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On an exotic PL automorphism of some 4-manifold and its application

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§1. Statement of the results.

Kirby and Siebenmann [5] proved that there are exotic PL structures on T^n ($n \geqq 5$). It is also known [2] that there is an exotic PL structure on $S^3 \times T^{n-3}$ ($n \geqq 5$). For $n \geqq 5$, there are "exotic" PL automorphisms of T^n and $S^2 \times T^{n-2}$ associated with the exotic PL structures on T^{n+1} and $S^3 \times T^{n-2}$.

In this paper, at first, we shall study the following problem:

Is there an "exotic" PL automorphism of some 4-manifold?

DEFINITION. Let M be a PL manifold and f a PL automorphism of M, i.e. a PL homeomorphism from M to itself. Then we say that f is exotic if f is topologically pseudo-isotopic to the identity, but not PL pseudo-isotopic to the identity.

We let:

$$M(k) = S^2 \times T^2 \# k(S^2 \times S^2),$$

$$V(k) = D^3 \times T^2 \natural k(D^3 \times S^2),$$

$$N(k) = S^2 \times S^1 \times I \# k(S^2 \times S^2).$$

Then one of our results is as follows:

THEOREM 1. *For some $k \geqq 0$, there is an exotic PL automorphism f of $M(k)$. Furthermore, any covering of f does not extend to a PL automorphism of the corresponding covering manifold of $V(k)$.*

This theorem means that we can realize the difference between the TOP category and the PL one on the 4-manifold.

Next, using f in Theorem 1, we shall construct a non-trivial element of certain 4-dimensional homotopy triangulation. We have:

THEOREM 2. *For some $k \geqq 0$, there is a non-trivial element in $hT(N(k), \partial N(k))$.*

This theorem is a partial answer to Shaneson's problem [4].

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§ 2. Proof of Theorem 1.

To prove Theorem 1, we use the following unknotting theorem moving the boundary by Hudson [3].

THEOREM 3. *Let M^m, Q^q be PL manifolds, $M, \partial M$ compact and connected. If $f, g: M \to Q$ are proper PL embeddings, f, g homotopic as maps of pairs $(M, \partial M) \to (Q, \partial Q)$; and if $q-m \geqq 3$, $(M, \partial M)$ is $(2m-q+1)$-connected, and if $(Q, \partial Q)$ is $(2m-q+2)$-connected, then f and g are ambient isotopic.*

PROOF OF THEOREM 1. It is known that we can give two different PL structures on $S^3 \times T^2$ [2]. We denote the standard one by α and the exotic one by β. Then "id": $(S^3 \times T^2)_\alpha \to (S^3 \times T^2)_\beta$ is a homeomorphism, but not PL. Let $D_0^3 \times T^2 \cup S^2 \times T^2 \times I \cup D_1^3 \times T^2$ be a decomposition of $(S^3 \times T^2)_\alpha$. We shall use the following notation:

$$\bar{X} = D_0^3 \times T^2,$$

$$\bar{H} = S^2 \times T^2 \times I,$$

$$\bar{Y} = D_1^3 \times T^2,$$

$$\partial_- \bar{H} = S^2 \times T^2 \times 0 = \partial D_0^3 \times T^2,$$

$$\partial_+ \bar{H} = S^2 \times T^2 \times 1 = \partial D_1^3 \times T^2.$$

By straightening the handles with index $\leqq 2$, we obtain a homeomorphism $g: (S^3 \times T^2)_\alpha \to (S^3 \times T^2)_\beta$ such that $g | (\bar{X} \cup \bar{Y}) = PL$. We define:

$$X = g(\bar{X}),$$

$$H = g(\bar{H}),$$

$$Y = g(\bar{Y}),$$

$$\partial_- H = g(\partial_- \bar{H}),$$

$$\partial_+ H = g(\partial_+ \bar{H}).$$

Then H is a PL h-cobordism of $S^2 \times T^2$ with itself. By the handlebody argument (see, e. g. [3]), H has the following decomposition:

$$H = (\partial_- H \cup h_1^2 \cup \cdots \cup h_n^2) \cup M(n) \times I \cup (\partial_+ H \cup k_1^2 \cup \cdots \cup k_n^2)$$

where h_i^2 and k_j^2 $(1 \leqq i, j \leqq n)$ are 2-handles attached trivially to $\partial_- H$ and $\partial_+ H$ respectively. We may assume $h_i^2 \cap h_j^2 = k_i^2 \cap k_j^2 = \emptyset$ $(i \neq j)$. Let

$$h_i: (D^2 \times D^3, S^1 \times D^3) \longrightarrow (h_i^2, h_i^2 \cap \partial_- H)$$

