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(3/2)-dimensional measure of singular sets of some Kleinian groups

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§ 0. Introduction.

The writer showed the existence of Kleinian groups with fundamental domains bounded by four circles whose singular sets have positive 1-dimensional measure ([4]). Further in former paper [5], he investigated the properties about the computing functions on some Kleinian groups and gave the results with respect to the local properties of the singular sets of these groups by using the computing function. Now in the natural way the following problem arises; to what extent does the Hausdorff dimension of the singular sets of Kleinian groups climb up, when the number N of the boundary circles increases? It is conjectured and seems still open that the 2-dimensional measure of the singular sets of general finitely generated Kleinian groups is always zero ([1]).

The purpose of this paper is to show the existence of the Kleinian groups whose singular sets have positive (3/2)-dimensional measure. We shall state preliminaries and notations about Kleinian groups in § 1. In § 2 we shall introduce the subcomputing function on some Kleinian groups which have the analogous properties to the computing functions and investigate the relations between this function and the Hausdorff dimension of the singular set of some Kleinian group. By using this function we shall give the example of the Kleinian group whose singular set has positive (3/2)-dimensional measure in § 3.

§ 1. Preliminaries and notations.

1. Let B_0 be a domain bounded by $N\,(\geqq 3)$ mutually disjoint circles $\{H_i, H_i'\}_{i=1}^p$ and $\{K_j\}_{j=1}^q$ $(2p+q=N)$. Let S_i $(1\leqq i\leqq p)$ be a hyperbolic or loxodromic transformation which transforms the outside of H_i onto the inside of H_i'. Let S_j^* $(1\leqq j\leqq q)$ be an elliptic transformation of period 2 which transforms the outside of K_j onto the inside of K_j. Then $\{S_i\}_{i=1}^p\cup\{S_j^*\}_{j=1}^q$ generates a discontinuous group G with B_0 as a fundamental domain. We use the notation $\mathfrak{G}$ to denote the set of $\{S_i\}_{i=1}^p$, their inverses and $\{S_j^*\}_{j=1}^q$.

Denote by ST the composed transformation obtained by S and T contained in G, that is, $ST(z)=S(T(z))$. Then any element of G has the form

$$S=S^{(\nu_k)}S_{j_k}^{*}\cdots S^{(\nu_1)}S_{j_1}^{*}S^{(\nu_0)},$$

where the indices ν_i $(i=0,\cdots,k)$ are non-negative integers and $S^{(\nu_i)}$ denotes the product of ν_i generators of $\{S_i\}$ or their inverses and $S_{j_1}^{*}$ denotes the generator of $\{S_j^{*}\}$. We call the sum

$$m=\sum_{i=0}^{k}\nu_i+k$$

the grade of S and for simplicity we denote by $S^{(m)}$ the element of grade m in G.

The image $S^{(m)}(B_0)$ of the fundamental domain B_0 under $S^{(m)}$ $(\in G)$ with grade m $(\geqq 1)$ is bounded by N circles $S^{(m)}(H_i)$, $S^{(m)}(H_i')$ and $S^{(m)}(K_j)$ $(i=1,\cdots,p;\ j=1,\cdots,q)$. For the sake of simplicity, we call the outer boundary circle $C^{(m)}$ of $S^{(m)}(B_0)$ a circle of grade m. The number of circles of grade m is obviously equal to $N(N-1)^{m-1}$.

Denote by D_m the $N(N-1)^{m-1}$-ply connected domain bounded by all the circles of grade m. Evidently $\{D_m\}_{m=1}^{\infty}$ is a monotone increasing sequence of domains. The complementary set D_m^c of D_m with respect to the extended z-plane consists of $N(N-1)^{m-1}$ mutually disjoint closed discs. The set $E=\bigcap_{m=1}^{\infty}D_m^c$ is perfect and nowhere dense. We call E the singular set of G.

2. For a linear transformation of the form

$$T(z)=\frac{az+b}{cz+d},\qquad ad-bc=1,\qquad c\neq 0,$$

the circle $I_T: |cz+d|=1$ is called the isometric circle of the transformation (see [8]). The radius of I_T equals $1/|c|$.

Consider two arbitrary transformations T and S of G. We assume that $S\neq T^{-1}$. Denote by I_S, $I_{T^{-1}}$ and I_{ST} the isometric circles of S, T^{-1} and ST, respectively. Let g_S, $g_{T^{-1}}$ and g_{ST} be their centers, and let R_S, R_T and R_{ST} be radii of I_S, I_T and I_{ST}, respectively. As for these values, the relation

$$R_{ST}=\frac{R_S\cdot R_T}{|g_{T^{-1}}-g_S|}=\frac{R_S\cdot R_T}{|T(\infty)-S^{-1}(\infty)|} \tag{1.1}$$

holds, since $g_{T^{-1}}$ and g_S are the poles of $T^{-1}(z)$ and $S(z)$.

3. Denoting by $r^{(m)}$ and $r_i^{(m+1)}$ $(i=1,\cdots,N-1)$ the radius of the outer boundary circle $C^{(m)}$ and the radii of $N-1$ inner boundary circles $C_i^{(m+1)}$ $(i=1,\cdots,N-1)$ of the image B_m of the fundamental domain B_0 under a transformation $S^{(m)}$ $(\in G)$, we have the following propositions ([2]).

PROPOSITION 1. *There exist positive constants K_0 (<1) and k_0 depending only on B_0 such that*

$$k_0 r^{(m)} \leqq r_i^{(m+1)} \leqq K_0 r^{(m)} \qquad (i=1, \cdots, N-1)\,. \tag{1.2}$$

PROPOSITION 2. *There exist positive constants $k(G)$ and $K(G)$ depending only on G such that*

$$k(G)(R_{S^{(m)}})^{\mu} \leqq (r^{(m)})^{-\frac{\mu}{2}} \leqq K(G)(R_{S^{(m)}})^{\mu}\,, \tag{1.3}$$

where μ is any positive number and $R_{S^{(m)}}=1/|c^{(m)}|$ is the radius of the isometric circle of $S^{(m)}$.

Denote by F_{n_0} the family of all closed discs bounded by circles of grade n ($\geqq n_0$). It is easily seen that F_{n_0} is a covering of the singular set of our Kleinian group G and by Proposition 1 we see that the diameter of any disc of F_{n_0} is less than a given δ (>0) for a sufficiently large n_0.

For such a covering F_{n_0} we have the following important proposition ([2]).

PROPOSITION 3. *Let $F_{n_0}^{\delta/k_0}$ be a covering of E constructed by discs in F_{n_0} whose radii are not greater than $\delta/(2k_0)$ and let r_C be the radius of a disc C in $F_{n_0}^{\delta/k_0}$, where k_0 is a positive constant in Proposition 1. Then it holds*

$$L_\eta(E)=\lim_{\delta\to 0}\inf_{\{F_{n_0}^{\delta/k_0}\}}\sum_{C\in F_{n_0}^{\delta/k_0}}(2r_C)^\eta \leqq \mathcal{K}\left(\frac{k_0}{2}\right)^{-\eta} M_\eta(E) \tag{1.4}$$

where $\mathcal{K}$ is an absolute constant and $M_\eta(E)$ denotes the η-dimensional measure of E.

4. Let T be any fixed generator or its inverse in $\mathfrak{G}$. Denote by H_T and $H_{T^{-1}}$ the boundary circles of B_0 which are equivalent, that is, $H_T=T(H_{T^{-1}})$ and further by D_T the closed disc bounded by H_T. If H_T is one of K_j $(1\leqq j\leqq q)$, then $H_{T^{-1}}=H_T$.

