Approximation of nonlinear semigroups and evolution equations

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§ 1. Introduction.

Consider a sequence of abstract Cauchy problems

$$(d/dt)u_n(t) \in A_n(t)u_n(t) \qquad (t \geq 0), \ u_n(0) = x_n, \ n = 0, 1, 2, \cdots \qquad (1.1)$$

in an arbitrary Banach space X. Here $A_n(t)$ is for each t a multi-valued function defined on a subset of X. We shall show that under suitable hypotheses, if x_n converges to x_0 and if $A_n(t)$ converges to $A_0(t)$ (in a sense to be made precise below), then $u_n(t)$ converges to $u_0(t)$.

We first deal with the case when the multi-valued function A_n does not depend on t and determines a strongly continuous semigroup of Lipschitzian operators on a subset of X. In Section 3, using a generation theorem of Crandall and Liggett [4], we obtain nonlinear generalizations of the Trotter-Neveu-Kato approximation theorem for semigroups. These extend results of a number of authors, including Brezis and Pazy [2], Mermin [14], Miyadera [15], and Miyadera and Ôharu [17]. Moreover, our result is best possible in the sense that our sufficient condition is necessary in the linear case.

In Section 4 we establish existence and uniqueness criteria for a special class of time dependent multi-valued Cauchy problems of the form

$$(d/dt)u(t) \in A(t)u(t) \qquad (t \geq 0), \ \ u(0) = x \qquad (1.2)$$

in a Hilbert space. We also prove an approximation theorem in this situation.

Finally, using existence theorems of Crandall and Liggett [4] and Martin [12], we establish in Section 5 approximation theorems for a class of problems of the form (1.1) in an arbitrary Banach space setting.

§ 2. Notation.

Let X be a Banach space with norm $\|\cdot\|$. When X is a Hilbert space, its inner product will be denoted by $\langle \cdot, \cdot \rangle$. "lim" [resp. "$w$-lim"] refers to limit in the norm [resp. weak] topology of X. $\mathscr{P}(X)$ denotes the set of all subsets of X. R denotes the real numbers, R^+ the nonnegative reals, Z^+ the non-

negative integers, and N the positive integers. $cl(S)$ denotes the closure of a set S.

For A, $B \subset X \times X$, $k \in R$ set

$$A^{-1} = \{(y, x) : (x, y) \in A\}, \quad kA = \{(x, ky) : (x, y) \in A\},$$

$$A + B = \{(x, y+z) : (x, y) \in A, (x, z) \in B\},$$

$$\mathcal{D}(A) = \{x : (x, y) \in A \quad \text{for some } y\},$$

$$\mathcal{R}(A) = \{y : (x, y) \in A \quad \text{for some } x\}, \quad Ax = \{y : (x, y) \in A\},$$

$$A \mid D = \{(x, y) \in A : x \in D\} \quad \text{where } D \subset X,$$

$$|D| = \inf \{\|x\| : x \in D\} \quad \text{where } D \subset X.$$

We do not distinguish between (single-valued or multi-valued) functions and their graphs; for instance, we identify the identity operator on X with the diagonal: $I = \{(x, x) : x \in X\}$. We do not distinguish between the singleton $\{x\} \in \mathcal{P}(X)$ and the element x. For $C_n \in X$, $n \in Z^+$, $\lim_{n \to \infty} C_n \supset C_0$ means for each $x_0 \in C_0$, there is an $x_n \in C_n$ for each $n \in N$ such that $\lim_{n \to \infty} x_n = x_0$.

Let $C \in X$. A function $T : C \to X$ is *Lipschitzian* if and only if there is a constant K such that $\|Tx - Ty\| \leq K \|x - y\|$ for all $x, y \in C$. The smallest possible value of K will be denoted by $\|T\|_{\text{Lip}}$.

A set $A \subset X \times X$ is called *dissipative* if and only if for each $\alpha > 0$, $(I - \alpha A)^{-1}$ is a function and $\|(I - \alpha A)^{-1}\|_{\text{Lip}} \leq 1$. If X is a real Hilbert space, then $A \subset X \times X$ is dissipative if and only if for all $(x_i, y_i) \in A$, $i = 1, 2$, $\langle x_1 - x_2, y_1 - y_2 \rangle \leq 0$. In other words, A is dissipative if and only if $-A$ is monotone. Also, in the general Banach space case, A is dissipative if and only if $-A$ is accretive, and a characterization of dissipativity can be given in terms of the duality map (or the semi-inner products) of X (cf. Kato [10, p. 141]).

§ 3. Nonlinear semigroups in a Banach space.

Let X be a Banach space, let $C \subset X$, and let $\omega \in R$. $T = \{T(t) : t \in R^+\}$ is a *strongly continuous semigroup of type ω on C*, or $T \in Q_\omega(C)$ for short, if and only if for all $s, t \in R^+$, $x \in C$,

(i) $T(t) : C \to C$,

(ii) $T(t)T(s)x = T(t+s)x$, $T(0)x = x$,

(iii) $\|T(t)\|_{\text{Lip}} \leq e^{\omega t}$,

(iv) $\lim_{t \to 0} T(t)x = x$.

Crandall and Liggett [4] recently established the following definitive generation theorem for semigroups of type ω.

THEOREM 3.1. *Let $A \subset X \times X$, $\omega \in R$ be such that $A - \omega I$ is dissipative and*

$\mathscr{R}(I-\alpha(A-\omega I)) \supset C$ for all sufficiently small $\alpha > 0$, where $C = cl(\mathscr{D}(A))$. Then A determines a semigroup $T \in Q_\omega(C)$ given by the exponential formula

$$T(t)x = \lim_{n\to\infty} (I-tn^{-1}A)^{-n}x$$

for all $x \in C$, $t \in R^+$.

For our approximation theorems we shall make the following hypothesis.

(*) Let $\omega \in R$. For each $n \in Z^+$ let $A_n \subset X \times X$, let $C_n = cl(\mathscr{D}(A_n))$, and suppose $A_n - \omega I$ is dissipative and $\mathscr{R}(I-\alpha(A_n-\omega I)) \supset C_n$ for $0 < \alpha < \alpha_0$, where α_0 is independent of n.

In addition to (*) we shall make assumptions about the sense in which A_n is to converge to A_0. Our conclusion will then be:

(**) Let $T_n = \{T_n(t): t \in R^+\} \in Q_\omega(C_n)$ be the semigroup determined by A_n as in Theorem 3.1. Then

$$\lim_{n\to\infty} T_n(t)x_n = T_0(t)x_0 \tag{3.1}$$

whenever $x_n \in C_n$, $\lim_{n\to\infty} x_n = x_0 \in C_0$; moreover, the convergence is uniform for t in compact subsets of R^+.