and

$$k_j: (D^2 \times D^3, S^1 \times D^3) \longrightarrow (k_j^2, k_j^2 \cap \partial_+ H) \qquad (1 \leqq i, j \leqq n)$$

be PL homeomorphisms. We shall use the following notation:

$$W_i=\partial_- H\cup h_1^2\cup\cdots\cup h_i^2 \qquad (1\leqq i\leqq n),$$
$$W=W_n\cup M(n)\times I,$$
$$V_j=\partial_+ H\cup k_1^2\cup\cdots\cup k_j^2 \qquad (1\leqq j\leqq n),$$
$$V=V_n\cup M(n)\times I,$$
$$\partial_+ V=\partial_+ H,$$
$$\partial_- V=\partial V-\partial_+ V.$$

On the other hand, we consider the decomposition of $\bar{H}=S^2\times T^2\times I$ as follows:

$$\bar{H}=(\partial_-\bar{H}\cup\bar{h}_1^2\cup\cdots\cup\bar{h}_n^2)\cup M(n)\times I\cup(\partial_+\bar{H}\cup\bar{k}_1^2\cup\cdots\cup\bar{k}_n)$$

where $\bar{h}_i^2$ and $\bar{k}_j^2$ $(1\leqq i,\ j\leqq n)$ are 2-handles attached trivially to $\partial_-\bar{H}$ and $\partial_+\bar{H}$ respectively. We may assume that $\bar{h}_i^2$ and $\bar{k}_i^2$ $(1\leqq i\leqq n)$ are a pair of complementary handles and $\bar{h}_i^2\cap\bar{h}_j^2=\bar{k}_i^2\cap\bar{k}_j^2=\emptyset$ $(i\neq j)$. Let

$$\bar{h}_i:\ (D^2\times D^3,\ S^1\times D^3)\longrightarrow(\bar{h}_i^2,\ \bar{h}_i^2\cap\partial_-\bar{H})$$

and

$$\bar{k}_j:\ (D^2\times D^3,\ S^1\times D^3)\longrightarrow(\bar{k}_j^2,\ \bar{k}_j^2\cap\partial_+\bar{H}) \qquad (1\leqq i,\ j\leqq n)$$

be PL homeomorphisms. We shall use the following notation:

$$\bar{W}_i=\partial_-\bar{H}\cup\bar{h}_1^2\cup\cdots\cup\bar{h}_i^2 \qquad (1\leqq i\leqq n),$$
$$\bar{W}=\bar{W}_n\cup M(n)\times I,$$
$$\bar{V}_j=\partial_+\bar{H}\cup\bar{k}_1^2\cup\cdots\cup\bar{k}_j^2 \qquad (1\leqq j\leqq n),$$
$$\bar{V}=\bar{V}_n\cup M(n)\times I,$$
$$\partial_+\bar{V}=\partial_+\bar{H},$$
$$\partial_-\bar{V}=\partial\bar{V}-\partial_+\bar{V}.$$

Now, by straightening the 2-handles, we obtain a homeomorphism g_0: $(S^3\times T^2)_\beta\to(S^3\times T^2)_\alpha$ such that:

(1) $g_0|(X\cup W_n\cup V_n\cup Y)=PL$,
(2) $g_0|(X\cup Y)=g^{-1}|(X\cup Y)$.

Clearly $g_0\circ h_1|D^2\times 0$ and $\bar{h}_1|D^2\times 0$ are homotopic as maps of pairs $(D^2\times 0,\ S^1\times 0)\to(\bar{H},\ \partial_-\bar{H})$. Hence, by Theorem 3 and the uniqueness of regular neighbourhoods, we get a homeomorphism g_0': $H\to\bar{H}$ such that:

(1) g_0' is isotopic to $g_0|H$,
(2) $g_0'(W_1)=\bar{W}_1$,
(3) $g_0'|(W_n\cup V_n)=PL$,

(4) $g_0'|\partial_-H$ is PL isotopic to $g_0|\partial_-H$,

(5) $g_0'|\partial_+H=g_0|\partial_+H$.

Using g_0', we can easily construct a homeomorphism $g_1:(S^3\times T^2)_\beta\to(S^3\times T^2)_\alpha$ such that:

(1) $g_1(X\cup W_1)=\bar{X}\cup\bar{W}_1$,

(2) $g_1|Y=g_0|Y$,

(3) $g_1|(X\cup W_n\cup V_n\cup Y)=PL$.

