J. Math. Soc. Japan
Vol. 24, No. 3, 1972

n-dimensional complex space forms immersed in $\left\{n+\frac{n(n+1)}{2}\right\}$-dimensional complex space forms

Dedicated to Professor Shigeo Sasaki on his 60th birthday

By Koichi OGIUE

(Received Jan. 12, 1972)
(Revised March 28, 1972)

§ 1. Introduction.

A Kaehler manifold of constant holomorphic sectional curvature is called a *complex space form*. A *Kaehler immersion* is an isometric immersion which is complex analytic. B. O'Neill ([2]) proved the following result.

Let M and $\tilde{M}$ be complex space forms of dimension n and $n+p$, respectively. If $p<\frac{n(n+1)}{2}$ and if M is a Kaehler submanifold of $\tilde{M}$, then M is totally geodesic in $\tilde{M}$.

He also gave the following example: There is a Kaehler imbedding of an n-dimensional complex projective space of constant holomorphic sectional curvature 1/2 into an $\left\{n+\frac{n(n+1)}{2}\right\}$-dimensional complex projective space of constant holomorphic sectional curvature 1. This shows that the dimensional restriction in the above result is the best possible.

We have proved in [1] the following result.

Let M be an n-dimensional complex space form of constant holomorphic sectional curvature c and $\tilde{M}$ be an $(n+p)$-dimensional complex space form of constant holomorphic sectional curvature $\tilde{c}$. If $p\geqq\frac{n(n+1)}{2}$ and if M is a Kaehler submanifold of $\tilde{M}$ with parallel second fundamental form, then either $c=\tilde{c}$ (i.e., M is totally geodesic in $\tilde{M}$) or $c=\tilde{c}/2$, the latter case arising only when $\tilde{c}>0$.

The purpose of this paper is to prove the following

THEOREM. *Let M be an n-dimensional complex space form of constant holomorphic sectional curvature c and $\tilde{M}$ be an $\left\{n+\frac{n(n+1)}{2}\right\}$-dimensional*

Work done under partial support by the Sakko-kai Foundation and the Matsunaga Science Foundation.

complex space form of constant holomorphic sectional curvature $\tilde{c}$. If M is a Kaehler submanifold of $\tilde{M}$, then either $c=\tilde{c}$ (i. e., M is totally geodesic in $\tilde{M}$) or $c=\tilde{c}/2$, the latter case arising only when $\tilde{c}>0$. Moreover, the immersion is rigid.

§ 2. Proof of Theorem.

We use the same notation as in [1] unless otherwise stated.

Let M be an n-dimensional complex space form of constant holomorphic sectional curvature c and $\tilde{M}$ be an $\left\{n+\frac{n(n+1)}{2}\right\}$-dimensional complex space form of constant holomorphic sectional curvature $\tilde{c}$. We assume that M is a Kaehler submanifold of $\tilde{M}$. First we note that $c\leqq\tilde{c}$.

If $c=\tilde{c}$, then M is totally geodesic in $\tilde{M}$. From now on we may therefore assume that $c<\tilde{c}$. We have proved in [1] that the second fundamental form σ of the immersion satisfies

(1) $$\|\nabla'\sigma\|^2=n(n+1)(n+2)(\tilde{c}-c)\left(\frac{\tilde{c}}{2}-c\right),$$

where ∇' denotes the covariant differentiation with respect to the connection in (tangent bundle) $\oplus$ (normal bundle). Therefore to prove our Theorem, it suffices to show that $\nabla'\sigma=0$.

We choose a local field of orthonormal frames*) $e_1, \cdots, e_n, e_{1^*}=\tilde{J}e_1, \cdots, e_{n^*}=\tilde{J}e_n, e_{\tilde{1}}, \cdots, e_{\tilde{p}}, e_{\tilde{1}^*}=\tilde{J}e_{\tilde{1}}, \cdots, e_{\tilde{p}^*}=\tilde{J}e_{\tilde{p}}$ in $\tilde{M}$ in such a way that, restricted to M, $e_1, \cdots, e_n, e_{1^*}, \cdots, e_{n^*}$ are tangent to M and**)

$$e_{\tilde{a}}=\frac{\sqrt{2}}{\sqrt{\tilde{c}-c}}\sigma(e_a, e_a),$$

$$e_{\widetilde{(a,b)}}=\frac{2}{\sqrt{\tilde{c}-c}}\sigma(e_a, e_b),$$

where

$$(a, b)=\min\{a, b\}+\frac{|a-b|(2n+1-|a-b|)}{2} \qquad \text{for} \quad a\neq b.$$

*) Hereafter we denote $\frac{n(n+1)}{2}$ by p.