Let $S^{(n)}=T_nT_{n-1}\cdots T_2T_1$ $(T_i\in\mathfrak{G})$ be any element of G with grade n and take its inverse transformation of $S^{(n)}$:

$$S^{-(n)}(z)=\frac{-d_nz+b_n}{c_nz-a_n}\,, \qquad a_nd_n-b_nc_n=1\,, \tag{1.5}$$

where $S^{-(n)}$ denotes the inverse $(S^{(n)})^{-1}=T_1^{-1}\cdots T_n^{-1}$ of $S^{(n)}$. If we take the derivative of $S^{-(n)}(z)$, we obtain easily

$$\left|\frac{dS^{-(n)}(z)}{dz}\right|^{\frac{\mu}{2}}=\left(\frac{1}{|c_nz-a_n|}\right)^{\mu}=\left(\frac{R_{S^{(n)}}}{|S^{(n)}(\infty)-z|}\right)^{\mu}. \tag{1.6}$$

Forming the sum of $(N-1)^n$ terms with respect to all $S^{(n)}=T_nT_{n-1}\cdots T_2T_1$ such that $T_n\neq T^{-1}$ and $T_i\neq T_{i+1}^{-1}$ $(1\leqq i\leqq n-1)$, we have the following function denoted by $\chi_n^{(\mu;T)}(z)$:

$$\chi_n^{(\mu;T)}(z)=\sum_{S^{(n)}}\left(\frac{R_{S^{(n)}}}{|S^{(n)}(\infty)-z|}\right)^{\mu}=\sum_{S^{(n)}}\left|\frac{dS^{-(n)}(z)}{dz}\right|^{\frac{\mu}{2}}. \tag{1.7}$$

The domain of definition of $\chi_n^{(\mu;T)}(z)$ is D_T. Since z moves on D_T and $T_n\neq T^{-1}$, the $(N-1)^n$ denominators of (1.7) do not vanish, and hence $\chi_n^{(\mu;T)}(z)$ is uniformly continuous in D_T. Let $S^{(m)}$ $(\in G)$ be an element with grade m of the form $S^{(m)}=S^{(m-1)}T$. Noting that $g_{S^{(m)}}=S^{-(m)}(\infty)$ and using the relation (1.1), we obtain easily

$$\chi_n^{(\mu;T)}(g_{S^{(m)}})=\sum_{S^{(n)}}\left(\frac{R_{S^{(n)}}}{|S^{(n)}(\infty)-S^{-(m)}(\infty)|}\right)^{\mu}=\sum_{S^{(n)}}\left(\frac{R_{S^{(m)}S^{(n)}}}{R_{S^{(m)}}}\right)^{\mu}. \tag{1.8}$$

We call $\chi_n^{(\mu;T)}(z)$ the μ-dimensional computing function of order n on T. There exist N computing functions $\chi_n^{(\mu;T)}(z)$ corresponding to the choice of T from $\mathfrak{G}$.

By using $\chi_n^{(\mu;T)}(z)$ we have the following result.

THEOREM 1 ([5]). *The following four propositions are equivalent to each other.*

(1) *For any sufficiently small* ε *there exists a positive integer* n_0 *such that*

$$\chi_{n_0}^{(\mu;T)}(z)>1+\varepsilon\,, \qquad (or\ <1-\varepsilon)\,, \tag{1.9}$$

for any $T\in\mathfrak{G}$ *and any* $z\in E\frown D_T$.

(2) *A subsequence* $\{\chi_{n_i}^{(\mu;T)}(z)\}$ $(i=1,2,\cdots)$ *of* $\{\chi_n^{(\mu;T)}(z)\}$ $(n=1,2,\cdots)$ *on some fixed* T $(\in\mathfrak{G})$ *diverges (or converges to zero) at some point* z_0 *of the singular set contained in* D_T, *that is,*

$$\lim_{i\to\infty}\chi_{n_i}^{(\mu;T)}(z_0)=\infty\,,\quad (or\ 0)\qquad for\ some\ z_0\in E\frown D_T\,. \tag{1.10}$$

(3) *It holds*

$$\lim_{n\to\infty}\chi_n^{(\mu;T)}(z)=\infty\,,\qquad (or\ 0) \tag{1.11}$$

for any T $(\in\mathfrak{G})$ *uniformly on* D_T.

(4) *It holds* $M_{\frac{\mu}{2}}(E)=\infty$ (*or* 0) *for the singular set* E *of* G.

§ 2. Relation between the computing function and the Hausdorff dimension of the singular set of the Kleinian group.

5. Let us investigate relations between the computing function and the Hausdorff dimension of the singular set of the Kleinian group. Given a set F in the z-plane, the Hausdorff dimension of F is the unique non-negative number $d(F)$ satisfying

$$M_d(F)=0\,,\qquad \text{if}\quad d>d(F)$$

and

$$M_d(F)=\infty\,,\qquad \text{if}\quad 0\leqq d<d(F)\,,$$

where $M_d(F)$ denotes the d-dimensional Hausdorff measure of F.

We get from Theorem 1 and the definition of the Hausdorff dimension the following propositions ([5]).

PROPOSITION 4. *Let T be any generator or its inverse of the Kleinian group G, that is, $T\in\mathfrak{G}$. Then the Hausdorff dimension $d(E)$ of the singular set E of G coincides with the following number:*

$$\sup\left\{\frac{\mu}{2};\ \lim_{n\to\infty}\chi_n^{(\mu;T)}(z)=\infty,\ {}^{\exists}z\in E\cap D_T\right\}$$

$$=\inf\left\{\frac{\mu'}{2};\ \lim_{n\to\infty}\chi_n^{(\mu';T)}(z)=0,\ {}^{\exists}z\in E\cap D_T\right\}.$$

PROPOSITION 5. *Let μ_0 be the Hausdorff dimension of E. Then $M_{\frac{\mu_0}{2}}(E)$ is positive and finite.*

In the former paper [5], the writer gave the following problem. Let $\frac{\mu^*}{2}=d(E)$ be the Hausdorff dimension of E. Does $\bar{\chi}^{(\mu^*;T)}(z)=\overline{\lim_{n\to\infty}}\chi_n^{(\mu^*;T)}(z)$ equal $\underline{\chi}^{(\mu^*;T)}(z)=\underline{\lim}_{n\to\infty}\chi_n^{(\mu^*;T)}(z)$ for any T ($\in\mathfrak{G}$)? If it is true, is the function $\chi^{(\mu^*;T)}(z)=\lim_{n\to\infty}\chi_n^{(\mu^*;T)}(z)$ always a positive constant on the singular set $E\cap D_T$? It is conjectured that this will be 1.

Since three cases that $M_{\frac{\mu}{2}}(E)$ is zero, finite and infinite, can occur at any μ and they are mutually exclusive, we have from Theorem 1 and Proposition 5 the following theorem.

THEOREM 2. *Let $\frac{\mu^*}{2}$ be the Hausdorff dimension of E. Then there exists some positive small ε such that*

$$|\chi_n^{(\mu^*;T)}(z)-1|<\varepsilon \tag{2.1}$$

for any n, any T ($\in\mathfrak{G}$) and any $z\in E\cap D_T$. Therefore it holds

$$0<1-\varepsilon\leqq\underline{\chi}^{(\mu^*;T)}(z)\leqq\bar{\chi}^{(\mu^*;T)}(z)\leqq 1+\varepsilon \tag{2.2}$$

for any T ($\in\mathfrak{G}$) and any $z\in E\cap D_T$.