Now we let:

$$H(i)=H-\bigcup_{r=1}^{i}h_r(D^2\times\mathrm{int}\,D^3)\,,$$

$$\partial_-H(i)=\partial H(i)-\partial_+H\,,$$

$$\bar{H}(i)=\bar{H}-\bigcup_{r=1}^{i}\bar{h}_r(D^2\times\mathrm{int}\,D^3)\,,$$

$$\partial_-\bar{H}(i)=\partial\bar{H}(i)-\partial_+\bar{H}\,.$$

Then, in particular, $g_1(H(1))=\bar{H}(1)$.

Since $\pi_2(\bar{H}(1),\ \partial_-\bar{H}(1))=\pi_2(\partial_-\bar{H}(1)\cup 3\text{-handle},\ \partial_-\bar{H}(1))=0$, $g_1\circ h_2|D^2\times 0$ and $\bar{h}_2|D^2\times 0$ are homotopic as maps of pairs $(D^2\times 0,\ S^1\times 0)\to(\bar{H}(1),\ \partial_-\bar{H}(1))$. Hence, by Theorem 3 and the uniqueness of regular neighbourhoods, we get a homeomorphism g_1': $H(1)\to\bar{H}(1)$ such that:

(1) g_1' is isotopic to $g_1|H(1)$,

(2) $g_1'(h_2^2)=\bar{h}_2^2$,

(3) $g_1'|(\partial_-H(1)\cup h_2^2\cup\cdots\cup h_n^2\cup V_n)=PL$,

(4) $g_1'|\partial_-H(1)$ is PL isotopic to $g_1|\partial_-H(1)$,

(5) $g_1'|\partial_+H=g_1|\partial_+H$.

Using g_1', we can construct a homeomorphism g_2: $(S^3\times T^2)_\beta\to(S^3\times T^2)_\alpha$ such that:

(1) $g_2(X\cup W_2)=\bar{X}\cup\bar{W}_2$,

(2) $g_2|Y=g_0|Y$,

(3) $g_2|(X\cup W_n\cup V_n\cup Y)=PL$.

By succeeding the same process as above, observing $\pi_2(\bar{H}(i),\ \partial_-\bar{H}(i))=\pi_2(\partial_-\bar{H}(i)\cup 3\text{-handle},\ \partial_-\bar{H}(i))=0$, we obtain a homeomorphism g_n: $(S^3\times T^2)_\beta\to(S^3\times T^2)_\alpha$ such that:

(1) $g_n(X\cup W_n)=\bar{X}\cup\bar{W}_n$,

(2) $g_n|Y=g_0|Y$,

(3) $g_n|(X\cup W_n\cup V_n\cup Y)=PL$.

Moreover, by the general position argument and straightening a 1-handle, without loss of generality, we may impose the following condition (4) on g_n:

(4) For some properly embedded PL arc l in V which connects a point of $\partial_+ V$ and a point of $\partial_- V$, and for some regular neighbourhood $N(l)$ of l in V such that $N(l)\cap(\bigcup_{i=1}^{n} k_i^2)=g_n(N(l))\cap(\bigcup_{i=1}^{n}\bar{k}_i^2)=\emptyset$, $g_n|N(l)$ is PL.

Define $\bar{g}=g_n|V: V\to\bar{V}$. We connect a point in $k_i^2\cap\partial_+ V$ and the base point p in $\partial_+ V$ with an arc l_i in $\partial_+ V$, and regard $k_i^2\cup l_i$ as an element of $\pi_2(V, \partial_+ V, p)$ to be denoted by a_i $(1\leqq i\leqq n)$. Similarly we connect a point in $\bar{k}_i^2\cap\partial_+\bar{V}$ and $\bar{g}(p)$ with an arc $\bar{l}_i$ in $\partial_+\bar{V}$, and regard $\bar{k}_i^2\cup\bar{l}_i$ as an element of $\pi_2(\bar{V}, \partial_+\bar{V}, \bar{g}(p))$ to be denoted by b_i $(1\leqq i\leqq n)$.

By the isomorphism $(\bar{g})_*: \pi_1(\partial_+ V)\to\pi_1(\partial_+\bar{V})$, we identify $\pi_1(\partial_+\bar{V})$ with $\pi_1(\partial_+ V)$, and these will be denoted simply by π.

