J. Math. Soc. Japan
Vol. 24, No. 3, 1972

Decomposition theorem for Hopf algebras and pro-affine algebraic groups

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(Received July 26, 1971)
(Revised March 4, 1972)

§ 0. Introduction.

Generalizing the natural correspondence between affine algebraic groups over an algebraically closed field k and their coordinate rings, we have an anti-equivalent correspondence between the category of pro-affine algebraic groups over k and the category of commutative reduced Hopf algebras (i. e. which have no nilpotent elements other than 0) over k. (See [**1**], [**2**], [**3**].)

The purpose of the paper is to discuss relations between the properties of these two objects and especially to obtain certain properties of groups from those of their Hopf algebras.

In the first two sections, we reproduce some known relations between the properties of commutative reduced Hopf algebras and pro-affine (or affine) algebraic groups (cf. [**3**]), and in § 3 give a certain property of the co-radical of a Hopf algebra. Sections § 4 and § 5 are devoted to discuss general commutative Hopf algebras over an arbitrary field. In § 4, we give the definition of semi-direct product of Hopf algebras which is the dual of smash product in the sense of Sweedler [**3**]. In § 5, we give a decomposition theorem for Hopf algebras. In § 6, we give definitions and properties of exact sequences of reduced Hopf algebras which are dual of those of groups. When the sequence splits, we may apply to it the decomposition theorem given in § 5. It is well known that a connected affine algebraic group over an algebraically closed field of characteristic zero is the semi-direct product of the unipotent radical and a linearly reductive subgroup. Applying the decomposition theorem for Hopf algebras, the decomposition theorem for pro-affine (or affine) algebraic groups can be described in terms of Hopf-algebra theory.

Throughout the paper, all Hopf algebras are commutative over a field k.

§ 1. Preliminaries.

(1.1) Let V be a vector space over a fixed ground field k, and let $V^* = \mathrm{Hom}_k(V, k)$ be the linear dual space. For $f \in V^*$, $v \in V$, we will usually write $\langle f, v\rangle$ for $f(v)$. For a subset S of V, we denote by $S^{\perp}$ the subset $\{f\in V^* \mid \langle f, S\rangle = 0\}$ of V^*, and also for a subset T of V^*, we denote by $T^{\perp}$ the subset $\{v\in V \mid \langle T, v\rangle = 0\}$ of V. We use the same notations and definitions as in [3].

(1.2) LEMMA. *Let C be a coalgebra over k.*

(1) *If $D\subseteq C$ is a sub-coalgebra, then $D^{\perp}\subseteq C^*$ is a two-sided ideal of C^* and $C^*/D^{\perp}\cong D^*$ as algebras.*

(2) *If $D\subseteq C$ is a coideal, then $D^{\perp}\subseteq C^*$ is a subalgebra of C^* and $D^{\perp}\cong (C/D)^*$ as algebras.*

PROOF. See [3], Chap. I.

(1.3) PROPOSITION. *Let $\mathfrak{M}$ be the set of all simple sub-coalgebras (resp. all minimal right coideals, or all minimal left coideals) of C, $\mathfrak{J}$ the set of all maximal two-sided ideals (resp. all maximal right ideals, or all maximal left ideals) I of C^* such that $I^{\perp}\neq (0)$. Then the sets $\mathfrak{M}$ and $\mathfrak{J}$ are in one-to-one correspondence by the following mappings:*

$$\mathfrak{M}\ni M\longmapsto M^{\perp}\in\mathfrak{J}$$

$$\mathfrak{J}\ni I\longmapsto I^{\perp}\in\mathfrak{M}.$$

PROOF. If $M\in\mathfrak{M}$, then $\dim M<\infty$. Since $C^*/M^{\perp}\cong M^*$ and M^* is a simple k-algebra, $M^{\perp}$ is a maximal ideal of C^* and $(M^{\perp})^{\perp}=M\neq(0)$. Conversely if $I\in\mathfrak{J}$, let M be a simple sub-coalgebra of C contained in $I^{\perp}\neq(0)$. Then we have $I\subseteq I^{\perp\perp}\subseteq M^{\perp}\neq C^*$. Hence $I=I^{\perp\perp}=M^{\perp}$, because I is a maximal ideal. Thus we have $I^{\perp}=(M^{\perp})^{\perp}=M$. This completes the proof of (1.3).