6. Let us introduce the subcomputing function closely related to the computing function.

Let T_1 be any fixed generator of G or its inverse and $S^{(n)}=T_nT_{n-1}\cdots T_2T_1$ ($T_i\in\mathfrak{G}$) be any element of G with grade n (>1) for fixed T_1 defined in (1.5) of No. 4. Taking the derivative of $S^{-(n)}(z)$ and forming the sum of $(N-1)^{n-1}$ terms with respect to all $S^{(n)}$ for fixed T_1 such that $T_n\neq T^{-1}$ and $T_i\neq T_{i+1}^{-1}$ ($1\leqq i\leqq n-1$), we have the following function

$$\sum_{S^{(n-1)}}\left(\frac{R_{S^{(n-1)}T_1}}{|S^{(n-1)}T_1(\infty)-z|}\right)^{\mu}=\sum_{S^{(n-1)}}\left|\frac{dT_1^{-1}S^{-(n-1)}(z)}{dz}\right|^{\frac{\mu}{2}}. \tag{2.3}$$

The domain of definition of this function is D_T. We denote it by $\chi_n^{(\mu;T,T_1)}(z)$. Then $\chi_n^{(\mu;T,T_1)}(z)$ is uniformly continuous in D_T from the same reason as in No. 5. Further we obtain easily

(2.4)
$$\chi_n^{(\mu;T,T_1)}(g_{S^{(m)}})=\sum_{S^{(n-1)}}\Big(\frac{R_{S^{(n-1)}T_1}}{|S^{(n-1)}T_1(\infty)-S^{-(m)}(\infty)|}\Big)^{\mu}$$
$$=\sum_{S^{(n-1)}}\Big(\frac{R_{S^{(m)}S^{(n-1)}T_1}}{R_{S^{(m)}}}\Big)^{\mu}.$$

We call $\chi_n^{(\mu;T,T_1)}(z)$ the μ-dimensional subcomputing T_1-function of order n on T. There exist N^2 subcomputing functions $\chi_n^{(\mu;T,T_1)}(z)$ in all, since $n>1$ and the first element T_1 of $S^{(n)}$ and T ($\neq T_n$) are any generators or their inverses ($\in \mathfrak{G}$) of G, respectively.

7. Now let us present some properties of subcomputing function.

(i) At first we have from the definition

(2.5)
$$\sum_{i=1}^{N}\chi_n^{(\mu;T,T_i)}(z)=\chi_n^{(\mu;T)}(z)\,,\qquad \text{for any}\quad z\in D_T\,.$$

(2.6)
$$k(G)\chi_n^{(\mu;T)}(z)\leqq \chi_n^{(\mu;T,T_i^{-1})}(z)+\chi_{n+1}^{(\mu;T,T_i)}(z)\leqq K(G)\chi_n^{(\mu;T)}(z)\,.$$

PROOF. Since (2.5) is obvious, we shall prove (2.6) only. From the definition, we have

$$\chi_n^{(\mu;T,T_i^{-1})}(z)+\chi_{n+1}^{(\mu;T,T_i)}(z)=\sum_{S_1^{(n-1)}}\Big(\frac{R_{S_1^{(n-1)}T_i^{-1}}}{|S_1^{(n-1)}T_i(\infty)-z|}\Big)^{\mu}+\sum_{S_2^{(n)}}\Big(\frac{R_{S_2^{(n)}T_i}}{|S_2^{(n)}T_i(\infty)-z|}\Big)^{\mu}$$

for fixed T_i ($\in \mathfrak{G}$). Since it holds

$$\sum_{S_2^{(n)}}\Big(\frac{R_{S_2^{(n)}T_i}}{|S_2^{(n)}T_i(\infty)-z|}\Big)^{\mu}\leqq k_1(G)\sum_{S_3^{(n-1)}}\Big(\frac{R_{S_3^{(n-1)}T_k}}{|S_3^{(n-1)}T_k(\infty)-z|}\Big)^{\mu},\qquad (T_k\neq T_i^{-1})\,,$$

we obtain from (2.5)

$$\chi_n^{(\mu;T,T_i^{-1})}(z)+\chi_{n+1}^{(\mu;T,T_i)}(z)\leqq K(G)\chi_n^{(\mu;T)}(z)\,,$$

where $K(G)=\max(1, k_1(G))$.

Next we shall prove the left inequality. We obtain easily

$$\chi_n^{(\mu;T)}(z)=\sum_{S^{(n)}}\Big(\frac{R_{S^{(n)}}}{|S^{(n)}(\infty)-z|}\Big)^{\mu}$$
$$=\sum_{S_4^{(n-1)}}\Big(\frac{R_{S_4^{(n-1)}T_i^{-1}}}{|S^{(n-1)}T_i(\infty)-z|}\Big)^{\mu}+\sum_{S_5^{(n)}}\Big(\frac{R_{S_5^{(n)}}}{|S_5^{(n)}(\infty)-z|}\Big)^{\mu},$$

where $S_5^{(n)}=S_5^{(n-1)}T_k$ ($T_k\neq T_i^{-1}$). Since it holds

$$\sum_{S_5^{(n)}}\Big(\frac{R_{S_5^{(n)}}}{|S_5^{(n)}(\infty)-z|}\Big)^{\mu}\leqq k_2(G)\sum_{S_5^{(n)}}\Big(\frac{R_{S_5^{(n)}T_i}}{|S_5^{(n)}T_i(\infty)-z|}\Big)^{\mu},$$

we have

$$k(G)\chi_n^{(\mu;T)}(z)\leqq \chi_n^{(\mu;T,T_i^{-1})}(z)+\chi_{n+1}^{(\mu;T,T_i)}(z)\,,$$

where $k(G)=1/\max(1, k_2(G))$.

(ii) Fix the generators T and T_1 and take a transformation $S^{(l+n)}=S^{(l)}S^{(n)}=T_{l+n}T_{l+n-1}\cdots T_{n+1}T_n\cdots T_2T_1$ with grade $l+n$ such that $T_{l+n}\neq T^{-1}$, where T and $T_j\in\mathfrak{G}$ $(1\leqq j\leqq l+n)$. If we differentiate the inverse transformation $S^{-(l+n)}(z)=S^{-(n)}S^{-(l)}(z)$ $(z\in D_T)$ with respect to z, we have

$$\left|\frac{dS^{-(l+n)}(z)}{dz}\right|^{\frac{\mu}{2}}=\left(\left|\frac{dS^{-(n)}(z')}{dz'}\right|\cdot\left|\frac{dS^{-(l)}(z)}{dz}\right|\right)^{\frac{\mu}{2}},\qquad z'=S^{-(l)}(z)\,.$$

Hence we get from (1.6)

$$\left(\frac{R_{S^{(l+n)}}}{|S^{(l+n)}(\infty)-z|}\right)^{\mu}=\left(\frac{R_{S^{(n)}}}{|S^{(n)}(\infty)-S^{-(l)}(z)|}\cdot\frac{R_{S^{(l)}}}{|S^{(l)}(\infty)-z|}\right)^{\mu}. \tag{2.7}$$

Forming the sum of $(N-1)^{l+n-1}$ terms with respect to all $S^{(l+n)}$ of grade $l+n$ in G for fixed T_1 such that $T_{l+n}\neq T^{-1}$, we obtain from the definition of the subcomputing function the following relation

$$\chi_{l+n}^{(\mu;T,T_1)}(z)=\sum_{S^{(l)}}\left\{\chi_n^{(\mu;T_{n+1},T_1)}(S^{-(l)}(z))\left(\frac{R_{S^{(l)}}}{|S^{(l)}(\infty)-z|}\right)^{\mu}\right\}, \tag{2.8}$$

where the domain of definition of $\chi_{l+n}^{(\mu;T,T_1)}(z)$ and $\chi_n^{(\mu;T_{n+1},T_1)}(z)$ are the closed discs D_T and $D_{T_{n+1}}$ bounded by H_T and $H_{T_{n+1}}$, respectively and $S^{-(l)}=(T_{n+l}\cdots T_{n+1})^{-1}=T_{n+1}^{-1}\cdots T_{n+l}^{-1}$.

