J. Math. Soc. Japan
Vol. 24, No. 4, 1972

Theta series and automorphic forms on GL_2

By Hideo SHIMIZU

(Received Jan. 17, 1972)

The main purpose of the present paper is to give another proof of Jacquet-Langlands [5, Th. 14.4], the assertion of which is the following.

Let $\mathcal{K}$ be a division quaternion algebra over a global field F. To every irreducible admissible representation π of the Hecke algebra $\mathcal{H}(\mathcal{K}_A^\times)$, we can make correspond an irreducible admissible representation π^* of the Hecke algebra $\mathcal{H}(GL_2(\boldsymbol{A}))$ so that, if π is a constituent of the representation of $\mathcal{H}(\mathcal{K}_A^\times)$ in $\mathcal{A}(\eta, \mathcal{K}_A^\times)$ (the space of automorphic forms on $\mathcal{K}_A^\times$ with a character η), then π^* is a constituent of the representation of $\mathcal{H}(GL_2(\boldsymbol{A}))$ in $\mathcal{A}_0(\eta, GL_2(\boldsymbol{A}))$ (the space of cusp forms on $GL_2(\boldsymbol{A})$ with a character η) under the condition that the component π_v of π is infinite dimensional for all places v of F unramified in $\mathcal{K}$.

In view of various ideas in Jacquet-Langlands [5], and also of Shalika-Tanaka [7] and Weil [11], we find it natural to consider theta series made of Weil representation of $SL_2(\boldsymbol{A})$ in the Schwartz space on $\mathcal{K}_{\boldsymbol{A}}$, and in order to construct an irreducible subspace of $\mathcal{A}_0(\eta, GL_2(\boldsymbol{A}))$ from the space of theta series, to make use of a spherical function associated with automorphic forms on $\mathcal{K}_A^\times$. In this way we obtain a proof of the above theorem, somewhat more direct than the original one, under a weaker condition that π is not one-dimensional (in substance our proof is quite similar to that of [5, Th. 13.1]). The main theorem in our formulation is stated as Theorem 1 (§5, No. 12). Applied to the holomorphic automorphic forms, it gives a generalization of Eichler [1, 2]. It is stated as Theorem 2 (§6, No. 5).

For convenience sake we summarize in §1-§3 generalities on admissible representations, theta series, automorphic forms and spherical functions.

§1. Admissible representation of GL_2.

1. Definition (non-archimedean case). In No. 1—No. 4, F will be a non-archimedean local field. By an *admissible representation* π of $GL_2(F)$ we understand a representation π of $GL_2(F)$ in a vector space $\mathcal{V}$ over $\boldsymbol{C}$ satisfying the following conditions.

(1.1) For any $x \in \mathcal{V}$, the group of elements g in $GL_2(F)$ such that

$\pi(g)x=x$ is an open subgroup of $GL_2(F)$.

(1.2) For any open compact subgroup H of $GL_2(F)$, the space of elements x in $\mathcal{V}$ such that $\pi(h)x=x$ for all $h\in H$ is finite dimensional.

We say that π is irreducible if $\mathcal{V}$ has no proper invariant subspace.

2. Local Hecke algebra. Let $\mathcal{H}_F$ be the space of all $\boldsymbol{C}$-valued locally constant functions of compact support on $GL_2(F)$. It forms an associative algebra under the convolution:

$$f_1*f_2(g)=\int_{GL_2(F)}f_1(gh)f_2(h^{-1})dh.$$

We call $\mathcal{H}_F$ the Hecke algebra of $GL_2(F)$. For an admissible representation π of $GL_2(F)$ in $\mathcal{V}$, we define a representation π of $\mathcal{H}_F$ in $\mathcal{V}$ by

$$\pi(f)x=\int_{GL_2(F)}f(g)\pi(g)xdg \qquad (f\in\mathcal{H}_F,\ x\in\mathcal{V}).$$

For a fixed x, $f(g)\pi(g)x$ is a $\mathcal{V}$-valued locally constant function of compact support on $GL_2(F)$. Therefore, the integral in the above expression is actually a finite sum. Denote by $\rho(g)f$ or $\lambda(g)f$ the right or left translate of a function f on $GL_2(F)$ by an element g in $GL_2(F)$:

$$(\rho(g)f)(h)=f(hg), \qquad (\lambda(g)f)(h)=f(g^{-1}h).$$

By definition we have

$$\pi(\lambda(g)f)=\pi(g)\pi(f) \qquad \text{for } g\in GL_2(F) \text{ and } f\in\mathcal{H}_F. \tag{1.3}$$

Let $\mathfrak{o}$ be the ring of all integers in F, and put $K=GL_2(\mathfrak{o})$. By an *elementary idempotent* we understand a function ξ on K of the form

$$\xi(k)=\sum\dim\sigma_i\operatorname{tr}\sigma_i(k^{-1}),$$

σ_i being a finite number of inequivalent irreducible representations of K. ξ is in fact an idempotent in $\mathcal{H}_F$, if we regard ξ as a function on $GL_2(F)$, putting $\xi(g)=0$ for $g\notin K$.

By (1.1) and (1.2) the representation π of $\mathcal{H}_F$ has the following properties.

(1.4) For any $x\in\mathcal{V}$, there exists a function f in $\mathcal{H}_F$ such that $\pi(f)x=x$.

(1.5) For any elementary idempotent ξ, $\pi(\xi)\mathcal{V}$ is finite dimensional.

Conversely, for any representation π of $\mathcal{H}_F$ in $\mathcal{V}$ with these properties, there exists an admissible representation π of $GL_2(F)$ satisfying (1.3).

3. Principal series of representations. We denote by $|\alpha|_F$ the module of α in $F^\times$; namely, $d(\alpha\alpha_1)=|\alpha|_F d\alpha_1$, $d\alpha_1$ being the additive Haar measure of F. Let T be the group of all upper triangular elements in $GL_2(F)$. Every one-dimensional representation ζ of T can be written in the form

$$\zeta\left(\begin{pmatrix}\alpha & \beta \\ 0 & \delta\end{pmatrix}\right)=\mu_1(\alpha)\mu_2(\delta)\left|\frac{\alpha}{\delta}\right|_F^{1/2},$$

where μ_1, μ_2 are quasi-characters of $F^\times$. Let $\mathscr{B}(\mu_1, \mu_2)$ be the space of all locally constant functions f on $GL_2(F)$ satisfying

$$f(tg)=\zeta(t)f(g) \qquad (t\in T,\ g\in GL_2(F)).$$

The right translation ρ defines a representation of $GL_2(F)$ in $\mathscr{B}(\mu_1, \mu_2)$. It can be shown that ρ is admissible. By [**5**, Th. 3.3] the irreducible constituents of $\mathscr{B}(\mu_1, \mu_2)$ are the following.

i) If $\mu_1\mu_2^{-1}$ equals neither $|\ |_F$ nor $|\ |_F^{-1}$, $\mathscr{B}(\mu_1, \mu_2)$ is irreducible.

ii) If $\mu_1\mu_2^{-1}=|\ |_F$, $\mathscr{B}(\mu_1, \mu_2)$ contains the only one proper invariant subspace $\mathscr{B}_s(\mu_1, \mu_2)$, which is of codimension 1.

iii) If $\mu_1\mu_2^{-1}=|\ |_F^{-1}$, $\mathscr{B}(\mu_1, \mu_2)$ contains the only one proper invariant subspace $\mathscr{B}_f(\mu_1, \mu_2)$, which is of dimension 1.

In the case i) we write $\pi(\mu_1, \mu_2)$ for ρ. In the case ii) we write $\sigma(\mu_1, \mu_2)$ (resp. $\pi(\mu_1, \mu_2)$) for the representation of $GL_2(F)$ in $\mathscr{B}_s(\mu_1, \mu_2)$ (resp. $\mathscr{B}(\mu_1, \mu_2)/\mathscr{B}_s(\mu_1, \mu_2)$) induced by ρ. In the case iii) we write $\sigma(\mu_1, \mu_2)$ (resp. $\pi(\mu_1, \mu_2)$) for the representation of $GL_2(F)$ in $\mathscr{B}(\mu_1, \mu_2)/\mathscr{B}_f(\mu_1, \mu_2)$ (resp. $\mathscr{B}_f(\mu_1, \mu_2)$) induced by ρ.

$\pi(\mu_1, \mu_2)$ (resp. $\sigma(\mu_1, \mu_2)$) and $\pi(\mu_2, \mu_1)$ (resp. $\sigma(\mu_2, \mu_1)$) are equivalent, and there is no other equivalence relation among these representations (cf. [**4**, §1, Th. 7]).

By [**5**, Prop. 2.7] a finite dimensional irreducible admissible representation π of $GL_2(F)$ is necessarily one-dimensional, and we have $\pi(g)=\chi(\det g)$ with a quasi-character χ of $F^\times$. If $\mu_1(\alpha)=\chi(\alpha)|\alpha|_F^{1/2}$, $\mu_2(\alpha)=\chi(\alpha)|\alpha|_F^{-1/2}$, π is equivalent to $\pi(\mu_1, \mu_2)$.

4. Absolutely cuspidal representations. An irreducible admissible representation π of $GL_2(F)$ is called *absolutely cuspidal* if it is not a constituent of $\mathscr{B}(\mu_1, \mu_2)$ for any choice of μ_1, μ_2.

5. Definition (archimedean case). In No. 5-No. 7, we assume that F is an archimedean local field so that F is either the real number field $\boldsymbol{R}$ or the complex number field $\boldsymbol{C}$. Let K be a maximal compact subgroup of $GL_2(F)$. Let dg (resp. dk) be a fixed Haar measure of $GL_2(F)$ (resp. K). We denote by $\mathscr{H}_F$ the space of Radon measures on $GL_2(F)$ spanned by the following two kinds of measures:

i) $f(g)dg$; f is a C^∞ function of compact support on $GL_2(F)$, which is K-finite on both sides.

ii) $\xi(k)dk$; $\xi(k)$ is a matrix coefficient of some irreducible representation of K.

In the following we identify $f(g)dg$ (resp. $\xi(k)dk$) with a function f (resp. ξ).

Let $\mathcal{H}'_F$ be the space spanned by the measures of type i) and $\mathcal{H}''_F$ the space spanned by the measures of type ii). If $*$ denotes the convolution of measures, $\mathcal{H}_F$ forms an associative algebra under $*$. In fact, $*$ coincides on $\mathcal{H}'_F$ (resp. $\mathcal{H}''_F$) with the convolution of functions on $GL_2(F)$ (resp. K), and

$$f*\xi(g)=\int_K f(gk)\xi(k^{-1})dk\,,$$

$$\xi*f(g)=\int_K \xi(k^{-1})f(kg)dk\,.$$

We note that an elementary idempotent can be defined in the same way as in No. 2, and it is an element of $\mathcal{H}''_F$.

We say that a representation π of $\mathcal{H}_F$ in $\mathcal{V}$ is *admissible* if it satisfies the following conditions.

(1.6) For any $x\in\mathcal{V}$, we can find $f_i\in\mathcal{H}'_F$ and $x_i\in\mathcal{V}$ such that

$$x=\sum_{i=1}^{r}\pi(f_i)x_i\,.$$

(1.7) For any elementary idempotent ξ, $\pi(\xi)\mathcal{V}$ is finite dimensional.

(1.8) For any $x\in\mathcal{V}$ and for any elementary idempotent ξ, the mapping $f\to\pi(f)x$ of $\xi*\mathcal{H}'_F*\xi$ into $\pi(\xi)\mathcal{V}$ is continuous (the topology in $\pi(\xi)\mathcal{V}$ is the usual topology in a finite dimensional vector space over $\boldsymbol{C}$, and the topology in $\xi*\mathcal{H}'_F*\xi$ is the one induced by the Schwartz topology in the space of all C^∞ functions of compact support on $GL_2(F)$).

REMARK. If we limit ourselves to a special case where $\mathcal{V}$ is a space consisting of continuous functions on $GL_2(F)$ and π is defined by

$$\pi(\mu)\varphi(h)=\int\varphi(hg)d\mu(g)\qquad(\varphi\in\mathcal{V},\ \mu\in\mathcal{H}_F)\,,$$

then (1.6)-(1.8) can be replaced by the following conditions.

(1.6)′ For any $\varphi\in\mathcal{V}$, there is an elementary idempotent ξ such that $\pi(\xi)\varphi=\varphi$.

(1.7)′ For any elementary idempotent ξ, $\pi(\xi)\mathcal{V}$ is finite dimensional.

(1.8)′ Let φ, ξ be as in (1.8). Let f_i be a sequence of functions in $\xi*\mathcal{H}'_F*\xi$ such that the supports of f_i are all contained in a compact set of $GL_2(F)$, on which f_i converges uniformly to 0, together with all derivatives of higher order. Then $\pi(f_i)\varphi(g)$ converges to 0 for all $g\in GL_2(F)$.

In this situation, (1.8)′ is trivially satisfied. It can be shown that (1.8)′ implies (1.8), and that (1.6)′ and (1.7)′ imply (1.6).

6. Representation of Z, K or $\mathfrak{U}$ induced by an admissible representation. Let π be an admissible representation of $\mathcal{H}_F$ in $\mathcal{V}$. Let Z be the center of $GL_2(F)$. We can define a representation π of Z (resp. K) by the condition that $\pi(g)\pi(f)=\pi(\lambda(g)f)$ is satisfied for all f in $\mathcal{H}'_F$, if g is in Z (resp. K).

Let $\mathfrak{g}$ be the Lie algebra of $GL_2(F)$ and $\mathfrak{U}$ the universal enveloping algebra of $\mathfrak{g}_C=\mathfrak{g}\otimes_R C$. π being as above, we can define a representation π of $\mathfrak{U}$ in $\mathcal{V}$ so that we have

$$\pi(X)\pi(f)=\pi(X*f), \qquad \pi(f)\pi(X)=\pi(f*X)$$

for all $f\in\mathcal{H}'_F$ and $X\in\mathfrak{g}$. Here

$$X*f(g)=[(d/d\alpha)f(\exp(-\alpha X)g)]_{\alpha=0},$$

$$f*X(g)=[(d/d\alpha)f(g\exp(-\alpha X))]_{\alpha=0}.$$

If g is in Z or K, we have

$$\pi(\mathrm{Ad}(g)X)=\pi(g)\pi(X)\pi(g^{-1}).$$

7. Classification of admissible representations. Let T, ζ, μ_1, μ_2 be as in No. 3. Let $\mathcal{B}(\mu_1, \mu_2)$ be the space of all functions φ on $GL_2(F)$ which are K-finite on the right and satisfy

$$\varphi(tg)=\zeta(t)\varphi(g) \qquad \text{for } t\in T.$$

Note that any function in $\mathcal{B}(\mu_1, \mu_2)$ is necessarily a C^∞ function. If we put

$$\rho(\mu)\varphi(g_1)=\int\varphi(g_1g)d\mu(g)$$

for $\mu\in\mathcal{H}_F$ and $\varphi\in\mathcal{B}(\mu_1, \mu_2)$, we obtain a representation ρ of $\mathcal{H}_F$ in $\mathcal{B}(\mu_1, \mu_2)$. It is admissible. By [**5**, Th. 5.11 and Th. 6.2] every irreducible admissible representation of $\mathcal{H}_F$ is equivalent to a constituent of some $\mathcal{B}(\mu_1, \mu_2)$.

The case $F=\boldsymbol{R}$. If $F=\boldsymbol{R}$, the irreducible constituents of $\mathcal{B}(\mu_1, \mu_2)$ are the following ([**5**, Th. 5.11]).

i) If $\mu_1\mu_2^{-1}(\alpha)$ is not of the form $\alpha^p\operatorname{sgn}\alpha$ with a non-zero integer p, $\mathcal{B}(\mu_1, \mu_2)$ is irreducible.

ii) If $\mu_1\mu_2^{-1}(\alpha)=\alpha^p\operatorname{sgn}\alpha$ for a positive integer p, $\mathcal{B}(\mu_1, \mu_2)$ contains the only one proper invariant subspace $\mathcal{B}_s(\mu_1, \mu_2)$, which is of finite codimension.

iii) If $\mu_1\mu_2^{-1}(\alpha)=\alpha^p\operatorname{sgn}\alpha$ for a negative integer p, $\mathcal{B}(\mu_1, \mu_2)$ contains the only one proper invariant subspace $\mathcal{B}_f(\mu_1, \mu_2)$, which is of finite dimension.

In the case i) we write $\pi(\mu_1, \mu_2)$ for the representation ρ of $\mathcal{H}_F$ in $\mathcal{B}(\mu_1, \mu_2)$. In the case ii) we write $\sigma(\mu_1, \mu_2)$ (resp. $\pi(\mu_1, \mu_2)$) for the representation of $\mathcal{H}_F$ in $\mathcal{B}_s(\mu_1, \mu_2)$ (resp. $\mathcal{B}(\mu_1, \mu_2)/\mathcal{B}_s(\mu_1, \mu_2)$) induced by ρ. In the case iii) we write $\sigma(\mu_1, \mu_2)$ (resp. $\pi(\mu_1, \mu_2)$) for the representation of $\mathcal{H}_F$ in $\mathcal{B}(\mu_1, \mu_2)/\mathcal{B}_f(\mu_1, \mu_2)$ (resp. $\mathcal{B}_f(\mu_1, \mu_2)$) induced by ρ.

The equivalence relations of these representations are as follows. $\pi(\mu_1, \mu_2)$ and $\sigma(\mu_1', \mu_2')$ are not equivalent. $\pi(\mu_1, \mu_2)$ and $\pi(\mu_1', \mu_2')$ are equivalent if and only if $(\mu_1, \mu_2)=(\mu_1', \mu_2')$ or (μ_2', μ_1'). $\sigma(\mu_1, \mu_2)$ and $\sigma(\mu_1', \mu_2')$ are equivalent if and only if (μ_1, μ_2) is one of the four pairs (μ_1', μ_2'), (μ_2', μ_1'), $(\mu_1'\eta,$

$\mu_2'\eta$), $(\mu_2'\eta, \mu_1'\eta)$. Here $\eta(\alpha)=\text{sgn}\,\alpha$.

The case $F=\boldsymbol{C}$. If $F=\boldsymbol{C}$, the irreducible constituents of $\mathscr{B}(\mu_1, \mu_2)$ are the following ([**5**, Th. 6.2]).

i) If $\mu_1\mu_2^{-1}(\alpha)$ is not of the form $\alpha^p\bar{\alpha}^q$ or $\alpha^{-p}\bar{\alpha}^{-q}$, p and q being positive integers, then $\mathscr{B}(\mu_1, \mu_2)$ is irreducible.

ii) If $\mu_1\mu_2^{-1}(\alpha)=\alpha^p\bar{\alpha}^q$ with positive integers p, q, $\mathscr{B}(\mu_1, \mu_2)$ contains the only one proper invariant subspace $\mathscr{B}_s(\mu_1, \mu_2)$, which is of finite codimension.

iii) If $\mu_1\mu_2^{-1}(\alpha)=\alpha^{-p}\bar{\alpha}^{-q}$ with positive integers p, q, $\mathscr{B}(\mu_1, \mu_2)$ contains the only one proper invariant subspace $\mathscr{B}_f(\mu_1, \mu_2)$, which is of finite dimension.

We define $\pi(\mu_1, \mu_2)$ or $\sigma(\mu_1, \mu_2)$ in the same way as in the real case. Unlike the real case, every irreducible admissible representation is equivalent to some $\pi(\mu_1, \mu_2)$. $\pi(\mu_1, \mu_2)$ and $\pi(\mu_1', \mu_2')$ are equivalent if and only if $(\mu_1, \mu_2)=(\mu_1', \mu_2')$ or (μ_2', μ_1').

8. The case of quaternion algebras. We consider in this section the multiplicative group of a division quaternion algebra $\mathscr{K}$ over a local field F.

We define the Hecke algebra $\mathscr{H}(\mathscr{K}^\times)$ and admissible representations of $\mathscr{H}(\mathscr{K}^\times)$ exactly in the same way as in No. 2 or No. 5, taking $\mathscr{K}^\times$ (resp. the unique maximal compact subgroup of $\mathscr{K}^\times$) for $GL_2(F)$ (resp. K). In this case we still denote by K the maximal compact subgroup of $\mathscr{K}^\times$. Write $n(x)$ for the reduced norm of x in $\mathscr{K}$. Then K is the group of all $g\in\mathscr{K}^\times$ with $n(g)\in\mathfrak{o}^\times$ (resp. $n(g)=1$) if F is non-archimedean (resp. $F=\boldsymbol{R}$) (there is no division quaternion algebra over $\boldsymbol{C}$). However, for any admissible representation π of $\mathscr{H}(\mathscr{K}^\times)$, there exists always a representation π of $\mathscr{K}^\times$ satisfying (1.3) for $g\in\mathscr{K}^\times$ and $f\in\mathscr{H}(\mathscr{K}^\times)$ (even if F is archimedean, because K is a normal subgroup of $\mathscr{K}^\times$). If π is irreducible, the corresponding representation π of $\mathscr{K}^\times$ is an irreducible (continuous) representation of finite dimension.

9. Global Hecke algebra. In this section, we assume that F is a global field, i. e. an algebraic number field of finite degree or an algebraic function field over a finite field.

We write v for a place in F, F_v for the completion of F with respect to v, and $\boldsymbol{A}$ for the adele of F. Also we write $\mathfrak{o}_v$ and $\mathfrak{o}$ for the rings of all integers in F_v and F, respectively. (If F is a number field, denote by S_∞ the set of all archimedean places. If F is a function field, we fix a non-empty finite set S_∞ of places. By an integer in F we understand an element in F contained in $\mathfrak{o}_v$ for all $v\notin S_\infty$.)

Let $\mathscr{K}$ be a quaternion algebra over F and put $\mathscr{K}_v=F_v\otimes_F\mathscr{K}$. We say that v is ramified in $\mathscr{K}$ if $\mathscr{K}_v$ is a division algebra. The number of ramified places is finite and even. Conversely, if there is given a set S of even number of non-archimedean or real places, there exists a unique (up to isomorphism) quaternion algebra $\mathscr{K}$ over F such that S is exactly the set of places ramified in $\mathcal{K}$.

For all v unramified in $\mathcal{K}$, we define an isomorphism θ_v of $\mathcal{K}_v$ onto $M_2(F_v)$ in the following way. Take a maximal order $\mathfrak{O}$ in $\mathcal{K}$ with respect to $\mathfrak{o}$. For an unramified v not in S_∞, let $\mathfrak{O}_v$ be the $\mathfrak{o}_v$-module in $\mathcal{K}_v$ generated by $\mathfrak{O}$. There is an isomorphism of $\mathfrak{O}_v$ onto $M_2(\mathfrak{o}_v)$, which can be naturally extended to an isomorphism of $\mathcal{K}_v$ onto $M_2(F_v)$. Let θ_v be this isomorphism. For an unramified v in S_∞, take θ_v to be any isomorphism of $\mathcal{K}_v$ onto $M_2(F_v)$. If $\mathfrak{O}'$ is another maximal order, we have $\mathfrak{O}_v=\mathfrak{O}_v'$ for almost all v; hence the choice of $\{\theta_v\}$ is canonical so far as "almost all" v are concerned.