Here are our main results on semigroups.

THEOREM 3.2. Let (*) hold. Suppose that $\lim_{n\to\infty} C_n \supset C_0$; and if $x_n \in C_n$, $x_0 \in C_0$, and $\lim_{n\to\infty} x_n = x_0$, then

$$\lim_{n\to\infty} (I-\alpha(A_n-\omega I))^{-1}x_n = (I-\alpha(A_0-\omega I))^{-1}x_0 \tag{3.2}$$

for $0 < \alpha < \alpha_1$, α_1 being independent of n. Then (**) holds.

THEOREM 3.3. Let (*) hold. Suppose $C_n \supset C_0$ for each $n \in N$ and that

$$\lim_{n\to\infty} (I-\alpha(A_n-\omega I))^{-1}x_0 = (I-\alpha(A_0-\omega I))^{-1}x_0 \tag{3.3}$$

for all $x_0 \in C_0$ and $0 < \alpha < \alpha_1$, α_1 being independent of n. Then (**) holds.

Note that

$$(I-\alpha(A_n-\omega I))^{-1} = \beta(I-\alpha\beta A)^{-1}$$

where $\beta = (1+\alpha\omega)^{-1}$, and so (3.2) holds if and only if

$$\lim_{n\to\infty} (I-\alpha A_n)^{-1}x_n = (I-\alpha A_0)^{-1}x_0 \tag{3.2'}$$

for all x_n, x_0 as in (**) and $0 < \alpha < \alpha_2$, where α_2 is independent of n. The reason for preferring (3.2) to (3.2') is to emphasize that the study of the general case (i. e. $T_n \in Q_\omega(C_n)$) reduces to the contraction case (i. e. $T_n \in Q_0(C_n)$), or, equivalently, it suffices to treat the case that each A_n is dissipative.

We shall show that Theorem 3.3 is a special case of Theorem 3.2.

In the linear case (i. e. when each A_n and $T_n(t)$ are linear and each $C_n = X$), (3.1) (with $x_n = x_0$ for all n) implies (3.3), which in turn implies (3.2). This

can be seen by noting the formula

$$\alpha(I-\alpha A_n)^{-1}x = \int_0^\infty e^{-t/\alpha}T_n(t)x\,dt$$

for $0 < \alpha < 1/\omega$ (or $0 < \alpha$ if $\omega \leq 0$) and using the dominated convergence theorem (cf. Chernoff [3], Goldstein [6, p. 39]). It follows that (3.1) is necessary and sufficient for (3.3) (and also for (3.2)) in the linear case. In this sense, then, Theorem 3.2 gives the "best" sufficient condition for (**) to hold.

First we prove that Theorem 3.3 is a special case of Theorem 3.2. Let $B_n = A_n - \omega I$. Whenever $x_n \in C_n$, $\lim\limits_{n \to \infty} x_n = x_0 \in C_0$, we have

$$\|(I-\alpha B_n)^{-1}x_n - (I-\alpha B_0)^{-1}x_0\| \leq \|(I-\alpha B_n)^{-1}x_n - (I-\alpha B_n)^{-1}x_0\|$$

$$+ \|(I-\alpha B_n)^{-1}x_0 - (I-\alpha B_0)^{-1}x_0\| = J_1 + J_2,$$

where $J_1 \leq \|x_n - x_0\|$ and $J_2 \to 0$ as $n \to \infty$ by hypothesis. Hence the hypotheses of Theorem 3.3 imply those of Theorem 3.2, which is what we sought to show.

The point of considering the more complicated condition (3.2) rather than (3.3) is to allow the sets C_n to vary as much as possible.

Now we prove Theorem 3.2. The idea of the proof is to deduce the theorem from the Crandall-Liggett generation theorem (Theorem 3.1) by viewing a convergent sequence of semigroups on X as a single semigroup on the space of convergent sequences in X. This idea goes back to Kisyński [11].

Let

$$\mathcal{X} = \{\underline{x} = \{x_n\}_1^\infty \subset X : \lim_{n \to \infty} x_n \text{ exists}\}.$$

$\mathcal{X}$ is a (nonreflexive) Banach space with norm $\|\|\underline{x}\|\| = \sup\limits_n \|x_n\|$. Define $\mathcal{A} \subset \mathcal{X} \times \mathcal{X}$ by: $(\underline{x}, \underline{y}) \in \mathcal{A}$ (where $\underline{x} = \{x_n\}_1^\infty$, $\underline{y} = \{y_n\}_1^\infty \in \mathcal{X}$) if and only if $x_n \in \mathcal{D}(A_n)$, $y_n \in A_n x_n$ for all $n \in N$ and $\lim\limits_{n \to \infty} x_n \in C_0$. Let $B_n = A_n - \omega I$, $\mathcal{B} = \mathcal{A} - \omega I$. It is easy to see that $(I-\alpha \mathcal{B})^{-1}$ is a function for each $\alpha > 0$. To show that $\mathcal{B}$ is dissipative, let $\alpha > 0$, $\underline{x}, \underline{y} \in \mathcal{R}(I-\alpha \mathcal{B})$; then

$$\|\|(I-\alpha \mathcal{B})^{-1}\underline{x} - (I-\alpha \mathcal{B})^{-1}\underline{y}\|\| = \sup_n \|(I-\alpha \mathcal{B}_n)^{-1}x_n - (I-\alpha \mathcal{B}_n)^{-1}y_n\|$$

$$\leq \sup_n \|x_n - y_n\| = \|\|\underline{x} - \underline{y}\|\|$$

since each B_n is dissipative. Hence $\mathcal{B}$ is dissipative on $\mathcal{X}$.

Next we show that $\mathcal{R}(I-\alpha \mathcal{B}) \supset C$ for all sufficiently small $\alpha > 0$, where $C = cl(\mathcal{D}(\mathcal{B})) = cl(\mathcal{D}(\mathcal{A}))$. By hypothesis there is an $\alpha_2 > 0$ such that $\mathcal{R}(I-\alpha B_n) \supset C_n$ and (3.2) holds for $n \in \mathbf{Z}^+$ and $0 < \alpha < \alpha_2$. (Just take $\alpha_2 = \min(\alpha_0, \alpha_1)$.) Let $\underline{x} \in C$ and $0 < \alpha < \alpha_2$ be given. For each $n \in N$ choose $y_n = (I-\alpha B_n)^{-1}x_n$

$\in \mathcal{D}(A_n)$. Then $\underline{y} = \{y_n\}_1^\infty$ is convergent by (3.2), and hence $\underline{y} \in \mathcal{X}$. Thus there is a $\underline{y} \in \mathcal{D}(\mathcal{A})$ such that $(I-\alpha\mathcal{B})^{-1}\underline{x} = \underline{y}$. It follows from Theorem 3.1 that $\mathcal{A}$ determines a semigroup $\mathcal{T} = \{\mathcal{T}(t) : t \in \mathbf{R}^+\} \in Q_\omega(C)$. We have $\mathcal{T}(t)\underline{x} = \{T_n(t)x_n\}_1^\infty$ for $t \in \mathbf{R}^+$, $\underline{x} \in C$. Since $\mathcal{T}(t)\underline{x} \in \mathcal{X}$, it follows that (3.1) holds whenever $\underline{x} \in C$ and $x_0 = \lim\limits_{n \to \infty} x_n$.

