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Simply connected surgery of submanifolds in codimension two I

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§ 1. Introduction and statement of results.

In the paper we shall perform surgery on a compact locally flat m-submanifold in a compact 1-connected $(m+2)$-manifold so that a new submanifold is relatively highly connected to the ambient manifold. The technique may be regarded as a generalization of that originated by M. Kervaire ([5], Chapter III) and J. Levine [8] for studying higher dimensional knot cobordism in codimension two, and used by M. Kato [4] for embedding spheres in codimension two.

We shall work mainly in the PL category, although required results in the differentiable category may be obtained via the smoothing theory, which is now familiar, under appropriate modification, if meaningful. Thus all manifolds considered will be PL, compact and oriented, and homeomorphisms of manifolds will be PL and orientation preserving.

Our purpose of the paper is to classify certain locally flat closed m-submanifolds, called *m-knots*, in a compact 1-connected $(m+2)$-manifold up to concordance in terms of their homology classes.

Let W be a compact 1-connected $(m+2)$-manifold.

DEFINITION. By an *m-knot* in W we shall mean a locally flat closed m-submanifold M of W satisfying one of the following conditions (1) and (2):

(1) If $m=2n+1$, then $\pi_k(W, M)=0$ for $k\leqq n+1$ and the inclusion map induces an isomorphism

$$H_{n+1}(M)\cong H_{n+1}(W)\,.$$

(2) If $m=2n$, then $\pi_k(W, M)=0$ for $k\leqq n+1$.

By an *almost m-knot* in W we shall mean a closed m-submanifold M in the interior of W which is locally flat except for at a point and satisfies (1) or (2).

DEFINITION. A homology class $\mu\in H_m(W)$ is a *Poincaré class*, if the cap

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product with μ gives rise to homomorphisms $\cap\mu: H^{m-k}(W)\to H_k(W)$ satisfying one of the following conditions (1) and (2).

(1) If $m=2n+1$, then

$$\cap\mu: H^n(W)\cong H_{n+1}(W) \quad \text{and} \quad \cap\mu: H^{n+1}(W)\cong H_n(W).$$

(2) If $m=2n$, then

$$\cap\mu: H^n(W)\cong H_n(W) \quad \text{and} \quad \cap\mu: H^{n-1}(W)\longrightarrow H_{n+1}(W)$$

is surjective.

It is seen that by computation (Lemma 2.1) an m-knot in W represents a Poincaré class. We ask ourselves if the converse is true: namely, can one represent a Poincaré class $\mu\in H_m(W)$ by an m-knot or an almost m-knot in W?

THEOREM A (Existence theorem). *Let W be a compact 1-connected $(m+2)$-manifold.*

(1) *If $m=2n+1$, then any Poincaré class of $H_m(W)$ can be represented by an m-knot in W, provided that $n\geqq 1$.*

(2) *If $m=2n\geqq 6$, i.e., $m\neq 2$, 4, then any Poincaré class of $H_m(W)$ can be represented by an almost m-knot in W.*

This is a generalization of M. Kato [4].

DEFINITION. Locally flat closed m-submanifolds M and L in W are *locally flat concordant*, if there is a locally flat compact submanifold V in $W\times I$ which is homeomorphic with $M\times I$ and $\partial V=M\times 0\cup(-L)\times 1$. We shall say that M and L are *locally flat concordant modulo a spherical m-knot*, if there is a locally flat $(m+2, m)$-sphere pair (S^{m+2}, Σ), called a *spherical m-knot*, such that if one takes a relative connected sum of (W, M) and (S^{m+2}, Σ) along unknotted $(m+2, m)$-ball pairs in the interiors of them, then the result $M\#\Sigma$ is locally flat concordant to L in W.

THEOREM B (Classification theorem). *Let W be a compact 1-connected $(m+2)$-manifold.*

(1) *If $m=2n+1\geqq 5$, then m-knots in W are locally flat concordant modulo a spherical knot if and only if they represent the same homology class.*

(2) *If $m=2n\geqq 6$, then m-knots in W are locally flat concordant if and only if they represent the same homology class.*

This is an extension of Kervaire [5].

Part of our arguments works equally well in the non-simply connected case. Indeed, we perform surgery below the middle dimension without assuming 1-connectivity of manifolds involved.

Let W be a compact $(m+2)$-manifold which is an (oriented non-simply connected) Poincaré complex of dimension m with fundamental class $\mu\in H_m(W)$, (for non-simply connected Poincaré complex, see Wall [17]).

DEFINITION. A closed m-submanifold M in W is a *spine* of (W, μ), if M represents μ and the inclusion map $M \to W$ is a simple homotopy equivalence.

In order to find a locally flat spine of (W, μ), the obstruction to performing surgery in the middle dimension will be completely described by the oriented Wall's surgery obstruction, provided that $m = 2n+1 \geqq 5$. Namely, we have the following:

THEOREM C. *Let W be a compact $(m+2)$-manifold which is an (oriented non-simply connected) Poincaré complex of dimension m with fundamental class $\mu \in H_m(W)$. Suppose that $m = 2n+1 \geqq 5$. Then there is an obstruction $\theta(W, \mu)$ in the oriented surgery obstruction group $L_m(\pi_1 W)$ such that $\theta(W, \mu) = 0$ if and only if (W, μ) has a locally flat spine.*

Here is a striking example, which implies that Theorem A does not hold in case $m = 4$.

EXAMPLE D. *There is a compact 6-manifold N which is a regular neighbourhood of a 1-connected Poincaré complex of dimension 4 in S^6, but it has neither spine nor locally flat topological spine.*

In the forthcoming paper, part II, we will develop an obstruction theory to pursue a locally flat surgery in even dimensional case.

This will be done by defining an "Hermitian form" twisted by a generator of $\pi_1(W-L)$. Cf. [21], [22].

At this point, the authors would like to express their thanks to Professors William Browder and Dennis Sullivan for encouraging this work at the very beginning.

Added in proof: In the proof of the statement in Example D that N has no locally flat topological spine, we have used a result of Kirby [7] which asserts that any locally flat topological submanifold with codimension 2 has a topological normal bundle. However, later, the authors were informed that there are some gaps in the Kirby's proof. Of course, the other statements in Example D remain valid.

§ 2. Poincaré classes.

In the section we shall study homological property of a submanifold satisfying certain homological relative connectivity condition.

Let W be a compact $(m+c)$-manifold and M a closed m-submanifold of W representing a homology class $\mu \in H_m(W)$. Thus M is oriented by the fundamental class $[M]$ so that $i_*[M] = \mu$, where $i_* : H_*(M) \to H_*(W)$ is a homomorphism induced by the inclusion map $i: M \to W$.

DEFINITION. A submanifold M of W is a *Poincaré submanifold* if (1) $i_* : H_k(M) \cong H_k(W)$ for all $k \leqq n+1$ or (2) $H_k(W, M) = 0$ for all $k \leqq n+1$, according as $m = 2n+1$ or $m = 2n$.

EXAMPLE. An m-knot in a 1-connected $(m+2)$-manifold is a Poincaré submanifold.

The cap product with μ gives rise to a homomorphism $\cap\mu$: $H^{m-k}(W) \to H_k(W)$ and we have a commutative diagram:

$$\begin{array}{ccccccccc} & & H^{m-k-1}(M) & \xleftarrow{i^*} & H^{m-k-1}(W) & & H^{m-k}(M) & \xleftarrow{i^*} & H^{m-k}(W) \\ (2.0) & & \wr\|\downarrow\cap[M] & & \downarrow\cap\mu & & \wr\|\downarrow\cap[M] & & \downarrow\cap\mu \\ \cdots \longrightarrow & & H_{k+1}(M) & \xrightarrow{i_*} & H_{k+1}(W) \longrightarrow H_{k+1}(W, M) \longrightarrow & & H_k(M) & \xrightarrow{i_*} & H_k(W) \longrightarrow \cdots, \end{array}$$

where $\cap[M]$ is the so-called Poincaré duality isomorphism.

DEFINITION. A homology class $\mu \in H_m(W)$ is a *Poincaré class*, if

(1) $\cap\mu : H^n(W) \cong H_{n+1}(W)$ and $\cap\mu : H^{n+1}(W) \cong H_n(W)$, or

(2) $\cap\mu : H^n(W) \cong H_n(W)$ and $\cap\mu : H^{n-1}(W) \to H_{n+1}(W)$ is surjective, according as $m=2n+1$ or $m=2n$.