We fix $\{\theta_v\}$ once and for all and identify $\mathcal{K}_v$ with $M_2(F_v)$ and hence $\mathcal{K}_v^\times$ with $GL_2(F_v)$ by θ_v. Put

$$K_v=\begin{cases} GL_2(\mathfrak{o}_v) & \text{if } v \text{ is non-archimedean}, \\ O_2(\boldsymbol{R}) & \text{if } F_v=\boldsymbol{R}, \\ U_2(\boldsymbol{C}) & \text{if } F_v=\boldsymbol{C}, \end{cases}$$

and denote by $\mathcal{H}(\mathcal{K}_v^\times)$ the Hecke algebra of $\mathcal{K}_v^\times=GL_2(F_v)$.

If v is ramified in $\mathcal{K}$, we denote by K_v the maximal compact subgroup of $\mathcal{K}_v^\times$, and by $\mathcal{H}(\mathcal{K}_v^\times)$ the Hecke algebra of $\mathcal{K}_v^\times$ defined in No. 8.

Put $K=\prod_v K_v$. Let $\mathcal{H}(\mathcal{K}_A^\times)$ be the space spanned by all $\bigotimes_v f_v$ with $f_v\in\mathcal{H}(\mathcal{K}_v^\times)$, where almost all f_v are the characteristic functions of K_v. It forms an associative algebra (as a subalgebra of the tensor product of $\mathcal{H}(\mathcal{K}_v^\times)$).

Let π_v be an admissible representation of $\mathcal{H}(\mathcal{K}_v^\times)$ in $\mathcal{V}_v$ and assume

(1.9) for almost all v, the restriction of π_v to K_v contains the identity representation exactly once.

Take an element e_v in $\mathcal{V}_v$ such that $\pi_v(k)e_v=e_v$ for all $k\in K_v$. Let $\mathcal{V}$ be the restricted tensor product of $\mathcal{V}_v$ with respect to $\{e_v\}$, i.e. the space spanned by all $\bigotimes_v x_v$ ($x_v\in\mathcal{V}_v$) such that $x_v=e_v$ for almost all v. We can define a representation π of $\mathcal{H}(\mathcal{K}_A^\times)$ in $\mathcal{V}$ by putting

$$\pi(f)x=\otimes\pi_v(f_v)x_v$$

if $f=\otimes f_v$ and $x=\otimes x_v$. By the assumption (1.9) the equivalence class of π is independent of the choice of $\{e_v\}$. We call π the tensor product of π_v and write $\pi=\otimes\pi_v$ (note that (1.9) is implicitly assumed whenever we speak of the tensor product of admissible representations). π is irreducible if and only if all π_v are irreducible.

The tensor product of admissible representations of $\mathcal{H}(\mathcal{K}_v^\times)$ is an admissible representation of $\mathcal{H}(\mathcal{K}_A^\times)$ in the sense of [**5**, § 9], and every irreducible admissible representation of $\mathcal{H}(\mathcal{K}_A^\times)$ is the tensor product of admissible representations of $\mathcal{H}(\mathcal{K}^\times)$ ([**5**, Prop. 9.1]).

$\mathcal{H}(\mathcal{K}_A^\times)$ can be interpreted as an algebra of measures (of compact support) on $\mathcal{K}_A^\times$. If φ is a continuous function on $\mathcal{K}_A^\times$ and $\mu\in\mathcal{H}(\mathcal{K}_A^\times)$, we put

$$\rho(\mu)\varphi(h)=\int\varphi(hg)d\mu(g)\,.$$

In particular, if an element f in $\mathcal{H}(\mathcal{K}_A^\times)$ is of the form $\otimes f_v$, where f_v is a function on $\mathcal{K}_v^\times$, then f is identified with a function $f(g)=\prod f_v(g_v)$ on $\mathcal{K}_A^\times$, and we have

$$\rho(f)\varphi(h)=\int\varphi(hg)f(g)dg\,.$$

§ 2. Weil representations and theta series.

1. Weil representations (local case). Let us recall that the Schwartz space $\mathcal{S}(G)$ on a finite dimensional vector space G over a local field F is the space of all locally constant functions of compact support on G if F is non-archimedean, and $\mathcal{S}(G)$ is the space of all rapidly decreasing C^∞ functions on G if F is archimedean.

Let F be a local field and let $\mathcal{A}$ be either one of the following semisimple algebras over F:

a) $F\oplus F$,

b) a separable quadratic extension of F,

c) a quaternion algebra over F.

In each case, denote by $x\to x'$ the following involution of $\mathcal{A}$ over F:

a) $(\alpha,\beta)\to(\beta,\alpha)$,

b) the non-trivial automorphism of $\mathcal{A}$ over F,

c) the canonical involution of $\mathcal{A}$ over F.

Put $\mathrm{tr}\,(a)=a+a'$, $n(a)=aa'$ for $a\in\mathcal{A}$. $n(a)$ is a homomorphism of $\mathcal{A}^\times$ into $F^\times$.

Fix a non-trivial additive character ψ of F. Since $(x,y)\to\mathrm{tr}\,(xy)$ is non-degenerate bilinear form on $\mathcal{A}$, $\mathcal{A}$ can be identified with its dual by the pairing $\langle x,y\rangle=\psi(\mathrm{tr}\,(xy))$. Let dx be the unique Haar measure on $\mathcal{A}$ which equals its dual. For $M\in\mathcal{S}(\mathcal{A})$, the Fourier transform M' of M is by definition

$$M'(x)=\int_{\mathcal{A}}M(y)\langle x,y\rangle dy$$

and M' is again in $\mathcal{S}(\mathcal{A})$. By the self-duality of dx we have

$$M(x)=\int_{\mathcal{A}}M'(y)\langle x,-y\rangle dy\,.$$

Put $f(x)=\psi(n(x))=\psi(xx')$. By [**11**, Th. 2] there exists a constant $\gamma=\gamma(\mathcal{A}/F,\psi)$ such that $(M*f)'(x)=\gamma f(x')^{-1}M'(x)$ for all $M\in\mathcal{S}(\mathcal{A})$. $\gamma=1$ if

$\mathcal{A}=F\oplus F$ or $M_2(F)$ (cf. [11, Prop. 3]; note that the quadratic form $n(x)$ on $\mathcal{A}$ is then a kernel form). $\gamma=-1$ if $\mathcal{A}$ is a division quaternion algebra over F (cf. [11, Prop. 4]). If $\mathcal{A}$ is a separable quadratic extension of F, the value of γ is found in [5, Lemma 1.2] or [10] (in [10], it is assumed that the residue class field of F is not of characteristic 2).

Let r be a representation of $SL_2(F)$ in $\mathcal{S}(\mathcal{A})$ defined by

$$r\left(\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}\right)M(x)=\omega(\alpha)|\alpha|_{\mathcal{A}}^{1/2}M(\alpha x), \tag{2.1}$$

$$r\left(\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}\right)M(x)=\psi(\beta n(x))M(x), \tag{2.2}$$

$$r\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right)M(x)=\gamma(\mathcal{A}/F, \psi)M'(x^{\iota}). \tag{2.3}$$

Here ω is the non-trivial character of $F^{\times}/n(\mathcal{A}^{\times})$ if $\mathcal{A}$ is a separable quadratic extension of F, and $\omega=1$ otherwise. $|\ |_{\mathcal{A}}$ is the module in $\mathcal{A}$. Since the elements of the form $\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}$, $\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}$, $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ generate $SL_2(F)$, r is uniquely determined by (2.1)-(2.3). That r is actually a representation is proved in [5, Prop. 1.3].

LEMMA 1. *For an element a in $\mathcal{A}^{\times}$ and a function f on $\mathcal{A}$, write $\rho(a)f(x)=f(xa)$, $\lambda(a)f(x)=f(a^{-1}x)$, $\iota(a)f(x)=f(a^{-1}xa)$. Let $\mathcal{A}^1$ be the group of all elements in $\mathcal{A}$ with $n(a)=1$; let s be any element in $SL_2(F)$.*

i) *$r(s)$ commutes with $\rho(a)$ and $\lambda(a)$ for all $a\in\mathcal{A}^1$.*

ii) *$r(s)$ commutes with $\iota(a)$ for all $a\in\mathcal{A}^{\times}$.*

iii) *Put $s'=\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}^{-1}s\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}$ for $\alpha\in F^{\times}$. If there is an element a in $\mathcal{A}$ with $n(a)=\alpha$, we have $\rho(a)r(s)=r(s')\rho(a)$.*

PROOF. It is enough to prove i) and ii) when s is of the form $\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}$, $\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}$ or $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$. In the first two cases, this is immediately seen from definition. If $s=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$, this amounts to see that, for $M\in\mathcal{S}(\mathcal{A})$, $(\rho(a)M)'=\lambda(a)M'$, $(\lambda(a)M)'=\rho(a)M'$ $(a\in\mathcal{A}^1)$ and $(\iota(a)M)'=\iota(a)M'$ $(a\in\mathcal{A}^{\times})$. This is easy to prove. iii) can be proved in the same way.

2. Special or absolutely cuspidal representations. Let $\mathcal{A}$ be a separable quadratic extension or a division quaternion algebra over a local field F, and π an irreducible representation of $\mathcal{A}^{\times}$ in a finite dimensional vector space U over C. An element in the space $\mathcal{S}(\mathcal{A})\otimes_C U$ is regarded as a function on $\mathcal{A}$ taking values in U, whose coordinates (with respect to a basis of U) are Schwartz functions on $\mathcal{A}$. Denote again by r the representation $r\otimes 1$ of $SL_2(F)$ in $\mathcal{S}(\mathcal{A})\otimes U$, 1 being the identity representation of $SL_2(F)$ in U. Let

$\mathcal{S}(\mathcal{A}, \pi)$ be the space of all elements in $\mathcal{S}(\mathcal{A}) \otimes_{C} U$ such that

$$M(xg) = \pi(g^{-1})M(x)$$

for all $g \in \mathcal{A}^1$. It is invariant under the action of $SL_2(F)$ (Lemma 2, i)). Let G_+ be the group of all s in $GL_2(F)$ such that $\det s \in n(\mathcal{A}^\times)$. By [**5**, Prop. 1.5] the representation r of $SL_2(F)$ in $\mathcal{S}(\mathcal{A}, \pi)$ can be extended to a representation r_π of G_+ in $\mathcal{S}(\mathcal{A}, \pi)$ by setting

$$r_\pi\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\right)M(x) = |h|_{\mathcal{A}}^{1/2}\pi(h)M(xh)$$

if $\alpha = n(h)$ for $h \in \mathcal{A}^\times$.

If F is non-archimedean, there exists a unique division quaternion algebra $\mathcal{K}$ over F. Put $\mathcal{A} = \mathcal{K}$; then it is proved in [**5**, Th. 4.2] that the representation r_π of $G_+ = GL_2(F)$ in $\mathcal{S}(\mathcal{K}, \pi)$ is admissible and is a multiple of a single irreducible admissible representation π^*. If $\dim \pi = 1$, π is written as $\pi(g) = \chi(n(g))$ with a quasi-character χ of $F^\times$; then π^* is a special representation $\sigma(\chi|\ |_F^{1/2}, \chi|\ |_F^{-1/2})$. If $\dim \pi > 1$, π^* is an absolutely cuspidal representation. By [**5**, Th. 15.1], $\pi \to \pi^*$ gives a one to one correspondence between the equivalence classes of finite dimensional irreducible representations of $\mathcal{K}^\times$ and the equivalence classes of special or absolutely cuspidal representations of $GL_2(F)$.

Assume now that $F = \boldsymbol{R}$. Let $\mathcal{K}$ be a division quaternion algebra over $\boldsymbol{R}$. Identify $\mathcal{K}$ with the set of matrices of the form $\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ with $a, b \in \boldsymbol{C}$. Then $n(h) = \det h$ for $h \in \mathcal{K}$. Every irreducible finite dimensional representation π of $\mathcal{K}^\times$ is written as

$$\pi(h) = n(h)^r \rho_n(h)$$

with $r \in \boldsymbol{C}$, ρ_n being the n-th symmetric tensor representation of $GL_2(\boldsymbol{C})$. Let μ_1, μ_2 be quasi-characters of $\boldsymbol{R}^\times$ defined by

$$\mu_1(\alpha) = |\alpha|^{r+n+1/2}$$

$$\mu_2(\alpha) = |\alpha|^{r-1/2}(\operatorname{sgn} \alpha)^n$$

and put $\pi^* = \sigma(\mu_1, \mu_2)$. Every special representation of $\mathcal{H}(GL_2(\boldsymbol{R}))$ is obtained in this way. This correspondence of π and π^* is described in [**5**, § 5] by means of an intervening quasi-character of $\boldsymbol{C}^\times$.

3. Weil representations (global case) and theta series. Let F be a global field and $\mathcal{K}$ a quaternion algebra over F. We use the notation in § 1, No. 9. Let ψ be a non-trivial character of $\boldsymbol{A}/F$ and write $\psi(a) = \prod \psi_v(a_v)$ for $a = (a_v) \in \boldsymbol{A}$. (We shall fix this character throughout this paper.) Let $\mathfrak{a}_v$ be the largest $\mathfrak{o}_v$-lattice in F_v on which ψ_v is trivial. We call $\mathfrak{a}_v$ the *conductor* of

ψ_v. Almost all $\mathfrak{a}_v$ coincide with $\mathfrak{o}_v$.

Using the above ψ_v, we define the Weil representation r_v of $SL_2(F_v)$ in $\mathcal{S}(\mathcal{K}_v)$. Let $\mathcal{S}_0(\mathcal{K}_A)$ be the space spanned by all elements of the form $\bigotimes_v M_v$ with $M_v \in \mathcal{S}(\mathcal{K}_v)$, where for almost all v, M_v is the characteristic function M_v^0 of $\mathfrak{O}_v$. We shall prove in Lemma 7 that, for almost all v, M_v^0 is invariant under $r_v(s_v)$ for $s_v \in SL_2(\mathfrak{o}_v)$. Hence we get a representation r of $SL_2(A)$ in $\mathcal{S}_0(\mathcal{K}_A)$ by setting

$$r(s)(\otimes M_v) = \otimes r_v(s_v)M_v$$

for $s = (s_v) \in SL_2(A)$.

$\mathcal{S}_0(\mathcal{K}_A)$ is regarded as a subspace of the Schwartz space $\mathcal{S}(\mathcal{K}_A)$ on $\mathcal{K}_A$. By [**11**, Chap. III, No. 38, 39] the action of $SL_2(A)$ in $\mathcal{S}_0(\mathcal{K}_A)$ can be extended to $\mathcal{S}(\mathcal{K}_A)$, and the mapping $(s, M) \to r(s)M$ of $SL_2(A) \times \mathcal{S}(\mathcal{K}_A)$ into $\mathcal{S}(\mathcal{K}_A)$ is continuous. By [**11**, Chap. III, No. 41]

$$\Theta(M) = \sum_{\xi \in \mathcal{K}_F} M(\xi) \tag{2.4}$$

converges uniformly on any compact subset of $\mathcal{S}(\mathcal{K}_A)$. It follows that $\Theta(r(s)M)$ is, as a function of s, continuous on $SL_2(A)$.

PROPOSITION 1. *If* $\sigma \in SL_2(F)$, *then*

$$\Theta(r(\sigma)M) = \Theta(M)\,. \tag{2.5}$$

PROOF. We can assume that M is of the form $\otimes M_v$ with $M_v \in \mathcal{S}(\mathcal{K}_v)$. If $\sigma = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ with $\alpha \in F^\times$ or $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$ with $\beta \in F$, the left hand side of (2.5) is reduced to

$$|\alpha|_A \sum_{\xi \in \mathcal{K}_F} M(\alpha\xi)\,, \quad \text{or} \quad \sum_{\xi \in \mathcal{K}_F} \psi(\beta n(\xi))M(\xi)\,,$$

which is clearly $\sum_\xi M(\xi)$, since $|\alpha|_A = 1$ and $\psi(\beta n(\xi)) = 1$.

To prove (2.5) for $\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, note first the following. If dx_v are the self-dual measures on $\mathcal{K}_v$ with respect to $\langle x_v, y_v \rangle = \psi_v(\mathrm{tr}\,(x_v y_v))$, we can introduce the product measure dx of dx_v on $\mathcal{K}_A$, and dx is self-dual with respect to the pairing $\langle x, y \rangle = \psi(\mathrm{tr}\,(xy))$. If $M = \otimes M_v$, then $M' = \otimes M_v'$ is the Fourier transform of M. As is stated in No. 1, $\gamma(\mathcal{K}_v/F_v, \psi_v) = 1$ or -1 according as v is unramified or ramified in $\mathcal{K}$. Since the number of ramified v is even, we have $\prod_v \gamma(\mathcal{K}_v/F_v, \psi_v) = 1$. Consequently, (2.5) for $\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is reduced to the Poisson's formula

$$\sum_{\xi \in \mathcal{K}_F} M(\xi) = \sum_{\xi \in \mathcal{K}_F} M'(\xi)\,.$$

REMARK. The statement in the above is valid if we take, in place of

$\mathcal{K}$, an algebra over F of type a), b) or c) in No. 1. The case of separable quadratic extension of F is discussed in Shalika-Tanaka [7], where $\Theta(r(s)M)$ is used to construct cusp forms on $SL_2(A)$.

Assume for a moment that the characteristic of F is not 2. In the notation in Weil [11], $Ps(\mathcal{A})_A$ is isomorphic to $Sp(\mathcal{A})_A$ and there is an obvious embedding of $SL_2(A)$ into $Sp(\mathcal{A})_A$, and hence into $Ps(\mathcal{A})_A$. We see that $s \to (s, r(s))$ gives an isomorphism of $SL_2(A)$ into $Mp(\mathcal{A})_A$, and the restriction of this isomorphism to $SL_2(F)$ is the same as $\boldsymbol{r}_F$ defined in [**11**, Chap. III, No. 40]. Then, Proposition 1, together with the remark preceding it, is a consequence of [**11**, Th. 6].

§ 3. Automorphic forms and spherical functions.

1. Definition of automorphic forms. Let $\mathcal{K}$ be a quaternion algebra over a global field F and η a quasi-character of $A^\times/F^\times$. By an *automorphic form* (more precisely, an *automorphic form with a quasi-character* η), we understand a continuous function φ on $\mathcal{K}_F^\times \backslash \mathcal{K}_A^\times$ satisfying the following conditions.

(3.1) φ is K-finite on the right.

(3.2) For any elementary idempotent ξ in $\mathcal{H}(\mathcal{K}_A^\times)$, the space $\{\rho(\xi f)\varphi \mid f \in \mathcal{H}(\mathcal{K}_A^\times)\}$ is finite dimensional.

(3.3) $\varphi(zg) = \eta(z)\varphi(g)$ for all $z \in A^\times$ and $g \in \mathcal{K}_A^\times$.

(3.4) For any compact set Ω in $\mathcal{K}_A^\times$ and for any constant $c > 0$, there exist constants c_1, c_2 such that

$$\left|\varphi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right)\right| \leqq c_1 |a|_A^{c_2}$$

for all $g \in \Omega$ and for all $a \in A^\times$ with $|a|_A \geqq c$.

(The condition (3.4) should be neglected unless $\mathcal{K}_A^\times = GL_2(A)$.) Here the notation is the same as in § 1, No. 9 and $|\ |_A$ is the module in $\mathcal{K}_A$. We denote by $\mathcal{A}(\eta, \mathcal{K}_A^\times)$ the space of all automorphic forms with a quasi-character η.

Let $\mathcal{A}_0(\eta, GL_2(A))$ be the space of all φ in $\mathcal{A}(\eta, GL_2(A))$ such that

$$\text{(C)} \qquad \int_{A/F} \varphi\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g\right) du = 0$$

for all $g \in GL_2(A)$. Such a φ is called *cusp form*. To simplify the statement, we occasionally write $\mathcal{A}_0(\eta, \mathcal{K}_A^\times)$ for $\mathcal{A}(\eta, \mathcal{K}_A^\times)$ if $\mathcal{K}$ is a division algebra.

If $\varphi \in \mathcal{A}(\eta, \mathcal{K}_A^\times)$ and $\mu \in \mathcal{H}(\mathcal{K}_A^\times)$, then $\rho(\mu)\varphi \in \mathcal{A}(\eta, \mathcal{K}_A^\times)$; thus we obtain a representation ρ of $\mathcal{H}(\mathcal{K}_A^\times)$ in $\mathcal{A}(\eta, \mathcal{K}_A^\times)$. $\mathcal{A}_0(\eta, \mathcal{K}_A^\times)$ is invariant under ρ. It can be shown that the restriction of ρ to $\mathcal{A}_0(\eta, \mathcal{K}_A^\times)$ is the direct sum of irreducible admissible representations, each of which occurs with a finite multiplicity ([5, Prop. 10.5, Prop. 10.9, Lemma 14.1]). Moreover, each multiplicity is at most 1 if $\mathcal{K}_A^\times = GL_2(A)$ ([5, Prop. 11.1.1]).

REMARK. For any quasi-character η of $A^\times/F^\times$, we can find a quasi-character χ such that $\chi^2\eta$ is a character. Put $\varphi'(g)=\chi(n(g))\varphi(g)$ for $\varphi \in \mathcal{A}_0(\eta, \mathcal{K}_A^\times)$. Then $\varphi \to \varphi'$ gives an isomorphism of $\mathcal{A}_0(\eta, \mathcal{K}_A^\times)$ onto $\mathcal{A}_0(\chi^2\eta, \mathcal{K}_A^\times)$. If ρ is the representation of $\mathcal{H}(\mathcal{K}_A^\times)$ in the former space, the representation in the latter space is the tensor product of ρ and the one-dimensional representation $\chi\circ n$. For this reason we may assume that η is a character without losing generality.

2. The space $L_0^2(\eta, \mathcal{K}_A^\times)$. η being a character of $A^\times/F^\times$, let $L_0^2(\eta, \mathcal{K}_A^\times)$ be the space of all functions φ on $\mathcal{K}_A^\times$ satisfying the following conditions.

(3.5) $\varphi(z\gamma g)=\eta(z)\varphi(g)$ for $z\in A^\times$, $\gamma \in \mathcal{K}_F^\times$, $g\in \mathcal{K}_A^\times$,

(3.6) $|\varphi(g)|$ is square-integrable on $P(\mathcal{K}^\times)_F\backslash P(\mathcal{K}^\times)_A$, where $P(\mathcal{K}^\times)=\mathcal{K}^\times/F^\times$.