We next show that if $x_n \in C_n$ and $\lim\limits_{n \to \infty} x_n = x_0 \in C_0$, then $\underline{x} = \{x_n\}_1^\infty \in C$. Let $\varepsilon > 0$ be given. For each $n \in \mathbf{Z}^+$ choose $y_n = (I - t_n B_n)^{-1} x_n \in \mathcal{D}(A_n)$; here $\{t_n\}_0^\infty$ is a sequence of positive numbers which will be chosen presently.

Let $N_1 = N_1(\varepsilon, \underline{x})$ be such that $\|x_n - x_0\| < \varepsilon/3$ for $n > N_1$. Chooes and fix $\sigma = \sigma(\varepsilon, x_0) > 0$ such that $\|(I - \sigma B_0)^{-1} x_0 - x_0\| < \varepsilon/3$; this can be done by [4, Lemma 1.2 (ii)]. Choose $N_2 = N_2(\varepsilon, \underline{x})$ such that $\|(I - \sigma B_n)^{-1} x_n - (I - \sigma B_0)^{-1} x_0\| < \varepsilon/3$ for $n > N_2$; this can be done by (3.2). It follows that

$$\|(I - \sigma B_n)^{-1} x_n - x_n\| \leqq \|(I - \sigma B_n)^{-1} x_n - (I - \sigma B_0)^{-1} x_0\|$$

$$+ \|(I - \sigma B_0)^{-1} x_0 - x_0\| + \|x_n - x_0\| < \varepsilon$$

for $n > N_3 = \max(N_1, N_2)$. For $1 \leqq n \leqq N_3$ choose $t_n > 0$ so small (by [4, Lemma 1.2 (ii)]) that $\|(I - t_n B_n)^{-1} x_n - x_n\| < \varepsilon$, and set $t_n = \sigma$ for $n = 0$ and for $n > N_3$. It follows that $\underline{y} = \{y_n\}_1^\infty \in \mathcal{X}$ (since $\lim\limits_{n \to \infty} y_n = \lim\limits_{n \to \infty} (I - \sigma B_n)^{-1} x_n = (I - \sigma B_0)^{-1} x_0 = y_0$), $z_n = t_n^{-1}((I - t_n B_n)^{-1} x_n - x_n) \in B_n y_n$ for all $n \in \mathbf{Z}^+$, and $\lim\limits_{n \to \infty} z_n = \sigma^{-1}((I - \sigma B_0)^{-1} x_0 - x_0) \in B_0 y_0$, so that actually $\underline{y} \in \mathcal{D}(\mathcal{B}) = \mathcal{D}(\mathcal{A})$. Furthermore,

$$\|\|\underline{y} - \underline{x}\|\| = \sup_n \|(I - t_n B_n)^{-1} x_n - x_0\| \leqq \varepsilon$$

by the above estimates. Since $\varepsilon > 0$ is arbitrary, it follows that $\underline{x} \in C$, and in fact, $C = cl(\mathcal{D}(\mathcal{A}))$.

(We note, parenthetically, that we cannot take $y_n = (I - \sigma B_n)^{-1} x_n$ for all $n \in \mathbf{Z}^+$ and some $\sigma > 0$. To see this let $X = \mathbf{R}$ and define $A_n = B_n$ by $A_n(x) = -x/n$ for $|x| \leqq n$, $A_n(x) = -n^2 x + (n^2 - 1) x |x|^{-1}$ for $|x| > n$. Then all the hypotheses of Theorems 3.2 and 3.3 hold, and for $\underline{x} \in \mathcal{X}$, $\lim\limits_{n \to \infty} (I - \sigma A_n)^{-1} x_n = (I - \sigma A_0)^{-1} x_0$ for each $\sigma > 0$, $\lim\limits_{\sigma \to 0} (I - \sigma A_n)^{-1} x_n = x_n$ for each $\sigma > 0$, and yet $\inf\limits_{\sigma \geqq 0} \sup\limits_n \|(I - \sigma A_n)^{-1} x_n - x_n\| \geqq 1$ if $x_n \geqq n + 1$ for some $n \in N$.)

It only remains to prove the uniformity assertion in (∗∗). To that end, let $\varepsilon > 0$, $\tau > 0$, $\underline{x} \in C$ be given. Then by the strong continuity of $\mathcal{T}$,

$$\|\|\mathcal{T}(t)\underline{x} - \mathcal{T}(s)\underline{x}\|\| < \varepsilon/3$$

for $0 < s$, $t \leqq \tau$ and $|t - s| < 2\delta = 2\delta(\varepsilon, \underline{x}, \tau)$. Next,

$$\|T_n(t)x_n - T_0(t)x_0\| < \varepsilon/3$$

for $N > N_t = N(t, \varepsilon, \underline{x})$ since $\mathcal{T}(t)\underline{x} \in \mathcal{X}$. Therefore

$$\|T_n(s)x_n - T_0(s)x_0\|$$

$$\leqq \|T_n(s)x_n - T_n(t)x_n\| + \|T_n(t)x_n - T_n(t)x_0\| + \|T_n(t)x_0 - T_n(s)x_0\| < \varepsilon$$

if $0 \leqq s,\ t \leqq \tau,\ |s-t| < 2\delta$, and $n > N_t$. Let $k \in N$ be such that $(k-1)\delta < t \leqq k\delta$ and let $N = \max\{N_0, N_\delta, \cdots, N_{k\delta}\}$. Then we have

$$\|T_n(s)x_n - T_0(s)x_0\| < \varepsilon$$

whenever $0 \leqq s \leqq \tau,\ n > N$; and N depends only on ε, x, and τ. This completes the proof.

REMARK 3.4. Much of the literature on nonlinear semigroups deals with Banach spaces with certain smoothness properties. For instance, many authors only consider spaces with a uniformly convex dual. However, the space $\mathfrak{X}$ we constructed in the above proof is always a "bad" space (viz. it is nonreflexive), no matter how "good" the space X is. Thus our proof depends crucially on the availability of a generation theorem which takes place in an arbitrary Banach space.