**) We make use of the following convention on the range of indices:

$$A, B, C, D=1, \cdots, n, 1^*, \cdots, n^*, \tilde{1}, \cdots, \tilde{p}, \tilde{1}^*, \cdots, \tilde{p}^*$$

$$i, j, k, l=1, \cdots, n, 1^*, \cdots, n^*$$

$$a, b, c, d, e=1, \cdots, n$$

$$\alpha, \beta=\tilde{1}, \cdots, \tilde{p}, \tilde{1}^*, \cdots, \tilde{p}^*$$

$$\lambda, \mu=\tilde{1}, \cdots, \tilde{p}.$$

With respect to the frame field of $\tilde{M}$ chosen above, let $\omega^1, \cdots, \omega^n, \omega^{1^*}, \cdots, \omega^{n^*}, \omega^{\tilde{1}}, \cdots, \omega^{\tilde{p}}, \omega^{\tilde{1}^*}, \cdots, \omega^{\tilde{p}^*}$ be the field of dual frames. Then the structure equations of $\tilde{M}$ are given by

(2) $$d\omega^A = -\sum_B \omega^A_B \wedge \omega^B ,$$

(3) $$\omega^A_B + \omega^B_A = 0 ,$$

$$\omega^a_b = \omega^{a^*}_{b^*}, \qquad \omega^\lambda_\mu = \omega^{\lambda^*}_{\mu^*}, \qquad \omega^a_\mu = \omega^{a^*}_{\mu^*},$$

$$\omega^a_{b^*} = \omega^b_{a^*}, \qquad \omega^a_{\lambda^*} = \omega^\lambda_{a^*}, \qquad \omega^\lambda_{\mu^*} = \omega^\mu_{\lambda^*},$$

(4) $$d\omega^A_B = -\sum_C \omega^A_C \wedge \omega^C_B + \Phi^A_B ,$$

$$\Phi^A_B = \frac{1}{2} \sum_{C,D} K^A_{BCD} \omega^C \wedge \omega^D ,$$

(5) $$K^A_{BCD} = \frac{\tilde{c}}{4}(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} + J_{AC}J_{BD} - J_{AD}J_{BC} + 2J_{AB}J_{CD}) ,$$

where

$$(\tilde{J}_{AB}) = \left(\begin{array}{cc|cc} 0 & -I_n & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ I_n & 0 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & -I_p \\ & & I_p & 0 \end{array}\right),$$

I_s being the identity matrix of degree s.

Restricting these forms to M, we have the structure equations of the immersion:

(6) $$\omega^\alpha = 0 ,$$

(7) $$\omega^\alpha_i = \sum_j h^\alpha_{ij} \omega^j , \qquad h^\alpha_{ij} = h^\alpha_{ji} ,$$

(8) $$d\omega^i = -\sum_j \omega^i_j \wedge \omega^j ,$$

(9) $$d\omega^i_j = -\sum_k \omega^i_k \wedge \omega^k_j + \Omega^i_j ,$$

$$\Omega^i_j = \frac{1}{2} \sum_{k,l} R^i_{jkl} \omega^k \wedge \omega^l ,$$

(10) $$R^i_{jkl} = \frac{c}{4}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + J_{ik}J_{jl} - J_{il}J_{jk} + 2J_{ij}J_{kl}) ,$$

where

$$(J_{ij}) = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

Since $\sigma(e_i, e_j) = \sum h^\alpha_{ij} e_\alpha$, we can see the following (cf. [1]):

$$
(11)\quad
\begin{cases}
(h_{ij}^{\tilde{a}}) = \begin{pmatrix}
\begin{matrix} 0 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & \text{✡} & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{matrix} & 0 \\
0 & \begin{matrix} 0 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & -\text{✡} & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{matrix}
\end{pmatrix}
\begin{matrix} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ n+a \\ \\ \\ \\ \end{matrix}
\\[2ex]
(h_{ij}^{\widetilde{(a,b)}}) = \begin{pmatrix}
\begin{matrix} & & \\ & & \ast \\ & \ast & \end{matrix} & 0 \\
0 & \begin{matrix} & & \\ & & -\ast \\ & -\ast & \end{matrix}
\end{pmatrix}
\begin{matrix} \\ \\ \\ \\ n+a \\ n+b \end{matrix}
\end{cases}
$$