LEMMA 2.1. *A Poincaré submanifold represents a Poincaré class.*

PROOF. Let M be a Poincaré m-submanifold of W representing a homology class $\mu \in H_m(W)$. We observe the diagram (2.0). In case (1); $m=2n+1$, it follows immediately that i^*: $H^k(W) \cong H^k(M)$ for $k \leqq n$ and $\cap\mu$: $H^n(W) \cong H_{n+1}(W)$. In order to show that μ is a Poincaré class, it suffices to see that $\cap\mu$: $H^{n+1}(W) \cong H_n(W)$. From the condition that $H_{n+1}(M) \cong H_{n+1}(W)$ we have that by the universal coefficient theorem rank $H^{n+1}(M) =$ rank $H_{n+1}(M)$ $=$ rank $H_{n+1}(W) =$ rank $H^{n+1}(W)$. On the other hand, we have a commutative diagram:

$$\begin{array}{ccccccc} \cdots \longleftarrow H^{n+2}(W, M) \longleftarrow & H^{n+1}(M) & \xleftarrow{i^*} & H^{n+1}(W) & \longleftarrow 0 \\ & \wr\|\downarrow\cap[M] & & \downarrow\cap\mu & \\ 0 \longrightarrow & H_n(M) & \cong & H_n(W) & \longrightarrow 0. \end{array}$$

By the universal coefficient theorem, $H^{n+2}(W, M)$ is torsion free, since $H_{n+1}(W, M)=0$. Therefore, the cokernel of i^*: $H^{n+1}(W) \to H^{n+1}(M)$ must be torsion free. Since we have seen that rank $H^{n+1}(M) =$ rank $H^{n+1}(W)$, it follows that i^* is surjective and hence bijective. This implies that $\cap\mu$ is bijective, completing the proof of the case (1).

In case (2); $m=2n$, the result follows immediately from the diagram (2.0) together with the universal coefficient theorem, completing the proof.

By $K_k(M)$ and $K^k(M)$ we shall denote the kernel of i_*: $H_k(M) \to H_k(W)$ and the cokernel of i^*: $H^k(W) \to H^k(M)$, respectively. Observing the diagram (2.0) and applying general non-sense arguments due to Browder [**2**], we may deduce the following:

OBSERVATION 2.2. *Let M be a closed m-submanifold of W representing a*

Poincaré class μ of $H_m(W)$. Suppose that $H_k(W, M)=0$ for all $k\leqq n$, where $m=2n+1$, or $2n$.

(1) *In case $m=2n+1$, then*

(i) *there are direct sum decompositions*

$$H_k(M)\cong K_k(M)\oplus H_k(W),$$
$$H^k(M)\cong K^k(M)\oplus H^k(W) \qquad \text{for } k=n,\ n+1,$$

(ii) *$K_n(M)\cong H_{n+1}(W, M)$, $K^n(M)\cong H^{n+1}(W, M)$, and*

(iii) *$\cap\mu\colon K^n(M)\cong K_{n+1}(M)$.*

(2) *In case $m=2n$, then*

(i) *there are direct sum decompositions*

$$H_n(M)\cong K_n(M)\oplus H_n(W), \quad H^n(M)\cong K^n(M)\oplus H^n(W),$$

(ii) *$K_n(M)\cong H_{n+1}(W, M)$ is torsion free, and*

(iii) *$\cap\mu\colon K^n(M)\cong K_n(M)$ so that the intersection pairing $K_n(M)\otimes K_n(M)\to \boldsymbol{Z}$ is non-singular.*

Next we summarize homology property of an L-equivalence between Poincaré submanifolds. For this, let us recall the notion of L-equivalence defined by R. Thom [**14**].

DEFINITION. Two proper submanifolds M and L of W with $\partial M=\partial L$ (possibly $=\emptyset$) are *L-equivalent* relative to $(\partial W, \partial M)$, if there is a compact proper submanifold V in $W\times I$ such that $\partial V=M\times 0\cup(-L)\times 1\cup\partial M\times I$.

Assuming that M and L are Poincaré submanifolds in W, we examine the homology class ν represented by an L-equivalence $(V, \partial V)$ in $(W\times I, W\times\partial I)$. Since M and L represent homologous cycles in W, it follows from Lemma 2.1 that they represent the same Poincaré class $\mu\in H_m(W)$.

Thus $(V, \partial V)$ is oriented by the fundamental class $[V]$ so that $i_*[V]=\nu$ and $\partial\nu=\mu\times 0+(-\mu)\times 1$, where $i_*\colon H_{m+1}(V, \partial V)\to H_{m+1}(W\times I, W\times\partial I)$ and $\partial\colon H_{m+1}(W\times I, W\times\partial I)\to H_m(W\times\partial I)$ is the boundary homomorphism. Following Wall [**16**], we observe a commutative diagram with exact sequences in the rows:

$$(*)\qquad \begin{array}{ccccccccc}
\cdots\longrightarrow & H^{m-k}(W\times I) & \longrightarrow & H^{m-k}(W\times\partial I) & \longrightarrow & H^{m-k+1}(W\times I, W\times\partial I) & \longrightarrow\cdots \\
& \downarrow i^* & & \downarrow i^* & & \downarrow i^* & \\
\cdots\longrightarrow & H^{m-k}(V) & \longrightarrow & H^{m-k}(\partial V) & \longrightarrow & H^{m-k+1}(V, \partial V) & \longrightarrow\cdots \\
& \wr\!\!\!\parallel \downarrow \cap[V] & & \wr\!\!\!\parallel \downarrow \cap[\partial V] & & \wr\!\!\!\parallel \downarrow \cap[V] & \\
\cdots\longrightarrow & H_{k+1}(V, \partial V) & \longrightarrow & H_k(\partial V) & \longrightarrow & H_k(V) & \longrightarrow\cdots \\
& \downarrow i_* & & \downarrow i_* & & \downarrow i_* & \\
\cdots\longrightarrow & H_{k+1}(W\times I, W\times\partial I) & \longrightarrow & H_k(W\times\partial I) & \longrightarrow & H_k(W\times I) & \longrightarrow\cdots
\end{array}$$

where $[\partial V]=[M]+(-[L])$. Note that the top and bottom sequences are connected by $\cap\nu$ and $\cap(\partial\nu)$ preserving the commutativity and are reduced to splitting short exact sequences:

$$
(**)\qquad
\begin{tikzcd}
 & H^{m-k}(W) & & & \\
0 \arrow[r] & H^{m-k}(W\times I) \arrow[r] \arrow[u, "\cong"', "(\times 0)^*"] \arrow[d, "\cap\nu"] & H^{m-k}(W\times\partial I) \arrow[r] \arrow[ul, "(\times 0)^*"'] \arrow[d, "\cap\partial\nu"] & H^{m-k+1}(W\times I,\ W\times\partial I) \arrow[r] \arrow[d, "\cap\nu"] & 0 \\
0 \arrow[r] & H_{k+1}(W\times I,\ W\times\partial I) \arrow[r] & H_k(W\times\partial I) \arrow[r] & H_k(W\times I) \arrow[r] & 0 \\
 & & H_k(W) \arrow[u, "(\times 0)_*"] \arrow[ur, "\cong"', "(\times 0)_*"] & &
\end{tikzcd}
$$

We shall employ the following notations for kernels and cokernels:

$$
\begin{aligned}
&K_k(V,\partial V)=\text{kernel}\,(H_k(V,\partial V)\longrightarrow H_k(W\times I,\ W\times\partial I))\,,\\
&K_k(V)=\text{kernel}\,(H_k(V)\longrightarrow H_k(W\times I))\,,\\
&K_k(\partial V)=\text{kernel}\,(H_k(\partial V)\longrightarrow H_k(W\times\partial I))\,,\\
&K^k(V,\partial V)=\text{cokernel}\,(H^k(W\times I,\ W\times\partial I)\longrightarrow H^k(V,\partial V))\,,\\
&K^k(V)=\text{cokernel}\,(H^k(W\times I)\longrightarrow H^k(V))\,,\qquad \text{and}\\
&K^k(\partial V)=\text{cokernel}\,(H^k(W\times\partial I)\longrightarrow H^k(\partial V))\,.
\end{aligned}
$$

Then the following is deduced by general non-sense (Browder [2]).

OBSERVATION 2.3. *Suppose that M and L are Poincaré m-submanifolds of W.*

(1) *In case $m+1=2(n+1)$, suppose that $H_k(W\times I,\ V)=0$ for all $k\leqq n+1$. Then $\partial\nu$ is a Poincaré class of $H_m(W\times\partial I)$ and hence all columns in the diagram (**) are isomorphisms for $k=n,\ n+1$ so that all columns in the diagram (*) have splitting maps for $k=n,\ n+1$ to give a commutative diagram:*

$$
\begin{tikzcd}
0=K^n(\partial V) \arrow[r] \arrow[d, "\cap\partial\nu"] & K^{n+1}(V,\partial V) \arrow[r, "\cong"] \arrow[d, "\cap\nu", "\cong"'] & K^{n+1}(V) \arrow[r] \arrow[d, "\cap\nu", "\cong"'] & K^{n+1}(\partial V)=0 \arrow[d, "\cap\partial\nu"] \\
0=K_{n+1}(\partial V) \arrow[r] & K_{n+1}(V) \arrow[r, "\cong"] & K_{n+1}(V,\partial V) \arrow[r] & K_n(\partial V)=0\,.
\end{tikzcd}
$$

Thus $K_{n+1}(V)\cong K^{n+1}(V)\subset H^{n+2}(W\times I,\ V)$ is torsion free and the intersection pairing

$$K_{n+1}(V)\otimes K_{n+1}(V)\longrightarrow \boldsymbol{Z}$$

is non-singular.