In the following result we replace condition (3.2) (or (3.3)) by a condition of the form: $\lim_{n \to \infty} A_n y_0 = A_0 y_0$ for sufficiently many $y_0 \in \mathscr{D}(A_0)$; the above limit must be carefully interpreted since each A_n is in general multi-valued.

THEOREM 3.5. *Let* (*) *hold. Let* $D \subset \mathscr{D}(A_0)$ *be such that* $cl(D) = C_0$, $\mathscr{R}(I - \alpha(A_0 - \omega I)\,|\,D)$ *is dense in* C_0 *for all sufficiently small* $\alpha > 0$, *and* $D \subset \mathscr{D}(A_n)$ *for each* $n \in N$. *Suppose that for each* $y_0 \in D$, *each sufficiently small* $\alpha > 0$, *and each* $x_0 \in (I - \alpha(A_0 - \omega I))y_0$, *there is an* $x_n \in (I - \alpha(A_n - \omega I))y_0$ *such that* $\lim x_n = x_0$. *Then* (**) *holds.*

This follows from Theorem 3.3, once we show that (3.3) holds for all sufficiently small $\alpha > 0$ and all $x_0 \in D_\alpha$, where each D_α is dense in C_0. Again let $B_n = A_n - \omega I,\ n \in Z^+$. Let $\alpha > 0$ be sufficiently small, and let $D_\alpha = \mathscr{R}(I - \alpha B_0\,|\,D)$. Then D_α is dense in C_0 by hypothesis. Moreover, for each $x_0 \in D_\alpha$, choose $y_0 \in D$ such that $x_0 \in (I - \alpha B_0)y_0$. Next choose $x_n \in (I - \alpha B_n)y_0$ such that $\lim_{n \to \infty} x_n = x_0$. Then

$$\|(I - \alpha B_n)^{-1}x_0 - (I - \alpha B_0)^{-1}x_0\| = \|(I - \alpha B_n)^{-1}x_0 - (I - \alpha B_n)^{-1}x_n\|$$

$$\leqq \|x_n - x_0\| \to 0 \qquad \text{as } n \to \infty,$$

and thus (3.3) holds.

REMARK 3.6. The scope of Theorems 3.2 and 3.3 is wider than that of Theorem 3.5. This is true even in the linear case, since there exists a sequence of skew-adjoint (linear) operators $\{A_n: n \in Z^+\}$ on a Hilbert space X such that $\lim_{n \to \infty}(I - \alpha A_n)^{-1}x_0 = (I - \alpha A_0)^{-1}x_0$ for all $x_0 \in X$ and all $\alpha \in R \backslash \{0\}$, but nevertheless $\mathscr{D}(A_0) \cap \bigcup_{n=1}^{\infty} \mathscr{D}(A_n) = \{0\}$ (cf. Goldstein [7]).

We conclude this section with a result on the convergence of the solutions of the Cauchy problems (1.1) with $A_n(t)$ not depending on t.

THEOREM 3.7. *Let X be reflexive. Let the hypotheses of Theorem 3.2 (or 3.3 or 3.5) hold. Assume additionally that $A_n \subset X \times X$ is closed.*

(i) *If $x_n \in \mathcal{D}(A_n)$, $u_n(\cdot)$ defined by $u_n(t) = T_n(t)x_n$ is the unique function satisfying: $u_n(\cdot)$ is Lipschitz continuous on compact intervals in R^+, $u_n(\cdot)$ is strongly differentiable a. e., $u_n(0) = x_n$, and $(d/dt) u_n(t) \in A_n u_n(t)$ a. e.*

(ii) *If $x_n \in \mathcal{D}(A_n)$ for all $n \in Z^+$ and $\lim\limits_{n \to \infty} x_n = x_0$, then $\lim\limits_{n \to \infty} u_n(t) = u_0(t)$ for all $t \in R^+$, the convergence being uniform for t in compact subsets of R^+.*

(i) is due to Crandall and Liggett [4, Theorem II], Miyadera [30, Theorem 1]; (ii) follows immediately from Theorem 3.2 (or 3.3 or 3.5). We note, as did Crandall and Liggett [4], that we need not assume that X is reflexive if instead we assume that each A_n is single-valued and either linear or continuous. In fact, in the continuous case, we have the following result.

THEOREM 3.8. *Let $\{A_n\}_0^\infty$ be a sequence of continuous operators defined on all of X such that $A_n - \omega I$ is dissipative for some real ω and all $n \in Z^+$. The Cauchy problem*

$$(d/dt)u_n(t) = A_n u_n(t), \qquad u_n(0) = x_n$$

has a unique strongly continuously differentiable solution $u_n \colon R^+ \to X$ for each $x_n \in X$ and each $n \in Z^+$. If

$$\lim_{n \to \infty} (I - \alpha(A_n - \omega I))^{-1}x_0 = (I - \alpha(A_0 - \omega I))^{-1}x_0$$

for all $x_0 \in X$ and $0 < \alpha < \alpha_0$, α_0 being independent of x_0, then

$$\lim_{n \to \infty} u_n(t) = u_0(t)$$

for each $t \in R^+$ whenever $\lim\limits_{n \to \infty} x_n = x_0$; here u_n is as above. Moreover, the convergence is uniform for t in compact subset of R^+.

The proof is the same as that of Theorem 3.2, only instead of using the Crandall-Liggett generation theorem, we use one due to Martin [13] and Webb [22]. We omit the details.

§ 4. Some time dependent multi-valued equations in Hilbert space.

Let H be a real Hilbert space. A dissipative subset of $H \times H$ is maximal if it is not properly contained in any other dissipative set.

THEOREM 4.1. *Let $\omega \in R$. Let $A \subset H \times H$ be such that $A - \omega I$ is maximal dissipative. Let $F \colon R^+ \times \mathcal{D}(A) \to H$ satisfy a global Lipschitz condition:*

$$\|F(t, x) - F(s, y)\| \le K\{\|x - y\| + |t - s|\}$$

for some constant K and all (t, x), $(s, y) \in \mathbf{R}^+ \times \mathcal{D}(A)$. Then the initial value problem

$$(d/dt)u(t) \in Au(t) + F(t, u(t)) \qquad (t \in \mathbf{R}^+), \tag{4.1}$$

$$u(0) = x \in \mathcal{D}(A) \tag{4.2}$$

has a unique solution $u : \mathbf{R}^+ \to H$ in the following sense:

(i) *u is Lipschitz continuous on compact subintervals of $\mathbf{R}^+$, $u(\mathbf{R}^+) \subset \mathcal{D}(A)$, $u(0) = x$,*

(ii) *u has a strongly right continuous right derivative satisfying*

$$(d^+/dt^+)u(t) \in Au(t) - F(t, u(t)) \qquad (t \in \mathbf{R}^+),$$

(iii) *the strong derivative $(d/dt)u(t)$ exists and is strongly continuous on $\mathbf{R}^+$ except for at most countably many values of t.*

The idea behind the proof is the trivial classical trick which converts a nonautonomous ordinary differential equation to an autonomous one (see e. g., [21, p. 34]).