(3.7) If $\mathcal{K}_A^\times = GL_2(A)$,

$$\int_{A/F}\varphi\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}g\right)du=0 \qquad \text{for almost all } g.$$

$L_0^2(\eta, \mathcal{K}_A^\times)$ forms a Hilbert space, the inner product being

$$(\varphi_1, \varphi_2)=\int_{P(\mathcal{K}^\times)_F\backslash P(\mathcal{K}^\times)_A}\varphi_1(g)\overline{\varphi_2(g)}d\dot{g}\,.$$

The right translation ρ defines a unitary representation of $\mathcal{K}_A^\times$ in $L_0^2(\eta, \mathcal{K}_A^\times)$. The space $\mathcal{A}_0(\eta, \mathcal{K}_A^\times)$ coincides with the space of all K-finite functions in $L_0^2(\eta, \mathcal{K}_A^\times)$ (cf. Godement [4, § 3, No. 1]).

If $\mathcal{L}$ is a closed subspace of $L_0^2(\eta, \mathcal{K}_A^\times)$ invariant and irreducible (topologically) under the action of $\mathcal{K}_A^\times$, then $\mathcal{V}=\mathcal{L}\cap\mathcal{A}_0(\eta, \mathcal{K}_A^\times)$ is invariant and irreducible under the action of $\mathcal{H}(\mathcal{K}_A^\times)$; conversely if the subspace $\mathcal{V}$ of $\mathcal{A}_0(\eta, \mathcal{K}_A^\times)$ is irreducible under the action of $\mathcal{H}(\mathcal{K}_A^\times)$, its closure $\mathcal{L}$ is invariant and irreducible under the action of $\mathcal{K}_A^\times$, and $\mathcal{V}$ is the space of K-finite functions in $\mathcal{L}$ (cf. [4, § 3, No. 3]).

3. Spherical functions. We write $\mathcal{K}^1$ for the group of all elements in $\mathcal{K}^\times$ of reduced norm 1 and put $K_v^1=K_v\cap\mathcal{K}_v^1$, $K^1=K\cap\mathcal{K}_A^1$. Let $\mathcal{L}$ be an irreducible closed subspace of $L_0^2(\eta, \mathcal{K}_A^\times)$ and π a representation of $\mathcal{K}_A^\times$ in $\mathcal{L}$. For an irreducible representation $\mathfrak{d}$ of K^1, let $\mathcal{L}(\mathfrak{d})$ be the space of all φ in $\mathcal{L}$ such that

$$\int_{K^1}\chi_{\mathfrak{d}}(k_1^{-1})\pi(k_1)\varphi dk_1=\varphi\,,$$

where $\chi_{\mathfrak{d}}(k_1)=\dim\mathfrak{d}\,\mathrm{tr}\,\mathfrak{d}(k_1)$.

LEMMA 2. *$\mathcal{L}(\mathfrak{d})$ is finite dimensional.*

PROOF. Let $\mathcal{V}$ be the space of all K-finite vectors in $\mathcal{L}$. We first prove that $\mathcal{L}(\mathfrak{d})\subset\mathcal{V}$. By [4, § 3, Th. 2] π is the tensor product of irreducible unitary representations π_v of $\mathcal{K}_v^\times$ in $\mathcal{L}_v$. Denote again by π (resp. π_v) the admissible representation of $\mathcal{H}(\mathcal{K}_A^\times)$ (resp. $\mathcal{H}(\mathcal{K}_v^\times)$) in $\mathcal{V}$ (resp. $\mathcal{V}_v$), $\mathcal{V}_v$ being the space of all K_v-finite vectors in $\mathcal{L}_v$. Let S be a finite set of places such that for all $v\notin S$, the restriction of π_v to K_v contains the identity representation. If $v\notin S$, we have $\pi_v=\pi(\mu_1, \mu_2)$ with unramified quasi-characters μ_1, μ_2 of $F_v^\times$. Then it is easy to see that $\mathcal{V}_v$ contains the unique (up to a scalar multiple) vector invariant under $K_v{}^1$, which is still K_v-invariant (hence the same is true for $\mathcal{L}_v$). It follows that every element in $\mathcal{L}(\mathfrak{d})$ is H-finite if $H=Z(K)K^1\prod_{v\notin S}K_v$, $Z(K)$ being the center of K. Since H is of finite index in K, it is also K-finite.

It is evident that, if $\mathcal{L}(\mathfrak{d})\neq\{0\}$, an element in $\mathcal{L}(\mathfrak{d})$ transforms under the action of H according to an irreducible representation $\tilde{\mathfrak{d}}$ of H determined uniquely by $\mathfrak{d}$ and η. If $\xi(k)=\sum\dim\sigma_i\operatorname{tr}\sigma_i(k^{-1})$, where σ_i are all the irreducible constituents of the representation of K induced by $\tilde{\mathfrak{d}}$, then $\mathcal{L}(\mathfrak{d})$ is contained in $\pi(\xi)\mathcal{V}$. Hence $\mathcal{L}(\mathfrak{d})$ is finite dimensional.

By Lemma 2 we can define the spherical function $\omega_\mathfrak{d}$ of type $\mathfrak{d}$ of π (cf. Godement [3]). By definition we have

$$\omega_\mathfrak{d}(g)=\operatorname{tr}(E(\mathfrak{d})\pi(g)),$$

$E(\mathfrak{d})$ being the projection of $\mathcal{L}$ to $\mathcal{L}(\mathfrak{d})$. It follows that

$$\omega_\mathfrak{d}(g)=\sum_{i=1}^{N}(\pi(g)\varphi_i, \varphi_i) \tag{3.8}$$

if $\{\varphi_1, \cdots, \varphi_N\}$ is an orthonormal basis of $\mathcal{L}(\mathfrak{d})$. In a special case where the multiplicity of $\mathfrak{d}$ in $\mathcal{L}(\mathfrak{d})$ is 1, we have

$$\varphi(g_0)\omega_\mathfrak{d}(g)=\dim\mathfrak{d}\int_{K^1}\varphi(g_0kgk^{-1})dk \tag{3.9}$$

for any φ in $\mathcal{L}(\mathfrak{d})$ and for any g_0 in $\mathcal{K}_A^\times$ (cf. [3, Th. 8]).

Since $\pi=\bigotimes_v\pi_v$ and $\mathfrak{d}=\bigotimes_v\mathfrak{d}_v$ with irreducible unitary representations π_v of $\mathcal{K}_v^\times$ and irreducible representations $\mathfrak{d}_v$ of $K_v{}^1$, we have

$$\omega_\mathfrak{d}(g)=\prod_v\omega_{\mathfrak{d}_v}(g_v), \tag{3.10}$$

$\omega_{\mathfrak{d}_v}$ being the spherical function of type $\mathfrak{d}_v$ of π_v. Also we have

$$\omega_\mathfrak{d}(k_1gk_1{}^{-1})=\omega_\mathfrak{d}(g)\qquad\text{for } k_1\in K^1, \tag{3.11}$$

$$\int_{K^1}\chi_\mathfrak{d}(k_1{}^{-1})\omega_\mathfrak{d}(k_1g)dk_1=\omega_\mathfrak{d}(g). \tag{3.12}$$

These are immediately seen from definition.

§ 4. Construction of a space of automorphic forms.

1. Let $\mathcal{K}$ be a division quaternion algebra over a global field F and η a character of $A^\times/F^\times$. Write $\eta(a)=\prod \eta_v(a_v)$ for $a=(a_v)\in A^\times$. Let $\mathcal{V}$ be an irreducible subspace of $\mathcal{A}(\eta, \mathcal{K}_A^\times)$ and π the representation of $\mathcal{H}(\mathcal{K}_A^\times)$ in $\mathcal{V}$. Let $\mathcal{L}$ be the closure of $\mathcal{V}$ in $L_0^2(\eta, \mathcal{K}_A^\times)$ and write still π for the representation of $\mathcal{K}_A^\times$ in $\mathcal{L}$. In the notation in § 3, No. 2, let $\mathfrak{d}$ be any irreducible representation of K^1 such that $\mathcal{L}(\mathfrak{d})\neq\{0\}$ and let $\{\varphi_i\}_{i=1}^N$ be an orthonormal basis of $\mathcal{L}(\mathfrak{d})$.

Denote by $GL_2(A)_+$ the group of all $s\in GL_2(A)$ such that $\det s=n(h)$ for some $h\in\mathcal{K}_A^\times$ and put $GL_2(F)_+=GL_2(F)\cap GL_2(A)_+$. If s is in $GL_2(A)_+$, write $s=\begin{pmatrix}\det s & 0\\ 0 & 1\end{pmatrix}s_1$ and take an arbitrary h in $\mathcal{K}_A^\times$ with $n(h)=\det s$. For an element M in $\mathcal{S}(\mathcal{K}_A)$, let ϕ_M be a function on $GL_2(A)_+$ defined by

$$\phi_M(s)=\sum_{i=1}^N |\det s|_A \int_{P(\mathcal{K}^\times)_F\backslash P(\mathcal{K}^\times)_A} \Phi_i(M, s, g)\overline{\varphi_i(g)}d\dot{g}\,, \tag{4.1}$$

where

$$\Phi_i(M, s, g)=\int_{\mathcal{K}_F^1\backslash\mathcal{K}_A^1} \varphi_i(g_1hg)\Theta(\rho(g_1h)\iota(g)r(s_1)M)dg_1\,. \tag{4.2}$$

Since $\Theta(\rho(\gamma)M)=\Theta(\lambda(\gamma)M)=\Theta(M)$ for $\gamma\in\mathcal{K}_F^\times$, the integrand in (4.2) is, as a function of g_1, left $\mathcal{K}_F^1$-invariant, and the integral is independent of a choice of h. We see easily that $\Phi_i(M, s, g)$ is, as a function of g, left $\mathcal{K}_F^\times$-invariant.

LEMMA 3. *$\phi_M(s)$ is a continuous function on $GL_2(A)_+$, and $\phi_M(\sigma s)=\phi_M(s)$ for all $\sigma\in GL_2(F)_+$ and $s\in GL_2(A)_+$.*

PROOF. Since $x\to g^{-1}xhg$ is an automorphism of $\mathcal{K}_A$, the mapping $(h, g, s_1)\to\rho(h)\iota(g)r(s_1)M$ is a continuous mapping of $\mathcal{K}_A^\times\times\mathcal{K}_A^\times\times SL_2(A)$ into $\mathcal{S}(\mathcal{K}_A)$. Hence $\Theta(\rho(h)\iota(g)r(s_1)M)$ is a continuous function of h, g, s_1 (for a fixed ξ, $\rho(h)\iota(g)r(s_1)M(\xi)$ is a continuous function of h, g, s_1 and $\Theta(M)$ is uniformly convergent on a compact subset of $\mathcal{S}(\mathcal{K}_A)$). Since $P(\mathcal{K}^\times)_F\backslash P(\mathcal{K}^\times)_A$ and $\mathcal{K}_F^1\backslash\mathcal{K}_A^1$ are compact, the integrand in (4.1) is bounded if s stays in a compact set of $GL_2(A)_+$. It implies that ϕ_M is continuous.

Let σ be an element in $GL_2(F)_+$ of the form $\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}$. We can find an element δ in $\mathcal{K}_F^\times$ such that $n(\delta)=\alpha$. Substituting δh for h and then replacing g_1 by $\delta g_1\delta^{-1}$ in (4.2), we see that $\Phi_i(M, \sigma s, g)=\Phi_i(M, s, g)$. Hence $\phi_M(\sigma s)=\phi_M(s)$. Assume now that $\sigma\in SL_2(F)$. By Lemma 1, if $s=\begin{pmatrix}\det s & 0\\ 0 & 1\end{pmatrix}s_1=s_2\begin{pmatrix}\det s & 0\\ 0 & 1\end{pmatrix}$, we have $\rho(g_1h)\iota(g)r(s_1)M=\rho(g_1h)r(s_1)\iota(g)M=r(s_2)\rho(g_1h)\iota(g)M$, and by Proposition 1, $\Theta(r(s_2)\rho(g_1h)\iota(g)M)$ remains invariant if we replace s_2 by σs_2. Hence $\Phi_i(M, \sigma s, g)=\Phi_i(M, s, g)$ and $\phi_M(\sigma s)=\phi_M(s)$. This proves the lemma.

2. An element $s=(s_v)$ in $GL_2(A)$ belongs to $GL_2(A)_+$ if and only if $\det s_v$

is positive for all real places v ramified in $\mathcal{K}$. From this it follows that $GL_2(A)=GL_2(F)GL_2(A)_+$. By Lemma 3 ϕ_M can be extended to a function on $GL_2(A)$ invariant under the left translations by elements of $GL_2(F)$. Obviously ϕ_M is then continuous on $GL_2(A)$.

Consider an arbitrary continuous function ϕ on $GL_2(F)\backslash GL_2(A)$. For a fixed s, $\phi\left(\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}s\right)$ is a function of $a\in A$, invariant under the translation $a\to a+\alpha$ for $\alpha\in F$. Let ψ be as in § 2, No. 3. Every character of A/F can be written as $a\to\psi(\alpha a)$ with $\alpha\in F$. Hence the Fourier coefficients of the above function are

$$\hat{\phi}(\alpha, s)=\int_{A/F}\phi\left(\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}s\right)\psi(-\alpha a)da\,,$$

da being the Haar measure of A such that the total volume of A/F is 1. We see that

(4.3) $$\hat{\phi}(\alpha, s)=\hat{\phi}\left(1, \begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}s\right) \qquad \text{for } \alpha\in F^{\times}\,.$$

Let us now prove that

(4.4) $$\phi_M(s)=\sum_{\alpha\in F}\hat{\phi}_M(\alpha, s)$$

for $M\in\mathcal{S}(\mathcal{K}_A)$. Assume first that $s\in GL_2(A)_+$. The term by term integration of (4.2) gives (this is permitted, since $\Theta(\rho(h)\iota(g)r(s_1)M)$ converges uniformly while (h, g, s_1) stays in a compact subset of $\mathcal{K}_A^{\times}\times\mathcal{K}_A^{\times}\times SL_2(A)$)

$$\Phi_i(M, s, g)=\int_{\mathcal{K}_F^1\backslash\mathcal{K}_A^1}\varphi_i(g_1hg)r(s_1)M(0)dg_1$$
$$+\sum_{\xi\in\mathcal{K}_F^{\times}}\int_{\mathcal{K}_F^1\backslash\mathcal{K}_A^1}\varphi_i(g_1hg)r(s_1)\iota(g)M(\xi g_1h)dg_1\,.$$

Here the second term can be written as

$$\sum_{\xi\in\mathcal{K}_F^{\times}/\mathcal{K}_F^1}\int_{\mathcal{K}_A^1}\varphi_i(g_1\xi hg)r(s_1)\iota(g)M(g_1\xi h)dg_1\,.$$

Therefore, putting

(4.5) $$\phi_0(s)=\sum_{i=1}^{N}|\det s|_A\int_{P(\mathcal{K}^{\times})_F\backslash P(\mathcal{K}^{\times})_A}\int_{\mathcal{K}_F^1\backslash\mathcal{K}_A^1}\varphi_i(g_1hg)r(s_1)M(0)\overline{\varphi_i(g)}dg_1d\dot{g}\,,$$

and

(4.6) $$\phi_1(s)=\sum_{i=1}^{N}|\det s|_A\int_{P(\mathcal{K}^{\times})_F\backslash P(\mathcal{K}^{\times})_A}\int_{\mathcal{K}_A^1}\varphi_i(g_1hg)r(s_1)\iota(g)M(g_1h)\overline{\varphi_i(g)}dg_1d\dot{g}\,,$$

we have

$$\phi_M(s)=\phi_0(s)+\sum_{\alpha\in F_+^\times}\phi_1\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}s\right). \tag{4.7}$$

Here we have put $F_+^\times = F^\times \cap n(\mathcal{K}_F^\times)$.

By the same reasoning as before, the Fourier coefficients of ϕ_M can be calculated term by term. If $s=s_2\begin{pmatrix}\det s & 0\\ 0 & 1\end{pmatrix}$, we have

$$\begin{aligned}\int_{A/F}\Phi_i\left(M,\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}s,g\right)\psi(-\alpha a)da\\ =\sum_{\xi\in\mathcal{K}_F}\int_{A/F}\int_{\mathcal{K}_F^1\backslash\mathcal{K}_A^1}\varphi_i(g_1hg)r(s_2)\rho(g_1h)\iota(g)M(\xi)\\ \psi(a\,n(\xi)-\alpha a)dg_1da\,.\end{aligned}$$

This is not 0 if and only if there exists an element ξ in $\mathcal{K}_F$ with $\alpha=n(\xi)$, and if $\alpha=n(\xi)$ for $\xi\in\mathcal{K}_F^\times$, then it equals

$$\begin{aligned}&\int_{\mathcal{K}_A^1}\varphi_i(g_1hg)\rho(g_1h)r(s_1)\iota(g)M(\xi)dg_1\\ &=\int_{\mathcal{K}_A^1}\varphi_i(\xi g_1hg)r(s_1)\iota(g)M(\xi g_1h)dg_1\\ &=\int_{\mathcal{K}_A^1}\varphi_i(g_1\xi hg)r(s_1)\iota(g)M(g_1\xi h)dg_1\,.\end{aligned}$$

From this we see that $\hat\phi_M(0,s)=\phi_0(s)$ and

$$\hat\phi_M(\alpha,s)=\begin{cases}\phi_1\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}s\right) & \text{if } \alpha\in F_+^\times,\\ 0 & \text{otherwise}.\end{cases} \tag{4.8}$$

Hence (4.4) (for $s\in GL_2(A)_+$) follows from (4.7).

If s is not in $GL_2(A)_+$, find an element β in $F^\times$ such that $s'=\begin{pmatrix}\beta & 0\\ 0 & 1\end{pmatrix}s$ $\in GL_2(A)_+$. Then $\phi_M(s')=\phi_M(s)$ and $\hat\phi_M(\alpha,s')=\hat\phi_M(\alpha\beta,s)$ by (4.3) so that (4.4) is valid for s. Putting $\alpha=\beta^{-1}$ in the above equality, we see that $\hat\phi_M(1,s)=0$ if $s\notin GL_2(A)_+$ (cf. (4.8)).

Lemma 4. *If π is not a representation of dimension* 1, *we have* $\hat\phi_M(0,s)=0$.

Proof. For $\varphi\in\mathcal{V}$, put

$$H\varphi(g)=\int_{\mathcal{K}_F^1\backslash\mathcal{K}_A^1}\varphi(g_1g)dg_1\,.$$

It is easy to see that $H\varphi$ is a continuous function on $\mathcal{K}_A^\times$ belonging to $L_0^2(\eta,\mathcal{K}_A^\times)$. Furthermore, $H\varphi$ is right K-finite. Hence $H\varphi\in\mathcal{A}(\eta,\mathcal{K}_A^\times)$. Since $\varphi\to H\varphi$ commutes with the right translation, either $H(\mathcal{V})=0$ or the representation of $\mathcal{H}(\mathcal{K}_A^\times)$ in $H(\mathcal{V})$ is equivalent to π. In the latter case π is necessarily one-dimensional representation, for $H\varphi(g)$ depends only on $n(g)$. Hence we have $H(\mathcal{V})=\{0\}$, and $\hat{\phi}_M(0, s)=\phi_0(s)=0$ by (4.5).

3. In the notation in No. 2, we put $W_M(s)=\hat{\phi}_M(1, s)$. Evidently

$$W_M\left(\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}s\right)=\psi(a)W_M(s)$$

for $a\in \boldsymbol{A}$. Remark that $\hat{\phi}_M(1, s)$ is $\phi_1(s)$ if $s\in GL_2(\boldsymbol{A})_+$ and 0 otherwise. In view of (4.6) and (3.8) we obtain

$$W_M(s)=\begin{cases}|\det s|_A\int_{\mathcal{K}_A^1}\omega_{\mathfrak{d}}(g_1h)r(s_1)M(g_1h)dg_1 & \text{if } s\in GL_2(\boldsymbol{A})_+\\ 0 & \text{if } s\notin GL_2(\boldsymbol{A})_+ .\end{cases} \tag{4.9}$$

It follows from (3.11) and (3.12) that

$$W_M=W_{\tilde{M}}=W_{\bar{\chi}_{\mathfrak{d}}*M},$$

if we put

$$\tilde{M}(x)=\int_{K^1}M(k_1xk_1^{-1})dk_1,$$

$$\bar{\chi}_{\mathfrak{d}}*M(x)=\int_{K^1}\chi_{\mathfrak{d}}(k_1)M(k_1x)dk_1.$$

For this reason we may limit ourselves to the functions M such that $M=\tilde{M}=\bar{\chi}_{\mathfrak{d}}*M$.

In a special case where the multiplicity of $\mathfrak{d}$ in $\mathcal{L}(\mathfrak{d})$ is 1, we still obtain (4.9) if we put

$$\varphi(g)\phi_M(s)=\dim\mathfrak{d}\,|\det s|_A\int_{\mathcal{K}_F^1\backslash\mathcal{K}_A^1}\varphi(g_1hg)\Theta(\rho(g_1h)r(s_1)\iota(g)M)dg_1 \tag{4.10}$$

for $s\in GL_2(\boldsymbol{A})_+$ and for $M\in\mathcal{S}(\mathcal{K}_A)$ such that $\tilde{M}=M$, where φ is any non-zero function in $\mathcal{L}(\mathfrak{d})$ and g is any element in $\mathcal{K}_A^\times$ with $\varphi(g)\neq 0$ (cf. (3.9)).

4. Let $\mathcal{S}_1(\mathcal{K}_A)$ be the subspace of $\mathcal{S}(\mathcal{K}_A)$ spanned by all M satisfying the following conditions.

i) $M(x)=\prod M_v(x_v)$ with $M_v\in\mathcal{S}(\mathcal{K}_v)$.

ii) $\tilde{M}=M$.

iii) $\bar{\chi}_{\mathfrak{d}}*M=M$.

iv) If $F_v=\boldsymbol{R}$ and $\mathcal{K}_v$ is a division quaternion algebra over $\boldsymbol{R}$, $\mathcal{K}_v$ is identified with the set of all matrices of the form $\begin{pmatrix}a & b\\ -\bar{b} & \bar{a}\end{pmatrix}$ with $a, b\in\boldsymbol{C}$. We have $n(x)=\det x$ for $x\in\mathcal{K}_v$. Assume that π_v is written as

$$\pi_v(g)=n(g)^r\rho_n(g)$$

($r\in\boldsymbol{C}$, $\rho_n=$ the n-th symmetric tensor representation of $GL_2(\boldsymbol{C})$) and that $\psi_v(\alpha)=\exp(2\pi iu_v\alpha)$ with $u_v\in\boldsymbol{R}$. Let χ_n be the character of ρ_n. Then M_v is of the form

$$M_v(x)=\exp(-2\pi|u_v|n(x))P(n(x))\chi_n(x^\iota)$$

for $x\in\mathcal{K}_v$, P being a polynomial.

v) If $F_v=\boldsymbol{R}$ or $\boldsymbol{C}$ and $\mathcal{K}_v=M_2(F_v)$, and if $\psi_v(\alpha)=\exp(2\pi iu_v\operatorname{tr}_{F_v/\boldsymbol{R}}(\alpha))$ with $u_v\in\boldsymbol{R}$, M_v is of the form

$$M_v(x)=\exp(-\pi d_v|u_v|\operatorname{tr}(x\,{}^t\bar{x}))P(x),$$

where $d_v=[F_v:\boldsymbol{R}]$ and $P(x)$ is a polynomial of ξ_{ij}, $\bar{\xi}_{ij}$ if $x=\begin{pmatrix}\xi_{11} & \xi_{12}\\ \xi_{21} & \xi_{22}\end{pmatrix}$.