Then $(a_1, \cdots, a_n)$ and $(b_1, \cdots, b_n)$ are bases of free $Z[\pi]$-modules $\pi_2(V, \partial_+ V)$ and $\pi_2(\bar{V}, \partial_+\bar{V})$ respectively. We represent the $Z[\pi]$-module isomorphism $(\bar{g})_*: \pi_2(V, \partial_+ V)\to\pi_2(\bar{V}, \partial_+\bar{V})$ with a matrix $G=[g_{i,j}]$ by the above bases, where $g_{i,j}\in Z[\pi]$. It is well known that Whitehead group of $\pi=Z+Z$ is trivial. This implies that, for some $m\geqq 0$, $G\oplus 1_m=ED$ where E is a finite product of elementary matrices and

$$D=\begin{bmatrix} \pm\sigma & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} \quad \text{where } \sigma\in\pi\,.$$

We define h_{n+i}^2 and k_{n+j}^2 $(1\leqq i, j\leqq m)$ as 2-handles, in $N(l)$, attached trivially to $\partial_- V\cap N(l)$ and $\partial_+ V\cap N(l)$ respectively such that h_{n+i}^2 and k_{n+i}^2 are pair of complementary handles. Let

$$h_{n+i}: (D^2\times D^3, S^1\times D^3)\longrightarrow(h_{n+i}^2, h_{n+i}^2\cap\partial_- V)$$

and

$$k_{n+j}: (D^2\times D^3, S^1\times D^3)\longrightarrow(k_{n+j}^2, k_{n+j}^2\cap\partial_+ V) \quad (1\leqq i, j\leqq m)$$

be PL homeomorphisms. We shall use the following notation:

$$U=V-\bigcup_{i=1}^{m} h_{n+i}(D^2\times\operatorname{int} D^3)\,,$$

$$W_{n+i}=W_n\cup h_{n+1}^2\cup\cdots\cup h_{n+i}^2 \quad (1\leqq i\leqq m)\,,$$

$$\bar{U}=\bar{g}(U)\,,$$

$$\partial_+ U=\partial_+ V\,,$$

$$\partial_+\bar{U}=\partial_+\bar{V}\,,$$

$$\partial_- U=\partial U-\partial_+ U\,,$$

$$\bar{\bar{g}}=\bar{g}|U:\ U\longrightarrow\bar{U}\,,$$

$$\bar{k}^2_{n+i}=\bar{\bar{g}}(k^2_{n+i})\qquad(1\leqq i\leqq m)\,.$$

Now we connect a point in $k^2_{n+i}\cap\partial_+U$ and p with an arc l_{n+i} in ∂_+U, and regard $k^2_{n+i}\cup l_{n+i}$ as an element of $\pi_2(U,\partial_+U)$ to be denoted by a_{n+i} $(1\leqq i\leqq m)$. Similarly we regard $\bar{k}^2_{n+i}\cup\bar{\bar{g}}(l_{n+i})$ as an element of $\pi_2(\bar{U},\partial_+\bar{U})$ to be denoted by b_{n+i} $(1\leqq i\leqq m)$. We may regard a_i and b_i $(1\leqq i\leqq n)$ as elements of $\pi_2(U,\partial_+U)$ and $\pi_2(\bar{U},\partial_+\bar{U})$ naturally.

Then, by the bases $(a_1,\cdots,a_{n+m})$ and $(b_1,\cdots,b_{n+m})$, the $Z[\pi]$-module isomorphism $(\bar{\bar{g}})_*:\ \pi_2(U,\partial_+U)\to\pi_2(\bar{U},\partial_+\bar{U})$ is represented with the matrix $G\oplus 1_m=ED$.

By the similar argument in the handle addition theorem (see, e.g. [3], p. 228 and 250), we obtain a new handlebody decomposition of U satisfying the following conditions (1) and (2):

(1) $U=\partial_+U\cup(k^2_1)'\cup\cdots\cup(k^2_{n+m})'\cup M(n+m)\times I$ where $(k^2_i)'$ $(1\leqq i\leqq n+m)$ is a 2-handle attached trivially to ∂_+U and $(k^2_i)'\cap(k^2_j)'=\emptyset$ $(i\neq j)$.

(2) We connect a point of $(k^2_i)'\cap\partial_+U$ and p with an arc l'_i in ∂_+U, and regard $(k^2_i)'\cup l'_i$ as an element of $\pi_2(U,\partial_+U)$ to be denoted by a'_i $(1\leqq i\leqq n+m)$. Then, by the bases $(a'_1,\cdots,a'_{n+m})$ and $(b_1,\cdots,b_{n+m})$, $(\bar{\bar{g}})_*:\ \pi_2(U,\partial_+U)\to\pi_2(\bar{U},\partial_+\bar{U})$ is represented with the matrix
$$1_{n+m}=\begin{bmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{bmatrix}.$$