Since $S^{(l)}(\infty)$ and z are contained in $D_{T_{l+n}^{-1}}$ and D_T, respectively and $T\neq T_{l+n}^{-1}$, each denominator of the right hand side in (2.8) does not vanish and is greater than some positive constant from the assumption about B_0. Noting that $R_{S^{(l)}}$ tends to zero as $l\to\infty$, we have the following inequality:

$$\chi_{l+n}^{(\mu;T,T_1)}(z)>k_0(l)\sum_{S^{(l)}}\chi_n^{(\mu;T_{n+1},T_1)}(S^{-(l)}(z))\,, \tag{2.9}$$

where $k_0(l)$ is a constant depending only on l and tends to zero for $l\to\infty$.

In the same manner $S^{(n)}(\infty)$ and $S^{-(l)}(z)$ are contained in $D_{T_n^{-1}}$ and $D_{T_{n+1}}$, respectively and hence the factor $|S^{(n)}(\infty)-S^{-(l)}(z)|$ of the denominator of the right hand side in (2.7) does not vanish from $T_n^{-1}\neq T_{n+1}$ and then it is greater than some positive constant from the assumption about B_0. Noting the above consideration, we can easily find from (2.7) that there is the following relation between two computing functions on the same T with different orders:

$$K_1(n)\chi_l^{(\mu;T,T_{n+1})}(z)>\chi_{l+n}^{(\mu;T,T_1)}(z)>K_0(n)\chi_l^{(\mu;T,T_{n+1})}(z)\,, \tag{2.10}$$

where $K_i(n)$ $(i=0,1)$ denote the constants depending only on n.

(iii) In [5], we got the equi-continuity of the sequence of computing functions. We have also the same property with respect to the sequence of subcomputing functions. The proof is analogous to that of computing function.

PROPOSITION 6. (1) *Suppose that the sequence of subcomputing functions*

$\{\chi_n^{(\mu;T,T_1)}(z)\}$ $(n=1,2,\cdots)$ *on some* T *and* $T_1\in\mathfrak{G}$ *converges to a finite number or* 0 *as* $n\rightarrow\infty$ *at some point* z' *of the singular set contained in* D_T. *Then the sequence* $\{\chi_n^{(\mu;T,T_1)}(z)\}$ $(n=1,2,\cdots)$ *is equi-continuous on* D_T, *that is, there exists some positive number* δ *depending only on* ε *such that*

$$(2.11)\qquad |\chi_n^{(\mu;T,T_1)}(z)-\chi_n^{(\mu;T,T_1)}(z_0)|<\varepsilon$$

for any point z *contained in* $D_\delta(z_0)$ *satisfying* $|z-z_0|<\delta$, *where* $D_\delta(z_0)$ *denotes an open disc of radius* δ *with center* $z_0\in D_T$.

(2) *Suppose that the sequence of subcomputing functions* $\{\chi_n^{(\mu;T,T_1)}(z)\}$ $(n=1,2,\cdots)$ *on some* T *and* $T_1\in\mathfrak{G}$ *diverges for* $n\rightarrow\infty$ *at some point* z' *of the singular set contained in* D_T. *Put* $\eta_n^{(\mu;T,T_1)}(z)=1/\chi_n^{(\mu;T,T_1)}(z)$. *Then the sequence* $\{\eta_n^{(\mu;T,T_1)}(z)\}$ $(n=1,2,\cdots)$ *is equi-continuous on* D_T, *that is, there exists some positive number* δ *depending only on* ε *such that*

$$(2.12)\qquad |\eta_n^{(\mu;T,T_1)}(z)-\eta_n^{(\mu;T,T_1)}(z_0)|<\varepsilon$$

for any point z *contained in* $D_\delta(z_0)$ *defined in* (1) *satisfying* $|z-z_0|<\delta$.

8. Suppose that there exists a positive integer n_0 for any sufficiently small ε such that

$$(2.13)\qquad \chi_{n_0}^{(\mu;T,T_i)}(z)>\frac{1+\varepsilon}{N},\qquad \left(\text{or }<\frac{1-\varepsilon}{N}\right),$$

for any T and $T_i\in\mathfrak{G}$ and any $z\in E\cap D_T$. Then it holds from (2.5)

$$\chi_{n_0}^{(\mu;T)}(z)>1+\varepsilon,\qquad (\text{or }<1-\varepsilon)$$

for any $T\in\mathfrak{G}$ and any $z\in E\cap D_T$. Therefore we find that (2.13) is the sufficient condition for Theorem 1.

We shall prove that the converse of the above is also true.

Let us suppose that the third proposition in Theorem 1 establishes, that is, it holds

$$\lim_{n\to\infty}\chi_n^{(\mu;T)}(z)=\infty,\qquad (\text{or } 0)$$

for any T $(\in\mathfrak{G})$ uniformly on D_T.

At first we consider the case where the limit is 0. From (2.5) there exists a certain positive integer n_0 such that for any sufficiently small ε such that it holds

$$(2.14)\qquad \chi_{n_0}^{(\mu;T,T_i)}(z)<\frac{1-\varepsilon}{N},$$

for any T and $T_i\in\mathfrak{G}$ on $E\cap D_T$.

Next we treat the case where the limit is ∞. From (2.6) there exists a sequence of subcomputing functions $\{\chi_n^{(\mu;T,T_i)}(z)\}$ $(n=1,2,\cdots)$ such that it holds

$$(2.15)\qquad \lim_{n\to\infty}\chi_n^{(\mu;T,T_i)}(z)=\infty$$

for any T and T_i ($\in \mathfrak{G}$) uniformly on $E \cap D_T$. Hence there exists a certain positive integer n_0 for sufficiently small ε such that it holds

$$\chi_{n_0}^{(\mu;T,T_i)}(z) > \frac{1-\varepsilon}{N}, \tag{2.16}$$

for any T and $T_i \in \mathfrak{G}$ on $E \cap D_T$.

Thus we have the following theorem.

THEOREM 3. *The four propositions in Theorem* 1 *are equivalent to the following proposition: there exists a positive integer* n_0 *for any sufficiently small* ε *such that*

$$\chi_{n_0}^{(\mu;T,T_i)}(z) > \frac{1+\varepsilon}{N}, \quad \left(or \ \frac{1-\varepsilon}{N}\right), \tag{2.17}$$

for any T *and* $T_i \in \mathfrak{G}$ *on* $E \cap D_T$.

9. REMARK. Using the properties (i), (ii) and (iii), we can easily find that the propositions in Theorem 3 are also equivalent to the following two propositions (A) and (B). The proof is also analogous to that of the computing function in [5].

(A) The subsequence $\{\chi_{n_j}^{(\mu;T,T_i)}(z)\}$ $(j=1,2,\cdots)$ of $\{\chi_n^{(\mu;T,T_i)}(z)\}$ $(n=1,2,\cdots)$ on some T and T_i ($\in \mathfrak{G}$) diverges to ∞ (or converges to zero) at some point z_0 of the singular set contained in D_T, that is,

$$\lim_{j\to\infty} \chi_{n_j}^{(\mu;T,T_j)}(z_0) = \infty, \qquad (\text{or } 0)$$

for some $z_0 \in E \cap D_T$.