Define $F(t, x) = F(-t, x)$ for $t < 0$, $x \in \mathcal{D}(A)$. Let $H_1 = H \times \mathbf{R}$. Vectors in H_1 will be denoted by column vectors such as $\binom{x}{t}$, $x \in H$, $t \in \mathbf{R}$. With respect to the inner product

$$\left\langle \binom{x}{t}, \binom{y}{s} \right\rangle_1 = \langle x, y \rangle + ts,$$

H_1 becomes a real Hilbert space. The corresponding norm will be denoted by $\| \cdot \|_1$. Define $D \subset H_1 \times H_1$ by

$$D = \left\{ \left(\binom{x}{t}, \binom{y}{1} \right) : x \in \mathcal{D}(A), \ t \in \mathbf{R}, \ y \in Ax + F(t, x) \right\}.$$

Then for $\left(\binom{x_i}{t_i}, \binom{y_i}{1} \right) \in D$, $i = 1, 2$, our hypotheses imply

$$\left\langle \binom{x_1}{t_1} - \binom{x_2}{t_2}, \binom{y_1}{1} - \binom{y_2}{1} \right\rangle_1$$

$$= \langle x_1 - x_2, z_1 - z_2 \rangle + \langle x_1 - x_2, F(t_1, x_1) - F(t_2, x_2) \rangle$$

$$\text{for some } z_i \in Ax_i, \ i = 1, 2$$

$$\geqq -\omega \| x_1 - x_2 \|^2 - 2K(\| x_1 - x_2 \|^2 + |t_1 - t_2|^2)$$

$$\geqq -\sigma \left\| \binom{x_1}{t_1} - \binom{x_2}{t_2} \right\|_1^2$$

where $\sigma = \omega + 2K$. Hence $B - \sigma I$ is dissipative. Moreover, it is maximal dissipative by a perturbation theorem of Crandall and Pazy [5, p. 403]. It follows from a generation theorem of Crandall and Pazy [5], [18] that D determines a semigroup $S = \{S(t) : t \in \mathbf{R}^+\} \in Q_\omega(cl(\mathcal{D}(A)))$ given by the exponential

formula of Theorem 3.1. Moreover, it follows easily from the properties that Crandall and Pazy [5], [18] established for S that if we set $\binom{u(t)}{t} = S(t)\binom{x}{0}$ with $x \in \mathscr{D}(A)$, then u is the unique solution of (4.1), (4.2) in the sense of (i), (ii), and (iii).

The next result is a generalization of Theorem 4.1 to the case when F is multi-valued.

THEOREM 4.2. *Let* $\omega \in R$. *Let* $A \subset H \times H$ *be such that* $A - \omega I$ *is maximal dissipative. Let* $F: R^+ \times S \to \mathscr{P}(H)$ *where* $\mathscr{D}(A) \subset S \subset H$ *satisfy:*

(F1)   *There are constants* M_1, M_2, *a nondecreasing function* k_1 *on* R^+ *satisfying* $k_1(r) < 1$ *for each* $r \in R^+$, *and a function* k_2 *on* $R^+ \times R^+$ *to* R^+ *such that*

(i)   *for each* $t_i \in R^+$, $x_i \in S$, $y_i \in F(t_i, x_i)$, $i = 1, 2$, $\langle x_1 - x_2, y_1 - y_2 \rangle \geqq -M_1\{\|x_1 - x_2\|^2 + |t_1 - t_2|^2\}$,

(ii)   *for each* $t_i \in R^+$, $x_i \in \mathscr{D}(A)$, $z_i \in Ax_i$, $y_i \in F(t_i, x_i)$, $i = 1, 2$, $\|y_1 - y_2\| \leqq M_2\{|t_1 - t_2| + \|x_1 - x_2\|\} + k_1(\|x_1\| + \|x_2\|)\,\|z_1 - z_2\| + k_2(t_1, t_2)$.

(F2)   *For each* $t \in R^+$ *and each* $y \in H$, *there is a sequence* $\{n_i\}$ *in* R^+ *tending to infinity, and there is a sequence* $\{x_i\}$ *in* S *such that* $y \in F(t, x_i) + n_i x_i$, $i \in N$. *Then the initial value problem* (4.1), (4.2) *has a unique solution* u *in the sense of* (i), (ii), *and* (iii) *of Theorem 4.1.*

The proof is similar to that of Theorem 4.1, but a little more complicated. One shows that $D - \tau I$ is dissipative, where τ is a suitable constant. To show that $D - \tau I$ is maximal dissipative, one uses a perturbation theorem of Brezis, Crandall and Pazy [1, p. 233]. We omit the details.

REMARK 4.3. Condition (F2) is superfluous if F is single-valued by [5, p. 403]. It follows that Theorem 4.1 is a special case of Theorem 4.2.

The next two theorems are approximation theorems in the situation of Theorem 4.1. (We omit the analogous result based on Theorem 4.2.) The basic situation is described as follows:

(+)   *Let* $\omega \in R$. *For each* $n \in Z^+$ *let* $A_n \subset H \times H$ *be such that* $A_n - \omega I$ *is maximal dissipative. For each* $n \in Z^+$ *let* $F_n: R^+ \times \mathscr{D}(A_n) \to H$ *satisfy a global Lipschitz condition:*

$$\|F(t, x) - F(s, y)\| \leqq K\{\|x - y\| + |t - s|\}$$

for some constant K *(independent of* n*) and all* (t, x), $(s, y) \in R^+ \times \mathscr{D}(A_n)$.

Our conclusion will be:

(++)   *For* $n \in Z^+$ *let* u_n *be the unique solution of the initial value problem*

$$(d/dt)u_n(t) \in A_n u_n(t) + F_n(t, u_n(t)) \qquad (t \in R^+), \quad u_n(0) = x_n \in \mathscr{D}(A_n) \qquad (4.3)$$

in the sense of Theorem 4.1. If $\lim_{n \to \infty} x_n = x_0$, *then* $\lim_{n \to \infty} u_n(t) = u_0(t)$ *for all* $t \in R^+$, *the convergence being uniform on compact subsets of* R^+.

Note that if (+) holds, then u_n as described in (++) exists by Theorem 4.1.