Now we define:

$$U_i=\partial_+U\cup(k^2_1)'\cup\cdots\cup(k^2_i)'\qquad(1\leqq i\leqq n+m)\,,$$
$$\bar{U}_i=\partial_+\bar{U}\cup\bar{k}^2_1\cup\cdots\cup\bar{k}^2_i\qquad(1\leqq i\leqq n+m)\,,$$
$$\overline{W}_{n+m}=\overline{W}_n\cup\bar{h}^2_{n+1}\cup\cdots\cup\bar{h}^2_{n+m}$$

where $\bar{h}^2_{n+i}=g_n(h^2_{n+i})$ $(1\leqq i\leqq m)$. Let

$$\bar{k}_{n+i}:\ (D^2\times D^3,\ S^1\times D^3)\longrightarrow(\bar{k}^2_{n+i},\ \bar{k}^2_{n+i}\cap\partial_+\bar{U})\qquad(1\leqq i\leqq m)$$

and

$$(k_i)':\ (D^2\times D^3,\ S^1\times D^3)\longrightarrow((k^2_i)',\ (k^2_i)'\cap\partial_+U)\qquad(1\leqq i\leqq n+m)$$

be PL homeomorphisms. Then $\bar{\bar{g}}\circ(k_1)'|D^2\times 0$ and $\bar{k}_1|D^2\times 0$ are homotopic as maps of pairs $(D^2\times 0,\ S^1\times 0)\to(U,\partial_+U)$ because of the condition (2) above. Hence, by Theorem 3 and the uniqueness of regular neighbourhoods, we get a homeomorphism $f_0:\ U\to\bar{U}$ such that:

(1) f_0 is isotopic to $\bar{\bar{g}}$,

(2) $f_0((k^2_1)')=\bar{k}^2_1$,

(3) $f_0 | U_{n+m} = PL$,
(4) $f_0 | \partial_+ U$ is PL isotopic to $\bar{g} | \partial_+ U$,
(5) $f_0 | \partial_- U = \bar{g} | \partial_- U$.

Then, using f_0, we can construct a homeomorphism $f_1 : (S^3 \times T^2)_\beta \to (S^3 \times T^2)_\alpha$ such that:

(1) $f_1(X \cup W_{n+m}) = \bar{X} \cup \bar{W}_{n+m}$,
(2) $f_1(Y \cup U_1) = \bar{Y} \cup \bar{U}_1$,
(3) $f_1 | (X \cup W_{n+m} \cup U_{n+m} \cup Y) = PL$.

We define:

$$U(i) = U - \bigcup_{r=1}^{i} (k_r)'(D^2 \times \operatorname{int} D^2),$$

$$\bar{U}(i) = \bar{U} - \bigcup_{r=1}^{i} \bar{k}_r(D^2 \times \operatorname{int} D^2),$$

$$\partial_+ U(i) = \partial U(i) - \partial_- U,$$

$$\partial_+ \bar{U}(i) = \partial \bar{U}(i) - \partial_- \bar{U}.$$

By a_i'' and b_i' $(2 \leqq i \leqq n+m)$, we denote the elements in $\pi_2(U(1), \partial_+ U(1))$ and $\pi_2(\bar{U}(1), \partial_+ \bar{U}(1))$ which corresponds to a_i' and b_i respectively. Then we get

$$(f_1 | U(1))_* a_i'' = b_i' \quad (2 \leqq i \leqq n+m) \quad \text{in } \pi_2(\bar{U}(1), \partial_+ \bar{U}(1))$$

from the following diagram,

$$\begin{array}{ccccc}
0 \longrightarrow & \pi_2(U_1, \partial_+ U) & \xrightarrow{\ i\ } & \pi_2(U, \partial_+ U) & \\
& \Big\downarrow (f_1 | U_1)_* = 1 & & \Big\downarrow (f_1 | U)_* = 1_{n+m} & \\
0 \longrightarrow & \pi_2(\bar{U}_1, \partial_+ \bar{U}) & \xrightarrow{\ i\ } & \pi_2(\bar{U}, \partial_+ \bar{U}) &
\end{array}$$

$$\begin{array}{ccccc}
& & & & \pi_2(U(1), \partial_+ U(1)) \\
& & \swarrow^{k} & & \\
\xrightarrow{\ j\ } & \pi_2(U, U_1) & \longrightarrow 0 \text{ (exact)} & & \\
& \Big\downarrow (f_1 | U)_* = 1_{n+m-1} & & & \Big\downarrow (f_1 | U(1))_* \\
\xrightarrow{\ \bar{j}\ } & \pi_2(\bar{U}, \bar{U}_1) & \longrightarrow 0 \text{ (exact)} & & \\
& & \nwarrow_{\bar{k}} & & \\
& & & & \pi_2(\bar{U}(1), \partial_+ \bar{U}(1))
\end{array}$$

where i, $\bar{i}$, j, $\bar{j}$, k and $\bar{k}$ are induced from the inclusion maps.