(In the first matrix the entry ✡ stands in row and column a, and $-\text{✡}$ in row and column $n+a$. In the second matrix $\ast$ stands at positions (a,b) and (b,a), and $-\ast$ at positions $(n+a,n+b)$ and $(n+b,n+a)$.)

where $\text{✡}=\dfrac{\sqrt{\tilde{c}-c}}{\sqrt{2}}$ and $\ast=\dfrac{\sqrt{\tilde{c}-c}}{2}$.

It is easily seen that (11) is equivalent to

$$
(11)'\quad
\begin{cases}
\omega_a^{\tilde{a}}=\dfrac{\sqrt{\tilde{c}-c}}{\sqrt{2}}\,\omega^a, \qquad \omega_{a^*}^{\tilde{a}}=-\dfrac{\sqrt{\tilde{c}-c}}{\sqrt{2}}\,\omega^{a^*}, \\[1ex]
\omega_b^{\tilde{a}}=\omega_{b^*}^{\tilde{a}}=0 \qquad (b\neq a), \\[1ex]
\omega_a^{\widetilde{(a,b)}}=\dfrac{\sqrt{\tilde{c}-c}}{2}\,\omega^b, \qquad \omega_{a^*}^{\widetilde{(a,b)}}=-\dfrac{\sqrt{\tilde{c}-c}}{2}\,\omega^{b^*}, \\[1ex]
\omega_c^{\widetilde{(a,b)}}=\omega_{c^*}^{\widetilde{(a,b)}}=0 \qquad (c\neq a,\ c\neq b).
\end{cases}
$$

If we define h_{ijk}^{α} by

(12) $$\sum_k h^{\alpha}_{ijk}\omega^k = dh^{\alpha}_{ij} - \sum_l h^{\alpha}_{il}\omega^l_j - \sum_l h^{\alpha}_{lj}\omega^l_i + \sum_\beta h^{\beta}_{ij}\omega^{\alpha}_{\beta}\,,$$

then from (4), (5), (6) and (7) we have $h^{\alpha}_{ijk} = h^{\alpha}_{ikj}$ so that

(13) h^{α}_{ijk} *is symmetric with respect to* i, j *and* k.

Moreover we can see that

(14) $$h^{\alpha}_{a^*b^*k} = -h^{\alpha}_{abk}\,, \qquad h^{\alpha}_{ab^*k} = h^{\alpha}_{a^*bk}\,.$$

Therefore we have the following

LEMMA. *The following three conditions are mutually equivalent:*

(i) $\nabla'\sigma = 0$.

(ii) $h^{\alpha}_{ijk} = 0$ *for all* α, i, j *and* k.

(iii) $\omega^{\tilde a}_{a^*} = 2\omega^{a}_{a^*}$, $\omega^{\tilde a}_{\tilde b} = 0$, $\omega^{\tilde a}_{\tilde b^*} = 0$,

$\omega^{\widetilde{(a,b)}}_{\tilde a} = \sqrt{2}\,\omega^b_a$, $\omega^{\widetilde{(a,b)}}_{\tilde a^*} = \sqrt{2}\,\omega^b_a$,

$\omega^{\tilde a}_{\widetilde{(b,c)}} = 0$, $\omega^{\tilde a}_{\widetilde{(b,c)}^*} = 0$,

$\omega^{\widetilde{(a,b)}}_{\widetilde{(a,b)}^*} = \omega^a_{a^*} + \omega^b_{b^*}$,

$\omega^{\widetilde{(a,b)}}_{\widetilde{(a,c)}} = \omega^b_c$, $\omega^{\widetilde{(a,b)}}_{\widetilde{(a,c)}^*} = \omega^b_{c^*}$,