(B) It holds

$$\lim_{n\to\infty} \chi_n^{(\mu;T,T_i)}(z) = \infty, \qquad (\text{or } 0)$$

for any T and T_j ($\in \mathfrak{G}$) uniformly on D_T.

10. We have the following theorem analogous to Theorem 2 with respect to the subcomputing function from Theorem 3.

THEOREM 4. *Let* $\mu^*/2$ *be the Hausdorff dimension of E. Then there exists some positive small* ε *such that*

$$\left|\chi_n^{(\mu^*;T,T_i)}(z) - \frac{1}{N}\right| < \frac{\varepsilon}{N} \tag{2.18}$$

for any n, any T *and* T_i ($\in \mathfrak{G}$) *and any* $z \in E \cap D_T$. *Therefore it holds*

$$\frac{1-\varepsilon}{N} \leqq \underline{\chi}^{(\mu^*;T,T_i)}(z) \leqq \bar{\chi}^{(\mu^*;T,T_i)}(z) \leqq \frac{1+\varepsilon}{N}, \qquad (\varepsilon < 1), \tag{2.19}$$

for any T *and* T_i ($\in \mathfrak{G}$) *and any* $z \in E \cap D_T$, *where* $\underline{\chi}^{(\mu^*;T,T_i)}(z) = \varliminf_{n\to\infty} \chi_n^{(\mu^*;T,T_i)}(z)$ *and* $\bar{\chi}^{(\mu^*;T,T_i)}(z) = \varlimsup_{n\to\infty} \chi_n^{(\mu^*;T,T_i)}(z)$.

COROLLARY. *Let* $\mu^*/2$ *be the Hausdorff dimension of E. Then it holds*

(2.20) $$\frac{1}{K} \leqq \frac{\chi_n^{(\mu^*; T_1, T_2)}(z)}{\chi_n^{(\mu^*; T_3, T_4)}(z')} \leqq K$$

for any T_i $(i=1,2,3,4)$ $(\in \mathfrak{G})$ *and any* $z, z' \in E \cap D_T$, *where* $K=(1+\varepsilon)/(1-\varepsilon)$ (>1) *is a constant independent of* n, z, z' *and* T_i $(i=1,2,3,4)$ $(\in \mathfrak{G})$.

§ 3. Example of the Kleinian groups whose singular sets have positive (3/2)-dimensional measure.

11. We shall give some theorems which show not only the existence of the Kleinian groups with title stated in the above, but also interest in themselves.

At first we shall give the following

THEOREM 5. *Let* $P_1=P(R, 0)$ *and* $P_2=P(R, \pi)$ *be fixed points on the real axis in the complex z-plane and* $P=P(r, \theta)$ *move on the fixed circle* $C_r: |z|=r$ $(r \neq R)$. *Then the function* $f(P)=\sum_{i=1}^{2} \frac{1}{\overline{PP_i}^3}$ *of P attains its minimum at the points on the real axis, where* $\overline{PP_i}$ *denotes the distance between P and* P_i.

PROOF. By using the polar coordinates, we obtain

$$f(r, \theta)=(R^2+r^2-2Rr\cos\theta)^{-\frac{3}{2}}+(R^2+r^2+2Rr\cos\theta)^{-\frac{3}{2}}, \qquad (r\,;\ \text{fixed})\,.$$

If we differentiate $f(r, \theta)$ with respect to θ for fixed r, we have

$$\frac{\partial f}{\partial \theta}=\frac{-3Rr\sin\theta\{(R^2+r^2+2Rr\cos\theta)^{\frac{5}{2}}-(R^2+r^2-2Rr\cos\theta)^{\frac{5}{2}}\}}{(R^2+r^2+2Rr\cos\theta)^{\frac{5}{2}}(R^2+r^2-2Rr\cos\theta)^{\frac{5}{2}}}\,.$$

Hence the values which satisfy the equation $\frac{\partial f}{\partial \theta}=0$ in $0 \leqq \theta \leqq 2\pi$ are the ones which satisfy the following equations

(i) $\sin\theta=0$ and (ii) $(1+a\cos\theta)^5=(1-a\cos\theta)^5$, where $a=2Rr/(R^2+r^2)$ $(\leqq 1)$. From (i) we obtain the solutions 0 and π. From (ii) we obtain

(3.1) $$\cos\theta\,(a^4\cos^4\theta+10a^4\cos^2\theta+5)=0\,.$$

Hence the values which satisfy the equation (3.1) are those satisfying $\cos\theta=0$ (hence $\theta=\pi/2$ and $3\pi/2$) or

(3.2) $$a^4\cos^4\theta+10a^2\cos^2\theta+5=0\,.$$

If we put $x=a^2\cos^2\theta$ $(0 \leqq x \leqq 1)$, we obtain, as the values which satisfy (3.2), $x=-5\pm 2\sqrt{5}$ and these contradict $0 \leqq x \leqq 1$. Observing the variation of $\frac{\partial f}{\partial \theta}$ at $\theta=0$, $\frac{\pi}{2}$, π and $\frac{3}{2}\pi$, we can show the fact that $f(P)$ attains its minimum $\theta=\frac{\pi}{2}$ and $\frac{3}{2}\pi$ and its maximum at $\theta=0$ and π. q. e. d.

Next we shall give the following

THEOREM 6 ([6]). *Let* $P_1=P(2,\pi)$, $P_2=P\left(2,\frac{\pi}{3}\right)$ *and* $P_3=P\left(2,-\frac{\pi}{3}\right)$ *be fixed in the complex z-plane, and* $P=P(r,\theta)$ *move in the fixed closed unit disc* $U:|z|\leqq 1$. *Then the function* $f(P)=\sum_{i=1}^{3}\frac{1}{\overline{PP_i}^3}$ *attains its minimum at the origin.*

The proof of this theorem is given in [6] and [7].

12. The Example. Now let us give the example of Kleinian groups whose singular sets have positive (3/2)-dimensional measure.

At first we consider the unit circle and denote it by $C_{0,0}$. Next we describe the six circles C_{1,j_1} $(j_1=1,\cdots,6)$ with equal radii 1 so that C_{1,j_1} $(j_1=1,\cdots,6)$ are tangent externally with each other around $C_{0,0}$ and hence the segments, which join the centers of C_{1,j_1} successively, constitute a regular hexagon R_1, where the center of $C_{1,1}$ has a real coordinate (2, 0). Further we describe the twelve circles C_{2,j_2} $(j_2=1,2,\cdots,12)$ with equal radii 1 so that C_{2,j_2} $(j_2=1,2,\cdots,12)$ are tangent externally with each other around C_{1,j_1} and hence the segments, which join the centers of C_{2,j_2} successively, constitute a regular hexagon R_2, where the center of $C_{2,1}$ has a real coordinate (4, 0). We continue such procedure by turns. Generally, we describe the C_{n,j_n} $(j_n=1,2,\cdots,6n)$ with equal radii 1 so that C_{n,j_n} $(j_n=1,2,\cdots,6n)$ are tangent externally with each other around $C_{n-1,j_{n-1}}$ and hence the segments, which join the centers of C_{n,j_n} successively, constitute a regular hexagon R_n, where the center of $C_{n,1}$ has a real coordinate $(2n, 0)$. We call the first index i and the second j_i of C_{i,j_i} i-th rank and j_i-th emission, respectively. It is obvious that the total number of circles $C_{0,0}$ and C_{i,j_i} $(i=1,2,\cdots,n;\ j_i=1,2,\cdots,6i)$ is equal to $3n(n+1)+1$.