5. Let $\mathcal{V}^*$ be the space spanned by all $\rho\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\right)\phi_M$ for $M\in\mathcal{S}_1(\mathcal{K}_A)$ and $a\in E$, E being a representative system of $A^\times/(A^\times)^2$. By (4.4) the mapping $\phi_M(s)\to W_M(s)=\hat{\phi}_M(1,s)$ is injective and commutes with the right translation. Let $\mathcal{W}^*$ be the space spanned by all $\rho\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\right)W_M$ for $M\in\mathcal{S}_1(\mathcal{K}_A)$ and $a\in E$.

PROPOSITION 2. *If π is not one-dimensional, $\mathcal{V}^*$ is a subspace of $\mathcal{A}_0(\eta, GL_2(\boldsymbol{A}))$.*

PROOF. In §5 we shall see that $\mathcal{W}^*$ is invariant under $\rho(\mu)$ for all $\mu\in\mathcal{H}(GL_2(\boldsymbol{A}))$ and the representation of $\mathcal{H}(GL_2(\boldsymbol{A}))$ in $\mathcal{W}^*$ is admissible. This implies the conditions (3.1) and (3.2) for all functions in $\mathcal{V}^*$. So far we see that ϕ_M is continuous on $GL_2(\boldsymbol{A})$, left $GL_2(F)$-invariant and cuspidal (i.e. $\hat{\phi}_M(0,s)=0$). Therefore it is enough to prove that ϕ_M satisfies (3.3) and (3.4) (then, every right translate of ϕ_M will also satisfy these conditions).

Let z be in $\boldsymbol{A}^\times$. Since

$$zs=\begin{pmatrix}z^2\det s & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}z^{-1} & 0\\ 0 & z\end{pmatrix}s_1,$$

$$r\left(\begin{pmatrix}z^{-1} & 0\\ 0 & z\end{pmatrix}\right)M=|z^{-1}|_A\rho(z^{-1})M,$$

we see that $\Phi_i(M,zs,g)=|z^{-1}|_A\eta(z)\Phi_i(M,s,g)$ and hence that $\phi_M(zs)=\eta(z)\phi_M(s)$ (cf. (4.1), (4.2)).

To prove (3.4) in our case, we may assume that Ω is a compact subset of $GL_2(\boldsymbol{A})_+$ and a varies within $\boldsymbol{A}_+^\times=n(\mathcal{K}_A^\times)$. Let us substitute $\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}s$ for s in (4.1). Write $s=\begin{pmatrix}\det s & 0\\ 0 & 1\end{pmatrix}s_1$ and let h be an element in $\mathcal{K}_A^\times$ such that $n(h)=a\det s$. If s varies in Ω and g varies in a compact fundamental domain Ω_1 of $P(\mathcal{K}^\times)_F$ in $P(\mathcal{K}^\times)_A$, $\iota(g)r(s_1)M$ stays in a compact subset of $\mathcal{S}(\mathcal{K}_A)$. By [**11**, Lemma 5] there exists a function M_0 in $\mathcal{S}(\mathcal{K}_A)$ such that

$$|\iota(g)r(s_1)M(x)|\leqq M_0(x)$$

for all $x\in\mathcal{K}_A$, $s\in\Omega$, $g\in\Omega_1$. On the other hand, the functions φ_i are bounded on $\mathcal{K}_A^\times$. Hence we get an estimate

$$\left|\phi_M\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} s\right)\right| \leqq c_1 |n(h)|_A \int_{\mathcal{K}_F^1\backslash \mathcal{K}_A^1} \Theta(\rho(g_1 h)M_0) dg_1 ,$$

c_1 being a constant independent of a and s. Now (3.4) is a consequence of the following lemma.

LEMMA 5. *Let c be a constant >0, and M an element in $\mathcal{S}(\mathcal{K}_A)$. Then $\Theta(\rho(h)M)$ is bounded for all $h \in \mathcal{K}_A^\times$ such that $|n(h)|_A > c$.*

PROOF. Assume first that F is a number field. Let S_∞ be the set of all archimedean places in F and put $\mathcal{K}_\infty = \prod_{v \in S_\infty} \mathcal{K}_v$. We can assume that $M(x) = M_\infty(x_\infty) \prod_{v \notin S_\infty} M_v(x_v)$, where $M_\infty \in \mathcal{S}(\mathcal{K}_\infty)$ and M_v is the characteristic function of a $\mathfrak{o}_v$-lattice L_v in $\mathcal{K}_v$, and almost all L_v are $\mathfrak{O}_v$. Clearly $\Theta(\rho(h)M)$ does not change if we replace h by δh for $\delta \in \mathcal{K}_F^\times$. Let $\mathcal{K}_A^0$ be the group of all $g \in \mathcal{K}_A^\times$ with $|n(h)|_A = 1$. Identify an element $\alpha \in \boldsymbol{R}^\times$ with an element $g \in \mathcal{K}_A^\times$ such that $g_v = 1$ for $v \notin S_\infty$ and $g_v = \alpha$ for $v \in S_\infty$. We have $\mathcal{K}_A^\times = \boldsymbol{R}^\times \mathcal{K}_A^0$ and $\mathcal{K}_F^\times \backslash \mathcal{K}_A^0$ is compact. Hence we may assume that $h = \alpha \in \boldsymbol{R}^\times$ applying [**11**, Lemma 5] again.

Let L be the set of all $\xi \in \mathcal{K}_F$ such that $\xi \in L_v$ for all $v \notin S_\infty$. Projecting L to $\mathcal{K}_\infty$, we get a $\boldsymbol{Z}$-lattice in $\mathcal{K}_\infty$. We have

$$\Theta(\rho(\alpha)M) = \sum_{\xi \in L} M_\infty(\alpha\xi) .$$

Let M'_∞ be the Fourier transform of M_∞ and L' the dual lattice of L. By Poisson's formula

$$\sum_{\xi \in L} M_\infty(\alpha\xi) = |\alpha|^{-m} \sum_{\xi \in L'} M'_\infty(\alpha^{-1}\xi) ,$$

m being the dimension of $\mathcal{K}_\infty$ over $\boldsymbol{R}$. Letting $|\alpha| \to \infty$, the right hand side converges to a constant multiple of $\int M'_\infty(x_\infty) dx_\infty$. This proves our assertion.

If F is a function field, it is easy to show that the support of $\rho(h)M$ is contained in a fixed compact subset of $\mathcal{K}_A$ for all h with $|n(h)|_A > c$. Then the lemma follows immediately.

§ 5. Whittaker spaces.

1. We shall prove that the representation of $\mathcal{H}(GL_2(A))$ in $\mathcal{W}^*$ introduced in § 4, No. 4 is admissible, and determine its equivalence class.

From the definition of W_M and (3.10) it follows that, if $M(x) = \prod M_v(x_v)$ is an element of $\mathcal{S}_1(\mathcal{K}_A)$, then we have

$$W_M(s) = \prod_v W_{M_v}(s_v) , \tag{5.1}$$

where

(5.2) $$W_{M_v}(s)=|\det s|_{F_v}\int_{\mathcal{K}_v^1}\omega_{\mathfrak{d}_v}(g_1h)r_v(s_1)M_v(g_1h)dg_1$$

for $s=\begin{pmatrix}\det s & 0\\ 0 & 1\end{pmatrix}s_1\in GL_2(F_v)_+$, h being an element of $\mathcal{K}_v^\times$ with $n(h)=\det s$, and

(5.3) $$W_{M_v}(s)=0 \qquad \text{for } s\notin GL_2(F_v)_+ .$$

Clearly we have

(5.4) $$W_{M_v}(ss')=W_{r_v(s')M_v}(s) \qquad \text{for } s'\in SL_2(F_v) .$$

Let η_v be as in § 4, No. 1. By the same proof as in Proposition 2 we get:

(5.5) $$W_{M_v}(sz)=\eta_v(z)W_{M_v}(s) \qquad \text{for } z\in F_v^\times .$$

Denote by $\mathcal{S}_1(\mathcal{K}_v)$ the space of all $M_v\in\mathcal{S}(\mathcal{K}_v)$ satisfying

$$M_v(k_1xk_1^{-1})=M_v(x)\quad(k_1\in K_v^1), \qquad \tilde{\chi}_{\mathfrak{d}_v}*M_v=M_v$$

as well as the conditions iv), v) in § 4, No. 4. Let $\mathcal{W}_v^*$ be the space spanned by all $\rho\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right)W_{M_v}$ for $M_v\in\mathcal{S}_1(\mathcal{K}_v)$ and $\alpha\in E_v$, E_v being a representative system of $F_v^\times/(F_v^\times)^2$. Let M_v^0 be the characteristic function of $\mathfrak{O}_v$ and write $W_v^0=W_{M_v}$ for $M_v=M_v^0$. We shall prove in Lemma 7 that, for almost all v, W_v^0 is invariant under the right translations by elements of $GL_2(\mathfrak{o}_v)$. By (5.1) we see that $\mathcal{W}^*$ is the restricted tensor product of $\mathcal{W}_v^*$ with respect to $\{W_v^0\}$ and that, if we let $\mathcal{H}(GL_2(A))$ (resp. $\mathcal{H}(GL_2(F_v))$) act on $\mathcal{W}^*$ (resp. $\mathcal{W}_v^*$) by right translation, the representation of $\mathcal{H}(GL_2(A))$ in $\mathcal{W}^*$ is the tensor product of the representations of $\mathcal{H}(GL_2(F_v))$ in $\mathcal{W}_v^*$ (granted that $\mathcal{W}^*$ or $\mathcal{W}_v^*$ is invariant under this action, which we are going to prove).

π being as in § 4, No. 1, write $\pi=\otimes\pi_v$. Let S be the set of all places in F ramified in $\mathcal{K}$.

PROPOSITION 3. *$\mathcal{W}_v^*$ is invariant under the action of $\mathcal{H}(GL_2(F_v))$ and the representation ρ_v of $\mathcal{H}(GL_2(F_v))$ in $\mathcal{W}_v^*$ is admissible. If $v\notin S$ and π_v is infinite dimensional, ρ_v is equivalent to π_v. If $v\in S$, ρ_v is equivalent to π_v^** (§ 2, No. 2).

The proof of this proposition will be given in No. 2–No. 11. Since all the arguments in the following are purely local, we write for simplicity r for r_v. $\mathfrak{d}_v$ denotes always an irreducible representation of K_v^1 contained in the restriction of π_v to K_v^1.

2. In No. 2–No. 6, v denotes a non-archimedean place in F unramified in $\mathcal{K}$ so that $\mathcal{K}_v=M_2(F_v)$ and $\mathcal{K}_v^\times=GL_2(F_v)$. Assume first that $\pi_v=\pi(\mu_1,\mu_2)$ with quasi-characters μ_1, μ_2 of $F_v^\times$ such that $\mu_1\mu_2^{-1}$ is neither $|\ |_{F_v}$ nor $|\ |_{F_v}^{-1}$.

By Godement [3, No. 16] the spherical function $\omega_{\mathfrak{d}_v}$ is obtained in the following way.*) Let T, ζ be as in § 1, No. 3 (take F to be F_v). Put $U=T\cap K_v^1$; then $\mathcal{K}_v^\times=TK_v^1$. If we put

$$\chi\xi_v(tk)=\zeta(t)\int_U\zeta(u^{-1})\chi_{\mathfrak{b}_v}(uk)du$$

for $t\in T$ and $k\in K_v^1$, we have

$$\omega_{\mathfrak{b}_v}(g)=\int_{K_v^1}\chi\xi_v(k_1gk_1^{-1})dk_1 \tag{5.6}$$

for $g\in\mathcal{K}_v^\times$. Here dk_1 (resp. du) is the Haar measure of K_v^1 (resp. U) with the total volume 1.

Let us calculate W_M for $M\in\mathcal{S}_1(\mathcal{K}_v)$. We put $T^1=T\cap\mathcal{K}_v^1$. For $\alpha\in F_v^\times$ set $h=\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}$. Since $M(k_1xk_1^{-1})=M(x)$ and $\bar{\chi}_{\mathfrak{b}_v}*M=M$, we have

$$\begin{aligned}|\alpha|_{F_v}^{-1}W_M\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right)&=\int_{\mathcal{K}_v^1}\omega_{\mathfrak{b}_v}(hg_1)M(hg_1)dg_1\\
&=\int_{\mathcal{K}_v^1}\chi\xi_v(hg_1)M(hg_1)dg_1\\
&=\int_{T^1/U}\int_{K_v^1}\int_U\zeta(ht_1u^{-1})\chi_{\mathfrak{b}_v}(uk_1)M(ht_1k_1)d\dot{t}_1dk_1du\\
&=\int_{T^1/U}\int_U\zeta(ht_1u^{-1})M(ht_1u^{-1})d\dot{t}_1du\\
&=\int_{T^1}\zeta(ht_1)M(ht_1)dt_1\,.\end{aligned}$$

Here dt_1 is the left invariant measure of T^1 and $dt_1=d\dot{t}_1du$. If we write $t_1=\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}\begin{pmatrix}\gamma & 0\\ 0 & \gamma^{-1}\end{pmatrix}$, we have $dt_1=|\gamma|_{F_v}^{-2}d\beta d^\times\gamma$. Then the last expression in the above equals

$$\mu_1(\alpha)|\alpha|_{F_v}^{-1/2}\int_{F_v^\times}\int_{F_v}\mu_1\mu_2^{-1}(\gamma)M\left(\begin{pmatrix}\alpha\gamma & \beta\\ 0 & \gamma^{-1}\end{pmatrix}\right)d\beta d^\times\gamma\,.$$

Therefore, if $s=\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}s_1$ with $\det s=\alpha$, we get

$$W_M(s)=\mu_1(\alpha)|\alpha|_{F_v}^{1/2}\int_{F_v^\times}\int_{F_v}\mu_1\mu_2^{-1}(\gamma)r(s_1)M\left(\begin{pmatrix}\alpha\gamma & \beta\\ 0 & \gamma^{-1}\end{pmatrix}\right)d\beta d^\times\gamma\,. \tag{5.7}$$

For $M\in\mathcal{S}(\mathcal{K}_v)$ and $(\alpha_1,\alpha_2)\in F_v\times F_v$, we put

$$f(M)(\alpha_1,\alpha_2)=\int_{F_v}M\left(\begin{pmatrix}\alpha_1 & \xi\\ 0 & \alpha_2\end{pmatrix}\right)d\xi\,.$$

---

*) In [3, No. 16] the space $\mathfrak{H}^\zeta$ of the induced representation consists of all continuous functions on $GL_2(F_v)$ satisfying $f(tg)=\zeta(t)f(g)$ $(t\in T)$. The space $\mathcal{B}(\mu_1,\mu_2)$ of $\pi(\mu_1,\mu_2)$ is the space of all locally constant functions in $\mathfrak{H}^\zeta$. However, the spherical functions of both representations are the same, for all K_v^1-finite functions in $\mathfrak{H}^\zeta$ are locally constant.

Clearly $f(M)\in \mathcal{S}(F_v\oplus F_v)$. Denote by r_0 the Weil representation of $SL_2(F_v)$ in $\mathcal{S}(F_v\oplus F_v)$ (with respect to the character ψ_v of F_v). By [**5**, Prop. 1.6] r_0 can be extended to a representation of $GL_2(F_v)$ such that

$$r_0\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right)m(\alpha_1, \alpha_2)=m(\alpha\alpha_1, \alpha_2)$$

for $m\in \mathcal{S}(F_v\oplus F_v)$.

LEMMA 6. *For $M\in \mathcal{S}(\mathcal{K}_v)$ and $s_1\in SL_2(F_v)$ we have*

$$f(r(s_1)M)=r_0(s_1)f(M)\,.$$

The proof is immediate.

It follows from Lemma 6 and (5.7) that

$$W_M(s)=\mu_1(\det s)|\det s|_{F_v}^{1/2}\int_{F_v^\times}\mu_1\mu_2^{-1}(\gamma)r_0(s)f(M)(\gamma, \gamma^{-1})d^\times\gamma \tag{5.8}$$

so that W_M is contained in the space $W(\mu_1, \mu_2\,;\psi_v)$ in the notation in [**5**, § 3]. By the assumption on μ_1, μ_2 this space is the Whittaker space of $\pi(\mu_1, \mu_2)$ (cf. [**5**, Prop. 3.5]).

For $s_1\in SL_2(F_v)$ and $M\in \mathcal{S}_1(\mathcal{K}_v)$, we have $\rho(s_1)W_M=W_N$ with $N=r(s_1)M$, and $\mathcal{S}_1(\mathcal{K}_v)$ is invariant under $r(s_1)$, because $r(s_1)$ commutes with $\rho(k_1)$ and $\lambda(k_1)$ for $k_1\in K_v^1$ (Lemma 1). By (5.4) and (5.5) the space $\mathcal{W}_v{}^*$ spanned by all $\rho\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right)W_M$ ($M\in \mathcal{S}_1(\mathcal{K}_v)$ and $\alpha\in E_v$) is invariant under $\rho(s)$ for all $s\in GL_2(F_v)$. $\mathcal{W}_v{}^*$ is clearly non-zero. Since $W(\mu_1, \mu_2\,;\psi_v)$ is irreducible, we have $\mathcal{W}_v{}^*=W(\mu_1, \mu_2\,;\psi_v)$ and hence ρ_v is equivalent to $\pi(\mu_1, \mu_2)$.

3. Let π_v be $\pi(\mu_1, \mu_2)$ with quasi-characters μ_1, μ_2 of $F_v^\times$ such that $\mu_1\mu_2^{-1}=|\ |_{F_v}^{-1}$. Write $\mu_1(\alpha)=\chi(\alpha)|\alpha|_{F_v}^{-1/2}$, $\mu_2(\alpha)=\chi(\alpha)|\alpha|_{F_v}^{1/2}$. Then π_v is the one-dimensional representation $\chi\circ n$. Obviously $\mathfrak{d}_v$ is the identity representation and $\omega_{\mathfrak{d}_v}(g)=\chi(n(g))$. By a simple calculation we again obtain (5.8) for $M\in \mathcal{S}_1(\mathcal{K}_v)$. As in No. 2, we see that $\mathcal{W}_v{}^*$ is an invariant subspace of $W(\mu_1, \mu_2\,;\psi_v)$. Consequently ρ_v is admissible.

4. For almost all v, the restriction of π_v to K_v contains the identity representation. By [**5**, Lemma 3.9], if $\mathcal{K}_v^\times=GL_2(F_v)$ and v is non-archimedean, such a π_v is of the form $\pi(\mu_1, \mu_2)$ with unramified ($=$trivial on $\mathfrak{o}_v^\times$) quasi-characters μ_1, μ_2 of $F_v^\times$.

LEMMA 7. *Assume that $\pi_v=\pi(\mu_1, \mu_2)$ with unramified quasi-characters μ_1, μ_2 of $F_v^\times$. Let $\mathfrak{a}_v$ be the conductor of ψ_v. Put*

$$L_v=\mathfrak{o}_v e_{11}+\mathfrak{o}_v e_{12}+\mathfrak{a}_v e_{21}+\mathfrak{a}_v e_{22}\,,$$

e_{ij} being a 2 by 2 matrix such that (i, j)-coefficient is 1 and the other coefficients are 0. If N is the characteristic function of L_v, then $r(s_1)N=N$ for $s_1\in SL_2(\mathfrak{o}_v)$. Furthermore, if $M=\int_{K_v{}^1}\lambda(k_1)Ndk_1$, then $\rho(s)W_M=W_M$ for $s\in GL_2(\mathfrak{o}_v)$.

PROOF. $GL_2(\mathfrak{o}_v)$ is generated by $\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}, \begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}$ ($\alpha\in \mathfrak{o}_v^\times$, $\beta\in\mathfrak{o}_v$). We note that L_v is a $\mathfrak{o}_v$-lattice and $n(x)\in\mathfrak{a}_v$ for all $x\in L_v$. It follows from definition that $r(s)N=N$ if $s=\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}$ or $\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}$. Let $L_v{}^*$ be the set of all $x\in\mathcal{K}_v$ such that $\mathrm{tr}\,(xL_v)\subset\mathfrak{a}_v$. Evidently $L_v{}^*=\mathfrak{a}_ve_{11}+\mathfrak{o}_ve_{12}+\mathfrak{a}_ve_{21}+\mathfrak{o}_ve_{22}$ and the Fourier transform N' of N is the characteristic function of $L_v{}^*$ up to a positive constant. Hence $r\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right)N(x)=N'(x')=cN(x)$. Since $r\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}^2\right)N=r(-1)N=N$, c must be 1. This proves the first assertion. By Lemma 1, i), the same assertion is valid also for M. It follows from (5.8) that

$$W_M\left(s\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right)=\mu_1(\det s)|\det s|_{F_v}^{1/2}$$

$$\int\mu_1\mu_2{}^{-1}(\gamma)r_0(s)f\left(\rho\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}M\right)(\gamma,\gamma^{-1})d^\times\gamma\,.$$

If $\alpha\in\mathfrak{o}_v^\times$, we have $\rho\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right)M=M$ so that $\rho\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right)W_M=W_M$. Together with what we have proved above, this proves the second assertion.

5. We assume now that $\pi_v=\sigma(\mu_1,\mu_2)$ with quasi-characters μ_1, μ_2 of $F_v^\times$ such that $\mu_1\mu_2{}^{-1}=|\ |_{F_v}$. Write $\mu_1(\alpha)=\chi(\alpha)|\alpha|_{F_v}^{1/2}$, $\mu_2(\alpha)=\chi(\alpha)|\alpha|_{F_v}^{-1/2}$. In the notation in § 1, No. 3, put $\mathcal{V}=\mathcal{B}(\mu_1,\mu_2)$ and $\mathcal{V}_s=\mathcal{B}_s(\mu_1,\mu_2)$.

We first note that the restriction of π_v to $K_v{}^1$ does not contain the identity representation. If this is not true, there would be a non-zero function f in $\mathcal{V}_s$ invariant under $K_v{}^1$. By [**5**, § 3]

$$\langle\varphi_1,\varphi_2\rangle=\int_{K_v}\varphi_1(k)\varphi_2(k)dk$$

is a non-degenerate bilinear form on $\mathcal{B}(\mu_1,\mu_2)\times\mathcal{B}(\mu_1{}^{-1},\mu_2{}^{-1})$ invariant under the right translation, and $\mathcal{B}_s(\mu_1,\mu_2)$ is the space of all $\varphi\in\mathcal{B}(\mu_1,\mu_2)$ orthogonal to the function $\chi^{-1}\circ\det$ in $\mathcal{B}(\mu_1{}^{-1},\mu_2{}^{-1})$. Hence

$$\int_{K_v}\chi^{-1}(\det k)f(k)dk=\int_{\mathfrak{o}_v^\times}\chi^{-1}(\alpha)f\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right)d^\times\alpha=0\,.$$

It implies that $f(1)=0$ so that f is identically 0. This is a contradiction.