$\omega^{\widetilde{(a,b)}}_{\widetilde{(c,d)}} = 0$, $\omega^{\widetilde{(a,b)}}_{\widetilde{(c,d)}^*} = 0$,

where a, b, c *and* d *are different.*

From (11) and (12) we have

(15) $$h^{\tilde a}_{aak} = h^{\tilde a^*}_{aa^*k} = 0\,.$$

From (13), (14) and (15) we have

$$\begin{aligned}\sum_k h^{\tilde a}_{aa^*k}\omega^k &= \sum_b h^{\tilde a}_{aa^*b}\omega^b + \sum_b h^{\tilde a}_{aa^*b^*}\omega^{b^*}\\ &= \sum_b h^{\tilde a}_{aab^*}\omega^b - \sum_b h^{\tilde a}_{aab}\omega^{b^*}\\ &= 0\,,\end{aligned}$$

that is,

$$h^{\tilde a}_{aa^*k} = 0\,.$$

This, together with (11) and (12), implies

(16) $$\omega^{\tilde a}_{a^*} = 2\omega^{a}_{a^*}\,.$$

From (4), (5), (6), (11)′ and (16), we have (for example, $(17)_1$ is obtained by putting $A = \tilde a$ and $B = a$ in (4))

(17) $$\begin{cases}\displaystyle\sum_{b\neq a}(\omega^{\tilde a}_{\widetilde{(a,b)}} - \sqrt{2}\,\omega^a_b)\wedge\omega^b + \sum_{b\neq a}(\omega^{\tilde a}_{\widetilde{(a,b)}^*} - \sqrt{2}\,\omega^a_{b^*})\wedge\omega^{b^*} = 0\\ \displaystyle\sum_{b\neq a}(\omega^{\tilde a}_{\widetilde{(a,b)}^*} - \sqrt{2}\,\omega^a_{b^*})\wedge\omega^b - \sum_{b\neq a}(\omega^{\tilde a}_{\widetilde{(a,b)}} - \sqrt{2}\,\omega^a_b)\wedge\omega^{b^*} = 0\end{cases}$$

$$
(18)\quad
\begin{cases}
(\omega^{\tilde a}_{\widetilde{(a,b)}} - \sqrt{2}\,\omega^{a}_{\tilde b})\wedge\omega^{a} + (\omega^{\tilde a}_{\widetilde{(a,b)}*} - \sqrt{2}\,\omega^{a}_{\tilde b*})\wedge\omega^{a*} + \sqrt{2}\,\omega^{\tilde a}_{\tilde b}\wedge\omega^{b} + \sqrt{2}\,\omega^{\tilde a}_{\tilde b*}\wedge\omega^{b*} \\
\qquad + \sum_{c\neq a,b}\omega^{\tilde a}_{\widetilde{(b,c)}}\wedge\omega^{c} + \sum_{c\neq a,b}\omega^{\tilde a}_{\widetilde{(b,c)}*}\wedge\omega^{c*} = 0 \\
(\omega^{\tilde a}_{\widetilde{(a,b)}*} - \sqrt{2}\,\omega^{a}_{\tilde b*})\wedge\omega^{a} - (\omega^{\tilde a}_{\widetilde{(a,b)}} - \sqrt{2}\,\omega^{a}_{\tilde b})\wedge\omega^{a*} + \sqrt{2}\,\omega^{\tilde a}_{\tilde b*}\wedge\omega^{b} - \sqrt{2}\,\omega^{\tilde a}_{\tilde b}\wedge\omega^{b*} \\
\qquad + \sum_{c\neq a,b}\omega^{\tilde a}_{\widetilde{(b,c)}*}\wedge\omega^{c} - \sum_{c\neq a,b}\omega^{\tilde a}_{\widetilde{(b,c)}}\wedge\omega^{c*} = 0
\end{cases}
$$