(2) *In case* $m+1=2n+1$, *suppose that* $H_k(W\times I, V)=0$ *for* $k\leqq n$. *Then* $\partial\nu$ *is a Poincaré class and hence all columns in the diagram* (**) *are isomorphisms for* $k=n$ *and epimorphisms for* $k=n+1$ *so that we have a commutative diagram with splitting exact sequences in the rows:*

$$\begin{array}{ccccccccc}
0 & \longleftarrow & K^{n+1}(V,\partial V) & \longleftarrow & H^{n+1}(V,\partial V) & \longleftarrow & H^{n+1}(W\times I, W\times\partial I) & \longleftarrow & 0 \\
 & & \wr\Vert\downarrow\cap[V] & & \wr\Vert\downarrow\cap[V] & & \downarrow\cap\nu & & \\
0 & \longrightarrow & K_n(V) & \longrightarrow & H_n(V) & \longrightarrow & H_n(W\times I) & \longrightarrow & 0
\end{array}$$

and that $K_n(V)\cong H_{n+1}(W\times I, V)$.

Thus $K_n(V)$ *is a direct summand of* $H_n(V)$ *and, if one takes a double* $D(V)$ *of* V *in* $W\times S^1=D(W\times I)$, *then the kernel of* $H_n(D(V))\to H_n(W\times S^1)$ *is exactly a direct sum* $K_n(V)\oplus K_n(V)$.

§ 3. Surgery below the middle dimension.

In the section we shall prove a number of lemmas on surgery below the middle dimension of a locally flat submanifold of codimension $c\geqq 2$.

Let W be a compact $(m+c)$-manifold and L a locally flat compact m-submanifold of W. We shall employ the following notations:

N denotes a normal block bundle of L in W,

E denotes the closure of the complement $W-N$ of L in W, called *the exterior of* N *in* W *or an exterior of* L *in* W,

$\mathfrak{F}N$ stands for the frontier of N in W so that $N\cap E=\mathfrak{F}N$. As usual, ∂W and $\operatorname{Int} W$ denote the boundary of W and the interior $W-\partial W$ of W.

DEFINITION. A locally flat manifold pair (W, L) is *exterior k-connected*, if $\pi_i(E, \mathfrak{F}N)=0$ for $i\leqq k$, namely, $(E, \mathfrak{F}N)$ is k-connected.

It is clear from the uniqueness of normal block bundles that the definition does not depend on the choice of (E, N). In codimension $c=2$ case, we would like to find an exterior $[m/2]$-connected locally flat m-submanifold M in W which is locally flat L-equivalent to L relative to $(\partial W, \partial L)$. For this we shall perform surgery on L in W. First, let us recall the exchanging handle process due to Browder ([1], p. 338).

Putting $m=p+q+1$, let $H=D^{p+1}\times D^{q+c}$ be a handle of index $p+1$ contained in $E\cap\operatorname{Int} W$ with $H\cap\partial E=H\cap\partial N=S^p\times D^{q+c}$, where D^k stands for the k-fold cartesian product of $D=[-1, 1]$ and $S^k=\partial D^{k+1}$. Then we have new submanifolds $N'=N\cup H$ and $E'=\operatorname{cl}(E-H)$. We shall say that (E', N') *is obtained from* (E, N) *by exchanging the handle* H. We investigate the effect of the exchanging handle process.

Putting $K=\mathfrak{F}N\cup D^{p+1}\times 0$, let U be a regular neighborhood of K in E. We may assume that U is a union of the handle H and a collar neighborhood of $\mathfrak{F}N-S^p\times\mathrm{Int}\,D^{q+c}$ in $E-D^{p+1}\times\mathrm{Int}\,D^{q+c}$. We consider a homotopy group $\pi_k(E;E-K,U)$ of a triad. $(E;E-K,U)$ with an exact sequence:

$$\cdots\to\pi_{k+1}(E;E-K,U)\to\pi_k(E-K,U-K)\to\pi_k(E,U)\to\pi_k(E;E-K,U)\to\cdots.$$

Note that the homotopy group $\pi_k(E;E-K,U)$ may be defined for only $k\geqq 2$. However, we make use of the notation in case $k\leqq 1$, whenever, provided it vanishes, the exact sequence above still makes sense as an exact sequence of pointed sets under a similar convention for the homotopy group of a pair.

By the general position we have that

$$\pi_k(E;E-K,U)\cong 0 \quad \text{for } k+(p+1)+1\leqq m+c; \quad \text{i.e. } k\leqq q+c-1.$$

It follows from the exact sequence of the triad that

$$\pi_k(E-K,U-K)\cong\pi_k(E,U) \qquad \text{for } k\leqq q+c-2$$

and $\pi_{q+c-1}(E-K,U-K)\to\pi_{q+c-1}(E,U)$ is surjective. Since U is a regular neighbourhood of K in E, we have that

$$\pi_k(E-K,U-K)\cong\pi_k(E',\mathfrak{F}N')$$

and

$$\pi_k(E,U)\cong\pi_k(E,K) \qquad \text{for all } k.$$

Therefore, we may conclude the following:

LEMMA 3.1. *We have isomorphisms*

$$\pi_k(E',\mathfrak{F}N')\cong\pi_k(E,K) \qquad \textit{for all } k\leqq q+c-2$$

and an epimorphism $\pi_{q+c-1}(E',\mathfrak{F}N')\to\pi_{q+c-1}(E,K)\to 0$.

Next we transfer the exchanging handle process to a surgery on L in W.

We shall identify $2D^k-\mathrm{Int}\,D^k$ with $S^{k-1}\times D$ so that $(\partial D^k=)S^{k-1}\equiv S^{k-1}\times 1$, where $2D^k$ stands for the k-fold cartesian product of $[-2,2]$. Thus we may think of $2D^k$ as to be obtained from D^k and $S^{k-1}\times D$ by identifying S^{k-1} with $S^{k-1}\times 1$ via essentially the identity map. Suppose that we have an $(m+c)$-ball $2H=2D^{p+1}\times D^{q+c}$ in $\mathrm{Int}\,W$ such that

$$E\cap 2H=H=D^{p+1}\times D^{q+c},$$

$$N\cap 2H=\mathrm{cl}(2H-H)=(S^p\times D)\times D^{q+c},$$

$$\partial E\cap 2H=\partial N\cap 2H=(S^p\times\partial D)\times D^{q+c}\cup(S^p\times D)\times D^{q+1}\times\partial D^{c-1}$$

and

$$L\cap 2H=(S^p\times 0)\times D^{q+1}\times 0^{c-1},$$

where 0^{c-1} stands for the origin of D^{c-1}.

Then the second D and the forth D^{c-1} in $(S^p\times D)\times D^{q+1}\times D^{c-1}$ are transversal to L, and we have a handle $H_0=((S^p\times[0, 1])\cup D^{p+1})\times D^{q+1}\times 0^{c-1}$ of index $p+1$ attached transversally to L in W. We shall say that the handle $H=D^{p+1}\times D^{q+c}$ can be *extended normally to* L to give rise to a *normally embedded handle* H_0.

We are ready to perform surgery on L in W via the normally embedded handle H_0 to kill an element of $\pi_{p+1}(E, \mathfrak{F}N)$ represented by $(D^{p+1}\times 0, S^p\times 0)$. We put $N'=N\cup H$, $E'=\mathrm{cl}(E-H)$, as before, and

$$N_*=N'-2D^{p+1}\times(\mathrm{Int}\,(1/3)D^{q+1})\times D^{c-1},$$

$$E_*=\mathrm{cl}(W-N_*)=E'\cup 2D^{p+1}\times(1/3)D^{q+1}\times D^{c-1}$$

and

$$L_*=(L-(S^p\times 0)\times(1/2)D^{q+1}\times 0^{c-1})\cup((S^p\times[0, 1])\cup D^{p+1})\times(1/2)S^q\times 0^{c-1}.$$

We shall say that L_* is obtained from L by performing surgery on $(S^p\times 0)\times 0$ in W via the normally embedded handle H_0 of index $p+1$. It is not hard to see that L_* is locally flat. Since $2D^{p+1}\times(D^{q+1}-\mathrm{Int}\,(1/3)D^{q+1})\times D^{c-1}$ collapses

$$(S^p\times[0, 1]\cup D^{p+1})\times(1/2)S^q\times 0^{c-1}\cup(S^p\times 0)\times(D^{q+1}-\mathrm{Int}\,(1/2)D^{q+1})\times 0^{c-1}$$
$$\cup(S^p\times D)\times S^q\times D^{c-1},$$

it follows that N_* is a regular neighborhood of a locally flat submanifold L_* in W so that N_* admits a normal block bundle structure.

LEMMA 3.2. *Suppose that* $c\geqq 2$ *and* $m\geqq 2p+2$ *so that* $p+1\leqq q\leqq q+c-2$. *Then we have isomorphisms* $\pi_k(E, \mathfrak{F}N)\cong\pi_k(E_*, \mathfrak{F}N_*)$ *for* $k\leqq p$ *and a surjection* $\pi_{p+1}(E, K)\to\pi_{p+1}(E_*, \mathfrak{F}N_*)$, *where* $K=\mathfrak{F}N\cup D^{p+1}\times 0$ *as in Lemma* 3.1.