THEOREM 4.3. *Let* (+) *hold. Suppose that* $\lim\limits_{n\to\infty} C_n \supset C_0$ *(where* $C_n = cl(\mathcal{D}(A_n))$*) and suppose that*

$$\lim_{n\to\infty} (I - \alpha(A_n + F_n(t, \cdot) - \sigma I))^{-1} x_0 = (I - \alpha(A_0 + F_0(t, \cdot) - \sigma I))^{-1} x_0$$

for all $x_0 \in C_0$ *and all sufficiently small* $\alpha > 0$*; here* $\sigma = 2K + \omega$*. Then* (++) *holds.*

Let $F_n(t, x) = F_n(-t, x)$ for $t < 0$, $x \in \mathcal{D}(A_n)$, and let H_1 be as in the proof of Theorem 4.1. Define $D_n \subset H_1 \times H_1$ by $D_n = \left\{ \left(\binom{x}{t}, \binom{y}{1} \right) : x \in \mathcal{D}(A_n), t \in R, y \in A_n x + F_n(t, x) \right\}$. Let $E_n = D_n - \sigma I$. Then E_n is maximal dissipative by the proof of Theorem 4.1. It is easy to see that $\left(\binom{x}{t}, \binom{y}{s} \right) \in (I - \alpha E_n)$ if and only if $s = t - \alpha$, $y \in x - \alpha(A_n x + F_n(t, x) - \sigma x)$; hence $(I - \alpha E_n)^{-1} \binom{y}{s} = \binom{z}{t}$ where $t = x + \alpha$, $z = (1 - \alpha \sigma)^{-1} (I - \beta(A_n + F_n(t, \cdot) - \sigma I))^{-1} y$; here $\beta = \alpha(1 - \alpha \sigma)^{-1}$. It follows from Theorem 3.2 that the semigroup determined by D_n converges strongly to the semigroup determined by D_0. Since these semigroups govern (4.3), the conclusion of the theorem follows easily (cf. the last sentence of the proof of Theorem 4.1).

THEOREM 4.4. *Let* (+) *hold. Let* $D \subset \mathcal{D}(A_0)$ *be such that* $cl(D) = C_0$ $(= cl(\mathcal{D}(A_0)))$, $\mathcal{R}(I - \alpha(A_0 - \omega I) | D)$ *is dense in* C_0 *for all sufficiently small* $\alpha > 0$, *and* $D \in \mathcal{D}(A_n)$ *for each* $n \in N$*. Suppose that for each* $y_0 \in D$*, each sufficiently small* $\alpha > 0$*, and each* $x_0 \in (I - \alpha(A_0 - \omega I)) y_0$*, there is an* $x_n \in (I - \alpha(A_n - \omega I)) x_0$ *such that* $\lim\limits_{n\to\infty} x_n = x_0$*. Suppose further that* $\lim\limits_{n\to\infty} F_n(t, x_0) = F_0(t, x_0)$ *for each* $t \in R^+$ *and each* $x_0 \in D$*. Then* (++) *holds.*

The proof is essentially the same as that of Theorem 4.3, but it is based on Theorem 3.5 rather than Theorem 3.2. We omit the details.

§ 5. Time dependent equations in an arbitrary Banach space.

The proof of the approximation theorems of Section 3 can be easily adapted to the case of time dependent evolution equations. The existence theorem we shall use is the following result of Crandall and Liggett [4, Appendix to Section 1].

THEOREM 5.1. *Let* $\omega \in R$*. For each* $t \in J = [0, T]$ *let* $A(t) \subset X \times X$ *be such that* $A(t) - \omega I$ *is dissipative. In addition, assume the following conditions.*

(i) $\mathcal{D}(A(t))$ *is independent of* t.

(ii) $\mathcal{R}(I - \alpha(A(t) - \omega)) \supset C = cl(\mathcal{D}(A(0)))$ *for* $0 < \alpha < \alpha_0$ *and* $t \in J$.

(iii) $|A(t)x| \leq |A(\tau)x| + |t - \tau| L(\|x\|)(1 + |A(\tau)x|)$ *for* $t, \tau \in J$, $x \in \mathcal{D}(A(0))$.

(iv) $\|(I-\alpha(A(t)-\omega I))^{-1}x-(I-\alpha(A(\tau)-\omega I))^{-1}x\| \leq \alpha\,|t-\tau|\,L(\|x\|+|A(\tau)x|)$
for $t,\ \tau \in J,\ 0 < \alpha < \alpha_0,\ x \in \mathscr{D}(A(0))$.
Here $L:\ \mathbf{R}^+ \to \mathbf{R}^+$ is an increasing function. Then

$$U(t,\ s)x = \lim_{n\to\infty} \prod_{i=1}^{n} [I+(t-s)n^{-1}(A(s+i(t-s)n^{-1})-\omega I)]^{-1}x$$

exists for $x \in C$ and $0 \leq s \leq t \leq T$. Here the product is an ordered product:
$\prod_{i=1}^{n} T_i x = T_n T_{n-1} \cdots T_1 x$. For $x \in C$ and $0 \leq r \leq s \leq t \leq T$ we have

   (a) $U(t,\ s):\ C \to C$,

   (b) $U(t,\ s)U(s,\ r)x = U(t,\ r)x,\ U(t,\ t)x = x$,

   (c) $\|U(t,\ s)\|_{\mathrm{Lip}} \leq e^{\omega(t-s)}$,

   (d) $\lim_{t\to s} U(t,\ r)x = U(s,\ r)x$.

$U = \{U(t,\ x):\ 0 \leq s \leq t \leq T\}$ is called the *evolution operator* determined by $A(\cdot)$. u defined by $u(t) = U(t,\ 0)x$ is the candidate for the unique solution of (1.2), although (1.2) need not have a solution in general, even if $A(t)$ does not depend on t. (For counterexamples see [4, Section 4].)