$$
(19)\quad
\begin{cases}
\sqrt{2}\,(\omega^{\widetilde{(a,b)}}_{\tilde a} - \sqrt{2}\,\omega^{b}_{\tilde a})\wedge\omega^{a} + \sqrt{2}\,(\omega^{\widetilde{(a,b)}}_{\tilde a*} - \sqrt{2}\,\omega^{b}_{\tilde a*})\wedge\omega^{a*} \\
\qquad + (\omega^{\widetilde{(a,b)}}_{\widetilde{(a,b)}*} - \omega^{a}_{\tilde a*} - \omega^{b}_{\tilde b*})\wedge\omega^{b*} + \sum_{c\neq a,b}(\omega^{\widetilde{(a,b)}}_{\widetilde{(a,c)}} - \omega^{b}_{\tilde c})\wedge\omega^{c} + \sum_{c\neq a,b}(\omega^{\widetilde{(a,b)}}_{\widetilde{(a,c)}*} - \omega^{b}_{\tilde c*})\wedge\omega^{c*} \\
\qquad = 0 \\
\sqrt{2}\,(\omega^{\widetilde{(a,b)}}_{\tilde a*} - \sqrt{2}\,\omega^{b}_{\tilde a*})\wedge\omega^{a} - \sqrt{2}\,(\omega^{\widetilde{(a,b)}}_{\tilde a} - \sqrt{2}\,\omega^{b}_{\tilde a})\wedge\omega^{a*} \\
\qquad + (\omega^{\widetilde{(a,b)}}_{\widetilde{(a,b)}*} - \omega^{a}_{\tilde a*} - \omega^{b}_{\tilde b*})\wedge\omega^{b} + \sum_{c\neq a,b}(\omega^{\widetilde{(a,b)}}_{\widetilde{(a,c)}*} - \omega^{b}_{\tilde c*})\wedge\omega^{c} - \sum_{c\neq a,b}(\omega^{\widetilde{(a,b)}}_{\widetilde{(a,c)}} - \omega^{b}_{\tilde c})\wedge\omega^{c*} \\
\qquad = 0
\end{cases}
$$

$$
(20)\quad
\begin{cases}
(\omega^{\widetilde{(a,b)}}_{\widetilde{(a,c)}} - \omega^{b}_{\tilde c})\wedge\omega^{a} + (\omega^{\widetilde{(a,b)}}_{\widetilde{(a,c)}*} - \omega^{b}_{\tilde c*})\wedge\omega^{a*} + (\omega^{\widetilde{(a,b)}}_{\widetilde{(c,b)}} - \omega^{a}_{\tilde c})\wedge\omega^{b} \\
\qquad + (\omega^{\widetilde{(a,b)}}_{\widetilde{(c,b)}*} - \omega^{a}_{\tilde c*})\wedge\omega^{b*} + \sqrt{2}\,\omega^{\widetilde{(a,b)}}_{\tilde c}\wedge\omega^{c} + \sqrt{2}\,\omega^{\widetilde{(a,b)}}_{\tilde c*}\wedge\omega^{c*} \\
\qquad + \sum_{d\neq a,b,c}(\omega^{\widetilde{(a,b)}}_{\widetilde{(c,d)}}\wedge\omega^{d} + \omega^{\widetilde{(a,b)}}_{\widetilde{(c,d)}*}\wedge\omega^{d*}) = 0 \\
(\omega^{\widetilde{(a,b)}}_{\widetilde{(a,c)}*} - \omega^{b}_{\tilde c*})\wedge\omega^{a} - (\omega^{\widetilde{(a,b)}}_{\widetilde{(a,c)}} - \omega^{b}_{\tilde c})\wedge\omega^{a*} + (\omega^{\widetilde{(a,b)}}_{\widetilde{(c,b)}*} - \omega^{a}_{\tilde c*})\wedge\omega^{b} \\
\qquad - (\omega^{\widetilde{(a,b)}}_{\widetilde{(c,b)}} - \omega^{a}_{\tilde c})\wedge\omega^{b*} + \sqrt{2}\,\omega^{\widetilde{(a,b)}}_{\tilde c*}\wedge\omega^{c} - \sqrt{2}\,\omega^{\widetilde{(a,b)}}_{\tilde c}\wedge\omega^{c*} \\
\qquad + \sum_{d\neq a,b,c}(\omega^{\widetilde{(a,b)}}_{\widetilde{(c,d)}*}\wedge\omega^{d} - \omega^{\widetilde{(a,b)}}_{\widetilde{(c,d)}}\wedge\omega^{d*}) = 0\,,
\end{cases}
$$

where a, b, c and d are different.