If we make all radii of circles C_{i,j_i} of all ranks short by ε, we get a connected domain D_n bounded by $3n(n+1)+1$ circles whose radii are all $1-\varepsilon$. Now we shall form a Kleinian group G with fundamental domain D_n in the following; we let these $3n(n+1)+1$ circles $C_{0,0}$, C_{1,j_1} $(j_1=1,2,\cdots,6),\cdots,C_{n,j_n}$ $(j_n=1,2,\cdots,6n)$ correspond to elliptic transformations $S_{0,0}$, S_{1,j_1} $(j_1=1,\cdots,6)$, $\cdots,S_{n,j_n}$ $(j_n=1,\cdots,6n)$ with period 2. But we require some remarks: (i) two fixed points of S_{n,j_n} $(j_n=1,\cdots,6n)$ are the endpoints of diameters of circles C_{n,j_n} $(j_n=1,\cdots,6n)$ orthogonal to the sides of R_n except six transformations $S_{n,1+in}$ $(i=0,\cdots,5)$ and (ii) two fixed points of each of $S_{n,1+in}$ $(i=0,\cdots,5)$ are intersecting points of the half straight line emanating from the origin and going through each of the centers of circles $C_{n,1+in}$ $(i=0,\cdots,5)$ with these six circles.

Then the singular set E of such Kleinian group G generated by $S_{0,0}$, S_{1,j_1} $(j_1=1,\cdots,6),\cdots,S_{n,j_n}$ $(j_n=1,\cdots,6n)$ is contained in $R_n\cap\left[\left\{\bigcup_{i=1}^{n}\bigcup_{j_i=1}^{6i}D_{i,j_i}\right\}\cup D_{0,0}\right]$, where D_{i,j_i} denotes the closed disc bounded by C_{i,j_i}.

13. Let $\frac{\mu}{2}$ $(\mu>3)$ be the Hausdorff dimension of the singular set of the Kleinian group defined in No. 12. Then it holds from Theorem 2

$$|\chi_\nu^{(\mu;S_{0,0})}(z)-1|<\varepsilon \tag{3.3}$$

for some ε and any ν, where ε (<1) is a positive number independent of ν, $S_{0,0}$ and z $(\in E\cap D_{0,0})$. Hence if we can determine the number n of the rank of outermost circles C_{n,j_n} $(j_n=1,\cdots,6n)$ from the origin so that the computing functions of $C_{0,0}$, $\chi_\nu^{(\mu;S_{0,0})}(z)$, are bounded for any ν and any singular point z $(\in E\cap D_{0,0})$, we can show from Theorem 2 the existence of Kleinian group G with fundamental domain B_0 whose singular set E has positive (3/2)-dimensional measure.

Take any point $z\in E\cap D_{0,0}$ and some neighborhood $U(z)$ of z which will be determined later. Then there exists a center of isometric circle $g_{S^{(m)}}$ which is contained in $U(z)$ for sufficiently large m $(\geqq m_0)$, where $S^{(m)}=S^{(m-1)}S_{0,0}$.

Consider the μ-dimensional computing function of order ν on $S_{0,0}$, $\chi_\nu^{(\mu;S_{0,0})}(z)$, at $z\in E\cap D_{0,0}$ for $\mu>3$. From the definition of the computing function it holds that

$$\chi_\nu^{(\mu;S_{0,0})}(g_{S^{(m)}})=\frac{\sum_{S^{(\nu)}}(R_{S^{(m)}S^{(\nu)}})^\mu}{(R_{S^{(m)}})^\mu}. \tag{3.4}$$

We modify the right hand side of (3.4) in the following

$$\frac{\sum_{S^{(\nu)}}(R_{S^{(m)}S^{(\nu)}})^\mu}{(R_{S^{(m)}})^\mu}=\prod_{j=1}^{\nu}\left[\frac{\sum_{S^{(j)}}(R_{S^{(m)}S^{(j)}})^\mu}{\sum_{S^{(j-1)}}(R_{S^{(m)}S^{(j-1)}})^\mu}\right]. \tag{3.5}$$

Since $\chi_\nu^{(\mu;S_{0,0})}(z)$ $(\nu=1,2,\cdots)$ is equi-continuous in $D_{0,0}$ (No. 8), there exists a number δ depending only on any sufficiently small ε' such that it holds

$$|\chi_\nu^{(\mu;S_{0,0})}(z)-\chi_\nu^{(\mu;S_{0,0})}(g_{S^{(m)}})|<\varepsilon', \qquad \text{for} \quad |z-g_{S^{(m)}}|<\delta. \tag{3.6}$$

Then we have from (3.3) and (3.6)

$$1+(\varepsilon+\varepsilon')>\chi_\nu^{(\mu;S_{0,0})}(g_{S^{(m)}})>1-(\varepsilon+\varepsilon')>0. \tag{3.7}$$

We take a closed disc $D_\delta(z)$ with center z and radius δ defined in the above as a neighborhood $U(z)$ of z and we can determine $g_{S^{(m)}}$ such that it is contained in $D_\delta(z)$.

Now we know that each term of the product in (3.5) tends to 1 from the well-known theorem about the infinite product. Hence there exists an integer j_0 depending only on any given sufficiently small ε^* such that it holds

$$(3.8)\qquad \frac{\sum\limits_{S^{(j)}}(R_{S^{(m)}S^{(j)}})^{\mu}}{\sum\limits_{S^{(j-1)}}(R_{S^{(m)}S^{(j-1)}})^{\mu}} > 1-\varepsilon^*, \qquad ({}^{\forall}j>j_0)$$

for all terms of the product (3.5) except finite j_0 terms from the first term. This inequality is equivalent to

$$(3.9)\qquad \sum_{S^{(j-1)}}(R_{S^{(m)}S^{(j-1)}})^{\mu}\Big\{\frac{\sum\limits_{S_k}(R_{S^{(m)}S^{(j-1)}S_k})^{\mu}}{(R_{S^{(m)}S^{(j-1)}})^{\mu}}-(1-\varepsilon^*)\Big\}>0, \qquad (j>j_0).$$

By the definition of the computing function we see easily that

$$(3.10)\qquad \frac{\sum\limits_{S_k}(R_{S^{(m)}S^{(j-1)}S_k})^{\mu}}{(R_{S^{(m)}S^{(j-1)}})^{\mu}}=\frac{\sum\limits_{S_k}(R_{S^{(m+j-1)}S_k})^{\mu}}{(R_{S^{(m+j-1)}})^{\mu}}=\chi_1^{(\mu;S_{i,j_i})}(g_{S^{(m+j-1)}}),$$

$$(i=0,\ j_i=0) \text{ or } (i=1,\cdots,n;\ j_i=1,\cdots,6i),$$

where $S^{(m+j-1)}=S^{(m+j-2)}S_{i,j_i}$.

14. In this section we shall estimate the values of $\chi_1^{(\mu;S_{i,j_i})}(g_{S^{(m+j-1)}})$ from below for all S_{i,j_i} $(i=0,\ j_i=0$ or $i=1,\cdots,n;\ j_i=1,\cdots,6i)$. For convenience of calculation we suppose that the all radii of the boundary circles C_{i,j_i} $(i=0,\ j_i=0$ or $i=1,\cdots,n;\ j_i=1,\cdots,6i)$ are equal to 1. Moreover, if we suppose that μ is very close to 3, we may calculate the values of 3-dimensional computing functions of order 1 on S_{i,j_i} $\chi_1^{(3;S_{i,j_i})}(g_{S^{(m+j-1)}})$ instead of $\chi_1^{(\mu;S_{i,j_i})}(g_{S^{(m+j-1)}})$, as $\chi_1^{(\mu;S_{i,j_i})}(g_{S^{(m+j-1)}})$ is continuous.