THEOREM 5.2. *For each* $n \in \mathbf{Z}^+$ *and each* $t \in J = [0,\ T]$, *let* $A_n(t) \subset X \times X$ *be such that all the hypotheses of Theorem 5.1 hold with* $\omega,\ L,\ and\ \alpha_0$ *not depending on* n. *Let* $C_n = cl(\mathscr{D}(A_n(t)))$, *assume* $\lim_{n\to\infty} C_n \supset C_0$, *and suppose that for* $x_n \in C_n,\ \lim_{n\to\infty} x_n = x_0 \in C_0$ *we have*

$$\lim_{n\to\infty} (I-\alpha(A_n(t)-\omega I))^{-1}x_n = (I-\alpha(A_0(t)-\omega I))^{-1}x_0 \qquad (5.1)$$

for $t \in J,\ 0 < \alpha < \alpha_0$. *Finally assume that if* $x_n \in \mathscr{D}(A_n(0))$, $\lim_{n\to\infty} x_n \in C_0$, *and* $t \in J$, *then there is a* $y_n \in A_n(0)x_n$ *such that* $\lim_{n\to\infty} y_n$ *exists if and only if there is a* $z_n \in A_n(t)x_n$ *such that* $\lim_{n\to\infty} z_n$ *exists. Let* $U_n = \{U_n(t,\ s):\ 0 \leq s \leq t \leq T\}$ *be the evolution operator determined by* $A_n(\cdot)$. *Then*

$$\lim_{n\to\infty} U_n(t,\ s)x_n = U_0(t,\ s)x_0 \qquad (5.2)$$

whenever $x_n \in C_n,\ \lim_{n\to\infty} x_n = x_0$; *moreover, the limit is uniform in* t *on* $[s,\ T]$.

We shall use the notation of the proof of Theorem 3.2. For $t \in J$ define $\mathscr{A}(t) \subset \mathscr{X} \times \mathscr{X}$ by: $(\underline{x},\ \underline{y}) \in \mathscr{A}(t)$ (where $\underline{x},\ \underline{y} \in \mathscr{X}$) if and only if $x_n \in \mathscr{D}(A_n(t))$, $y_n \in A_n(t)x_n$ for all $n \in N$, and $\lim_{n\to\infty} x_n \in C_0$. Let $\mathscr{B}(t) = \mathscr{A}(t)-\omega I$; then it follows from the proof of Theorem 3.2 that $\mathscr{B}(t)$ is dissipative and $\mathscr{R}(I-\alpha\mathscr{B}(t)) \supset C = cl(\mathscr{D}(\mathscr{B}(0)))$ for all sufficiently small $\alpha > 0$. Moreover, $\mathscr{D}(\mathscr{B}(t))$ is independent of t by the last hypothesis of the theorem.

Next, let $N(B) = \inf\{\|\|\underline{x}\|\|:\ \underline{x} \in B\}$ for $B \subset \mathscr{H}$. For $t,\ \tau \in J,\ \underline{x} \in \mathscr{D}(\mathscr{A}(0))$ we have

$$N(\mathcal{A}(t)\underline{x}) = \sup_n |A_n(t)x_n|$$

$$\leqq \sup_n \left(|A_n(\tau)x_n| + |t-\tau|L(\|x_n\|)(1+|A_n(\tau)x_n|)\right) \quad \text{by (iii)}$$

$$\leqq N(\mathcal{A}(\tau)\underline{x}) + |t-\tau|L(\|\underline{x}\|)\ (1+N(\mathcal{A}(\tau)\underline{x})).$$

Next, for t, $\tau \in J$, $0 < \alpha < \alpha_0$, and $\underline{x} \in \mathcal{D}(\mathcal{A}(0))$ we have

$$\|(I-\alpha\mathcal{B}(t))^{-1}\underline{x} - (I-\alpha\mathcal{B}(\tau))^{-1}\underline{x}\|$$

$$= \sup_n \|(I-\alpha B_n(t))^{-1}x_n - (I-\alpha B_n(\tau))^{-1}x_n\|$$

$$\leqq \sup_n \left(\alpha|t-\tau|L(\|x_n\| + |A_n(\tau)x_n|)\right) \quad \text{by (iv)}$$

$$\leqq \alpha|t-\tau| + L(\|\underline{x}\| + N(\mathcal{A}(\tau)\underline{x})).$$

It follows from Theorem 5.1 that $\mathcal{U} = \{\mathcal{U}(t, s): 0 \leqq s \leqq t \leqq T\}$, the evolution operator determined by $\mathcal{A}(\cdot)$, exists and satisfies (a)-(d).

We have $\mathcal{U}(t, s)\underline{x} = \{U_n(t, s)x_n\}_1^\infty$ for $\underline{x} \in C$, and thus (5.2) holds for $\underline{x} \in C$ (with $x_0 = \lim_{n\to\infty} x_n$). The rest of the proof is very much like that of Theorem 3.2, so it will be omitted.

The next approximation theorem will be one in which the solution of the Cauchy problems exist and converge. We shall require an existence theorem of Martin [12], which is a generalization of an earlier result of Kato [8]. The basic assumption is as follows.

(§) (α) *Let* $\{A(t): t \in J = [0, T]\}$ *be a family of (single-valued) functions on* X *such that* $A(t) - \omega(t)I$ *is dissipative, where* ω *is continuously differentiable on* J, $\mathcal{R}(I - \alpha(A(t) - \omega(t)I)) = X$ *for all* $\alpha > 0$, *and* $D = \mathcal{D}(\mathcal{A}(t))$ *does not depend on* t.

(β) *Furthermore, if* $\{x_n\}_1^\infty \subset D$, $\lim_{n\to\infty} x_n = x_0$, $\sup_n \|A(t)x_n\| < \infty$, *then* $x_0 \in D$ *and* $w\text{-}\lim_{n\to\infty} A(t)x_n = A(t)x_0$.

(γ) *There is a constant* K *such that if* Q *is a bounded subset of* D *with* $\sup\{\|A(t)x\|: t \in J, x \in Q\} < \infty$, *then for every* $\varepsilon > 0$, $\beta > 0$ *there exists a* $\delta = \delta(\varepsilon, \beta, Q) > 0$ *such that*

$$\|x-y-b(A(t)x-A(t)y)\| \leqq (1+Kb)\|x-y\| + \varepsilon b$$

for $0 < b \leqq \delta$, $\|x-y\| > \beta$, $x, y \in Q$, $t \in J$.

THEOREM 5.3 [12]. *Let* (§) *hold. Suppose further there is a continuous function* $d: J \times J \times R^+ \to R^+$ *such that* $d(t, s, \cdot)$ *is nondecreasing for each* $t, s \in J$ *and such that*

$$\|A(t)x - A(s)x\| \leqq |t-s|d(t, s, \|x\|)\ (1+\|A(t)x\|)$$

for all $t, s \in J$, $x \in D$. *Then there exists an evolution operator* U *satisfying* (a).

(b), (d) *of Theorem* 5.1 *and*

$$\text{(c')} \quad \|U(t, s)\|_{\text{Lip}} \le \exp\left(\int_s^t \omega(\tau) d\tau\right) \quad \text{for} \quad 0 \le s \le t \le T.$$

If $s \in J$, $x \in D$, *then* $u(\cdot) = U(\cdot, s)x$ *is the unique weakly continuously differentiable solution of*

$$u_w'(t) = A(t)u(t) \quad (s \le t \le T), \quad u(s) = x$$

[*here* u_w' *denotes the weak derivative of* u]; *furthermore,* $A(\cdot)u(\cdot)$ *is Bochner integrable,*

$$u(t) = x + \int_s^t A(\tau)u(\tau) d\tau$$

for all $t \in [s, T]$, *and the strong derivative* $u'(t)$ *exist for almost all* t *and* $u'(t) = A(t)u(t)$ *a. e. in* $[s, T]$.