PROOF. Since (E', N') is obtained from (E, N) by exchanging the handle H of index $p+1$, we have that by Lemma 3.1

$$\pi_k(E', \mathfrak{F}N')\cong\pi_k(E, K)\qquad \text{for } k\leqq p+1\leqq q+c-2.$$

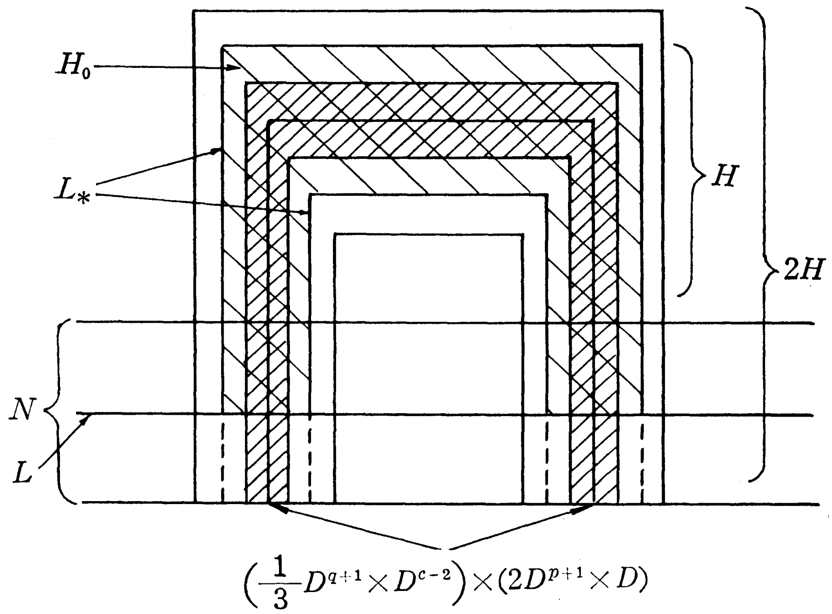
Since $\pi_k(K, \mathfrak{F}N)=0$ for $k\leqq p$, we have that

$$\pi_k(E, \mathfrak{F}N)\cong\pi_k(E, K)\qquad \text{for } k\leqq p.$$

Hence we have that

$$(*)\qquad \begin{aligned}&\pi_{p+1}(E, K)\cong\pi_{p+1}(E', \mathfrak{F}N')\qquad \text{and}\\ &\pi_k(E, \mathfrak{F}N)\cong\pi_k(E', \mathfrak{F}N')\qquad \text{for } k\leqq p.\end{aligned}$$

On the other hand, (E', N') is obtained from (E_*, N_*) by exchanging a handle $(1/3)D^{q+1}\times(2D^{p+1}\times D^{c-1})$ of index $q+1$. Putting $K_*=\mathfrak{F}N_*\cup(1/3)D^{q+1}\times 0$, from Lemma 3.1 we have isomorphisms $\pi_k(E', \mathfrak{F}N')\cong\pi_k(E_*, K_*)$ for $k\leqq p+c-2$, and a surjection $\pi_{p+c-1}(E', \mathfrak{F}N')\to\pi_{p+c-1}(E_*, K_*)\to 0$. Since $\pi_k(K_*, \mathfrak{F}N_*)=0$ for



$k\leqq q$, we have that $\pi_k(E_*, K_*)\cong\pi_k(E_*, \mathcal{F}N_*)$ for $k\leqq q$. Since $p+1\leqq q$, we have isomorphisms $\pi_k(E', \mathcal{F}N')\cong\pi_k(E_*, \mathcal{F}N_*)$ for $k\leqq p$ and a surjection $\pi_{p+1}(E', \mathcal{F}N')\to\pi_{p+1}(E_*, \mathcal{F}N_*)\to 0$. This together with (*) completes the proof.

Let L_* be obtained from L by performing surgery on S^p in W via a normally embedded handle $H_0=D^{p+1}\times D^{q+1}$. Then we have a locally flat L-equivalence $V=L\times[0, 1/2]\cup H_0\times(1/2)\cup L_*\times[1/2, 1]$ in $W\times I$ from L to L_* relative to $(\partial W, \partial L)$. Hence if M is obtained from L by surgery on L in W via a finite number of normally embedded handles, then M and L are locally flat L-equivalent relative to $(\partial W, \partial L)$.

Thirdly, we extend a handle H in $\operatorname{Int} W\cap E$ to a normally embedded handle. For this, following Thom [14], we take a Thom map $T(L): W\to MS\widetilde{PL}_c$ so that $T(L)|L: L\to BS\widetilde{PL}_c$ is a classifying map of a normal block bundle N of L in W.

We shall say that a Thom map $T(L): (W, L)\to(MS\widetilde{PL}_c, BS\widetilde{PL}_c)$ is trivial at k, if

$$T(L)_*: \pi_k(W, L)\longrightarrow\pi_k(MS\widetilde{PL}_c, BS\widetilde{PL}_c)$$

is the zero map.

EXAMPLE. Since $MS\widetilde{PL}_2$ and $BS\widetilde{PL}_2$ are $K(\boldsymbol{Z}, 2)$ so that the inclusion map $BS\widetilde{PL}_2\to MS\widetilde{PL}_2$ is a homotopy equivalence, it follows that any Thom map $T(L): (W, L)\to(MS\widetilde{PL}_2, BS\widetilde{PL}_2)$ is trivial at each k, called simply to be *trivial*. On the other hand, if (W, L) is k-connected, then $T(L)$ is obviously trivial at each $i\leqq k$.

LEMMA 3.3. *Suppose that* $c \geqq 2$, $m \geqq 2p+1$, *and* $T(L)$ *is trivial at* $p+1$. *Then any handle* $H = D^{p+1} \times D^{q+c}$ *of index* $p+1$ *contained in* $E \cap \operatorname{Int} W$ *with* $\partial E \cap H = \partial N \cap H = S^p \times D^{q+c}$ *can be extended normally to* L.

PROOF. Let $\pi : N \to L$ be a projection map such that $\pi | \mathfrak{F}N : \mathfrak{F}N \to L$ is a spherical fiber space over L. Note that $S^p \times D^{q+c} \subset \operatorname{Int} \mathfrak{F}N$. Since $m \geqq 2p+1$, we may homotope $\pi | S^p \times 0 : S^p \times 0 \to L$ to a locally flat embedding $S^p \subset L$. Then we will see that the restricted block bundle $N | S^p$ is trivial. Since $S^p \subset L$ is homotopic to $S^p \times 0 \subset \mathfrak{F}N$ bounded by $D^{p+1} \times 0$ in W, S^p is homotopic to zero in W, and hence there is a map $\alpha : (D^{p+1}, S^p) \to (W, L)$ with $\alpha | S^p = id$. However, $T(L) : (W, L) \to (MS\widetilde{PL}_c, BS\widetilde{PL}_c)$ is trivial at $p+1$, which implies that $T(L) \circ \alpha : (D^{p+1}, S^p) \to (MS\widetilde{PL}_c, BS\widetilde{PL}_c)$ is homotopic to a map $\beta : D^{p+1} \to BS\widetilde{PL}_c$ relative to $\alpha | S^p$. Since $T(L) \circ \alpha | S^p = T(L) | S^p$ classifies the restricted bundle $N | S^p$ and is homotopic to zero in $BS\widetilde{PL}_c$ with null-homotopy β, it follows that $N | S^p$ is a trivial block bundle. Putting $N | S^p = (S^p \times D^c)$, we may assume that $\pi | (S^p \times D^c)$ is the projection onto the first factor. By the covering homotopy theorem, a homotopy from $\pi | S^p \times 0 : S^p \times 0 \to L$ to $S^p \subset L$ is covered by a homotopy of the spherical fiber space and hence $S^p \times 0 \subset \mathfrak{F}N$ is homotopic to a cross-section of $\mathfrak{F}N | S^p = (S^p \times S^{c-1})$ over S^p. Since $m+c-1 \geqq 2p+2$, homotopic embeddings in $\operatorname{Int} \mathfrak{F}N$ are ambient isotopic. Hence, after changing the normal bundle structure, if necessary, we may assume that $S^p \times 0$ is just a cross-section of $\mathfrak{F}N | S^p = (S^p \times S^{c-1})$ over $S^p \subset L$. Let U be a normal block bundle of S^p in L. Then $(N | U) | S^p$ is a Whitney sum of $N | S^p = (S^p \times D^c)$ and U. On the other hand, $S^p \times 0$ has a trivial normal block bundle $S^p \times D^{q+c+1}$ in W. Since $S^p \times 0 \subset \mathfrak{F}N$ and $S^p \subset L$ are homotopic and hence ambient isotopic in W, they have isomorphic normal block bundles. Therefore, a normal block bundle $(N | U) | S^p$ of S^p in W has to be trivial. This implies that U is stably trivial and, therefore, U is itself trivial, since the suspension map $\pi_{p-1}(S\widetilde{PL}_{q+1}) \to \pi_{p-1}(S\widetilde{PL}_{q+c+1})$ is an isomorphism for each $p \leqq q$.

