}{\{F(x)\}\cup\Gamma'\rightarrow\Theta}}{\{(\exists v)F(v)\}\cup\Gamma'\rightarrow\Theta},$$

where $\Gamma=\{(\exists v)F(v)\}\cup\Gamma'$. By the hypothesis of induction, we can assume that $\mathfrak{D}_1$ is normal. Let $x_1, \cdots, x_n$ be free individual variables which do not appear in $\mathfrak{D}_1$. (Without loss of generality, we can assume that there are such variables.) Then by (c), (g), in the Lemma 4.7, for any $G(u_1, \cdots, u_n, x)\in K$, there is a normal derivation of $\{G(x_1, \cdots, x_n, x)\}\cup\Gamma'\rightarrow\Theta$, because $(\exists u_1)\cdots(\exists u_n)F_1(u_1, \cdots, u_n, x)\notin\Gamma', \cdots, F_1(x_1, \cdots, x_n, x)\notin\Gamma'$.

Then

$$\begin{array}{ll}
 & \downarrow \\
(\vee\rightarrow) & \dfrac{\{G(x_1, x_2, \cdots, x_n, x)\}\cup\Gamma'\rightarrow\Theta\quad (G(x_1, x_2, \cdots, x_n, x)\in K(x_1, \cdots, x_n, x))}{\{F_1(x_1, \cdots, x_n, x)\}\cup\Gamma'\rightarrow\Theta} \\
(\exists\rightarrow)\left\{\begin{array}{l}\\ \\ \\ \end{array}\right. & \qquad\vdots \\
 & \overline{\{(\exists v)F(v)\}\cup\Gamma'\rightarrow\Theta}
\end{array}$$

is a normal derivation of $\Gamma\rightarrow\Theta$.

SUBCASE 3.6. R is $(\rightarrow\forall, F(x), (\forall v)F(v))$. Similar to subcase 3.3.

Gakushuin University

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