This theorem was proved by Kato [8] assuming that X^* is uniformly convex, but not assuming (β), (γ). The above version, due to Martin [12], is a generalization of Kato's result since a dissipative operator A satisfying $\mathcal{R}(I - \alpha A) = X$ for all $\alpha > 0$ automatically satisfies (β), (γ) if X^* is uniformly convex (cf. [8, p. 512], [12, p. 416]). However, as Kato has noted, a linear dissipative operator A satisfying $\mathcal{R}(I - \alpha A) = X$ for all $\alpha > 0$ need not satisfy (β), (γ) if X is an arbitrary Banach space.

THEOREM 5.4. *For each* $n \in \mathbf{Z}^+$ *let* $\{A_n(t): t \in J\}$ *be a family of operators satisfying the hypothesis of Theorem* 5.3 *with* ω, K, δ, d *being independent of* n. *Denote* $\mathcal{D}(A_n(t))$ *by* D_n. *Assume*

(i) $\lim_{n \to \infty} (I - \alpha(A_n(t) - \omega(t)I))^{-1} x_0 = (I - \alpha(A_0(t) - \omega(t)I))^{-1} x_0$ *for all* $x_0 \in X$, $t \in J$, $\alpha > 0$,

(ii) *if for* $n \in N$, $t \in J$, $x_n \in D_n$, $\lim_{n \to \infty} x_n = x_0$ *exists and* $\sup_n \|A_n(t)x_n\| < \infty$, *then* $x_0 \in D_0$ *and* $\lim_{n \to \infty} A_n(t)x_n = A_0(t)x_0$. *Then* $\lim_{n \to \infty} U_n(t, s)x_n = U_0(t, s)x_0$, *uniformly for* $t \in [s, T]$, *where* U_n *is the evolution operator determined by* $A_n(\cdot)$, *whenever* $\{x_n\}_1^\infty$ *is a sequence in* X *converging to* x_0 *and satisfying: for each* $\varepsilon > 0$ *there is a convergent sequence* $\{x_n^\varepsilon\}_1^\infty$ *such that* $\|x_n - x_n^\varepsilon\| < \varepsilon$, $x_n \in D_n$ *for all* $n \in N$, *and* $\sup_n \|A_n(0)x_n^\varepsilon\| < \infty$.

The unsatisfactory feature of the above theorem is condition (ii) which is rather restrictive. Theorem 5.4, like all of our other approximation theorems, is deduced from an existence theorem (Theorem 5.3 in this case) by viewing a convergent sequence of evolution operators on X as a single evolution operator on $\mathcal{X}$. We omit the details of proof except for the observation that the dual space of $\mathcal{X}$ is

$$\mathcal{X}^* = \{\underline{\phi}: \{\phi_n\}_0^\infty \subset X^*: \|\underline{\phi}\| = \sum_{n=0}^\infty \|\phi_n\| < \infty\},$$

and for $\underline{\phi} \in \mathcal{X}^*$, $\underline{x} \in \mathcal{X}$,

$$\underline{\phi}(\underline{x}) = \sum_{n=1}^{\infty} \phi_n(x_n) + \phi_0(\lim_{n\to\infty} x_n) \,.$$

This follows from a representation theorem of Singer [19], [20].

We thank Robert H. Martin, Jr., for showing us how to fix up an error in an early version of the proof of Theorem 5.4.

Finally, our method of proof in the linear case yields the following result.

THEOREM 5.5. *For each $n \in \mathbf{Z}^+$ let $\{A_n(t): t \in J\}$ be a family of linear operators on X satisfying (α) of (§), where ω does not depend on n. Suppose $B_n = A_n(\cdot)(I - (A_n(0) - \omega_0 I))^{-1}$ is strongly continuously differentiable on J, where $\omega_0 = \sup \{|\omega(t)| : t \in J\}$; and the strong derivative of B_n is bounded independent of $t \in J$, $n \in \mathbf{Z}^+$. Finally assume that*

$$\lim_{n\to\infty} (I - \alpha(A_n(t) - \omega(t)I))^{-1}x = (I - \alpha(A_0(t) - \omega(t)I))^{-1}x$$

for all $x \in X$, $t \in J$. Let U_n denote the evolution operator determined by A_n. Then

$$\lim_{n\to\infty} U_n(t, s)x_n = U_0(t, x)x_0 \,,$$

uniformly for $t \in [s, T]$, whenever $\{x_n\}_1^\infty$ is a convergent sequence in X with limit x_0. Let $\{f_1\}_0^\infty$ be a boundedly convergent sequence of continuous X-valued functions on J, and let $\lim_{n\to\infty} x_n = x_0$. Then

$$\lim_{n\to\infty} u_n(t) = u_0(t)$$

for $t \in [s, T]$, where u_n is the unique mild solution of

$$u_n'(t) = A_n(t)u_n(t) + f_n(t) \qquad (s \le t \le T), \quad u_n(s) = x_n \,;$$

and the convergence is uniform if the convergence of $\{f_n\}_1^\infty$ is uniform.

" Mild solution " means solution of the associated integral equation. This theorem complements some approximation theorems of Kato [9].

REMARKS. (1) After this paper was submitted we learned of some related preprints. Brezis and Pazy [25] overlaps with Section 3. Fitzgibbon [27], [28] and Crandall and Pazy [26] overlap with Section 5. Our proofs are quite different from those of the above authors. We thank these authors for sending us their papers, and we thank Crandall and Pazy for some useful comments concerning the present paper.

(2) If $T_n(t) = \lim_{m\to\infty} (I - tm^{-1}A_n)^{-1}$ and if (3.1) holds, then we cannot expect (3.2) to hold since T_n does not determine A_n uniquely (cf. [4]). Thus the sufficient condition (3.2) cannot be necessary in general. However, if the dual space X^* is uniformly convex, then T_n determines A_n uniquely (cf. Brezis [24]), and the necessity of (3.2) might be valid in this case. Bénilan

[23] established the necessity of (3.2) in the Hilbert space case.

(3) Finally, in the case when both X and X^* are uniformly convex, Martin [29] has characterized the generators G of semigroups $T \in Q_\omega(C)$ (where C is a closed subset of X) having the property that $cl(\mathcal{D}(G)) = C$.

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