From (17) and Cartan's lemma we may write

$$
(21)\quad
\begin{cases}
\omega^{\tilde a}_{\widetilde{(a,b)}} - \sqrt{2}\,\omega^{a}_{\tilde b} = \sum_{c\neq a}(\varphi^{a}_{bc}\omega^{c} + \varphi^{a}_{bc*}\omega^{c*}) \\
\omega^{\tilde a}_{\widetilde{(a,b)}*} - \sqrt{2}\,\omega^{a}_{\tilde b*} = \sum_{c\neq a}(\varphi^{a}_{bc*}\omega^{c} - \varphi^{a}_{bc}\omega^{c*})\,,
\end{cases}
$$

where

$$
(22)\qquad \varphi^{a}_{bc} = \varphi^{a}_{cb}\,, \qquad \varphi^{a}_{bc*} = \varphi^{a}_{cb*}\,.
$$

From (18), (21) and Cartan's lemma we may write

$$\begin{cases} \sqrt{2}\,\delta_c^b\omega_{\bar b}^{\bar a}-\varphi_{bc}^a\omega^a-\varphi_{bc^*}^a\omega^{a^*}+\omega_{\widetilde{(b,c)}}^{\bar a}=\sum_{d\neq a}(\psi_{bcd}^a\omega^d+\psi_{bcd^*}^a\omega^{d^*}) \\ \sqrt{2}\,\delta_c^b\omega_{\bar b^*}^{\bar a}-\varphi_{bc^*}^a\omega^a+\varphi_{bc}^a\omega^{a^*}+\omega_{\widetilde{(b,c)}^*}^{\bar a}=\sum_{d\neq a}(\psi_{bcd^*}^a\omega^d-\psi_{bcd}^a\omega^{d^*}), \end{cases}$$

or

$$(23)\qquad \begin{cases} \sqrt{2}\,\omega_{\bar b}^{\bar a}-\varphi_{bb}^a\omega^a-\varphi_{bb^*}^a\omega^{a^*}=\sum_{d\neq a}(\psi_{bbd}^a\omega^d+\psi_{bbd^*}^a\omega^{d^*}) \\ \sqrt{2}\,\omega_{\bar b^*}^{\bar a}-\varphi_{bb^*}^a\omega^a+\varphi_{bb}^a\omega^{a^*}=\sum_{d\neq a}(\psi_{bbd^*}^a\omega^d-\psi_{bbd}^a\omega^{d^*}) \end{cases}$$

$$(24)\qquad \begin{cases} \omega_{\widetilde{(b,c)}}^{\bar a}-\varphi_{bc}^a\omega^a-\varphi_{bc^*}^a\omega^{a^*}=\sum_{d\neq a}(\psi_{bcd}^a\omega^d+\psi_{bcd^*}^a\omega^{d^*}) \\ \omega_{\widetilde{(b,c)}^*}^{\bar a}-\varphi_{bc^*}^a\omega^a+\varphi_{bc}^a\omega^{a^*}=\sum_{d\neq a}(\psi_{bcd^*}^a\omega^d-\psi_{bcd}^a\omega^{d^*}), \end{cases}$$

where

(25) *ψ_{bcd}^a and $\psi_{bcd^*}^a$ are symmetric with respect to b, c and d.*

Since $\omega_{\bar b}^{\bar a}+\omega_{\bar a}^{\bar b}=0$ and $\omega_{\bar b^*}^{\bar a}=\omega_{\bar a^*}^{\bar b}$, we can see from (23) that

$$(26)\qquad \varphi_{bb}^a=0\,,\qquad \varphi_{bb^*}^a=0\,,\qquad \psi_{bbd}^a=0\,,\qquad \psi_{bbd^*}^a=0\,,$$

and hence

$$(27)\qquad \omega_{\bar b}^{\bar a}=0\,,\qquad \omega_{\bar b^*}^{\bar a}=0\qquad (a\neq b)\,.$$