Let $h : D^{p+1} \times D^{q+c} \to H$ $(= D^{p+1} \times D^{q+c})$ be a homeomorphism defining the handle H such that $h(S^p \times D^{q+c}) = S^p \times D^{q+c}$. We take a trivialization $\varphi : S^p \times D^{q+c+1} \to N | U$ so that

$$\varphi | S^p \times D^{q+1} \times D^{c-1} \times 1 = h | S^p \times D^{q+c}$$

and

$$\varphi(S^p \times D^{q+1} \times 0) = U .$$

For this, first we take a trivialization

$$\varphi_0 : S^p \times D^c \longrightarrow N | S^p$$

of $N | S^p$ so that $\varphi_0(S^p \times 0^{c-1} \times 1) = h(S^p \times 0)$ $(= S^p \times 0)$. This is done by smoothing the cross-section $h(S^p \times 0)$ as a vector field and then taking the orthogonal complement. Next, we extend this trivialization φ_0 to a trivialization

$$\varphi_1: S^p\times D^{q+1}\times D^c \longrightarrow (N|U)|S^p$$

by taking a Whitney sum of φ_0 and a trivialization of U so that $\varphi_1(S^p\times D^{q+1}\times 0)=U$. Now we have a normal block bundle $\varphi_1(S^1\times D^{q+1}\times D^{c-1}\times 1)$ of $h(S^p\times 0)$ in $\mathfrak{F}N$. From the uniqueness of normal block bundles we may assume that $\varphi_1(S^p\times D^{q+1}\times D^{c-1}\times 1)=h(S^p\times D^{q+c})$ keeping $\varphi_1(S^p\times 0\times 1)$ fixed. Thus we have a $(q+1)$-frame

$$\varphi_1^{-1}\cdot h|S^p\times D^{q+1}\times 0: S^p\times D^{q+1}\times 0 \longrightarrow S^p\times D^{q+1}\times D^{c-1}\times 1\,.$$

Since the suspension homomorphism $\pi_p(S\widetilde{PL}_{q+1})\rightarrow\pi_p(S\widetilde{PL}_{q+c})$ is surjective for $p\leqq q$, we may further assume that

$$\varphi_1^{-1}\circ h(S^p\times D^{q+1}\times 0)=S^p\times D^{q+1}\times 0^{c-1}\times 1\,.$$

Regarding $S^p\times D^{q+1}\times D^{c-1}\times 1$ as $S^p\times D^{q+1}\times D^{c-1}$ we may extend $\varphi_1^{-1}\circ h|S^p\times D^{q+1}\times D^{c-1}$ to a homeomorphism $\phi: S^p\times D^{q+1}\times D^c\rightarrow S^p\times D^{q+1}\times D^c$ by setting, if $(x, t)\in(S^p\times D^{q+1}\times D^{c-1})\times D$, then $\phi(x, t)=(\varphi_1^{-1}\circ h(x), t)$. A composition $\varphi=\varphi_1\circ\phi$ is now the required trivialization of $(N|U)|S^p$. Indeed, $\varphi(x, 1)=(h(x), 1)=h(x)$ for $x\in S^p\times D^{q+c}$ and

$$\varphi(S^p\times D^{q+1}\times 0)=(h(S^p\times D^{q+1}\times 0^{c-1}), 0)=\varphi_1(S^p\times D^{q+1}\times 0)=U\,.$$

Therefore, a handle $2H=\varphi(S^p\times(D^{q+1}\times D^{c-1})\times D)\cup H$ is the required handle giving a normally embedded handle $H_0=\varphi(S^p\times(D^{q+1}\times 0^{c-1})\times[0,1])\cup D^{p+1}\times D^{q+1}\times 0^{c-1}$, completing the proof.

We are ready to complete surgery below the middle dimension of a locally flat submanifold in codimension $c=2$.

LEMMA 3.4. *Let W be a compact $(m+2)$-manifold and L a locally flat compact m-submanifold of W. Then L is L-equivalent to an exterior $[m/2]$-connected locally flat compact m-submanifold M in W relative to $(\partial W, \partial L)$.*

PROOF. Putting $L_0=L$, we have that by the general position (W, L_0) is exterior 0-connected. Assuming inductively that we have already obtained an exterior p-connected locally flat m-submanifold L_p $(0\leqq p\leqq[m/2]-1)$ which is L-equivalent to L relative to $(\partial W, \partial L)$, we shall find L_{p+1} by surgery on L_p in W so that (W, L_{p+1}) is exterior $(p+1)$-connected. Since $(E_p, \mathfrak{F}N_p)$ is p-connected, it follows that there are only a finite number of $(p+1)$-cells of E_p relative to $\mathfrak{F}N_p$. Since $2(p+1)+1\leqq m+2$, by the general position, one can represent them by locally flat mutually disjoint $(p+1)$-balls D_i^{p+1}, $i=1,\cdots, V$, contained in $E_p\cap\operatorname{Int}W$ with $D_i^{p+1}\cap\partial E_p=D_i^{p+1}\cap\partial N_p=S_i^p$. By thickening them, we have handles $H_i=D_i^{p+1}\times D^{q+2}$ contained in $E_p\cap\operatorname{Int}W$ with $H_i\cap\partial E_p=H_i\cap\partial N_p=S_i^p\times D^{q+2}$, where $p+q+1=m$. Recall that $T(L_p): (W, L_p)\rightarrow$

$(MS\widetilde{PL}_2, BS\widetilde{PL}_2)$ is trivial(1). Therefore, since $p+1 \leqq [m/2]$, by Lemma 3.3, these handles can be extended normally to L_p to give normally embedded handles H_{0i}, simultaneously. Let $L_{p+1}=(L_p)_*$ be a locally flat m-submanifold obtained from L_p by performing surgery on $S_i^p \subset L_p$ in W via the handles H_{0i}. Then by Lemma 3.2 we have that

$$\pi_k(E_{p+1}, \mathcal{F}N_{p+1}) \cong \pi_k(E_p, \mathcal{F}N_p)=0 \qquad \text{for } k \leqq p$$

and

$$\pi_{p+1}(E_p, K_p) \longrightarrow \pi_{p+1}(E_{p+1}, \mathcal{F}N_{p+1}) \qquad \text{is surjective},$$

where $K_p=\mathcal{F}N_p \cup (\bigcup_i D_i^{p+1})$. Since $\pi_p(K_p, \mathcal{F}N_p)=0$ and $\pi_{p+1}(K_p, \mathcal{F}N_p) \to \pi_{p+1}(E_p, \mathcal{F}N_p)$ is surjective, it follows from the homotopy exact sequence of a triple $(E_p ; K_p, \mathcal{F}N_p)$ that $\pi_{p+1}(E_p, K_p)=0$ and hence $\pi_{p+1}(E_{p+1}, \mathcal{F}N_{p+1})=0$. Therefore, (W, L_{p+1}) is exterior $(p+1)$-connected. As observed below the proof of Lemma 3.2, L_p and L_{p+1} are L-equivalent relative to $(\partial W, \partial L)$. This completes the inductive step and the proof of Lemma 3.4.

§ 4. Surgery in the middle dimension.

First of all, we study exterior connectivity of a pair (W, L).

LEMMA 4.1. *Let P be a p-subpolyhedron of a compact $(p+c)$-manifold W. Let N be a derived neighborhood of P in W, and $E=\mathrm{cl}(W-N)$.*

(1) *If $c \geqq 3$, then $\pi_i(W, P)=0$ for $i \leqq k$ if and only if $\pi_i(E, \mathcal{F}N)=0$ for $i \leqq k$.*

(2) *If $c=2$ and $\pi_i(E, \mathcal{F}N)=0$ for $i \leqq k$, then $\pi_i(W, P)=0$ for $i \leqq k$ and*

$$\pi_{k+1}(E, \mathcal{F}N) \longrightarrow \pi_{k+1}(W, N) \cong \pi_{k+1}(W, P) \qquad \textit{is surjective.}$$

PROOF. This is essentially proved by Hudson [3]. For completeness, we shall give the proof. We observe the homotopy exact sequence of a triad $(W ; W-P, N)$:

$$\cdots \longrightarrow \pi_{i+1}(W ; W-P, N) \longrightarrow \pi_i(W-P, N-P)$$
$$\longrightarrow \pi_i(W, N) \longrightarrow \pi_i(W ; W-P, N) \longrightarrow \cdots .$$

Since N is a derived neighborhood of P in N, we have that

$$\pi_i(W-P, N-P) \cong \pi_i(E, \mathcal{F}N)$$

and

$$\pi_i(W, N) \cong \pi_i(W, P) \qquad \text{for all } i .$$

Then by virtue of Hudson's result ([3], Lemma 12.3) we have that $\pi_i(W ; W-P, N)=0$ for $i \leqq k+c-1$. This implies that if $c \geqq 3$, then

(1) At this step, we use the codimension condition $c=2$.

$$\pi_i(E, \mathcal{T}N) \cong \pi_i(W, P) = 0 \qquad \text{for } i \leqq k$$

and

$$\pi_i(E, \mathcal{T}N) \cong \pi_i(W, P) \qquad \text{for } i \leqq k+c-2,$$

and if $c=2$, then $\pi_i(E, \mathcal{T}N) \cong \pi_i(W, P)=0$ for $i \leqq k$ and

$$\pi_{k+1}(E, \mathcal{T}N) \longrightarrow \pi_{k+1}(W, P) \qquad \text{is surjective}.$$

Conversely, in case $c \geqq 3$, suppose that

$$\pi_i(W, P) = 0 \qquad \text{for } i \leqq k.$$

By the general position

$$\pi_i(W; W-P, N) = 0 \qquad \text{for at least } i \leqq c-1.$$

This implies that $\pi_i(W, P) \cong \pi_i(E, \mathcal{T}N) = 0$ for $i \leqq 1$. Thus if $k \leqq 1$, this completes the proof. If $k \geqq 2$, then the fact that $\pi_i(E, \mathcal{T}N)=0$ for $i \leqq j \leqq k-1$ implies that by Hudson $\pi_i(W; W-P, N)=0$ for $i \leqq j+c-1$. This together with $\pi_i(W, P)=0$ for $i \leqq k$ implies that $\pi_i(E, \mathcal{T}N)=0$ for $i \leqq j+c-2$ $(>j)$. Thus we have that by induction

$$\pi_i(E, \mathcal{T}N) \cong \pi_i(W, P) = 0 \qquad \text{for } i \leqq k.$$

This completes the proof of Lemma 4.1.