We divide these calculations into the various cases according as the various ranks of S_{i,j_i}.

(a) With respect to the generators S_{i,j_i} $(i=0,\ j_i=0$ or $i=1,\cdots,n-4,\ j_i=1,\cdots,6i)$ from 0-th rank to $(n-4)$-th rank, we use the estimate of $\chi_1^{(3;S_{0,0})}(z)$ in $D_{0,0}$. We obtain from (1.1) and Theorem 6 the following inequality

$$(3.11)\qquad \begin{aligned}\chi_1^{(3;S_{0,0})}(z)&=\sum_{S_k\neq S_{0,0}}\frac{R^3_{S_k}}{|g_{S_k}-z|^3}\geqq\chi_1^{(3;S_{0,0})}(0)=\sum_{S_k\neq S_{0,0}}\frac{1}{|g_{S_k}|^3}\\ &>\frac{6}{2^3}+\Big(\frac{6}{4^3}+\frac{6}{(2\sqrt{3})^3}\Big)+\Big(\frac{1}{6^2}+\frac{12}{(2\sqrt{7})^3}\Big)\\ &\quad+\Big(\frac{6}{8^3}+\frac{\sqrt{3}}{96}+\frac{3\sqrt{13}}{338}\Big)>1.15862, \qquad ({}^{\forall}z\in E\cap D_{0,0}),\end{aligned}$$

where the above first, second and third parentheses denote the sum with respect to circles of second, third and fourth rank, respectively.

(b) Next we shall estimate the values of the computing functions corresponding to the circles $C_{n-3,j_{n-3}}$ $(j_{n-3}=1,\cdots,6n-18)$ of $(n-3)$-th rank. For this purpose we take the computing function $\chi_1^{(3;S_{n-3,1})}(z)$ of the circle $C_{n-3,1}$ of $(n-3)$-th rank and the first emission as the representative. From Theorems

5 and 6 we have the following inequality:

$$\chi_1^{(3;s_{n-3,1})}(z)=\sum_{s_k\neq s_{n-3,1}}\frac{R_{s_k}^3}{|g_{s_k}-z|^3}$$

$$(3.12)\qquad >\frac{6}{2^3}+\left(\frac{6}{4^3}+\frac{6}{(2\sqrt{3})^3}\right)+\left(\frac{1}{6^2}+\frac{12}{(2\sqrt{7})^3}\right)$$

$$+\left(\frac{1}{9^3}+\frac{2\sqrt{67}}{67^2}+\frac{2\sqrt{61}}{61^2}+\frac{2\sqrt{7}}{27\times 7^2}+\frac{2\sqrt{73}}{73^2}+\frac{\sqrt{13}}{338}+\frac{2}{7^3}\right)$$

$$>1.12978,\qquad ({}^{\forall}z\in E\cap D_{n-3,1}),$$

where Theorem 6 is used in the value $6/2^3$, the first and second parentheses and Theorem 5 is used in the last parentheses.

(c) With respect to the computing functions corresponding to the circles $C_{n-2,j_{n-2}}$ ($j_{n-2}=1,\cdots,6n-12$) of $(n-2)$-th rank, we shall estimate the value of $\chi_1^{(3;s_{n-2,1})}(z)$ as the representative. We obtain from Theorems 5 and 6

$$\chi_1^{(3;s_{n-2,1})}(z)=\sum_{s_k\neq s_{n-2,1}}\frac{R_{s_k}^3}{|g_{s_k}-z|^3}$$

$$>\frac{6}{2^3}+\left(\frac{6}{4^3}+\frac{6}{(2\sqrt{3})^3}\right)$$

$$(3.13)\qquad +\left(\frac{1}{7^3}+\frac{2\sqrt{39}}{39^2}+\frac{2\sqrt{37}}{37^2}+\frac{2\sqrt{43}}{43^2}+\frac{\sqrt{7}}{98}\right)$$

$$+\left(\frac{1}{9^3}+\frac{2\sqrt{67}}{67^2}+\frac{2}{7^3}+\frac{6\sqrt{7}}{63^2}+\frac{2\sqrt{73}}{73^2}+\frac{2\sqrt{57}}{57^2}+\frac{2\sqrt{61}}{61^2}\right)$$

$$>1.06909,\qquad ({}^{\forall}z\in E\cap D_{n-2,1}),$$

where Theorem 6 is used in the value $6/2^3$ and the first parentheses and Theorem 5 is used in the second and the third parentheses.

(d) With respect to the computing functions corresponding to the circles $C_{n-1,j_{n-1}}$ ($j_{n-1}=1,\cdots,6n-6$) of $(n-1)$-th rank, we shall estimate the value of $\chi_1^{(3;s_{n-1,1})}(z)$ as the representative. We obtain from Theorems 5 and 6

$$\chi_1^{(3;s_{n-1,1})}(z)=\sum_{s_k\neq s_{n-1,1}}\frac{R_{s_k}^3}{|g_{s_k}-z|^3}$$

$$>\frac{6}{2^3}+\left(\frac{2\sqrt{13}}{13^2}+\frac{2\sqrt{21}}{21^2}+\frac{2\sqrt{19}}{19^2}+\frac{1}{5^3}\right)$$

$$(3.14)\qquad +\left(\frac{1}{7^3}+\frac{2\sqrt{31}}{31^2}+\frac{2\sqrt{43}}{43^2}+\frac{2\sqrt{37}}{37^2}+\frac{2\sqrt{39}}{39^2}\right)$$

$$+\left(\frac{1}{9^3}+\frac{2\sqrt{57}}{57^2}+\frac{2\sqrt{73}}{73^2}+\frac{2\sqrt{7}}{27\times 7^3}+\frac{2\sqrt{61}}{61^2}+\frac{2\sqrt{67}}{67^2}\right)$$

$$+\left(\frac{1}{11^3}+\frac{4\sqrt{91}}{91^2}+\frac{2\sqrt{111}}{111^2}+\frac{2\sqrt{97}}{97^2}+\frac{2\sqrt{93}}{93^2}+\frac{2\sqrt{103}}{103^2}\right)$$
$$>0.91867\,,\qquad ({}^{\forall}z\in E\cap D_{n-1,1})\,,$$

where Theorem 6 is used in the value $6/2^3$ and Theorem 5 is used in the values from the first to the fourth parentheses.