Then, in particular, $f_1 \circ (k_2)' | D^2 \times 0$ and $\bar{k}_2 | D^2 \times 0$ are homotopic as maps of pairs $(D^2 \times 0, S^1 \times 0) \to (U(1), \partial_+ U(1))$. Hence, as before, we obtain a homeomorphism $f_2 : (S^3 \times T^2)_\beta \to (S^3 \times T^2)_\alpha$ such that:

(1) $f_2(X \cup W_{n+m}) = \bar{X} \cup \bar{W}_{n+m}$,
(2) $f_2(Y \cup U_2) = \bar{Y} \cup \bar{U}_2$,
(3) $f_2 | (X \cup W_{n+m} \cup U_{n+m} \cup Y) = PL$.

We can repeat this process untill we obtain a homeomorphism f_{n+m}: $(S^3 \times T^2)_\beta \to (S^3 \times T^2)_\alpha$ such that:

(1) $f_{n+m}(X \cup W_{n+m}) = \bar{X} \cup \bar{W}_{n+m}$,
(2) $f_{n+m}(Y \cup U_{n+m}) = \bar{Y} \cup \bar{U}_{n+m}$,
(3) $f_{n+m} | (X \cup W_{n+m} \cup U_{n+m} \cup Y) = PL$.

We put:

$$f = (f_{n+m} | M(n+m) \times 1)^{-1} \circ (f_{n+m} | M(n+m) \times 0) \quad \text{and} \quad k = n+m\,.$$

We may regard f a PL automorphism of $M(k)$. Then f is exotic, since $(S^3 \times T^2)_\beta$ is not PL homeomorphic to $(S^3 \times T^2)_\alpha$.

Observing that any covering of $(S^3 \times T^2)_\beta$ is also exotic [2], we can easily prove the latter part of Theorem 1. Q. E. D.

§ 3. Proof of Theorem 2.

To prove Theorem 2, we use the following theorem by Shaneson [6].

THEOREM 4. *Let M be an oriented, closed 4-manifold with $\pi_1(M) = Z$. Then every PL automorphism of M, homotopic to the identity, is PL pseudo-isotopic to the identity.*

PROOF OF THEOREM 2. By Theorem 1 and Siebenmann's weak pseudo-isotopy theorem [7], we obtain a PL automorphism f of $M(k)$ and a topological isotopy $F_t : M(k) \times S^1 \to M(k) \times S^1$ $(t \in I)$ such that:

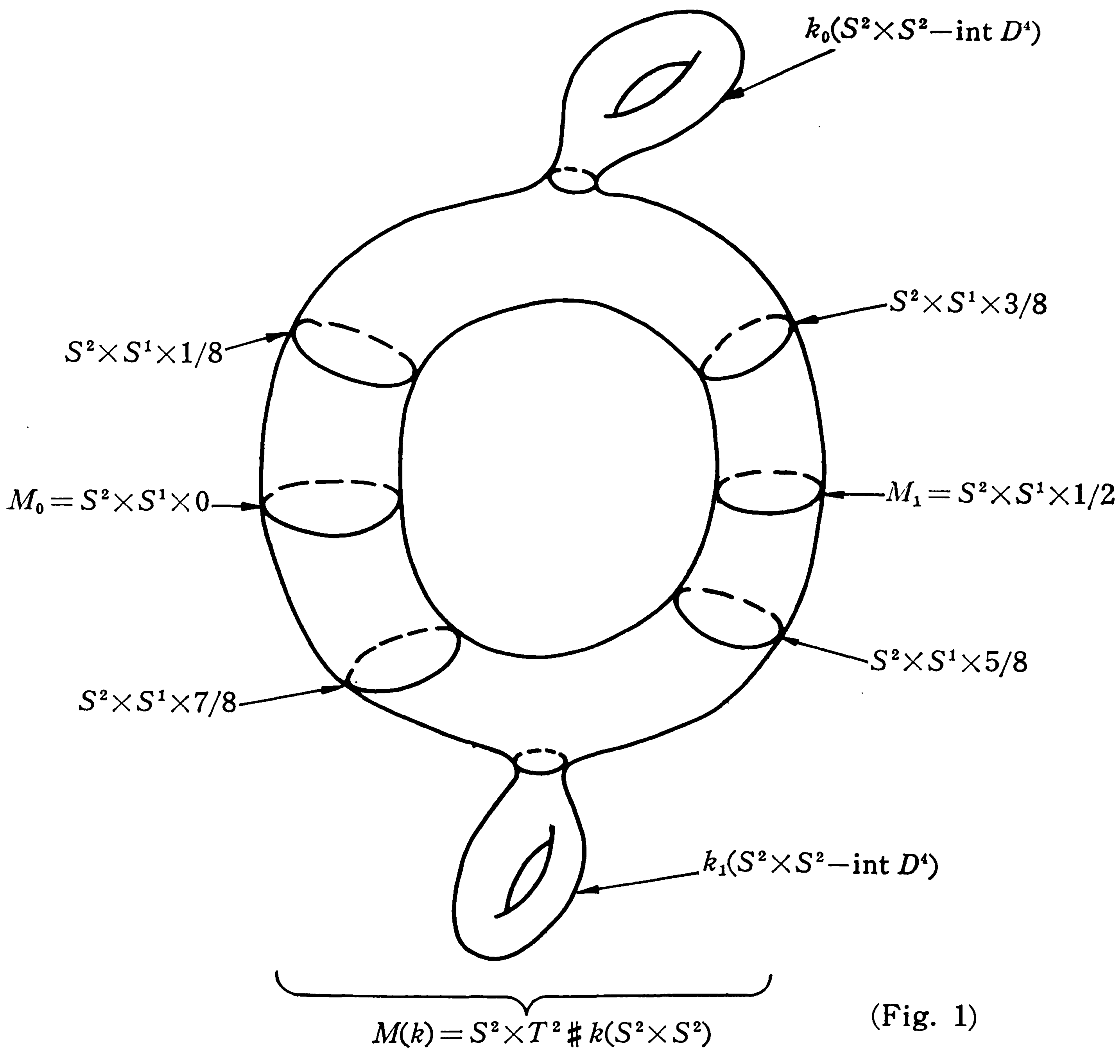
(1) f is exotic,
(2) f does not extend to a PL automorphism of $V(k)$,
(3) $F_0 = id_{M(k) \times S^1}$ and $F_1 = f \times id_{S^1}$.