The technique of simply connected surgery in the middle dimension will be divided into two cases; even and odd dimensional cases.

In the even dimensional case, it is done by the engulfing technique.

THEOREM 4.2 (*The even dimensional case*). *Let W be a compact* 1-*connected* $(m+c)$-*manifold and L a locally flat m-submanifold of W. Suppose that*

(i) $m=2n \geqq 6$, $c \geqq 2$,

(ii) *Thom map* $T(L): (W, L) \to (MS\widetilde{PL}_c, BS\widetilde{PL}_c)$ *is trivial at* $n+1$,

(iii) (W, L) *is exterior n-connected,*

(iv) $K_n(L)$ *is torsion free and a direct summand of* $H_n(L)$, *and*

(v) *the intersection pairing* $K_n(L) \otimes K_n(L) \to \boldsymbol{Z}$ *is defined and non-singular.*

Then one can find a proper m-submanifold M in W, which is locally flat except for at one point of $\operatorname{Int} M$, *so that (W, M) is n-connected, $K_n(M)=0$ and M represents the homology class represented by L.*

PROOF. We would like to kill the torsion free group $K_n(L)$, which by n-connectivity of (W, L) is the image of the map $\pi_{n+1}(W, L) \to H_{n+1}(W, L) \to H_n(L)$. For this let $\xi_1, \cdots, \xi_\alpha$ be a basis of $K_n(L)$. Since L is 1-connected, it follows that by the PL embedding theory there is an embedding $f\colon \bigvee_{i=1}^{\alpha} S_i^n \to \operatorname{Int} L$ from a wedge (one point union) of n-spheres into $\operatorname{Int} L$ such that each $f(S_i^n)$ represents ξ_i. Putting $K=f\left(\bigvee_{i=1}^{\alpha} S_i^n\right)$, let P be a regular neighborhood of K

in Int L. Since the intersection pairing $K_n(L)\otimes K_n(L)\to \boldsymbol{Z}$ is non-singular hence the intersection matrix of ξ_i is unimodular, and $m-1\geqq 5$, it follows that ∂P is an $(m-1)$-sphere. Notice that if we put $M=(L-\mathrm{Int}\,P)\cup(0*\partial P)$, then M is the expected manifold. However, we have to take the cone $0*\partial P$ in W. For this we make use of the engulfing method. Since P contracts into a point in W and $T(L): (W, L)\to(MS\widetilde{PL}_c, BS\widetilde{PL}_c)$ is trivial at $n+1$, the restricted normal bundle $N|P$ is trivial, say $N|P=(P\times D^c)$. Let U be the second derived star of the base point p of $K=f\Big(\bigvee_{i=1}^{\alpha} S_i{}^n\Big)$ in Int P such that P collapses $K\cup U$; written $P\searrow K\cup U$. In case $c\geqq 3$, putting $C=(p\times 0^{c-1}\times 1)\in(p\times S^{c-1})$, we have that $(K\times(0^{c-1}, 1))$ is C-inessential, since by Lemma 4.1 we have that $\pi_{n+1}(E, \mathcal{F}N)\cong\pi_{n+1}(W, L)\cong H_{n+1}(W, L)$, which implies that $K\times(0^{c-1}, 1)$ is contractible in E. Then C is a 0-collapsible 1-core, since E is 1-connected. Therefore, by the codimension reason, one can find a polyhedron J in E such that

$$(K\times(0^{c-1}, 1))\subset J\searrow C$$

and

$$J\cap\partial E=(K\times(0^{c-1}, 1))\cap\partial E=(K\times(0^{c-1}, 1))\,.$$

We may take a regular neighborhood A of J in E so that $\partial E\cap A=\mathcal{F}N\cap A=(P\times(\varepsilon D^{c-1}, 1))$, where ε is a small positive number. In case $c=2$, putting $C=(p\times S^1)$, we have that $(K\times 0\times 1)$ is C-inessential, since by Lemma 4.1, $\pi_{n+1}(E, \mathcal{F}N)\to\pi_{n+1}(W, L)\cong H_{n+1}(W, L)$ is surjective, which implies that $(K\times 0\times 1)$ is contractible in E. Then C is a 1-collapsible 1-core. Because a fiber C of a circle bundle $\mathcal{F}N$ over L represents a generator of $\pi_1(\mathcal{F}N)\cong\pi_1(E)$, since from the exact sequence of fiber spaces we have that

$$\pi_1(S^1)\longrightarrow\pi_1(\mathcal{F}N)\longrightarrow\pi_1(L)=0\,.$$

It follows from the codimension reason of $(K\times 0\times 1)$ in ∂E and E with respect to C that by Zeeman ([**20**], Theorem 21) there is a subpolyhedron J in E such that $(K\times 0\times 1)\subset J\searrow C$ and $J\cap\partial E=(K\times 0\times 1\cup C)\cap\partial E=K\times 0\times 1\cup C$. We may take a regular neighborhood A of J in E so that

$$\partial E\cap A=N\cap A=(P\times\varepsilon D\times 1)\cup(U\times S^1)\,.$$

Now for $c\geqq 2$, putting $B=(P\times D^c)\cup A$, we have that

$$\begin{aligned} B&\searrow(P\times D^c)\cup J\searrow(K\times 0^{c-1}\times 1)\cup U\times D^c\cup J\\ &=U\times D^c\cup J\searrow U\times D^c\cup C=U\times D^c\searrow p\,.\end{aligned}$$

This shows that B is an $(m+c)$-ball with $(P, \partial P)\subset(B, \partial B)$ so that ∂P is a locally flat $(m-1)$-sphere in an $(m+c-1)$-sphere ∂B. Let $M=(L-P)\cup(0*\partial P)$ be a manifold obtained from L by removing P and attaching a cone $0*\partial P$

from a point 0 in Int B. Since $0*\partial P$ kills $\xi_1, \cdots, \xi_\alpha$ generating $K_n(L)$ which is a direct summand of $H_n(L)$ and the modification has been done in the $(m+2)$-ball B, it follows that $H_k(W, M)=0$ for $k\leqq n$ and $K_n(M)=0$. From the construction, it is seen that M is locally flat except for the point $0\in \mathrm{Int}\, M$ and represents the homology class represented by L, completing the proof.

In the odd dimensional case, we may discuss the non-simply connected case as well as the 1-connected case.

Let W be a compact $(m+c)$-manifold and L a compact exterior n-connected locally flat m-submanifold of W.

Suppose that $m=2n+1\geqq 5$ and Thom map $T(L):(W, L)\rightarrow(MS\widetilde{PL}_c, BS\widetilde{PL}_c)$ is trivial at $n+1$. By $K_n(L)$ we shall denote the kernel of $H_n(L)\rightarrow H_n(W)$. In the 1-connected case any element of $K_n(L)$ is spherical. Let ξ be an element of $K_n(L)$. By the general position, Lemmas 4.1 and 3.3, there is a normally embedded handle $H_0=(D^{n+1}\times D^{n+1})$ in W with $H_0\cap L=(S^n\times D^{n+1})$, and $(S^n\times 0)$ represents ξ. Let $F: D^{n+1}\times D^{n+1}\rightarrow H_0$ be a homeomorphism with $F(S^n\times D^{n+1})=H_0\cap L=(S^n\times D^{n+1})$. Then $F|S^n\times D^{n+1}$ gives a framing of $(S^n\times 0)$ in L. If $H_0'=(D^{n+1}\times D^{n+1})'$ is a second normally embedded handle such that $(S^n\times 0)'$ represents ξ and $F': D^{n+1}\times D^{n+1}\rightarrow H_0'$ is a homeomorphism with $F'(S^n\times D^{n+1})=H_0'\cap L=(S^n\times D^{n+1})'$, then $F|(D^{n+1}, S^n)\times 0$ and $F'|(D^{n+1}, S^n)\times 0$ are regularly homotopic in (W, L). By the stability of $\pi_n(S\widetilde{PL})$, we may conclude that two framings $F|S^n\times D^{n+1}$ and $F'|S^n\times D^{n+1}$ are regular homotopic modulo a homeomorphism $h: S^n\times D^{n+1}\rightarrow S^n\times D^{n+1}$ representing an element of the kernel of the suspension homomorphism $\pi_n(S\widetilde{PL}_{n+1})\rightarrow\pi_n(S\widetilde{PL}_{n+c})$.