From (21) and (26) we have

$$(28)\qquad \begin{cases} \omega_{\widetilde{(a,b)}}^{\bar a}-\sqrt{2}\,\omega_b^a=\sum_{c\neq a,b}(\varphi_{bc}^a\omega^c+\varphi_{bc^*}^a\omega^{c^*}) \\ \omega_{\widetilde{(a,b)}^*}^{\bar a}-\sqrt{2}\,\omega_{b^*}^a=\sum_{c\neq a,b}(\varphi_{bc^*}^a\omega^c-\varphi_{bc}^a\omega^{c^*}), \end{cases}$$

which implies that $\omega_{\widetilde{(a,b)}}^{\bar a}-\sqrt{2}\,\omega_b^a$ and $\omega_{\widetilde{(a,b)}^*}^{\bar a}-\sqrt{2}\,\omega_{b^*}^a$ do not contain ω^a, ω^{a^*}, ω^b and ω^{b^*}. Moreover from (24), (25) and (26) we have

$$(29)\qquad \begin{cases} \omega_{\widetilde{(b,c)}}^{\bar a}-\varphi_{bc}^a\omega^a-\varphi_{bc^*}^a\omega^{a^*}=\sum_{d\neq a,b,c}(\psi_{bcd}^a\omega^d+\psi_{bcd^*}^a\omega^{d^*}) \\ \omega_{\widetilde{(b,c)}^*}^{\bar a}-\varphi_{bc^*}^a\omega^a+\varphi_{bc}^a\omega^{a^*}=\sum_{d\neq a,b,c}(\psi_{bcd^*}^a\omega^d-\psi_{bcd}^a\omega^{d^*}), \end{cases}$$

which implies that $\omega_{\widetilde{(b,c)}}^{\bar a}$ and $\omega_{\widetilde{(b,c)}^*}^{\bar a}$ do not contain ω^b, ω^{b^*}, ω^c and ω^{c^*}.

From (19) and (28) we can see that $\omega_{\widetilde{(a,b)}^*}^{\widetilde{(a,b)}}-\omega_{a^*}^a-\omega_{b^*}^b$ does not contain ω^a, ω^{a^*}, ω^b and ω^{b^*} so that we may write

$$(30)\qquad \omega_{\widetilde{(a,b)}^*}^{\widetilde{(a,b)}}-\omega_{a^*}^a-\omega_{b^*}^b=\sum_{c\neq a,b}(A_c^{ab}\omega^c+A_{c^*}^{ab}\omega^{c^*})\,.$$

From (20) and (29) we can see that $\omega_{\widetilde{(a,c)}}^{\widetilde{(a,b)}}-\omega_c^b$ and $\omega_{\widetilde{(a,c)}^*}^{\widetilde{(a,b)}}-\omega_{c^*}^b$ do not contain ω^c and ω^{c^*}. By symmetry, they do not also contain ω^b and ω^{b^*}. Therefore we may write

(31)
$$\begin{cases} \omega^{(\widetilde{a,b})}_{(\widetilde{a,c})} - \omega^b_c = \sum_{d\neq b,c} (B^{ab}_{cd}\,\omega^d + B^{ab}_{cd^*}\omega^{d^*}) \\ \omega^{(\widetilde{a,b})}_{(\widetilde{a,c})^*} - \omega^b_{c^*} = \sum_{d\neq b,c} (B^{ab}_{cd^*}\omega^d - B^{ab}_{cd}\,\omega^{d^*}) . \end{cases}$$

Since $\omega^{(\widetilde{a,b})}_{(\widetilde{a,c})} - \omega^b_c + \omega^{(\widetilde{a,c})}_{(\widetilde{a,b})} - \omega^c_b = 0$ and $\omega^{(\widetilde{a,b})}_{(\widetilde{a,c})^*} - \omega^b_{c^*} = \omega^{(\widetilde{a,c})}_{(\widetilde{a,b})^*} - \omega^c_{b^*}$, we can see from (31) that $B^{ab}_{cd} = B^{ab}_{cd^*} = 0$ and hence

(32)
$$\begin{cases} \omega^{(\widetilde{a,b})}_{(\widetilde{a,c})} = \omega^b_c \\ \omega^{(\widetilde{a,b})}_{(\widetilde{a,c})^*} = \omega^b_{c^*} . \end{cases}$$