(e) With respect to the computing functions corresponding to the circles C_{n,j_n} $(n=1,\cdots,6n)$ of n-th rank, we shall estimate the values of $\chi_1^{(3;S_{n,1})}(z)$ as the representative. We use only Theorem 5 and then we have the following

$$\chi_1^{(3;S_{n,1})}(z)=\sum_{S_k\neq S_{n,1}}\frac{R_{S_k}^3}{|g_{S_k}-z|^3}$$
$$(3.15)\qquad >\frac{3}{2^3}+\left(\frac{3}{4^3}+\frac{\sqrt{3}}{36}\right)+\left(\frac{3}{6^3}+\frac{\sqrt{7}}{98}\right)+\left(\frac{3}{8^3}+\frac{\sqrt{3}}{288}+\frac{\sqrt{13}}{338}\right)$$
$$+\left(\frac{3}{10^3}+\frac{\sqrt{21}}{882}+\frac{\sqrt{19}}{722}\right)+\left(\frac{3}{12^3}+\frac{\sqrt{3}}{972}+\frac{\sqrt{31}}{1922}+\frac{\sqrt{7}}{784}\right)$$
$$>0.55744\,,\qquad ({}^{\forall}z\in E\cap D_{n,1})\,.$$

15. Now let us return to the inequality (3.9) Since the first term $\sum_{S_k}(R_{S^{(m)}S^{(j-1)}S_k})^{\mu}/R(_{S^{(m)}S^{(j-1)}})^{\mu}$ in the parentheses of (3.9) is approximately equal to $\chi_1^{(3;S_{i,j_i})}(g_{S^{(m+j-1)}})$ from (3.10) and we found estimates of these values from below in all cases (a)—(e) in No. 14, if we substitute these values into the inequality (3.9), we obtain the following inequality:

$$(3.16)\qquad \begin{aligned}&0.15863\sum_{S^{(j-1)}}{}^{(1)}(R_{S^{(m+j-1)}})^{\mu}+0.12979\sum_{S^{(j-1)}}{}^{(2)}(R_{S^{(m+j-1)}})^{\mu}\\&+0.06910\sum_{S^{(j-1)}}{}^{(3)}(R_{S^{(m+j-1)}})^{\mu}-0.08132\sum_{S^{(j-1)}}{}^{(4)}(R_{S^{(m+j-1)}})^{\mu}\\&-0.44255\sum_{S^{(j-1)}}{}^{(5)}(R_{S^{(m+j-1)}})^{\mu}>0\,,\end{aligned}$$

where $\sum^{(i)}$, $(i=1,2,3,4,5)$ denotes the sum of radii of the isometric circles of transformations whose last element S_{i,j_i} of $S^{(j-1)}=S^{(j-2)}S_{i,j_i}$ corresponds to the cases of (a)—(e) in No. 14, respectively and further we take ε^* sufficiently small, for example, 10^{-5}.

If the left hand side of (3.16) is divided by $(R_{S^{(m)}})^{\mu}$, and the definition of the subcomputing function is recalled, the following inequality is given:

$$(3.17)\qquad \begin{aligned}&0.15863\sum_{S_{i,j_i}}{}^{(1)}\chi_{j-1}^{(\mu;S_{0,0},S_{i,j_i})}(g_{S^{(m)}})+0.12979\sum_{S_{i,j_i}}{}^{(2)}\chi_{j-1}^{(\mu;S_{0,0},S_{i,j_i})}(g_{S^{(m)}})\\&+0.06910\sum_{S_{i,j_i}}{}^{(3)}\chi_{j-1}^{(\mu;S_{0,0},S_{i,j_i})}(g_{S^{(m)}})-0.08132\sum_{S_{i,j_i}}{}^{(4)}\chi_{j-1}^{(\mu;S_{0,0},S_{i,j_i})}(g_{S^{(m)}})\\&-0.44255\sum_{S_{i,j_i}}{}^{(5)}\chi_{j-1}^{(\mu;S_{0,0},S_{i,j_i})}(g_{S^{(m)}})>0\,,\end{aligned}$$

where $\sum_{S_{i,j_i}}^{(1)}$ denotes the sum of the subcomputing function with respect to the right element S_{i,j_i} of $S^{(m+j-1)}$ from 0 to $(n-4)$-th rank, $\sum_{S_{i,j_i}}^{(2)}$ the sum with respect to the element of $(n-3)$-th rank, $\sum_{S_{i,j_i}}^{(3)}$ the sum with respect to the element of $(n-2)$-th rank, $\sum_{S_{i,j_i}}^{(4)}$ the sum with respect to the element of $(n-1)$-th rank, and $\sum_{S_{i,j_i}}^{(5)}$ the sum with respect to the element of n-th rank, respectively.

Since $\mu/2$ is the Hausdorff dimension, we can easily prove from Corollary of Theorem 4 in No. 10 that it holds

$$\frac{1}{K} \leqq \frac{\chi_{j-1}^{(\mu; S_{0,0}, S_{i,j_i})}(g_{S^{(m)}})}{\chi_{j-1}^{(\mu; S_{0,0}, S_{k,j_k})}(g_{S^{(m)}})} \leqq K \tag{3.18}$$

for any two subcomputing functions $\chi_{j-1}^{(\mu; S_{0,0}, S_{i,j_i})}(z)$ and $\chi_{j-1}^{(\mu; S_{0,0}, S_{k,j_k})}(z)$.

Now we divide the inequality (3.17) by any term $\chi_{j-1}^{(\mu; S_{0,0}, S_{k,j_k})}(g_{S^{(m)}})$ among the summations $\sum_{S_{i,j_i}}^{(l)}$, $(l=1, \cdots, 5)$. Then we obtain from (3.18) the following inequality:

$$\begin{aligned} &\frac{1}{K}[0.15863\{3(n-4)(n-3)+1\}+0.12979\times 6(n-3)+0.06910\times 6(n-2)] \\ &\qquad -K[0.08132\times 6(n-1)+0.44255\times 6n] \\ &\qquad > 0 . \end{aligned} \tag{3.19}$$

If we take a sufficiently large number n, the above inequality is satisfied and thus the existence of Kleinian groups, whose singular sets have positive (3/2)-dimensional measure, is proved.

16. Completing the above results, we obtain the following results.

THEOREM 7. *Among Kleinian groups whose fundamental domains are bounded by mutually disjoint N circles, if N is sufficiently large, there exist ones whose singular sets have positive* (3/2)*-dimensional measure.*

Recalling our results about Poincaré theta-series [2], we have the following theorem.

THEOREM 8. *Among Kleinian groups whose fundamental domains are bounded by mutually disjoint N circles, if N is sufficiently large, there exist ones, with respect to which the* (-3)*-dimensional Poincaré theta-series* $\Theta_3(z)$ *do not converge in* D^**, where* D^* *denotes the compact subdomain of* E^c *given by deleting the suitable neighborhoods of poles of* $\Theta_3(z)$ *and their transforms on G from any compact subdomain* $D \subset E^c$.

Considering the Schottky subgroup G^* of G given by inversion method (see [4]), we have the following

THEOREM 9. *Among Kleinian groups whose fundamental domains are bounded by mutually disjoint N circles, if N is sufficiently large, there exist*

Schottky groups whose singular sets have positive (3/2)*-dimensional measure. The* (−3)*-dimensional Poincaré theta-series* $\Theta_3(z)$ *with respect to such Schottky group does not converge in* D^*.

REMARK. 1. The above example shows that there exist Kleinian groups whose singular sets have the Hausdorff dimension greater than 3/2.

2. It is not necessary to estimate the values of $\chi_1^{(3:s_i,j_i)}(g_{s(m+j-1)})$ from below in the five cases (a)—(e) in No. 14 in order to show only the existence of Kleinian groups whose singular sets have positive (3/2)-dimensional measure. For this purpose it is enough to use the estimates of the values in the cases (c) and (e). It is conjectured that in the Corollary of Theorem 4

$$\lim_{n\to\infty}\frac{\chi_n^{(\mu^*;T_1,T_2)}(z)}{\chi_n^{(\mu^*;T_3,T_4)}(z)}=1,$$

that is, the value of K is equal to 1, but we can not prove it yet. If it is true, and K is replaced by 1 in (3.19), we get the solution $n>10.4\cdots$ and hence that the least total number N of the boundary circles of B_0 satisfying (3.19) is $3\times 11\times 12+1=397$. If we can estimate the values of computing functions more exactly in No. 14, we may obtain the solution $n>9\cdots$. Then the total number N will become $3\times 10\times 11+1=331$.

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