Therefore, ξ determines a unique regular homotopy class of a framing $F(\xi): S^n\times D^{n+1}\rightarrow L$ modulo an action of the kernel of $\pi_n(S\widetilde{PL}_{n+1})\rightarrow\pi_n(S\widetilde{PL}_{n+c})$, which will be called an *admissible* framing. If one can attach to L handles of index $n+1$ via admissible framings to kill $K_n(L)$, abstractly, then we shall call such abstract surgery to be *admissible*. On the other hand, Lemma 3.3 guarantees us that abstract admissible surgery may be realized by surgery on L in W. Therefore, we have reduced surgery on L in W to an abstract admissible surgery. Namely we have the following:

THEOREM 4.3. *One can attach normally embedded handles of index $n+1$ to L in W to kill $K_n(L)$ if and only if one can perform abstract admissible surgery on L to kill $K_n(L)$.*

§ 5. Proof of Theorems and Examples.

By $\Lambda_n(W)$ we shall denote a set of locally flat L-equivalence classes of locally flat closed m-submanifolds of W. Since locally flat L-equivalent submanifolds represent the same homology class, it follows that there is a well-defined map $\Lambda_m(W)\to H_m(W)$.

Our proof of Theorems will be based on the Thom's L-equivalence classification of codimension two locally flat submanifolds in terms of homology classes.

THOM'S THEOREM ([**14**], see also Williamson [**19**], Rourke and Sanderson [**12**] and Wall [**15**]). *Let W be a compact $(m+2)$-manifold. Then the natural map $\Lambda_m(W)\to H_m(W)$ is a bijection.*

Here is an outline of the proof. If M is a locally flat closed m-submanifold representing $\mu\in H_m(W)$, then we have a Thom map $T(M)\colon W/\partial W\to MS\widetilde{PL}_2$. Since L-equivalent locally flat closed m-submanifolds give rise to homotopic Thom maps, and by the t-regular approximation the converse is also true, it follows that the Thom construction gives rise to a bijection

$$T\colon \Lambda_m(W)\longrightarrow [W/\partial W,\, MS\widetilde{PL}_2]\,.$$

On the other hand, we have that $MS\widetilde{PL}_2=K(\boldsymbol{Z},2)$ and hence $[W/\partial W,\, MS\widetilde{PL}_2]$ may be identified with $H^2(W,\partial W)$. (To be precise, we shall specify the identification as follows: Let $U\in H^2(MS\widetilde{PL}_2\,;\boldsymbol{Z})$ be the universal Thom class, $g\colon W/\partial W\to MS\widetilde{PL}_2$ a map. We define an element $S(\{g\})\in H^2(W,\partial W)$ by

$$S(\{g\})=g^*(U)\,.$$

In this way we have a map $S\colon [W/\partial W,\, MS\widetilde{PL}_2]\to H^2(W,\partial W)$. Clearly S is an isomorphism.) Furthermore, by Poincaré-Lefschetz duality we have that $H^2(W,\partial W)\cong H_m(W)$ and the image of $T(M)$ coincides with μ. Thus we have a commutative diagram

$$\begin{tikzcd}
\Lambda_m(W) \arrow[rr, "T"] \arrow[dr] & & H^2(W,\partial W) \arrow[dl, "\cong"] \\
& H_m(W)\,. &
\end{tikzcd}$$

Since T is a bijection, it follows that so is the natural map $\Lambda_m(W)\to H_m(W)$, completing the outline of the proof. Recall that a Thom map $T(M)\colon (W,M)\to(MS\widetilde{PL}_2,\, BS\widetilde{PL}_2)$ is trivial, since $BS\widetilde{PL}_2=K(\boldsymbol{Z},2)$ so that the inclusion map $BS\widetilde{PL}_2\to MS\widetilde{PL}_2$ is a homotopy equivalence.

PROOF OF THEOREM A. Let $\mu\in H_m(W)$ be a Poincaré class of a compact 1-connected $(m+2)$-manifold W. By Thom's theorem, we may represent μ by a locally flat closed m-submanifold L in W. Since $T(L)\colon (W,L)\to(MS\widetilde{PL}_2,\, BS\widetilde{PL}_2)$ is trivial for any locally flat closed m-submanifold L of W, it follows from Lemma 3.4 that (W,L) is exterior n-connected, where $m=2n$ or $2n+1$

so that $n=[m/2]$.

Since μ is a Poincaré class and W is 1-connected, L satisfies the conclusions in Observation 2.2.

In the even dimensional case, since $m\geqq 6$, L satisfies the hypothesis in Theorem 4.2 and the condition that $K_n(L)\cong H_{n+1}(W, L)$. Therefore, by Theorem 4.2 we may kill $K_n(L)\cong H_{n+1}(W, L)$ and we have an almost m-knot M in W representing μ. In the odd dimensional case, if $m\geqq 5$, by Novikov [**11**] or Browder [**2**], one can perform an abstract admissible surgery on L killing $K_n(L)\cong H_{n+1}(W, L)$. By Theorem 4.3, this may be realized by surgery of L in W. Thus we have a locally flat closed m-submanifold M in W representing μ such that $H_i(W, M)=0$ for $i\leqq n+1$. From (i), (ii) and (iii) in Observation 2.2, we have that $H^{n+1}(W, M)\cong K^n(M)\cong K_{n+1}(M)=0$, since by the universal coefficient theorem $H^{n+1}(W, M)=0$. This implies that $H_{n+1}(M)\cong H_{n+1}(W)$ and hence that M is an m-knot in W. If $m=3$, then since the compact 5-manifold W is 1-connected, and W admits a Poincaré class $\mu\in H_3(W)$, it follows that $\cap\mu: H^1(W)\cong H_2(W)=0$, and hence that W is 2-connected. It follows from Kato [**4**] that the class $\mu\in H_3(W)$ can be represented by a locally flat 3-sphere in W.

This completes the proof of Theorem A.

PROOF OF THEOREM B. Let W be a compact 1-connected $(m+2)$-manifold, and M_0 and M_1 m-knots representing the same homology class μ. By Lemma 2.1 μ is a Poincaré class. By Thom's theorem, there is a locally flat L-equivalence V in $W\times I$ from M_0 to M_1. By Lemma 3.4, we may assume that $(W\times I, V)$ is exterior $(n+1)$-connected or exterior n-connected according as $m=2n+1$ or $m=2n$. Thus the homology class $\nu\in H_{m+1}(W\times I, W\times\partial I)$ represented by $(V, \partial V)$ satisfies the conclusions in Observation 2.3. In the case of $m=2n$, by Browder [**2**], we may perform abstract admissible surgery on V making the inclusion map $V\to W$ $(n+1)$-connected.

By Theorem 4.3, this may be realized by surgery on V in $W\times I$. Thus we have a locally flat L-equivalence U in $W\times I$ from M_0 to M_1 such that $(W\times I, U)$ is $(n+1)$-connected. We will see that $(U; M_0, M_1)$ is h-cobordism. For this, by virtue of the Poincaré-Lefschetz duality and the h-cobordism theorem it suffices to show that $H_k(U, M_0)=0$ for $k\leqq n$, since $\dim U=2n+1=m+1>5$. This follows immediately from the exact sequence:

$$\cdots\to H_{k+1}(W\times I, U)\to H_k(U, M_0)\to H_k(W\times I, M_0)\to H_k(W\times I, U)\to\cdots,$$

and the fact that $H_{k+1}(W\times I, U)=H_k(W\times I, M_0)=0$ for $k\leqq n$. Therefore, we have obtained a locally flat concordance U from M_0 to M_1. In the case of $m=2n+1$, by Theorem 4.2, we may find an L-equivalence U in $W\times I$ from

M_0 to M_1, which is locally flat except for at one point p in Int U, and $K_{n+1}(U) = 0$. Then it is not hard to see that $(U; M_0, M_1)$ is an h-cobordism and hence is a concordance from M_0 to M_1, since $m+1 \geqq 7$. If U is locally flat at p, then M_0 and M_1 are locally flat concordant. Suppose that U is locally knotted at p. Taking a suitable arc in U connecting p with a point q of M_1 and then taking a second derived star pair (A, B) of it in $\mathrm{Int}(W\times I, U)$, one can get as $(m+3, m+1)$-ball pair (A, B) in $\mathrm{Int}(W\times I, U)$ such that $\partial(A, B)$, say (S^{m+2}, Σ^m), is a locally flat $(m+2, m)$-sphere pair; namely a spherical m-knot, $(A, B)\cap(W\times 0, M_0\times 0)=\emptyset$, and $(A, B)\cap(W\times 1, M_1\times 1)=(S^{m+2}, \Sigma^m)\cap(W\times 1, M_1\times 1)$ is an unknotted $(m+2, m)$-ball pair, say, (C, D). Therefore, if we put $(W', M')=((W_1, M_1\times 1)\cup(S^{m+2}, \Sigma^m))-\mathrm{Int}(C, D)$, then $(W', M')=(W, M_1)\#(S^{m+2}, \Sigma^m)$. It is not difficult to construct a homeomorphism h: $\mathrm{cl}(W\times I-A)\rightarrow W\times I$ such that

$$h|W\times 0\cup \mathrm{cl}(W\times 1-C)=id\,,$$

$$h(W')=W\times 1\,,$$

$$h(S^m-\mathrm{Int}\,C)=C\,.$$

Since $\mathrm{cl}(U-B)$ is homeomorphic with $M_0\times I$ and locally flat in $\mathrm{cl}(W\times I-A)$, it follows that $h(\mathrm{cl}(U-B))$ is a locally flat concordance from $M_0\times 1$ to $h(M_1')$ in $W\times I$. This completes the proof.