Let $\mathcal{V}(\mathfrak{d}_v)$ be the space of all $f\in\mathcal{V}$ such that

$$\int_{K_v^1}\chi_{\mathfrak{d}_v}(k_1{}^{-1})\rho(k_1)f=f\,,$$

and put $\mathcal{V}_s(\mathfrak{d}_v)=\mathcal{V}_s\cap\mathcal{V}(\mathfrak{d}_v)$. By the above remark $\mathfrak{d}_v$ is not the identity representation. On the other hand, the representation of $GL_2(F_v)$ in $\mathcal{V}/\mathcal{V}_s$ is equivalent to $\chi\circ\det$, whose restriction to $K_v{}^1$ is the identity representation.

Hence $\mathcal{V}(\mathfrak{d}_v)=\mathcal{V}_s(\mathfrak{d}_v)$. From the definition of spherical functions it follows that (5.6) is still valid in our case.

As in No. 2, we have $W_M \in W(\mu_1, \mu_2; \psi_v)$ for $M \in \mathcal{S}_1(\mathcal{K}_v)$. By [**5**, Prop. 3.6] W_M belongs to the Whittaker space $W(\sigma(\mu_1, \mu_2); \psi_v)$ of $\sigma(\mu_1, \mu_2)$ if

$$\int_{F_v} f(M)(\xi_1, 0)d\xi_1 = \int_{F_v \times F_v} M\left(\begin{pmatrix}\xi_1 & \xi_2\\ 0 & 0\end{pmatrix}\right)d\xi_1 d\xi_2 = 0. \tag{5.9}$$

This condition is certainly satisfied by $M \in \mathcal{S}_1(\mathcal{K}_v)$, for

$$\begin{aligned}&\int_{F_v \times F_v} M\left(\begin{pmatrix}\xi_1 & \xi_2\\ 0 & 0\end{pmatrix}\right)d\xi_1 d\xi_2\\ &=\int_{F_v \times F_v}\int_{K_v^1} M\left(\begin{pmatrix}\xi_1 & \xi_2\\ 0 & 0\end{pmatrix}k_1\right)\chi_{\mathfrak{d}_v}(k_1)dk_1 d\xi_1 d\xi_2\\ &=\int_{F_v \times F_v}\int_{K_v^1} M\left(\begin{pmatrix}\xi_1' & \xi_2'\\ 0 & 0\end{pmatrix}\right)\chi_{\mathfrak{d}_v}(k_1)dk_1 d\xi_1' d\xi_2' = 0,\end{aligned}$$

since $\int \chi_{\mathfrak{d}_v}(k_1)dk_1 = 0$. By the same reasoning as in No. 2, we see that $\mathcal{W}_v^*$ is the Whittaker space of $\sigma(\mu_1, \mu_2)$ so that ρ_v is equivalent to $\sigma(\mu_1, \mu_2)$.

6. Let us assume that π_v is absolutely cuspidal, and is realized in its Kirillov model ([**5**, § 2]). The representation space of π_v is then the space $\mathcal{S}(F_v^\times)$ of all locally constant functions of compact support on $F_v^\times$. Let Ψ be any non-trivial additive character of F_v. We may assume that

$$\pi_v\left(\begin{pmatrix}\alpha & \beta\\ 0 & 1\end{pmatrix}\right)\varphi(\xi) = \Psi(\beta\xi)\varphi(\alpha\xi),$$

$$\pi_v\left(\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}\right)\varphi(\xi) = \eta_v(\alpha)\varphi(\xi)$$

for $\alpha \in F_v^\times$, $\beta \in F_v$ and $\varphi \in \mathcal{S}(F_v^\times)$. Hence π_v is determined by the action of $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$.

μ being a character of $F_v^\times$, we set

$$\hat{\varphi}(\mu) = \int_{F_v^\times} \varphi(\xi)\mu(\xi)d^\times\xi$$

for $\varphi \in \mathcal{S}(F_v^\times)$. Transforming the action of $\pi_v(g)$ $(g \in \mathcal{K}_v^\times)$ by the mapping $\varphi \to \hat{\varphi}$, we obtain (cf. [**5**, Prop. 2.10])

$$\pi_v\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right)\hat{\varphi}(\mu) = \mu^{-1}(\alpha)\hat{\varphi}(\mu), \tag{5.10}$$

$$\pi_v\left(\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}\right)\hat{\varphi}(\mu) = \int_{F_v^\times} \mu(\xi)\Psi(\beta\xi)\varphi(\xi)d^\times\xi, \tag{5.11}$$

$$(5.12)\qquad \pi_v\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right)\hat{\varphi}(\mu)=C(\mu)\hat{\varphi}(\mu^{-1}\eta_v^{-1})$$

with a constant $C(\mu)$ depending only on μ.

Here we sketch a proof. (5.10) and (5.11) are immediate. To see (5.12), let ν be a character of $\mathfrak{o}_v^\times$ and φ_ν an element in $\mathcal{S}(F_v^\times)$ such that $\varphi_\nu(\xi)=\nu^{-1}(\xi)$ if $\xi\in\mathfrak{o}_v^\times$ and 0 outside of $\mathfrak{o}_v^\times$. If ϖ is a prime element in F_v, the functions $\pi_v\left(\begin{pmatrix}\varpi^{-n} & 0\\ 0 & 1\end{pmatrix}\right)\varphi_\nu$ (for all integers n and for all characters ν of $\mathfrak{o}_v^\times$) form a basis of $\mathcal{S}(F_v^\times)$. Let ν_0 be the restriction of η_v to $\mathfrak{o}_v^\times$. Write $w=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$. Since $w\begin{pmatrix}\alpha^{-1} & 0\\ 0 & 1\end{pmatrix}w^{-1}=\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}$, we have

$$\pi_v\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right)\pi_v(w)\varphi_\nu=\nu\nu_0(\alpha)\pi_v(w)\varphi_\nu$$

for $\alpha\in\mathfrak{o}_v^\times$. Therefore, we can write

$$\pi_v(w)\varphi_\nu=\sum_n C_n(\nu^{-1}\nu_0^{-1})\pi_v\left(\begin{pmatrix}\varpi^{-n} & 0\\ 0 & 1\end{pmatrix}\right)\varphi_{\nu^{-1}\nu_0^{-1}}.$$

Taking the Fourier transforms of the both sides, we get

$$\pi_v(w)\hat{\varphi}_\nu(\mu)=\sum_n C_n(\nu^{-1}\nu_0^{-1})\mu(\varpi^n)\hat{\varphi}_{\nu^{-1}\nu_0^{-1}}(\mu).$$

Clearly $\hat{\varphi}_{\nu^{-1}\nu_0^{-1}}(\mu)=\hat{\varphi}_\nu(\eta_v^{-1}\mu^{-1})$ and this is not 0 if and only if the restriction μ_0 of μ to $\mathfrak{o}_v^\times$ equals $\nu^{-1}\nu_0^{-1}$. Hence, if we put

$$C(\mu)=\sum_n C_n(\mu_0)\mu(\varpi^n),$$

(5.12) holds for $\varphi=\varphi_\nu$. It is now easy to see that (5.12) holds for all $\pi_v\left(\begin{pmatrix}\varpi^{-n} & 0\\ 0 & 1\end{pmatrix}\right)\varphi_\nu$.

In the following we take ψ_v for Ψ. It is shown in [**5**, Prop. 2.21.2] that the hermitian form

$$(\varphi_1,\varphi_2)=\int_{F_v}\varphi_1(\xi)\overline{\varphi_2(\xi)}d^\times\xi$$

on $\mathcal{S}(F_v^\times)$ is invariant under π_v. Write $\mathcal{V}=\mathcal{S}(F_v^\times)$ and define $\mathcal{V}(\mathfrak{d}_v)$ as in No. 5. Let $\{\varphi_i\}_{i=1}^N$ be an orthonormal basis of $\mathcal{V}(\mathfrak{d}_v)$. By definition

$$\omega_{\mathfrak{d}_v}(g)=\sum_{i=1}^N\int_{K_v{}^1}(\pi_v(k_1g)\varphi_i,\varphi_i)\chi_{\mathfrak{d}_v}(k_1^{-1})dk_1.$$

Hence, if $M\in\mathcal{S}_1(\mathcal{K}_v)$, we have

$$(5.13)\qquad W_M(g)=|\det s|_{F_v}\sum_{i=1}^N\int_{\mathcal{K}_v{}^1}(\pi_v(g_1h)\varphi_i,\varphi_i)r(s_1)M(g_1h)dg_1,$$

where $s=\begin{pmatrix}\det s & 0\\ 0 & 1\end{pmatrix}s_1$ and $n(h)=\det s$.

As in No. 2 we see that $\mathcal{W}_v{}^*$ is invariant under $\rho(s)$ for $s\in GL_2(F_v)$. For $W\in\mathcal{W}_v{}^*$ and $\xi\in F_v^\times$, put $\varphi_W(\xi)=W\left(\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix}\right)$.

LEMMA 8. *$\varphi_W\in\mathcal{S}(F_v^\times)$ for $W\in\mathcal{W}_v{}^*$.*

PROOF. It is enough to prove this in case $W=W_M$ for $M\in\mathcal{S}_1(\mathcal{K}_v)$. Then it follows immediately from (5.13) that φ_W is locally constant and the support of φ_W is contained in a compact set of F_v. We now prove that $\varphi_W=0$ in a neighbourhood of 0. We shall prove in Lemma 14 that $\mathfrak{d}_v$ is not the identity representation. Hence

$$M(0)=\int_{K_v{}^1}\chi_{\mathfrak{d}_v}(k_1)M(0)dk_1=0$$

and hence there exists a neighbourhood V of 0 in $\mathcal{K}_v$ on which M is identically 0.

It is easy to see (cf. the proof of [5, Prop. 2.20]) that the support of the function $(\pi_v(g)\varphi,\varphi)$ is compact modulo $F_v^\times$, if $\varphi\in\mathcal{S}(F_v^\times)$. Hence there is a compact set C in $\mathcal{K}_v^\times$ such that $F_v^\times C$ contains the support of $(\pi_v(g)\varphi_i,\varphi_i)$ for $i=1,\cdots,N$.

Set $s=h=\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}$ in (5.13). g_1h is written in the form $\begin{pmatrix}\delta & 0\\ 0 & \delta\end{pmatrix}k_1\begin{pmatrix}\gamma & \beta\\ 0 & 1\end{pmatrix}$ with $k_1\in K_v{}^1$. Then $\alpha=\gamma\delta^2$. Assume that $k_1\begin{pmatrix}\gamma & \beta\\ 0 & 1\end{pmatrix}\in C$. Then γ is contained in a compact subset of $F_v^\times$. Consequently, we can find a small neighbourhood V' of 0 in F_v such that if $\alpha\in V'$, then δC is contained in V so that $M(g_1h)=0$, and hence $\varphi_W(\alpha)=0$. q. e. d.

By Lemma 8, $W\to\varphi_W$ is a linear mapping of $\mathcal{W}_v{}^*$ into $\mathcal{S}(F_v^\times)$. It is easily seen that

$$\varphi_X(\xi)=\psi_v(\beta\xi)\varphi_W(\alpha\xi)\qquad\text{if } X=\rho\left(\begin{pmatrix}\alpha & \beta\\ 0 & 1\end{pmatrix}\right)W. \tag{5.14}$$

We now assert that

$$\pi_v(s)\varphi_W=\varphi_{\rho(s)W}\qquad\text{for } s\in GL_2(F_v). \tag{5.15}$$

In view of (5.14) it is enough to prove (5.15) for $s=w=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$. To do this, let us calculate $\hat{\varphi}_W$. If $W=W_M$, we have

$$\begin{aligned}\hat{\varphi}_W(\mu)=\int_{F_v^\times}&\mu(\xi)d^\times\xi\\ &\int_{\mathcal{K}_v^1}\sum_{i=1}^N|\xi|_{F_v}\left(\pi_v\left(g_1\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix}\right)\varphi_i,\varphi_i\right)M\left(g_1\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix}\right)dg_1\\ &=\int_{\mathcal{K}_v^\times}\mu(\det g)|\det g|_{F_v}\sum_{i=1}^N(\pi_v(g)\varphi_i,\varphi_i)M(g)dg.\end{aligned} \tag{5.16}$$

LEMMA 9. *Let dx be the self-dual measure of $\mathcal{K}_v$ (with respect to $\langle x,y\rangle=\psi_v(\mathrm{tr}(xy))$) and dg the Haar measure of $\mathcal{K}_v^\times$ such that $|\det g|_{F_v}{}^2dg$ coincides* *with dx on $\mathcal{K}_v^\times$. Then we have*

(5.17)
$$\int_{\mathcal{K}_v^\times} |\det g|_{F_v} \mu(\det g)(\pi_v(g)\varphi_1, \varphi_2)\psi_v(\operatorname{tr} g)dg$$
$$= C(\mu)(\varphi_1, \varphi_2)$$

for $\varphi_1, \varphi_2 \in \mathcal{S}(F_v^\times)$.

PROOF. We follow the method in [**5**, Lemma 13.1.1]. Write $g=\begin{pmatrix}\gamma_{11} & \gamma_{12}\\ \gamma_{21} & \gamma_{22}\end{pmatrix}$ in the form

$$g=\begin{pmatrix}\delta & 0\\ 0 & \delta\end{pmatrix}\begin{pmatrix}1 & -\beta'\\ 0 & 1\end{pmatrix}\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}w\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}$$

if $\gamma_{21}\neq 0$. If $d\alpha$ is the self-dual measure of F_v (with respect to $\langle\alpha, \beta\rangle=\psi_v(\alpha\beta)$) and if $d^\times\alpha=|\alpha|_{F_v}{}^{-1}d\alpha$, then $dg=|\gamma|_{F_v}{}^{-1}d\beta d\beta' d^\times\gamma d^\times\delta$. In the above notation we have

$$(\pi_v(g)\varphi_1, \varphi_2)=\eta_v(\delta)\left(\pi_v\left(\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}w\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}\right)\varphi_1, \pi_v\left(\begin{pmatrix}1 & \beta'\\ 0 & 1\end{pmatrix}\right)\varphi_2\right).$$

Put

$$f_1=\pi_v\left(\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}w\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}\right)\varphi_1, \qquad f_2=\pi_v\left(\begin{pmatrix}1 & \beta'\\ 0 & 1\end{pmatrix}\right)\varphi_2.$$

By (5.10)-(5.12) we have

$$\hat{f}_1(\mu')=\mu'^{-1}(\gamma)C(\mu')\int_{F_v^\times}\mu'^{-1}\eta_v{}^{-1}(\xi)\psi_v(\xi\beta)\varphi_1(\xi)d^\times\xi,$$

$$\hat{f}_2(\mu')=\int_{F_v^\times}\mu'(\xi)\psi_v(\xi\beta')\varphi_2(\xi)d^\times\xi.$$

If $d\mu$ is the dual-measure of $d^\times\alpha$,

$$(f_1, f_2)=\int\hat{f}_1(\mu')\overline{\hat{f}_2(\mu')}d\mu'.$$

Therefore, the left hand side of (5.17) equals

$$\int\Big[\int|\delta^2\gamma|_{F_v}\mu(\delta^2\gamma)\eta_v(\delta)\mu'^{-1}(\gamma)C(\mu')$$
$$\left\{\int\mu'^{-1}\eta_v{}^{-1}(\xi)\psi_v(\xi\beta)\varphi_1(\xi)d^\times\xi\right\}\overline{\left\{\int\mu'(\xi)\psi_v(\xi\beta')\varphi_2(\xi)d^\times\xi\right\}}$$
$$\psi_v(\delta(\beta'-\beta))\,d\mu'\Big]|\gamma|_{F_v}{}^{-1}d\beta d\beta' d^\times\gamma d^\times\delta.$$

Now we have (by Fourier's inversion formula)

$$\iint\mu'(\xi)\varphi_2(\xi)\psi_v(\xi\beta')\psi_v(-\delta\beta')d^\times\xi d\beta'=|\delta|_{F_v}{}^{-1}\mu'(\delta)\varphi_2(\delta),$$

$$\iint\mu'^{-1}\eta_v{}^{-1}(\xi)\varphi_1(\xi)\psi_v(\xi\beta)\psi_v(-\delta\beta)d^\times\xi d\beta=|\delta|_{F_v}{}^{-1}\mu'^{-1}\eta_v{}^{-1}(\delta)\varphi_1(\delta)$$

so that (after a change of a variable) the left hand side of (5.17) equals

$$\int \mu\mu'^{-1}(\gamma)C(\mu')d\mu' d^\times\gamma \int \varphi_1(\delta)\overline{\varphi_2(\delta)}d^\times\delta .$$

Write $\mu(\varepsilon\varpi^n)=\nu(\varepsilon)t^n$ and $\mu'(\varepsilon\varpi^n)=\nu'(\varepsilon)t'^n$ with characters ν, ν' of $\mathfrak{o}_v^\times$ and complex numbers t, t' of absolute value 1. Put $c=\int_{\mathfrak{o}_v^\times} d^\times\varepsilon$. If $\gamma=\varepsilon\varpi^m$ for $\varepsilon\in\mathfrak{o}_v^\times$, then

$$\begin{aligned}\int \mu\mu'^{-1}(\gamma)C(\mu')d\mu' &= 1/c \sum_{\nu'} \int_{|t'|=1} \nu\nu'^{-1}(\varepsilon)(tt'^{-1})^m \sum_n C_n(\nu')t'^n dt' \\ &= 1/c \sum_{\nu'} \nu\nu'^{-1}(\varepsilon)t^m C_m(\nu') .\end{aligned}$$

Hence, integrating it by $d^\times\gamma$, we get

$$1/c \sum_m \int_{\mathfrak{o}_v^\times} \sum_{\nu'} \nu\nu'^{-1}(\varepsilon)t^m C_m(\nu')d^\times\varepsilon = \sum_m t^m C_m(\nu) = C(\mu) .$$

This proves the lemma.

We assume in the following that dg is normalized as in Lemma 9, though the final result is independent of this normalization. Putting $W'=\rho(w)W$ in (5.16) in place of W, we obtain

$$\hat{\varphi}_{W'}(\mu)=\int_{\mathcal{K}_v^\times} \mu(\det g)|\det g|_{F_v} \sum_i (\pi_v(g)\varphi_i, \varphi_i)M'(g^\iota)dg .$$

Since

$$M'(g^\iota)=\int_{\mathcal{K}_v^\times} M(h)\psi_v(\mathrm{tr}\,(hg^\iota))|\det h|_{F_v}{}^2 dh ,$$

we get (replacing g by $gh^{\iota-1}$)

$$\begin{aligned}\hat{\varphi}_{W'}(\mu) &= \iint |\det(gh)|_{F_v}\mu(\det(gh^{-1})) \\ &\qquad \sum_i (\pi_v(gh^{\iota-1})\varphi_i, \varphi_i)M(h)\psi_v(\mathrm{tr}\, g)dhdg \\ &= C(\mu)\int |\det h|_{F_v}\mu^{-1}(\det h) \\ &\qquad \sum_i (\pi_v(h^{\iota-1})\varphi_i, \varphi_i)M(h)dh \qquad \text{(by Lemma 9)} \\ &= C(\mu)\hat{\varphi}_W(\mu^{-1}\eta_v{}^{-1}) .\end{aligned}$$

This proves (5.15) for $s=w$.

(5.15) shows in particular that the space of all $W\in\mathcal{W}_v{}^*$ such that $\varphi_W=0$ is $GL_2(F_v)$-invariant. If $\varphi_W=0$, then $\pi_v(s)\varphi_W=\varphi_{\rho(s)W}=0$ and hence $W\left(\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix}s\right)=0$ for all $s\in GL_2(F_v)$. Therefore, the mapping $W\to\varphi_W$ is injective. Its image is a non-zero π_v-invariant subspace of $\mathcal{S}(F_v^\times)$ so that it must be the whole space. It follows that $\mathcal{W}_v{}^*$ is the Whittaker space of π_v and that ρ_v is equivalent to π_v.

7. We assume that v is non-archimedean and ramified in $\mathcal{K}$ so that $\mathcal{K}_v$ is now a division quaternion algebra over F_v. In this case π_v is an irreducible finite dimensional representation of $\mathcal{K}_v^\times$. Let χ be the character of π_v. It follows from definition that

$$\omega_{\mathfrak{b}_v}(g)=\int_{K_v^1}\chi_{\mathfrak{b}_v}(k_1{}^{-1})\chi(k_1 g)dk_1$$

and hence that, if $M\in\mathcal{S}_1(\mathcal{K}_v)$,

$$W_M(s)=|\det s|_{F_v}\int_{\mathcal{K}_v^1}\chi(g_1 h)r(s_1)M(g_1 h)dg_1$$

for $s=\begin{pmatrix}\det s & 0\\ 0 & 1\end{pmatrix}s_1$ and for $h\in\mathcal{K}_v^\times$ with $n(h)=\det s$. Note that $\mathcal{K}_v{}^1=K_v{}^1$.

Let U be the space of functions on $\mathcal{K}_v^\times$ spanned by all the coefficients of π_v. Let Ω be the representation of $\mathcal{K}_v^\times$ in U defined by right translation:

$$\Omega(g)f(h)=f(hg) \qquad \text{for } f\in U\,.$$

Ω is the direct sum of d copies of π_v, if $d=\dim\pi_v$.

For $M\in\mathcal{S}_1(\mathcal{K}_v)$, $f\in U$, $g\in\mathcal{K}_v^\times$ and $x\in\mathcal{K}_v$, we put

$$\varphi_{M,f}(g,x)=\int_{\mathcal{K}_v^1}f(gg_1)M(xg_1)dg_1\,.$$

Since f, M are locally constant, this integral is in substance a finite sum and $\varphi_{M,f}(g,x)\in U$ for a fixed x. Furthermore, we have $\varphi_{M,f}(gg_1,x)=\varphi_{M,f}(g,xg_1{}^{-1})$ or $\Omega(g_1)\varphi_{M,f}(g,x)=\varphi_{M,f}(g,xg_1{}^{-1})$ for $g_1\in\mathcal{K}_v{}^1$. Hence $\varphi_{M,f}(g,x)$ is (as a U-valued function of x) an element of $\mathcal{S}(\mathcal{K}_v,\Omega)$ in the notation in § 3, No. 2. If we write r_Ω for the Weil representation of $GL_2(F_v)$ in $\mathcal{S}(\mathcal{K}_v,\Omega)$, we get

$$r_\Omega(s)\varphi_{M,f}=|\det s|_{F_v}\varphi_{\rho(h)r(s_1)M,\Omega(h)f}\,,$$

where s_1 and h are the same as before. Denote by $\mathcal{V}_1$ the space spanned by $r_\Omega(s)\varphi_{M,\chi}$ for all $s\in GL_2(F_v)$ and $M\in\mathcal{S}_1(\mathcal{K}_v)$. Let L be the linear map of $\mathcal{S}(\mathcal{K}_v,\Omega)$ into $\boldsymbol{C}$ defined by

$$L(\varphi)=(\varphi(1))(1)$$

for $\varphi\in\mathcal{S}(\mathcal{K}_v,\Omega)$ (this is the value of the function $\varphi(1)(g)$ in U at $g=1$). Let $\mathcal{V}_2$ be the space of all $\varphi\in\mathcal{S}(\mathcal{K}_v,\Omega)$ such that $L(r_\Omega(s)\varphi)=0$ for all $s\in GL_2(F_v)$. Clearly $\mathcal{V}_2$ is $GL_2(F_v)$-invariant. We see at once that

$$L(r_\Omega(s)\varphi_{M,\chi})=W_M(s)\,.$$

It follows that the space $\mathcal{W}_v{}^*$ coincides with the space of $L(r_\Omega(s)\varphi)$ for all $\varphi\in\mathcal{V}_1$ and that the representation ρ_v of $GL_2(F_v)$ in $\mathcal{W}_v{}^*$ is equivalent to the representation of $GL_2(F_v)$ in $\mathcal{V}_1/\mathcal{V}_1\cap\mathcal{V}_2$ induced by r_Ω. Consequently, ρ_v is the direct sum of representations equivalent to π_v^* (cf. § 3, No. 2). By the uniqueness of the Whittaker space ([5, Th. 2.14]) we see that $\mathcal{W}_v^*$ is irreducible and ρ_v is equivalent to π_v^*.