Substituting (28), (30) and (32) into (19), we can see

$$\varphi^a_{bc} = \varphi^a_{bc^*} = 0 , \qquad A^{ab}_c = A^{ab}_{c^*} = 0 ,$$

which, together with (28), (29) and (30), implies

(33)
$$\begin{cases} \omega^{\tilde{a}}_{(\widetilde{a,b})} = \sqrt{2}\,\omega^a_b \\ \omega^{\tilde{a}}_{(\widetilde{a,b})^*} = \sqrt{2}\,\omega^a_{b^*} \end{cases}$$

(34)
$$\begin{cases} \omega^{\tilde{a}}_{(\widetilde{b,c})} = \sum_{d\neq a,b,c} (\phi^a_{bcd}\omega^d + \phi^a_{bcd^*}\omega^{d^*}) \\ \omega^{\tilde{a}}_{(\widetilde{b,c})^*} = \sum_{d\neq a,b,c} (\phi^a_{bcd^*}\omega^d - \phi^a_{bcd}\omega^{d^*}) \end{cases}$$

(35)
$$\omega^{(\widetilde{a,b})}_{(\widetilde{a,b})^*} = \omega^a_{a^*} + \omega^b_{b^*} .$$

Moreover (20) implies that we may write

(36)
$$\begin{cases} \omega^{(\widetilde{a,b})}_{(\widetilde{c,d})} = \sum_e (C^{ab}_{cde}\omega^e + C^{ab}_{cde^*}\omega^{e^*}) \\ \omega^{(\widetilde{a,b})}_{(\widetilde{c,d})^*} = \sum_e (C^{ab}_{cde^*}\omega^e - C^{ab}_{cde}\omega^{e^*}) . \end{cases}$$

Since $\omega^{(\widetilde{a,b})}_{(\widetilde{c,d})} + \omega^{(\widetilde{c,d})}_{(\widetilde{a,b})} = 0$ and $\omega^{(\widetilde{a,b})}_{(\widetilde{c,d})^*} = \omega^{(\widetilde{c,d})}_{(\widetilde{a,b})^*}$, we can see from (36) that $C^{ab}_{cde} = C^{ab}_{cde^*} = 0$, and hence

(37)
$$\omega^{(\widetilde{a,b})}_{(\widetilde{c,d})} = \omega^{(\widetilde{a,b})}_{(\widetilde{c,d})^*} = 0 .$$

Substituting (32), (34) and (37) into (20), we can see

$$\phi^a_{bcd} = \phi^a_{bcd^*} = 0 ,$$

which, together with (34), implies

(38)
$$\omega^{\tilde{a}}_{(\widetilde{b,c})} = \omega^{\tilde{a}}_{(\widetilde{b,c})^*} = 0 .$$

Thus we have proved the following

PROPOSITION.

$$\omega_{\tilde{a}^*}^{\tilde{a}} = 2\omega_{a^*}^{a}, \qquad \omega_{\tilde{b}}^{\tilde{a}} = \omega_{\tilde{b}^*}^{\tilde{a}} = 0,$$

$$\omega_{\tilde{a}}^{\widetilde{(a,b)}} = \sqrt{2}\,\omega_a^b, \qquad \omega_{\tilde{a}^*}^{\widetilde{(a,b)}} = \sqrt{2}\,\omega_{a^*}^b.$$

$$\omega_{\widetilde{(a,b)}^*}^{\widetilde{(a,b)}} = \omega_{a^*}^{a} + \omega_{b^*}^{b},$$

$$\omega_{\tilde{c}}^{\widetilde{(a,b)}} = \omega_{\tilde{c}^*}^{\widetilde{(a,b)}} = 0,$$

$$\omega_{\widetilde{(a,c)}}^{\widetilde{(a,b)}} = \omega_c^b, \qquad \omega_{\widetilde{(a,c)}^*}^{\widetilde{(a,b)}} = \omega_{c^*}^b,$$

$$\omega_{\widetilde{(c,d)}}^{\widetilde{(a,b)}} = \omega_{\widetilde{(c,d)}^*}^{\widetilde{(a,b)}} = 0,$$

where a, b, c and d are different.

Our Theorem follows immediately from Lemma and Proposition.

Department of Mathematics
Tokyo Metropolitan University

Bibliography

[1] K. Ogiue, On Kaehler immersions, to appear in Canad. J. Math.

[2] B. O'Neill, Isotropic and Kaehler immersions, Canad. J. Math., 17 (1965), 907-915.