In order to prove Theorem C we shall clarify relation between the abstract surgery due to Novikov [11], Browder [2], Wall [17], [18] and our surgery in codimension two.

Let X be an oriented non-simply connected Poincaré complex of dimension m, and ξ a stable bundle over X. Here, and from now on, a bundle means a block bundle.

By a pre-tangential map $i: (L, \tau_L)\rightarrow(X, \xi)$ we shall mean a degree 1 map $i: L\rightarrow X$ from a closed m-manifold L into a Poincaré complex X of dimension m such that $i^*\xi=\tau_L$ where τ_L denotes the stable tangent bundle over L. A second pre-tangential map $j: (M, \tau_M)\rightarrow(X, \xi)$ is bordant to i, if there are a cobordism $(V; L, M)$ and a map $k: V\rightarrow X$ such that $k^*\xi=\tau_V$, $k|L=i$ and $k|M=j$. According to Wall [18], there is an invariant $\theta(i)\in L_m(\pi_1 X)$ of the pre-tangential map i under bordism such that i is bordant to a pre-tangential map $j: (M, \tau_M)\rightarrow(X, \xi)$ with $j: M\rightarrow X$ a simple homotopy equivalence if and only if $\theta(i)=0$, provided $m\geqq 5$. Now let W be a compact $(m+2)$-manifold which is an (oriented non-simply connected) Poincaré complex of dimension m with fundamental class $\mu\in H_m(W)$. Let T: $W/\partial W\rightarrow MS\widetilde{PL}_2$ $(=K(\boldsymbol{Z}, 2))$ be a map representing the Poincaré dual in $H^2(W, \partial W)$ of μ. Since $BS\widetilde{PL}_2$ is a deformation retract of $M\widetilde{SPL}_2$, the map T is homotopic into $B\widetilde{SPL}_2$ and induces a 2-block bundle $\nu = T^*\eta$ over W, where η is the universal 2-block bundle over $B\widetilde{SPL}_2$. We fix a pair $(W, \tau_W - \nu)$ consisting of a Poincaré complex W of dimension m and a stable bundle $\tau_W - \nu$.

Let L be a locally flat closed m-submanifold of W representing μ with a Thom map $T(L): W/\partial W \to M\widetilde{SPL}_2$. Then a normal 2-block bundle ν_L of L in W is classified by $T(L)|L: L \to B\widetilde{SPL}_2$. Let $i: L \to W$ be the inclusion map. Since T and $T(L)$ are homotopic, it follows that $i^*\nu = i^* \circ T^*\eta = i^* \circ T(L)^*\eta = (T(L)|L)^*\eta = \nu_L$. Thus we have a pre-tangential map $i: (L, \tau_L) \to (W, \tau_W - \nu)$.

Let M be a second locally flat closed m-submanifold of W representing μ with a pre-tangential map $j: (M, \tau_\mu) \to (W, \tau_W - \nu)$. Then by Thom's theorem there is a locally flat L-equivalence V from L to M with a Thom map $T(L)$: $(W \times I/\partial W \times I, V) \to (M\widetilde{SPL}_2, B\widetilde{SPL}_2)$. Let $k: V \to W$ be a composition of the inclusion map $V \to W \times I$ and the projection $W \times I \to W$ onto the first factor. Then $(V; L, M)$ is a cobordism and $k: V \to W$ is a map such that $k^*(\tau_W - \nu) = \tau_V$, $k|L = i$; $L \to W$ and $k|M = j$: $M \to W$. This implies that the bordism class of i does not depend on the choice of locally flat closed m-submanifolds representing μ. Thus we define an invariant $\theta(W, \mu) = \theta(i)$ in $L_m(\pi_1 W)$.

We have shown the following:

THEOREM 5.1. *Let W be a compact $(m+2)$-manifold which is an oriented non-simply connected Poincaré complex of dimension m with fundamental class $\mu \in H_m(W)$. Suppose that $m \geqq 5$. Then there is a well defined invariant $\theta(W, \mu)$ of (W, μ) in $L_m(\pi_1 W)$ such that for any locally flat closed m-submanifold L representing μ a pre-tangential map $i: (L, \tau_L) \to (W, \tau_W - \nu)$ with the inclusion map $i: L \to W$ is bordant to a pre-tangential map $f: (M, \tau_M) \to (W, \tau_W - \nu)$ with a simple homotopy equivalence if and only if*

$$\theta(W, \mu) = 0 .$$

In particular, if (W, μ) has a locally flat spine, then $\theta(W, \mu) = 0$.

PROOF OF THEOREM C. Suppose that $m = 2n+1 \geqq 5$. We have already defined $\theta(W, \mu)$ and seen that if (W, μ) has a locally flat spine, then $\theta(W, \mu) = 0$. Suppose that $\theta(W, \mu) = 0$. By Lemma 3.3, we may take a locally flat closed m-submanifold L of W representing μ such that (W, L) is exterior n-connected.

Since $\theta(W, \mu) = 0$, it follows that by Wall [18] one can attach to L handles of index $n+1$ making a pre-tangential map $i: (L, \tau_L) \to (W, \tau_W - \nu)$ bordant to a pre-tangential map $f: (M, \tau_M) \to (W, \tau_W - \nu)$ with $f: M \to W$ a simple homotopy equivalence.

We have to see that Wall's tangential surgery is equivalent to our admissible abstract surgery defined in § 4. For this, let $F: (D^{n+1} \times D^{n+1}, S^n \times D^{n+1})$

$\to(W, L)$ be an embedding representing a normally embedded handle of index $n+1$ so that $F|S^n\times D^{n+1}: S^n\times D^{n+1}\to L$ is an admissible framing. Then this handle gives rise to an L-equivalence from L and hence a bordism from $i:(L, \tau_L)\to(W, \tau_W-\nu)$. Thus $F|S^n\times D^{n+1}: S^n\times D^{n+1}\to L$ gives rise to tangential surgery. However, in tangential surgery the framing $F|S^n\times D^{n+1}$ is determined by the homotopy class of $F|S^n\times 0$ modulo an action of the kernel of $\pi_n(\widetilde{PL}_{n+1})\to\pi_n(\widetilde{PL})$ as well as the admissible framing is. It follows that $F|S^n\times D^{n+1}$ induces admissible surgery if and only if it induces tangential surgery. Hence admissible abstract surgery and tangential surgery are equivalent. Thus Theorem 4.3 completes the proof of Theorem C.

PROOF OF EXAMPLE D. It is known that there is a compact parallelizable 1-connected 4-manifold V in S^5 with a collar neighborhood such that index of V equals 8, V is a homotopy type of a wedge of 2-spheres and ∂V is a 3-dimensional homology 3-sphere. For example, if we take a Brieskorn variety $V(3, 5, 2)$ defined by a complex algebraic equation

$$Z_0^3+Z_1^5+Z_2^2=0\,,$$

then the intersection of $V(3, 5, 2)$ with a small 5-sphere S_ε^5 centered at the origin with radius $\varepsilon>0$ is a homology 3-sphere, which bounds such a 4-manifold in S_ε^5, (see Milnor [**9**]). Let $K=V\cup 0*\partial V$ be a closed 4-manifold obtained from V by attaching a cone along the boundary ∂V. Since ∂V is a homology 3-sphere, it follows that K is a homology 4-manifold and hence a Poincaré complex of dimension 4. Furthermore, K is clearly 1-connected. We will see that K is not of the homotopy type of a closed (PL) 4-manifold. Suppose that K is homotopy equivalent to a closed PL 4-manifold L. Note that since $K-\{0\}$ is PL homeomorphic to a parallelizable manifold $V-\partial V$, the second Stiefel-Whitney class $w_2(K)$ of K, which may be defined by the dual of Spivak normal fiber space, vanishes. By Rohlin's theorem, a closed almost parallelizable (or $w_2=0$) (PL) 4-manifold must have index divisible by 16. Since L is homotopy equivalent to K and hence $w_2(L)=0$, it follows that L has index 8. This contradiction proves that K is not of the homotopy type of a closed PL 4-manifold. On the other hand, K can be embedded in S^6. Indeed, we have seen that V is embedded in S^5. But we may take the cone cap $0*\partial V$ from the center of a 6-ball D^6 bounding S^5. Hence K is embedded in D^6 and hence S^6. Let N be a regular neighborhood of K in S^6. Then, since N is of the homotopy type of K, N is a 1-connected Poincaré complex of dimension 4. If N has a (PL) spine, then N and hence K has the homotopy type of a closed 4-manifold. This contradicts the assertion above. If N has a locally flat topological spine L, then by Kirby [7] L has a normal 2-disk bundle in S^6, which is trivial. By the topological t-regular approximation of a map with expected preimage of dimension 5 (Siebenmann [**13**]), one can find a compact topological 5-manifold in S^6 bounded by L. This implies that index of L equals 0. Since L is homotopy equivalent to K, index of L must equal 8. This contradiction proves that N has no locally flat topological spine, completing the proof.

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