Here we prove a lemma which will be used in § 6. $\mathfrak{p}$ denotes a prime ideal in $\mathfrak{o}_v$ and $\mathfrak{O}_v$ a maximal order in $\mathcal{K}_v$.

LEMMA 10. *Assume that the restriction of π_v to K_v contains the identity representation. Then π_v is of the form $\chi_v\circ n$, χ_v being an unramified quasi-character of $F_v^\times$. Let $\mathfrak{a}_v$ be the conductor of ψ_v. If M is the characteristic function of the two-sided $\mathfrak{O}_v$-ideal L_v of norm $\mathfrak{a}_v$, then $\rho(s)W_M=W_M$ for all $s=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ in $GL_2(\mathfrak{o}_v)$ such that $\gamma\in\mathfrak{p}$.*

PROOF. Since K_v is a normal subgroup of $\mathcal{K}_v^\times$ and $\mathcal{K}_v^\times/K_v$ is abelian, π_v must be one-dimensional. Therefore we can write $\pi_v=\chi_v\circ n$. That χ_v is unramified is obvious. Under the assumption of the lemma we get $W_M(s)=r_\Omega(s)M(1)$ with $\Omega=\pi_v$. By definition we have $r_\Omega(s)M=M$ if $s=\begin{pmatrix}\alpha & \beta\\ 0 & \delta\end{pmatrix}$ is in $GL_2(\mathfrak{o}_v)$. It is well known that the set of all $x\in\mathcal{K}_v$ such that $\mathrm{tr}\,(xL_v)\subset\mathfrak{a}_v$ is $\mathfrak{a}_vL_v^{-1}\mathfrak{P}^{-1}=L_v\mathfrak{P}^{-1}$, $\mathfrak{P}$ being a prime ideal of $\mathfrak{O}_v$. Hence the Fourier transform M' of M is a constant multiple of the characteristic function of $L_v\mathfrak{P}^{-1}$. Then $r_\Omega(w)M=M'$ is invariant under $r_\Omega\left(\begin{pmatrix}1 & -\gamma\\ 0 & 1\end{pmatrix}\right)$ for all $\gamma\in\mathfrak{p}$. Since the group of all elements $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ in $GL_2(\mathfrak{o}_v)$ with $\gamma\in\mathfrak{p}$ is generated by the elements of the form $\begin{pmatrix}\alpha & \beta\\ 0 & \delta\end{pmatrix}$ and $\begin{pmatrix}1 & 0\\ \gamma & 1\end{pmatrix}$, this proves our assertion.

8. In No. 8–No. 11, v is assumed to be archimedean. In this section we assume that $F_v=\boldsymbol{R}$ or $\boldsymbol{C}$, $\mathcal{K}_v=M_2(F_v)$ and π_v is infinite dimensional and of the form $\pi(\mu_1,\mu_2)$ with quasi-characters μ_1, μ_2 of $F_v^\times$.

Note that the representation π_v of $\mathcal{H}(\mathcal{K}_v^\times)$ is induced by a unitary representation (which we again denote by π_v) of $\mathcal{K}_v^\times$ in a Hilbert space $\mathcal{L}_v$. Obviously $\omega_{\mathfrak{b}_v}$ is uniquely determined by the values of

$$\int\omega_{\mathfrak{b}_v}(g)f(g)dg=\mathrm{tr}\,(E(\mathfrak{b}_v)\pi_v(f))$$

for $f\in\mathcal{H}(\mathcal{K}_v^\times)$ so that $\omega_{\mathfrak{b}_v}$ depends only on the representation of $\mathcal{H}(\mathcal{K}_v^\times)$ in the space of K_v^1-finite vectors in $\mathcal{L}_v$ (here K_v^1 is $SO_2(\boldsymbol{R})$ if $F_v=\boldsymbol{R}$ and $SU_2(\boldsymbol{C})$ if $F_v=\boldsymbol{C}$). From this we see that (5.6) is still valid in our case.

Let r_0 be the Weil representation of $SL_2(F_v)$ in $\mathcal{S}(F_v\oplus F_v)$ with respect to the character ψ_v of F_v. As in No. 2, if $M(x)=\exp(-\pi d_v|u_v|\,\mathrm{tr}\,(x\,{}^t\bar{x}))P(x)$ is in $\mathcal{S}_1(\mathcal{K}_v)$, we get

(5.18) $$W_M(s)=\mu_1(\det s)|\det s|_{F_v}^{1/2}$$

$$\int_{F_v^\times}\mu_1\mu_2^{-1}(\gamma)r_0(s)f(M)(\gamma,\gamma^{-1})d^\times\gamma,$$

for $s\in GL_2(F_v)$, where

(5.19) $$f(M)(\alpha_1,\alpha_2)=\exp(-\pi d_v|u_v|(\alpha_1\bar{\alpha}_1+\alpha_2\bar{\alpha}_2))P_0(\alpha_1,\alpha_2),$$

$$P_0(\alpha_1,\alpha_2)=\int_{F_v}\exp(-\pi d_v|u_v|\xi\bar{\xi})P\left(\begin{pmatrix}\alpha_1 & \xi\\ 0 & \alpha_2\end{pmatrix}\right)d\xi.$$

Clearly P_0 is a polynomial of α_1, α_2, $\bar{\alpha}_1$, $\bar{\alpha}_2$.

For $m\in\mathcal{S}(F_v\oplus F_v)$, put

$$f'(m)(\alpha_1,\alpha_2)=\int_{F_v}m(\alpha_1,\xi)\psi_v(\alpha_2\xi)d\xi.$$

If m is of the form

$$m(\alpha_1,\alpha_2)=\exp(-\pi d_v|u_v|(\alpha_1\bar{\alpha}_1+\alpha_2\bar{\alpha}_2))Q(\alpha_1,\alpha_2),$$

where Q is a polynomial of α_1, $\bar{\alpha}_1$, α_2, $\bar{\alpha}_2$, then $f'(m)$ is written as

$$f'(m)(\alpha_1,\alpha_2)=\exp(-\pi d_v|u_v|(\alpha_1\bar{\alpha}_1+\alpha_2\bar{\alpha}_2))Q'(\alpha_1,\alpha_2),$$

Q' being another polynomial of α_1, $\bar{\alpha}_1$, α_2, $\bar{\alpha}_2$. By [**5**, Prop. 1.6] we have

$$f'(r_0(s)m)(\alpha_1,\alpha_2)=f'(m)((\alpha_1,\alpha_2)s)$$

for $s\in GL_2(F_v)$. From this it follows that $f(M)$ ($M\in\mathcal{S}_1(\mathcal{K}_v)$) is $SO_2(\boldsymbol{R})$- or $SU_2(\boldsymbol{C})$-finite according as $F_v=\boldsymbol{R}$ or $\boldsymbol{C}$, if each group is made to act on $f(M)$ through r_0. Hence W_M belongs to the space $W(\mu_1,\mu_2;\psi_v)$, which is the Whittaker space of $\pi(\mu_1,\mu_2)$ (cf. [**5**, Th. 5.13] for $F_v=\boldsymbol{R}$ and [**5**, Th. 6.3] for $F_v=\boldsymbol{C}$).

LEMMA 11. *Let $\mathfrak{g}_1$ be the Lie algebra of $SL_2(F_v)$. Then, $\mathcal{S}_1(\mathcal{K}_v)$ is invariant under $r(X)$ for all $X\in\mathfrak{g}_1$.*

PROOF. Since $r(s)$ commutes with $\rho(k_1)$ and $\lambda(k_1)$ ($k_1\in K_v^1$), it is sufficient to show that the space $\mathcal{M}$ of all functions of the form $M(x)=\exp(-\pi d_v|u_v|\operatorname{tr}(x\,{}^t\bar{x}))P(x)$, P being an arbitrary polynomial of ξ_{ij}, $\bar{\xi}_{ij}$, is invariant under $r(X)$. Assume first that $F_v=\boldsymbol{R}$. $\mathfrak{g}_1=\mathfrak{sl}_2(\boldsymbol{R})$ is spanned by $X_1=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, $X_2=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$, $X_3=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$. M being as above, we have

$$\begin{aligned}r(X_1)M(x)&=[(d/d\alpha)r(\exp\alpha X_1)M(x)]_{\alpha=0}\\&=[(d/d\alpha)(e^\alpha M(e^\alpha x))]_{\alpha=0},\\ r(X_2)M(x)&=[(d/d\alpha)r(\exp\alpha X_2)M(x)]_{\alpha=0}\\&=[(d/d\alpha)(\psi_v(\alpha n(x))M(x))]_{\alpha=0}.\end{aligned}$$

A direct calculation shows that $r(X_1)M$ and $r(X_2)M$ are again in $\mathcal{M}$. Since

$\mathrm{Ad}(w)X_2=-X_3$, we have only to show that $\mathcal{M}$ is invariant under $r(w)$ or that $\mathcal{M}$ is invariant under the Fourier transformation $M\to M'$. This is easy to prove. The proof is the same in case $F_v=\boldsymbol{C}$, q. e. d.

Let $\mathfrak{g}$ be the Lie algebra of $GL_2(F_v)$. By Lemma 11, (5.4) and (5.5) the space of all W_M $(M\in\mathcal{S}_1(\mathcal{K}_v))$ is invariant under $\rho(X)$ for $X\in\mathfrak{g}$. If $F_v=\boldsymbol{R}$, $\mathcal{W}_v{}^*$ is spanned by all W_M $(M\in\mathcal{S}_1(\mathcal{K}_v))$ and their right translates by $\varepsilon=\begin{pmatrix}-1&0\\0&1\end{pmatrix}$. It is obviously invariant under $\rho(\varepsilon)$ and $\rho(X)$ for $X\in\mathfrak{g}$. If $F_v=\boldsymbol{C}$, $\mathcal{W}_v{}^*$ is the space of all W_M $(M\in\mathcal{S}_1(\mathcal{K}_v))$. In either case $\mathcal{W}_v{}^*$ is invariant under $\rho(f)$ for all $f\in\mathcal{H}(\mathcal{K}_v^\times)$ (cf. [**5**, Lemma 5.4]) so that $\mathcal{W}_v{}^*=W(\mu_1,\mu_2;\psi_v)$. Hence ρ_v is equivalent to $\pi(\mu_1,\mu_2)$.

9. Let the assumptions be the same as in No. 8 except that $\pi_v=\pi(\mu_1,\mu_2)$ is now finite dimensional. Since π_v is induced by a unitary representation, π_v is necessarily one-dimensional. Consequently we may assume that $\mu_1\mu_2^{-1}=|\ |_{F_v}^{-1}$, and putting $\chi(\alpha)=\mu_2(\alpha)|\alpha|_{F_v}^{-1/2}$, we get $\pi_v=\chi\circ n$. $\mathfrak{d}_v$ is the identity representation and $\omega_{\mathfrak{d}_v}(g)=\chi(n(g))$. We see that (5.18) is still valid for $M\in\mathcal{S}_1(\mathcal{K}_v)$. As in No. 8, we infer that $\mathcal{W}_v{}^*$ is an invariant subspace of $W(\mu_1,\mu_2;\psi_v)$. Hence ρ_v is admissible.

10. We assume that $F_v=\boldsymbol{R}$, $\mathcal{K}_v=M_2(\boldsymbol{R})$ and $\pi_v=\sigma(\mu_1,\mu_2)$ with quasi-characters μ_1,μ_2 of $\boldsymbol{R}^\times$ such that $\mu_1\mu_2^{-1}(\alpha)=\alpha^p(\operatorname{sgn}\alpha)$ for a positive integer p. In this case $K_v{}^1=SO_2(\boldsymbol{R})$. Write $\mathfrak{d}_v$ as

$$\mathfrak{d}_v\left(\begin{pmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{pmatrix}\right)=e^{in\theta}.$$

By [**5**, Th. 5.11] $\mathfrak{d}_v$ is contained in the restriction of $\sigma(\mu_1,\mu_2)$ if and only if $n\geqq p+1$ or $n\leqq -p-1$ and $n\equiv p+1 \pmod 2$. If this condition is satisfied, $\mathfrak{d}_v$ is not contained in the restriction to $K_v{}^1$ of the representation of $\mathcal{H}(\mathcal{K}_v^\times)$ in $\mathcal{B}(\mu_1,\mu_2)/\mathcal{B}_s(\mu_1,\mu_2)$. It follows that (5.6) is still valid in this case. Hence we obtain again (5.18).

By [**5**, Cor. 5.14] W_M is in the Whittaker space $W(\sigma(\mu_1,\mu_2);\psi_v)$ of $\sigma(\mu_1,\mu_2)$ if and only if

$$\int_{-\infty}^{\infty}\alpha_1{}^i\frac{\partial^j}{\partial\alpha_2{}^j}f(M)(\alpha_1,0)d\alpha_1=0 \tag{5.20}$$

for all (i,j) such that $i+j=p-1$, $i\geqq 0$, $j\geqq 0$.

We now prove (5.20) for $M\in\mathcal{S}_1(\mathcal{K}_v)$. Differentiating (5.19) by α_2, we get

$$\frac{\partial^j}{\partial\alpha_2{}^j}f(M)(\alpha_1,0)=\sum_{k=0}^{j}C_{jk}\int_{-\infty}^{\infty}\exp(-\pi|u_v|(\alpha_1{}^2+\xi^2))\frac{\partial^{j-k}}{\partial\xi_{22}{}^{j-k}}P\left(\begin{pmatrix}\alpha_1&\xi\\0&0\end{pmatrix}\right)d\xi$$

with

$$C_{jk}=\begin{cases}\binom{j}{k}(-\pi|u_v|)^{k/2}k!/(k/2)! & \text{if } k\equiv 0 \pmod 2,\\ 0 & \text{if } k\not\equiv 0 \pmod 2)\end{cases}$$

(we write $P(x)=P\left(\begin{pmatrix}\xi_{11} & \xi_{12}\\ \xi_{21} & \xi_{22}\end{pmatrix}\right)$). Since $M\in\mathcal{S}_1(\mathcal{K}_v)$, we have

$$P(x)=1/2\pi\int_0^{2\pi}P(xk_1)e^{in\theta}d\theta$$

so that

$$\frac{\partial^{j-k}}{\partial\xi_{22}{}^{j-k}}P(x)=1/2\pi\int_0^{2\pi}\left(-\sin\theta\frac{\partial}{\partial\xi_{21}}+\cos\theta\frac{\partial}{\partial\xi_{22}}\right)^{j-k}P(xk_1)e^{in\theta}d\theta .$$

Putting $x=\begin{pmatrix}\alpha_1 & \xi\\ 0 & 0\end{pmatrix}$ and $(\alpha_1, \xi)k_1=(\alpha, \beta)$, we can write

$$\int_{-\infty}^{\infty}\alpha_1{}^i\frac{\partial^j}{\partial\alpha_2{}^j}f(M)(\alpha_1, 0)d\alpha_1$$

$$=1/2\pi\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{2\pi}\exp(-\pi|u_v|(\alpha^2+\beta^2))\,(\alpha\cos\theta+\beta\sin\theta)^i$$

$$\sum_k C_{jk}\left(-\sin\theta\frac{\partial}{\partial\xi_{21}}+\cos\theta\frac{\partial}{\partial\xi_{22}}\right)^{j-k}P\left(\begin{pmatrix}\alpha & \beta\\ 0 & 0\end{pmatrix}\right)e^{in\theta}d\theta d\alpha d\beta .$$

This is a linear combination of the integrals of the form

$$\int_0^{2\pi}\cos^l\theta\sin^m\theta\, e^{in\theta}d\theta$$

with $l\geqq 0$, $m\geqq 0$, $l+m\leqq i+j=p-1$. It is easy to see that these integrals all vanish if $|n|\geqq p+1$. Hence we get (5.20).

We infer, as in No. 8, that $\mathcal{W}_v{}^*$ is the Whittaker space of $\sigma(\mu_1, \mu_2)$ and ρ_v is equivalent to $\sigma(\mu_1, \mu_2)$.

11. We assume that $F_v=\boldsymbol{R}$ and $\mathcal{K}_v$ is a division quaternion algebra over $\boldsymbol{R}$. We use the notation in §4, No. 4, iv). Let χ be the character of π_v. Since the restriction of π_v to $K_v{}^1=\mathcal{K}_v{}^1$ is irreducible, $\mathfrak{d}_v$ is necessarily this restriction. Hence

$$\omega_{\mathfrak{d}_v}(g)=\chi(1)\int_{\mathcal{K}_v^1}\chi(k_1{}^{-1})\chi(k_1g)dk_1 . \tag{5.21}$$

Let ω be a quasi-character of $\boldsymbol{C}^\times$ defined by

$$\omega(z)=(z\bar{z})^{r-1/2}z^{n+1}$$

and $\mathcal{S}_1(\boldsymbol{C})$ the space of all functions m on $\boldsymbol{C}$ of the form

$$m(z)=\exp(-2\pi|u_v|z\bar{z})P(z, \bar{z}),$$

where $P(z, \bar{z})$ is a polynomial of z and $\bar{z}$ such that $P(zu, \bar{z}\bar{u})=\omega(u^{-1})P(z, \bar{z})$ for all $u\in\boldsymbol{C}$ with $u\bar{u}=1$. $P(z, \bar{z})$ is then written as $P(z, \bar{z})=P(z\bar{z})\bar{z}^{n+1}$, P being an arbitrary polynomial.

Let f be a linear mapping of $\mathcal{S}_1(\mathcal{K}_v)$ onto $\mathcal{S}_1(\boldsymbol{C})$ defined by

$$f(M)(z)=\exp(-2\pi|u_v|z\bar{z})P(z\bar{z})\bar{z}^{n+1}$$

for $M(x)=\exp(-2\pi|u_v|n(x))P(n(x))\chi_n(x')$. The following lemma can be easily proved by using [5, Lemma 5.20.1].

LEMMA 12. *Let r_ω be the Weil representation of $GL_2(\boldsymbol{R})_+$ in $\mathcal{S}(\boldsymbol{C},\omega)$ with respect to the additive character $\psi_v(\mathrm{tr}_{\boldsymbol{C}/\boldsymbol{R}}(z))$ of $\boldsymbol{C}$. Then we have*

$$r_\omega(s)f(M)=f(r(s)M)$$

for all $M\in\mathcal{S}_1(\mathcal{K}_v)$ and $s\in SL_2(\boldsymbol{R})$.

Let s be in $GL_2(\boldsymbol{R})_+$ and h (resp. a) an element in $\mathcal{K}_v^\times$ (resp. $\boldsymbol{C}^\times$) such that $\det s=n(h)=a\bar{a}$. Write $s=\begin{pmatrix}\det s & 0\\ 0 & 1\end{pmatrix}s_1$. By (5.21) and Lemma 12 we get

$$\begin{aligned}W_M(s)&=|\det s|_{\boldsymbol{R}}\chi(1)\int_{v^1}\int_{\mathcal{K}_v^1}\chi(k_1^{-1})\chi(k_1g_1h)r(s_1)M(g_1h)dk_1dg_1\\&=|\det s|_{\boldsymbol{R}}^{1/2}\chi(1)r_\omega(s_1)f(M)(a)\omega(a)\\&\qquad\iint\chi(k_1^{-1})\chi(k_1g_1h)\chi(h^{-1}g_1^{-1})dk_1dg_1\\&=|\det s|_{\boldsymbol{R}}^{1/2}\omega(a)r_\omega(s_1)f(M)(a)\\&=r_\omega(s)f(M)(1)\,.\end{aligned}$$

It is proved in [5, Lemma 5.12, Th. 5.13] that the functions $r_\omega(s)m(1)$ $(m\in\mathcal{S}_1(C))$ and their right translates by $\varepsilon=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$ generate the Whittaker space of $\pi_v^*=\sigma(\mu_1,\mu_2)$ for

$$\mu_1(\alpha)=|\alpha|_{\boldsymbol{R}}^{r+n+1/2}\,,\qquad \mu_2(\alpha)=|\alpha|_{\boldsymbol{R}}^{r-1/2}(\mathrm{sgn}\,\alpha)^n\,.$$

It implies that $\mathcal{W}_v^*$ coincides with this Whittaker space and that ρ_v is equivalent to π_v^*.

For a later application we remark the following. Put $M(x)=\exp(-2\pi|u_v|n(x))\chi_n(x')$. Let $\mathfrak{g}$ be the Lie algebra of $GL_2(\boldsymbol{R})$ (identified with $M_2(\boldsymbol{R})$). We regard $U=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$, $V_+=\begin{pmatrix}1 & i\\ i & -1\end{pmatrix}$, $V_-=\begin{pmatrix}1 & -i\\ -i & -1\end{pmatrix}$ as elements in $\mathfrak{g}\otimes_{\boldsymbol{R}}\boldsymbol{C}$. For an integer $p\geqq 0$, put

$$\varphi_{n+2p+2}=\begin{cases}\rho(V_+)^pW_M & \text{if } u_v>0\,,\\ \rho(V_+)^p\rho\left(\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\right)W_M & \text{if } u_v<0\,,\end{cases}$$

$$\varphi_{-n-2p-2}=\begin{cases}\rho(V_-)^p\rho\left(\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\right)W_M & \text{if } u_v>0\,,\\ \rho(V_-)^pW_M & \text{if } u_v<0\,.\end{cases}$$

Then, these functions form a basis of $\mathcal{W}_v^*$ and

$$\rho(U)\varphi_m = im\,\varphi_m \qquad \text{for } m=\pm(n+2),\ \pm(n+4), \cdots$$

Thus we have seen that, in all cases, the assertion of Proposition 3 is true.