We regard $S^2 \times T^2$ as $S^2 \times S^1 \times R/Z$, and assume that any $S^2 \times S^2$ is connected to $S^2 \times S^1 \times R/Z$ at $S^2 \times S^1 \times ([1/8, 3/8] \cup [5/8, 7/8])$ whenever we consider $S^2 \times T^2 \# k(S^2 \times S^2) = M(k)$. We let:

$$M_0 = S^2 \times S^1 \times 0 \subset M(k)$$

and

$$M_1 = S^2 \times S^1 \times 1/2 \subset M(k) \qquad \text{(see Fig. 1)}\,.$$



(Fig. 1)

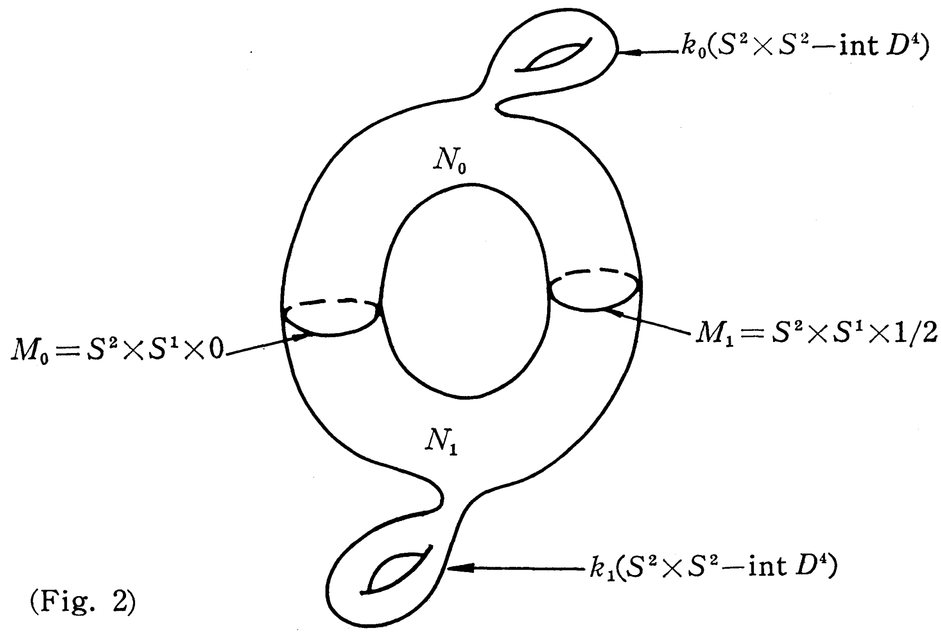
Then, without loss of generality, we may assume that $F_t(M_0 \times S^1) \cap M_1 \times S^1 = \emptyset$ for any $t \in I$, because of the latter part of Theorem 1. Hence, by Edward and Kirby's covering isotopy theorem [1], we obtain an isotopy $G_t : M(k) \times S^1 \to M(k) \times S^1$ $(t \in I)$ such that:

(1) $G_t | M_0 \times S^1 = F_t | M_0 \times S^1$,

(2) $G_t | M_1 \times S^1 = id_{M_1 \times S^1}$,

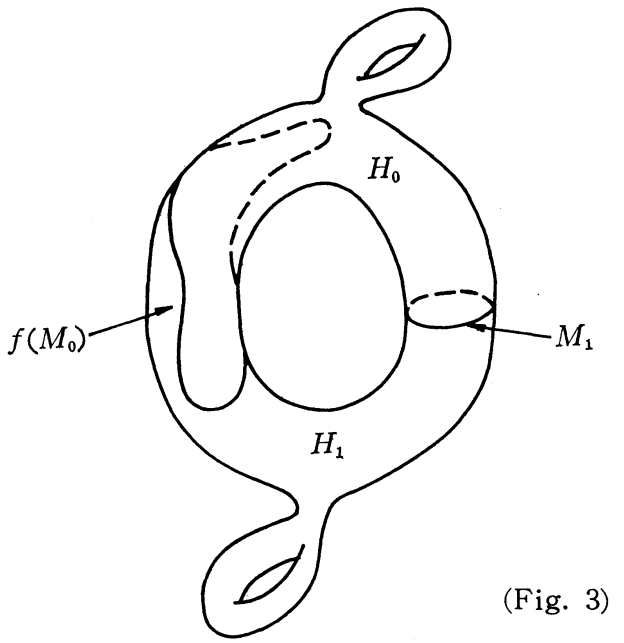
(3) $G_0 = id_{M(k) \times S^1}$.

By N_0 and N_1, we denote the manifold in $M(k)$ bounded by M_0 and M_1 (Fig. 2).



(Fig. 2)

Then $N_i=S^2\times S^1\times[i/2, (i+1)/2]\# k_i(S^2\times S^2)\subset M(k)$ $(i=0, 1)$ where $k_0+k_1=k$. Similarly, by H_0 and H_1, we denote the manifolds in $M(k)$ bounded by $f(M_0)$ and M_1 (Fig. 3).

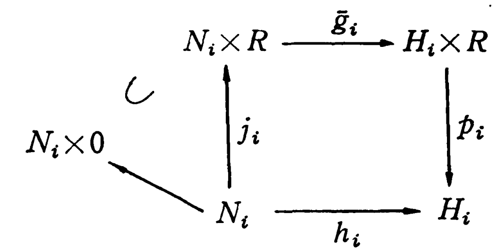


(Fig. 3)

Define

$$g_i=G_1|N_i\times S^1:\ N_i\times S^1\longrightarrow H_i\times S^1 \qquad (i=1, 0),$$

and let $\bar{g}_i:\ N_i\times R\to H_i\times R$ be an infinite cyclic covering of g_i $(i=1, 0)$. We define $h_i=p_i\circ\bar{g}_i\circ j_i$ $(i=1, 0)$ where j_i and p_i are the inclusion and projection maps as in the following diagram.



Then $h_i | \partial N_i = f | M_0 \cup id_{M_1}$, and, observing that $\bar{g}_i$ is a homeomorphism, we can regard (H_i, h_i) as an element of $hT(N(k_i), \partial N(k_i))$.

If we suppose that both (H_0, h_0) and (H_1, h_1) are trivial, then we obtain a PL automorphism h of $M(k)$ such that:

(1) $h | M_0 = f | M_0$,

(2) $h | M_1 = id_{M_1}$,

(3) h is homotopic to the identity fixing M_1,

(4) $f^{-1} \circ h$ is homotopic to the identity fixing M_0.

We define

$$M = D_0^3 \times S^1 \cup S^2 \times S^1 \times I \# k(S^2 \times S^2) \cup D_1^3 \times S^1$$

where $\partial(D_0^3 \times S^1)$, $\partial(D_1^3 \times S^1)$ are identified with $S^2 \times S^1 \times 0$ and $S^2 \times S^1 \times 1$ respectively, and

$$f_1 = id_{D_0^3 \times S^1} \cup f^{-1} \circ h \cup id_{D_1^3 \times S^1}.$$

Then f_1 is a PL automorphism of M which is homotopic to the identity. Hence f_1 is pseudo-isotopic to the identity. Thus, using f_1, we can construct a PL automorphism of $V(k)$ which is an extension of $f^{-1} \circ h$.

Similarly h extends to a PL automorphism of $V(k)$. Hence f itself extends to a PL automorphism of $V(k)$. This contradicts the latter part of Theorem 1. Hence (H_0, h_0) or (H_1, h_1) is non-trivial. Q. E. D.

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