12. PROPOSITION 4. *The notation being the same as in Proposition* 3, *assume that π is not one-dimensional. Then, π_v is infinite dimensional for all $v \notin S$.*

PROOF. Assume that π_v is finite dimensional (hence one-dimensional) for a place $v \notin S$. We use the notation in No. 3 or No. 9. The only proper invariant subspace of $W(\mu_1, \mu_2; \psi_v)$ is the Whittaker space $W(\sigma(\mu_1, \mu_2); \psi_v)$ of $\sigma(\mu_1, \mu_2)$. We shall show that $\mathcal{W}_v{}^*$ is not contained in this subspace so that $\mathcal{W}_v{}^* = W(\mu_1, \mu_2; \psi_v)$.

Let v be non-archimedean. By (5.8) and [**5**, Prop. 3.4, Prop. 3.6] W_M ($M \in \mathcal{S}_1(\mathcal{K}_v)$) is in $W(\sigma(\mu_1, \mu_2); \psi_v)$ if and only if

$$\int f(M)(0, \xi)d\xi = \iint M\left(\begin{pmatrix} 0 & \xi' \\ 0 & \xi \end{pmatrix}\right) d\xi d\xi' = 0. \tag{5.22}$$

The characteristic function $M_v{}^0$ of $\mathfrak{O}_v$ is in $\mathcal{S}_1(\mathcal{K}_v)$, but does not satisfy this condition.

Let v be archimedean. By [**5**, Cor. 5.14] and its analogue in the case of $\boldsymbol{C}$, we see that (5.22) is a necessary and sufficient condition for $W_M \in W(\sigma(\mu_1, \mu_2); \psi_v)$. The function $M(x) = \exp(-\pi d_v |u_v| \mathrm{tr}(x\,{}^t\bar{x}))$ is contained in $\mathcal{S}_1(\mathcal{K}_v)$, but does not satisfy this condition.

From this it follows that $\mathcal{W}_v{}^*$ has a one-dimensional constituent. Put $\mathcal{U}_v = W(\sigma(\mu_1, \mu_2); \psi_v)$ and let $\mathcal{U}$ be the restricted tensor product of $\mathcal{W}_{v'}{}^*$ ($v' \neq v$) and $\mathcal{U}_v$. $\mathcal{W}^*/\mathcal{U}$ is isomorphic to the restricted tensor product of $\mathcal{W}_{v'}{}^*$ ($v' \neq v$) and a one-dimensional space $\mathcal{W}_v{}^*/\mathcal{U}_v$.

On the other hand, by Proposition 2, $\mathcal{V}^*$ is an invariant subspace of $\mathcal{A}_0(\eta, GL_2(\boldsymbol{A}))$ so that $\mathcal{V}^*$ is a direct sum of irreducible subspaces. Since the representations of $\mathcal{H}(GL_2(\boldsymbol{A}))$ in $\mathcal{W}^*$ and $\mathcal{V}^*$ are equivalent, the representation of $\mathcal{H}(GL_2(\boldsymbol{A}))$ in $\mathcal{W}^*/\mathcal{U}$ is a direct sum of irreducible representations, each of which is equivalent to a constituent of $\mathcal{A}_0(\eta, GL_2(\boldsymbol{A}))$. Let $\sigma = \otimes \sigma_{v'}$ be any one of them. From what we have seen, σ_v must be one-dimensional. This is impossible (cf. [**5**, pp. 353-354]). q. e. d.

We resume Proposition 3 and Proposition 4 as follows.

THEOREM 1. *Let $\mathcal{K}$ be a division quaternion algebra over a global field F and S the set of all places in F ramified in $\mathcal{K}$. Let $\pi = \otimes \pi_v$ be an irreducible constituent of the representation ρ of $\mathcal{H}(\mathcal{K}_A^\times)$ in $\mathcal{A}(\eta, \mathcal{K}_A^\times)$, η being a character of $A^\times/F^\times$. For $v \in S$, let $\pi_v{}^*$ be as in* § 2, *No.* 2 *and define an admissible representation π^* of $\mathcal{H}(GL_2(\boldsymbol{A}))$ by*

$$\pi^* = \bigotimes_{v \notin S} \pi_v \bigotimes_{v \in S} \pi_v{}^*.$$

In the same notation as in § 4, *No.* 1, *let $\mathcal{V}^*$ be the space spanned by all*

$\rho\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\phi_M$ *for* $M \in \mathcal{S}_1(\mathcal{K}_A)$ *and* $a \in E$, E *being a representative system of* $A^\times/(A^\times)^2$.

Assume that π *is not one-dimensional. Then* $\mathcal{V}^*$ *is an invariant subspace of* $\mathcal{A}_0(\eta, GL_2(A))$ *and the representation of* $\mathcal{H}(GL_2(A))$ *in* $\mathcal{V}^*$ *is equivalent to* π^*.

§ 6. An application to the holomorphic automorphic forms.

1. In this section, F is a totally real algebraic number field. We denote by $\mathfrak{p}$ non-archimedean places in F and by v (exclusively) archimedean places in F. Also we write A_∞ (resp. A_f) the archimedean (resp. non-archimedean) part of A.

Let $\mathfrak{g}_v$ be the Lie algebra of $GL_2(F_v)$ and $\mathfrak{U}_v$ the universal enveloping algebra of $\mathfrak{g}_v \otimes_R C$. The universal enveloping algebra $\mathfrak{U}$ of $GL_2(A_\infty)$ is identified with $\bigotimes_v \mathfrak{U}_v$. In the notation in § 5, No. 11, regard

$$D = (1/4)(V_+V_- + V_-V_+) - (1/2)U^2$$

as an element in $\mathfrak{U}_v$. Put

$$D_v = \bigotimes_{v'} X_{v'} \qquad \text{with } X_{v'} = \begin{cases} D & \text{if } v' = v\,, \\ 1 & \text{if } v' \neq v\,. \end{cases}$$

For an integer m, let σ_m be the representation of $SO_2(\boldsymbol{R})$ defined by

$$\sigma_m\left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) = e^{im\theta}\,.$$

Let m_v be an integer $\geqq 2$ and $\mathfrak{n}$ an integral $\mathfrak{o}$-ideal. We denote by $U_\mathfrak{p}(\mathfrak{n})$ the group of all $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ in $GL_2(\mathfrak{o}_\mathfrak{p})$ such that $\gamma \equiv 0 \pmod{\mathfrak{n}\mathfrak{o}_\mathfrak{p}}$. Let $\mathcal{A}_0(GL_2(A))$ be the space of all continuous functions φ on $GL_2(F)\backslash GL_2(A)$ satisfying the conditions (3.1), (3.2), (3.4) and (C) in § 3. Consider the space H of all φ in $\mathcal{A}_0(GL_2(A))$ satisfying the following conditions.

$$\rho(D_v)\varphi = (1/2)((m_v-1)^2-1)\varphi\,,$$
$$\rho(k)\varphi = \prod \sigma_{m_v}(k_v)\varphi \qquad \text{for } k \in \prod U_\mathfrak{p}(\mathfrak{n}) \prod SO_2(F_v)\,,$$
$$\rho(z)\varphi = \prod (\operatorname{sgn} z_v)^{m_v}\varphi \qquad \text{for } z \in A_\infty^\times\,.$$

Evidently H is invariant under $\rho(z)$ for $z \in A^\times$, and ρ defines a representation ρ_A of $A^\times$ in H trivial on $F^\times(\prod \mathfrak{o}_\mathfrak{p}^\times)(A_\infty^\times)^0$, $(A_\infty^\times)^0$ being the group of all $z \in A_\infty^\times$ such that $z_v > 0$. Consequently ρ_A is actually a representation of a finite quotient group of $A^\times$ so that it is a direct sum of one-dimensional representations. Let Y be the set of all quasi-characters η of $A^\times/F^\times$ such that $\eta_v(\alpha) = (\operatorname{sgn}\alpha)^{m_v}$ and $\eta_\mathfrak{p}$ is unramified. It follows from the above argument that H is contained in $\mathscr{B}$, where $\mathscr{B}$ is the sum of the spaces $\mathscr{A}_0(\eta, GL_2(\boldsymbol{A}))$ for $\eta \in Y$.

Let φ be a non-zero element in H. Write $\varphi=\sum \varphi_i$, $\varphi_i \neq 0$, $\varphi_i \in \mathcal{V}_i$, $\mathcal{V}_i$ being a certain irreducible subspace of $\mathscr{B}$. It is immediate to see that, if $\pi=\otimes \pi_{\mathfrak{p}} \otimes \pi_v$ is the representation of $\mathscr{H}(GL_2(\boldsymbol{A}))$ in any one of $\mathcal{V}_i$, π has to satisfy the following conditions.

(6.1) π_v is equivalent to $\sigma(\mu_1, \mu_2)$, where μ_1 and μ_2 are quasi-characters of $F_v^\times$ such that $\mu_1(\alpha)=|\alpha|^{(m_v-1)/2}$, $\mu_2(\alpha)=|\alpha|^{-(m_v-1)/2}(\operatorname{sgn}\alpha)^{m_v}$.

(6.2) The restriction of $\pi_{\mathfrak{p}}$ to $U_{\mathfrak{p}}(\mathfrak{n})$ contains the identity representation.

2. LEMMA 13. *Let μ_1, μ_2 be quasi-characters of $F_{\mathfrak{p}}^\times$. Assume that $\pi_{\mathfrak{p}}$ is infinite dimensional and of the form $\pi(\mu_1, \mu_2)$ or $\sigma(\mu_1, \mu_2)$. Then, the restriction of $\pi_{\mathfrak{p}}$ to $U_{\mathfrak{p}}(\mathfrak{p})$ contains the identity representation if and only if μ_1, μ_2 are unramified. Suppose this condition is satisfied. If $\pi_{\mathfrak{p}}=\pi(\mu_1, \mu_2)$, the space of $U_{\mathfrak{p}}(\mathfrak{p})$-invariant vectors is spanned by two linearly independent vectors φ_1, φ_2, where φ_1 is $GL_2(\mathfrak{o}_{\mathfrak{p}})$-invariant and $\varphi_2=\pi_{\mathfrak{p}}\left(\begin{pmatrix}1 & 0\\ 0 & \varpi\end{pmatrix}\right)\varphi_1$. If $\pi_{\mathfrak{p}}=\sigma(\mu_1, \mu_2)$, the space of $U_{\mathfrak{p}}(\mathfrak{p})$-invariant vectors is of dimension* 1.

PROOF. First consider the case $\pi_{\mathfrak{p}}=\pi(\mu_1, \mu_2)$ and let $\pi_{\mathfrak{p}}$ act on the space $\mathscr{B}(\mu_1, \mu_2)$ (§ 1, No. 3). Let φ be a $U_{\mathfrak{p}}(\mathfrak{p})$-invariant function in $\mathscr{B}(\mu_1, \mu_2)$. Since $(T \cap GL_2(\mathfrak{o}_{\mathfrak{p}}))\backslash GL_2(\mathfrak{o}_{\mathfrak{p}})/U_{\mathfrak{p}}(\mathfrak{p})$ is represented by two elements $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, φ is determined by its values at these elements. If $\varphi \neq 0$, at least one of these two values is not 0. On the other hand, if α, $\delta \in \mathfrak{o}_{\mathfrak{p}}^\times$, we have

$$\varphi\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)=\varphi\left(\begin{pmatrix}\alpha & 0\\ 0 & \delta\end{pmatrix}\right)=\mu_1(\alpha)\mu_2(\delta)\varphi\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right),$$

$$\varphi\left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right)=\varphi\left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\alpha & 0\\ 0 & \delta\end{pmatrix}\right)=\mu_1(\delta)\mu_2(\alpha)\varphi\left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right).$$

Therefore μ_1, μ_2 must be trivial on $\mathfrak{o}_{\mathfrak{p}}^\times$. Assuming this is the case, let φ_1 be an element in $\mathscr{B}(\mu_1, \mu_2)$ such that $\varphi_1(u)=1$ for all $u \in GL_2(\mathfrak{o}_{\mathfrak{p}})$. Then $\varphi_2 =\rho\left(\begin{pmatrix}1 & 0\\ 0 & \varpi\end{pmatrix}\right)\varphi_1$ is $U_{\mathfrak{p}}(\mathfrak{p})$-invariant. Since

$$\varphi_2\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)=\mu_2(\varpi)|\varpi|_{F_{\mathfrak{p}}}^{-1/2}, \qquad \varphi_2\left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right)=\mu_1(\varpi)|\varpi|_{F_{\mathfrak{p}}}^{1/2},$$

φ_1 and φ_2 are linearly independent.

Next assume that $\mu_1\mu_2^{-1}=|\ |_{F_{\mathfrak{p}}}$ and $\pi_{\mathfrak{p}}=\sigma(\mu_1, \mu_2)$ acts on the space $\mathscr{B}_s(\mu_1, \mu_2)$. As is seen above, if $\mathscr{B}_s(\mu_1, \mu_2)$ contains a non-zero $U_{\mathfrak{p}}(\mathfrak{p})$-invariant vector, μ_1 and μ_2 must be trivial on $\mathfrak{o}_{\mathfrak{p}}^\times$. In this case, a function $\varphi \in \mathscr{B}(\mu_1, \mu_2)$ is in $\mathscr{B}_s(\mu_1, \mu_2)$ if and only if

(6.3) $$\int_{GL_2(\mathfrak{o}_{\mathfrak{p}})}\varphi(k)dk=0.$$

If φ is $U_{\mathfrak{p}}(\mathfrak{p})$-invariant, an easy calculation shows that (6.3) is reduced to

$$\varphi\left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right)+|\varpi|_{F_{\mathfrak{p}}}\varphi\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)=0.$$

Hence there is exactly one $U_{\mathfrak{p}}(\mathfrak{p})$-invariant function φ satisfying (6.3). This proves our assertion.

LEMMA 14. *Let $\pi_{\mathfrak{p}}$ be absolutely cuspidal. Then the restriction of $\pi_{\mathfrak{p}}$ to $U_{\mathfrak{p}}(\mathfrak{p})\cap K_{\mathfrak{p}}^1$ does not contain the identity representation.*

PROOF. The notation being the same as in § 5, No. 5, take for Ψ a character of $F_{\mathfrak{p}}$ whose conductor is $\mathfrak{o}_{\mathfrak{p}}$. Let φ be a $(U_{\mathfrak{p}}(\mathfrak{p})\cap K_{\mathfrak{p}}^1)$-invariant function in $\mathcal{S}(F_{\mathfrak{p}}^{\times})$. Since $\varphi(\xi)=\pi_{\mathfrak{p}}\left(\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}\right)\varphi(\xi)=\Psi(\beta\xi)\varphi(\xi)$ for all $\beta\in\mathfrak{o}_{\mathfrak{p}}$, the support of φ is contained in $\mathfrak{o}_{\mathfrak{p}}$. Putting $\pi_{\mathfrak{p}}(w)\varphi=\varphi'$, we get $\pi_{\mathfrak{p}}\left(\begin{pmatrix}1 & -\gamma\\ 0 & 1\end{pmatrix}\right)\varphi'$ $=\varphi'$ for all $\gamma\in\mathfrak{p}$, for $w\begin{pmatrix}1 & 0\\ \gamma & 1\end{pmatrix}=\begin{pmatrix}1 & -\gamma\\ 0 & 1\end{pmatrix}w$. Consequently, the support of φ' is contained in $\mathfrak{p}^{-1}$. By (5.12) we have

$$\hat{\varphi}'(\mu)=C(\mu)\hat{\varphi}(\mu^{-1}\eta_{\mathfrak{p}}^{-1}) \tag{6.4}$$

for all characters μ of $F_{\mathfrak{p}}^{\times}$. Write $\mu(\varepsilon\varpi^n)=\nu(\varepsilon)t^n$, where ν is a character of $\mathfrak{o}_{\mathfrak{p}}^{\times}$ and t is a complex number of absolute value 1. Then we have

$$\hat{\varphi}'(\mu)=\sum_{n=-1}^{\infty}t^n\int_{\mathfrak{o}_{\mathfrak{p}}^{\times}}\varphi'(\varpi^n\varepsilon)\nu(\varepsilon)d\varepsilon,$$

$$\hat{\varphi}(\mu^{-1}\eta_{\mathfrak{p}}^{-1})=\sum_{n=0}^{\infty}t^{-n}\eta_{\mathfrak{p}}(\varpi)^{-n}\int_{\mathfrak{o}_{\mathfrak{p}}^{\times}}\varphi(\varpi^n\varepsilon)\nu^{-1}\nu_0^{-1}(\varepsilon)d\varepsilon,$$

$$C(\mu)=\sum_{n=-\infty}^{-2}t^nC_n(\nu),$$

because $C_n(\nu)=0$ if $n\geqq -1$ by [**5**, Prop. 2.23]. Putting these expressions in (6.4), we get an equality which holds for all ν and t. This is possible only if $\varphi'=\varphi=0$. This proves the lemma.

3. From now on we assume that $\mathfrak{n}$ is square-free. Let $\mathcal{V}_0$ be the sum of all irreducible subspaces $\mathcal{V}$ in $\mathcal{B}$ such that the representation π of $\mathcal{H}(GL_2(\boldsymbol{A}))$ in $\mathcal{V}$ satisfies (6.1), (6.2) and $\pi_{\mathfrak{p}}$ is a special representation for all $\mathfrak{p}$ dividing $\mathfrak{n}$. Put $H_0=H\cap\mathcal{V}_0$. $\mathcal{V}$ being as above, $\mathcal{V}\cap H$ is one-dimensional (Lemma 13 and [**5**, Lemma 3.9]) so that $\dim H_0$ is the number of irreducible subspaces contained in $\mathcal{V}_0$.

Let us write for a moment $H=H(\mathfrak{n})$, $H_0=H_0(\mathfrak{n})$. Denote by $\mathfrak{p}_j$ $(j=1, 2, \cdots, \nu)$ all the prime divisors of $\mathfrak{n}$ and by ϖ_j a prime element of $\mathfrak{p}_j$. For a subset B of $A=\{1, 2, \cdots, \nu\}$, we put

$$\mathfrak{n}_B=\prod_{j\in B}\mathfrak{p}_j,\qquad p_B=\prod_{j\in B}\begin{pmatrix}1 & 0\\ 0 & \varpi_j\end{pmatrix}.$$

It follows from Lemma 13 that

$$H(\mathfrak{n})=\sum_{B\subset A}\sum_{C\subset A-B}\rho(p_C)H_0(\mathfrak{n}_B),$$

where the sum is direct. In other words, $H(\mathfrak{n})$ is the direct sum of $H_0(\mathfrak{n})$ and the space spanned by the right translates of elements in $H_0(\mathfrak{m})$, $\mathfrak{m}$ being a proper divisor of $\mathfrak{n}$. $H_0(\mathfrak{n})$ (to be precise, the intersection of $H_0(\mathfrak{n})$ and $\mathcal{A}_0(\eta, GL_2(\boldsymbol{A}))$) has been introduced in Miyake [6] as the orthogonal complement of the space $\sum_{B\subsetneq A}\sum_{C\subset A-B}\rho(p_C)H(\mathfrak{n}_B)$. We call any function in $H_0(\mathfrak{n})$ *properly of level* $\mathfrak{n}$.

4. Assuming that $\nu+[F:\boldsymbol{Q}]$ is even, let $\mathcal{K}$ be a definite quaternion algebra of discriminant $\mathfrak{n}$ over F.

Denote by $\mathcal{B}'$ the sum of $\mathcal{A}(\eta, \mathcal{K}_A^\times)$ for $\eta\in Y$. Let $\mathcal{V}'$ be an irreducible subspace in $\mathcal{B}'$ and π the representation of $\mathcal{H}(\mathcal{K}_A^\times)$ in $\mathcal{V}'$. Let $\mathcal{U}'$ be the sum of all $\mathcal{V}'$ such that

(6.5) π_v is equivalent to the representation $g\to n(g)^{-(m_v-2)/2}\rho_{m_v-2}(g)$,

(6.6) the restriction of $\pi_\mathfrak{p}$ to $K_\mathfrak{p}$ contains the identity representation.

Then it follows from Lemma 10 that, if $\mathfrak{p}$ divides $\mathfrak{n}$, we have $\pi_\mathfrak{p}=\chi_\mathfrak{p}\circ n$ with an unramified character $\chi_\mathfrak{p}$ of $F_\mathfrak{p}^\times$, and hence $\pi_\mathfrak{p}$ is trivial on $K_\mathfrak{p}$.

Denote by $\mathfrak{d}$ the irreducible representation of K^1 of the form $\otimes\mathfrak{d}_\mathfrak{p}\otimes\mathfrak{d}_v$, where $\mathfrak{d}_v$ is equivalent to ρ_{m_v-2} (we identify $K_v{}^1$ with $SU_2(\boldsymbol{C})$. cf. §4, No. 4) and $\mathfrak{d}_\mathfrak{p}$ is the identity representation. π being as above, $\mathfrak{d}$ is contained in the restriction of π to K^1 with the multiplicity 1.

Set $\mathcal{K}_\infty^\times=\prod\mathcal{K}_v^\times$, $\mathcal{K}_\infty^1=\prod\mathcal{K}_v{}^1$ and define the representation Λ of $\mathcal{K}_\infty^\times$ by

(6.7) $$\Lambda(g)=\bigotimes_v(n(g)^{-(m_v-2)/2}\rho_{m_v-2}(g_v)).$$

Let H' be the space of all φ in $\mathcal{U}'$ invariant under $\rho(k)$ for all $k\in\prod K_\mathfrak{p}$. It is easy to see that H' is the space of all functions φ on $\mathcal{K}_F^\times\backslash\mathcal{K}_A^\times$ satisfying the following conditions:

i) $\rho(k)\varphi=\varphi$ for $k\in\prod K_\mathfrak{p}$,

ii) $\varphi\to\rho(k)\varphi$ defines a representation of $\mathcal{K}_\infty^\times$ equivalent to a direct sum of Λ.

We consider the space U spanned by all matrix coefficients of Λ and the representation λ of $\mathcal{K}_\infty^\times$ in U defined by left translation. If $l=\dim\Lambda$, λ is a direct sum of l copies of Λ (since Λ is unitary). There is an isomorphism of H' onto the space of all functions φ' on $\mathcal{K}_F^\times\backslash\mathcal{K}_A^\times$ taking values in U such that

$$\varphi'(hkg)=\lambda(g^{-1})\varphi'(h)\qquad\text{for } g\in\mathcal{K}_\infty^\times,\ k\in\prod K_\mathfrak{p},\ h\in\mathcal{K}_A^\times.$$

This isomorphism is given by $\varphi\to\varphi'$, $(\varphi'(h))(g)=\varphi(hg)$.

We fix an arbitrary irreducible subspace V of U and denote by H'' the space of all φ in H' such that φ' takes its values in V.

5. Let $\mathfrak{a}_\mathfrak{p}$ be the conductor of $\psi_\mathfrak{p}$. Write $\psi_v(\alpha)=\exp(2\pi i u_v\alpha)$. Let $L_\mathfrak{p}$ denote a two-sided $\mathfrak{O}_\mathfrak{p}$-ideal of norm $\mathfrak{a}_\mathfrak{p}$ if $\mathfrak{p}$ divides $\mathfrak{n}$ ($\mathfrak{O}_\mathfrak{p}$ is the maximal order in $\mathcal{K}_\mathfrak{p}$) and $L_\mathfrak{p}=\mathfrak{o}_\mathfrak{p}e_{11}+\mathfrak{o}_\mathfrak{p}e_{12}+\mathfrak{a}_\mathfrak{p}e_{21}+\mathfrak{a}_\mathfrak{p}e_{22}$ if $\mathfrak{p}$ does not divide $\mathfrak{n}$. Put

$$M(x)=\prod M_\mathfrak{p}(x_\mathfrak{p})\prod M_v(x_v) \qquad \text{for } x\in\mathcal{K}_A\,,$$

where

$$M_\mathfrak{p}=\int_{K_\mathfrak{p}{}^1}\lambda(k_1)N_\mathfrak{p}dk_1\,,$$

$N_\mathfrak{p}$ being the characteristic function of $L_\mathfrak{p}$ ($M_\mathfrak{p}=N_\mathfrak{p}$ if $\mathfrak{a}_\mathfrak{p}=\mathfrak{o}_\mathfrak{p}$ or if $\mathfrak{p}$ divides $\mathfrak{n}$) and

$$M_v(x_v)=\exp(-2\pi|u_v|n(x_v))\chi_{m_v-2}(x_v')\,.$$

Let e be an element in $A^\times$ such that $e_v=-1$ whenever $u_v<0$ and all other components are 1.

Let φ be in H'' and g in $\mathcal{K}_A^\times$. If s is an element in $GL_2(A)$ such that $\det s=n(h)$ for $h\in\mathcal{K}_A^\times$, we denote by $\phi_{\varphi,g}(s)$ the right hand side of (4.10) in § 4, where M is the function defined just above. Extend $\phi_{\varphi,g}$ to a function on $GL_2(A)$, $GL_2(F)$-invariant on the left.

Put $\theta_{\varphi,g}(s)=\phi_{\varphi,g}\left(s\begin{pmatrix}e&0\\0&1\end{pmatrix}\right)$. Let $\{\varphi_i\}_{i=1}^n$ be a basis of H'' and $\{g_i\}_{i=1}^n$ a set of elements in $\mathcal{K}_A^\times$ such that $\det(\varphi_i(g_j))\neq 0$. Our aim is to prove the following theorem, which may be viewed as a generalization of Eichler [**1**, **2**].

THEOREM 2. *If $m_v>2$ for all v, H_0 is spanned by θ_{φ_i,g_j} $(i,j=1,\cdots,n)$.*

The proof will be given in No. 6–No. 10.

6. Let $\pi=\otimes\pi_\mathfrak{p}\otimes\pi_v$ be an irreducible constituent of $\mathcal{U}'$. Then $\pi_v{}^*$ is equivalent to $\sigma(\mu_1,\mu_2)$ defined in (6.1), and for all $\mathfrak{p}$ dividing $\mathfrak{n}$, we have $\pi_\mathfrak{p}=\chi_\mathfrak{p}\circ n$ so that $\pi_\mathfrak{p}{}^*=\sigma(\chi_\mathfrak{p}|\ |_{F_\mathfrak{p}}{}^{1/2},\chi_\mathfrak{p}|\ |_{F_\mathfrak{p}}{}^{-1/2})$ (cf. § 2, No. 2). Therefore, by Theorem 1,

$$\pi^*=\bigotimes_{\mathfrak{p}\nmid\mathfrak{n}}\pi_\mathfrak{p}\bigotimes_{\mathfrak{p}|\mathfrak{n}}\pi_\mathfrak{p}{}^*\bigotimes_v\pi_v{}^*$$

is an irreducible constituent of $\mathcal{U}_0$ if π is not one-dimensional. Denote by $\mathcal{V}^*$ the space of π^* defined in § 4, No. 4.

Let $\mathcal{U}'(\pi)$ be the sum of all irreducible subspaces $\mathcal{V}'$ in $\mathcal{U}'$ such that the representation of $\mathcal{H}(\mathcal{K}_A^\times)$ in $\mathcal{V}'$ is equivalent to π. Fix an irreducible subspace $\mathcal{V}'$ in $\mathcal{U}'(\pi)$. Take any non-zero element φ in $\mathcal{V}'\cap H'$ and an element g in $\mathcal{K}_A^\times$ such that $\varphi(g)\neq 0$. Obviously φ satisfies

$$\int_{K^1}\chi_\mathfrak{b}(k_1{}^{-1})\rho(k_1)\varphi dk_1=\varphi\,.$$

By definition we have $\theta_{\varphi,g}=\varphi(g)\rho\left(\begin{pmatrix}e&0\\0&1\end{pmatrix}\right)\phi_M$, ϕ_M being as in (4.10), and hence $\hat{\theta}_{\varphi,g}(1, s)=\varphi(g)\rho\left(\begin{pmatrix} e & 0 \\ 0 & 1\end{pmatrix}\right)W_M(s)$ (§ 4, No. 2). It follows from Lemma 7, Lemma 10 and the remark at the end of § 5, No. 10 that $\theta_{\varphi,g}$ is a non-zero element in $\mathcal{V}^* \cap H_0$ (that W_M is non-zero is clear in view of the argument in § 5). In the same notation, it is easy to see that

$$\hat{\theta}_{\Phi,g}(1, s)=\Phi(g)\rho\left(\begin{pmatrix} e & 0 \\ 0 & 1\end{pmatrix}\right)W_M(s) \qquad \text{for all } \Phi \in \mathcal{V}'(\pi) \cap H',\ g \in \mathcal{K}_A^\times .$$

Therefore, Theorem 2 follows if we prove that

(6.8) every irreducible constituent of $\mathcal{V}_0$ is equivalent to π^* for some irreducible constituent π of $\mathcal{V}'$, and

(6.9) π being as in (6.8), we have $\mathcal{V}'(\pi) \cap H'' \neq \{0\}$.

(6.8) will be a special case of [**5**, Th. 16.1] at the time its proof is completed. For the present we use [8, Prop. 4.1] instead.

7. For all $\mathfrak{p}$ prime to $\mathfrak{n}$, denote by $\mathcal{H}_\mathfrak{p}^0$ the subalgebra of $\mathcal{H}(GL_2(F_\mathfrak{p}))$ consisting of all right and left $GL_2(\mathfrak{o}_\mathfrak{p})$-invariant elements, and put

$$\mathcal{H}^0 = \bigotimes_{\mathfrak{p}\nmid\mathfrak{n}} \mathcal{H}_\mathfrak{p}^0 .$$

For all $\mathfrak{p}$ dividing $\mathfrak{n}$, let $\xi_\mathfrak{p}$ be an elementary idempotent in $\mathcal{H}(GL_2(F_\mathfrak{p}))$ such that $\xi_\mathfrak{p} * f_\mathfrak{p} = f_\mathfrak{p}$, $f_\mathfrak{p}$ being the characteristic function of $U_\mathfrak{p}(\mathfrak{n})$, and let $\xi_\mathfrak{p}'$ be the characteristic function of $K_\mathfrak{p}$. Then

$$f \longrightarrow f \otimes (\bigotimes_{\mathfrak{p}\mid\mathfrak{n}} \xi_\mathfrak{p} \bigotimes_v \bar{\sigma}_{m_v})$$

and

$$f \longrightarrow f \otimes (\bigotimes_{\mathfrak{p}\mid\mathfrak{n}} \xi_\mathfrak{p}' \bigotimes_v \bar{\chi}_{b_v})$$

define embeddings of $\mathcal{H}^0$ into $\mathcal{H}(GL_2(\boldsymbol{A}))$ and $\mathcal{H}(\mathcal{K}_A^\times)$, respectively. We identify $\mathcal{H}^0$ with the images of these embeddings. In this way $\mathcal{H}^0$ is made to act on $\mathcal{A}_0(GL_2(\boldsymbol{A}))$ as well as on $\mathcal{A}(\mathcal{K}_A^\times)$. It is obvious that H, H_0, H', H'' are invariant under $\rho(f)$ for $f \in \mathcal{H}^0$. Writing $T(f)$ (resp. $T_0(f)$, $T'(f)$, $T''(f)$) for the restriction of $\rho(f)$ to H (resp. H_0, H', H''), we obtain representations T, T_0, T', T'' of $\mathcal{H}^0$. We see immediately that T' is equivalent to the direct sum of l copies of T''.

8. H'' is isomorphic to the space $M(1, \{m_v-2\})$ defined in Shimizu [8, § 2.2] (if $\check{\varphi}(g)=\varphi(g^{-1})$, $\varphi \to \check{\varphi}'$ gives the isomorphism). Also T'' is equivalent to the representation $\mathfrak{T}$ defined in the same place, if $\mathfrak{T}$ is restricted to $\mathcal{H}^0$.

On the other hand, H is isomorphic to the space of holomorphic cusp forms introduced in Shimura [**9**]. Put $U(\mathfrak{n})=\prod_\mathfrak{p} U_\mathfrak{p}(\mathfrak{n})GL_2(\boldsymbol{A}_\infty)$ and let s_i $(i=1, \cdots, q)$ be the representatives in $GL_2(\boldsymbol{A})$ of $GL_2(F)\backslash GL_2(\boldsymbol{A})/U(\mathfrak{n})$. Put $\Gamma_i = GL_2(F) \cap s_i U(\mathfrak{n}) s_i^{-1}$. Let $\mathfrak{F}$ be the set of all $z=(z_v)$ with $z_v \in \boldsymbol{C}$, $\operatorname{Im} z_v \neq 0$.

For $s\in GL_2(\boldsymbol{A})$ we set

$$s(z)=\left(\frac{\alpha_v z_v+\beta_v}{\gamma_v z_v+\delta_v}\right),$$

$$j(s, z)=\prod_v |\det s_v|^{m_v/2}(\gamma_v z_v+\delta_v)^{-m_v} \quad \text{if } s_v=\begin{pmatrix}\alpha_v & \beta_v\\ \gamma_v & \delta_v\end{pmatrix}.$$

Let S_i be the space of all f satisfying the following conditions.

i) f is holomorphic on $\mathfrak{H}$.

ii) $f(\sigma(z))=f(z)j(\sigma, z)^{-1}$ for $\sigma\in\Gamma_i$.

iii) $\prod_v |\mathrm{Im}\, z_v|^{m_v/2}|f(z)|$ is bounded on $\mathfrak{H}$.

Let S be the direct product of $S_1, \cdots, S_q$. We can assume that $s_i\in GL_2(\boldsymbol{A}_f)$. For $\varphi\in H$, put

$$f_i(z)=j(s, z_0)^{-1}\varphi(s_i s),$$

where $z_0=(\sqrt{-1}, \cdots, \sqrt{-1})$ and s is an element in $GL_2(\boldsymbol{A}_\infty)$ such that $s(z_0)=z$. Then $\varphi\to(f_1, \cdots, f_q)$ gives an isomorphism of H onto S. Furthermore, the representation T of $\mathcal{H}^0$ in H is equivalent to the representation $\mathfrak{T}$ defined in [**9**, § 3], if it is restricted to $\mathcal{H}^0$.

9. We assert that T_0 is equivalent to T''. It is sufficient to show that $\mathrm{tr}\, T_0(f)=\mathrm{tr}\, T''(f)$ for all $f\in\mathcal{H}^0$ (cf. [**8**, § 4.4]). In the notation in No. 3, $H_0(\mathfrak{n}_B)$ is invariant under $T(f)$ and $\rho(p_C)$ commutes with $T(f)$. Consequently we have

$$\mathrm{tr}\,(T(f)\,|\,H(\mathfrak{n}))=\sum_{B\subset A} 2^{\#(A-B)}\,\mathrm{tr}\,(T(f)\,|\,H_0(\mathfrak{n}_B)), \tag{6.10}$$

where $T(f)|H_0(\mathfrak{n}_B)$ is the restriction of $T(f)$ to $H_0(\mathfrak{n}_B)$ and $\#(A-B)$ is the number of elements in $A-B$. On the other hand, the repeated application of [**8**, Prop. 4.1] yields

$$\mathrm{tr}\, T''(f)=\sum_{B\subset A}(-2)^{\#(A-B)}\,\mathrm{tr}\,(T(f)\,|\,H(\mathfrak{n}_B)). \tag{6.11}$$

Substituting (6.10) in (6.11), we see that $\mathrm{tr}\, T''(f)=\mathrm{tr}\,(T(f)|\,H_0(\mathfrak{n}))$, as asserted.

10. LEMMA 15. *Let μ_1, μ_2 be unramified quasi-characters of $F_\mathfrak{p}^\times$ and let φ be a $GL_2(\mathfrak{o}_\mathfrak{p})$-invariant element in the representation space of $\pi(\mu_1, \mu_2)$. Then φ is an eigenfunction of $\rho(f)$ for all $f\in\mathcal{H}_\mathfrak{p}^0$. Let f_1 (resp. f_2) be the characteristic function of*

$$GL_2(\mathfrak{o}_\mathfrak{p})\begin{pmatrix}\varpi & 0\\ 0 & 1\end{pmatrix}GL_2(\mathfrak{o}_\mathfrak{p}) \;\left(\textit{resp. } \begin{pmatrix}\varpi & 0\\ 0 & \varpi\end{pmatrix}GL_2(\mathfrak{o}_\mathfrak{p})\right).$$

If $\rho(f_i)\varphi=c_i\varphi$ $(i=1, 2)$, then

$$\mu_1(\varpi)+\mu_2(\varpi)=|\varpi|_{F_\mathfrak{p}}{}^{1/2}c_1,$$

$$\mu_1(\varpi)\mu_2(\varpi)=c_2.$$

The proof is straightforward if we let $\pi(\mu_1, \mu_2)$ act on the space $\mathcal{B}(\mu_1, \mu_2)$.

Take an irreducible subspace $\mathcal{W}$ of $\mathcal{U}_0$ and let $\sigma=\otimes\sigma_\mathfrak{p}\otimes\sigma_v$ be the representation of $\mathcal{H}(GL_2(\boldsymbol{A}))$ in $\mathcal{W}$. If $\varphi\in\mathcal{W}\cap H_0$, φ is an eigenfunction of $\rho(f)$ for all $f\in\mathcal{H}^0$. Write $\rho(f)\varphi=c_f\varphi$ for $f\in\mathcal{H}^0$. Since T_0 is equivalent to T'', there exists a non-zero function φ'' in H'' such that $\rho(f)\varphi''=c_f\varphi''$ for all $f\in\mathcal{H}^0$. It follows from Lemma 15 that there exists an irreducible representation $\pi=\otimes\pi_\mathfrak{p}\otimes\pi_v$ of $\mathcal{H}(\mathcal{K}_A^\times)$ contained in $\mathcal{U}'$ such that $\pi_\mathfrak{p}$ is equivalent to $\sigma_\mathfrak{p}$ for all $\mathfrak{p}$ prime to $\mathfrak{n}$. Note that $\pi_v{}^*$ is equivalent to σ_v. By [6, Corollary of Th. 1]*), π^* is necessarily equivalent to σ. Also it is clear that φ'' is contained in $\mathcal{U}'(\pi)\cap H''$. This proves (6.8) and (6.9), and completes the proof of Theorem 2.

11. We discuss a case where the situation seems the simplest. Assume that

i) $[F:\boldsymbol{Q}]$ is even,

ii) the class number of F is 1,

iii) every totally positive unit in F is a square of a unit in F.

Furthermore, we make a particular choice of ψ. Let $\psi_{\boldsymbol{Q}}$ be an additive character of the adele of $\boldsymbol{Q}$ trivial on $\boldsymbol{Q}$ such that $\psi_{\boldsymbol{Q},\infty}(\alpha)=e^{2\pi i\alpha}$ and the conductor of $\psi_{\boldsymbol{Q},p}$ is $\boldsymbol{Z}_p$ for all rational primes p. Put $\psi(x)=\psi_{\boldsymbol{Q}}(\mathrm{tr}_{F/\boldsymbol{Q}}(x))$. It implies that $u_v=1$ and $\mathfrak{a}_\mathfrak{p}\supset\mathfrak{o}_\mathfrak{p}$.

Put $\Gamma_1=GL_2(\mathfrak{o})$ and let S_1 be as in No. 8. Let $\mathcal{K}$ be a definite quaternion algebra of discriminant $\mathfrak{o}$ over F. Fix a maximal order $\mathfrak{O}$ in $\mathcal{K}$ and define the isomorphisms $\theta_\mathfrak{p}$ of $\mathcal{K}_\mathfrak{p}$ onto $M_2(F_\mathfrak{p})$ as in § 1, No. 8 (so that $K_\mathfrak{p}=\mathfrak{O}_\mathfrak{p}^\times$). It can be shown that if p is the class number of $\mathfrak{O}$, $\mathcal{K}_F^\times\backslash\mathcal{K}_A^\times/\prod K_\mathfrak{p}\mathcal{K}_\infty^\times$ is represented by the elements $x_1,\cdots,x_p$ in $\mathcal{K}_A^1$.

Let V be as in No. 4 and $\{\omega_1,\cdots,\omega_l\}$ a basis of V. Take elements $g_1,\cdots,g_l$ in $\mathcal{K}_\infty^1$ such that $\det(\omega_\lambda(g_\mu))\neq0$. Put $M'(x)=\prod_\mathfrak{p} M_\mathfrak{p}(x_\mathfrak{p})$.

By Theorem 2 we see that, if $m_v>2$ for all v, S_1 is spanned by $f_{ij\lambda\mu}$ $(i,j=1,\cdots,p;\lambda,\mu=1,\cdots,l)$ whose restrictions to $\mathfrak{F}^0=\{z\in\mathfrak{F}\,|\,\mathrm{Im}\,z_v>0\}$ are given by

$$f_{ij\lambda\mu}(z)=\sum_{\xi\in\mathcal{K}_F}\omega_\lambda(\xi^\iota g_\mu)M'(x_j{}^{-1}\xi x_i)$$
$$\prod_v[n(\xi_v)^{(m_v-2)/2}\exp(2\pi in(\xi_v)z_v)]\,.$$

If $\mathfrak{X}_i$ is the right $\mathfrak{O}$-ideal such that $\mathfrak{X}_{i\mathfrak{p}}=x_{i\mathfrak{p}}\mathfrak{O}_\mathfrak{p}$ and if $\mathfrak{a}=\prod\mathfrak{a}_\mathfrak{p}$, then the support of $M'(x_j{}^{-1}\xi x_i)$ is contained in $\mathfrak{a}\mathfrak{X}_j\mathfrak{X}_i{}^{-1}$, and its value depends only on $\xi \bmod \mathfrak{X}_j\mathfrak{X}_i{}^{-1}$.

REMARK. Let F be an algebraic number field of finite degree and $\mathfrak{d}$ the

*) It asserts that, if σ_i $(i=1,2)$ are irreducible constituents of $\mathcal{A}_0(\eta_i,GL_2(\boldsymbol{A}))$, respectively and if σ_{1v} is equivalent to σ_{2v} for almost all v including all archimedean valuations, then σ_1 is equivalent to σ_2.

different of F. It is proved in Hecke, Vorlesungen über die Theorie der algebraischen Zahlen, Satz 176 (as to a generalization to the function field case, see J. V. Armitage, On a theorem of Hecke in number fields and function fields, Inventiones Math., 2(1967), 238–246) that there exists a $\gamma \in F^{\times}$ such that $\mathfrak{d}\gamma$ is a square of an ideal in F. ψ_{Q} being the same as in § 6, No. 11, define a character ψ of A/F by

$$\psi(x) = \psi_{Q}(\mathrm{tr}_{F/Q}(\gamma x)) .$$

Then the conductor $\mathfrak{a}_{\mathfrak{p}}$ of $\psi_{\mathfrak{p}}$ is $\mathfrak{d}_{\mathfrak{p}}^{-1}\gamma_{\mathfrak{p}}^{-1}$ and hence it is a square of an ideal in $F_{\mathfrak{p}}$. In the discussions in § 6, No. 5, we can start with this particular character ψ. In this case, however, there is an alternative and simpler way of defining $\theta_{\varphi,g}$ or of defining M (cf. § 6, No. 5). Namely, for every $\mathfrak{p}$, we may take $M_{\mathfrak{p}}$ to be the characteristic function of the two-sided $\mathfrak{O}_{\mathfrak{p}}$-ideal $L_{\mathfrak{p}}$ of norm $\mathfrak{a}_{\mathfrak{p}}$ (if $\mathfrak{a}_{\mathfrak{p}} = \mathfrak{b}_{\mathfrak{p}}^{2}$, then $L_{\mathfrak{p}} = \mathfrak{b}_{\mathfrak{p}}\mathfrak{O}_{\mathfrak{p}}$). The statement in § 6, No. 11 can be modified accordingly. The space S_1 can be spanned by $f_{ij\lambda\mu}$ $(i, j = 1, \cdots, p;$ $\lambda, \mu = 1, \cdots, l)$ whose restrictions to $\{z \in \mathfrak{F} \mid \gamma_v \operatorname{Im} z_v > 0\}$ are given by

$$f_{ij\lambda\mu}(z) = \sum_{\xi \in \mathfrak{x}_j L \mathfrak{x}_i^{-1}} \omega_{\lambda}(\xi' g_{\mu}) \times \prod_{v} \left[n(\xi_v)^{(m_v-2)/2} \exp(2\pi i \gamma_v n(\xi_v) z_v) \right] .$$

Here L is a two-sided $\mathfrak{O}$-ideal of norm $\mathfrak{a}$.

College of General Education,
University of Tokyo

References

[1] M. Eichler, Über die Darstellbarkeit von Modulformen durch Thetareihen, J. Reine Angew. Math., **195** (1955), 156–171.

[2] M. Eichler, Quadratische Formen und Modulfunctionen, Acta Arith., **4** (1958), 217–239.

[3] R. Godement, A theory of spherical functions, I, Trans. Amer. Math. Soc., **73** (1952), 496–556.

[4] R. Godement, Notes on Jacquet-Langlands' theory, Lecture note, Institute for Advanced Study, Princeton, 1970.

[5] H. Jacquet and R. P. Langlands, Automorphic forms on $GL(2)$, Lecture note in Mathematics No. 114, Springer, 1970.

[6] T. Miyake, On automorphic forms on GL_2 and Hecke operators, Ann. of Math., **94** (1971), 174–189.

[7] J. A. Shalika and S. Tanaka, On an explicit construction of a certain class of automorphic forms, Amer. J. Math., **91** (1969), 1049–1076.

[8] H. Shimizu, On zeta functions of quaternion algebras, Ann. of Math., **81** (1965), 166–193.

[9] G. Shimura, On Dirichlet series and abelian varieties attached to automorphic forms, Ann. of Math., 76 (1962), 237-294.
[10] S. Tanaka, On irreducible unitary representations of some special linear groups of the second order, I, Osaka J. Math., 3 (1966), 217-227.
[11] A. Weil, Sur certaines groupes d'opérateurs unitaires, Acta Math., 111 (1964), 143-211.