(1.4) DEFINITIONS. Let C be a coalgebra over a field k. We denote by $R(C)$ the sum of all simple sub-coalgebras, and we call it the *co-radical* of C. C is called *co-semi-simple* if $C=R(C)$. C is called *irreducible* if it contains a unique simple sub-coalgebra. C is called *pointed* if all simple sub-coalgebras of C are 1-dimensional. We denote by $G(C)$ the subset $\{c\in C \mid c\neq 0,\ \Delta(c)=c\otimes c\}$ of C.

Now suppose C is a coalgebra so that C^* has a natural algebra structure. If N is a left C^*-module and there is a map $\psi: N\to N\otimes C$ such that

$$c^*\cdot n=(1\otimes c^*)\psi(n)\qquad \text{for all}\quad c^*\in C^*,\ n\in N,$$

then N is called a *rational* left C^*-module. The category of rational left C^*-modules is naturally equivalent to the category of right C-comodules. A subalgebra A of C^* is called *dense* if $A^{\perp}=(0)$. In this case, we can also define rational A-modules to which the right C-comodules correspond bijectively (see [3], sec. 2.1, chap. II).

(1.5) PROPOSITION. *Let C be a coalgebra over a field k. Then*

(1) *The followings are equivalent:*

(a) *Every rational C^*-module is completely reducible.*

(b) *C is co-semi-simple.*

(2) *The followings are equivalent:*

(a) *All simple rational left C^*-modules are 1-dimensional.*

(b) *All minimal right coideals of C are 1-dimensional.*

(c) *C is pointed.*

PROOF. (1) (Cf. [3], Lemma 14.0.1.)

(2) (a)$\Rightarrow$(b): Regard a right coideal of C as a right C-comodule via Δ_c.

(b)$\Rightarrow$(c): Let M be any simple sub-coalgebra of C. Let I be a minimal right coideal such that $I \subseteqq M$. By the assumption, we have $\dim I = 1$, and hence every element of I can be written in the form $\lambda \cdot m$, with $\lambda \in k$, where m is some fixed non-zero element in I. If we write $\Delta(m)$ for $m \otimes c$, we have then $m = \varepsilon(m)c$. Thus I is a subcoalgebra, and hence we must have $M = I$. This means that $\dim M = 1$.

(c)$\Rightarrow$(a): Let M be a simple rational left C^*-module. If m_0 is a non-zero element of M, then $M = \{zm_0 \mid z \in C^*\}$. We know that $\operatorname{ann} M$ is a maximal two-sided ideal of C^*, where $\operatorname{ann} M$ is the annihilator of M in C^*. Then we claim that $(\operatorname{ann} M)^{\perp} \neq (0)$. Since M is rational, there exists a map $\psi: M \to M \otimes C$ such that

$$c^* \cdot m = (1 \otimes c^*)\psi(m) \qquad \text{for all} \quad c^* \in C^*,\ m \in M.$$

We write $\sum_{i=1}^{n} m_i \otimes c_i$ for $\psi(m_0)$, where $\{m_i\}$ are linearly independent. Then,

$$\begin{aligned} a \in \operatorname{ann} M &\Leftrightarrow a(zm_0) = 0 \qquad \text{for all} \quad z \in C^* \\ &\Leftrightarrow \sum_{i=1}^{n} \langle az, c_i \rangle m_i = 0 \qquad \text{for all} \quad z \in C^* \\ &\Leftrightarrow \langle az, c_i \rangle = 0 \qquad \text{for all} \quad i \text{ and } z \in C^*. \end{aligned}$$

Now, let V be the subspace of C spanned by $\{c_i\}_{i=1,\cdots,n}$. Then V is obviously finite dimensional and $\neq (0)$. Thus, we have

$$a \in \operatorname{ann} M \Leftrightarrow az \in V^{\perp} \qquad \text{for all} \quad z \in C^*.$$

Therefore $\operatorname{ann} M \subseteqq V^{\perp}$ and $(\operatorname{ann} M)^{\perp} \supseteqq V^{\perp\perp} = V \neq (0)$, which shows that $(\operatorname{ann} M)^{\perp} \neq 0$.

From (1.3), $(\operatorname{ann} M)^{\perp}$ is a simple sub-coalgebra of C and $(\operatorname{ann} M)^{\perp\perp} = \operatorname{ann} M$. Therefore,

$$M \cong C^*/\operatorname{ann} M = C^*/(\operatorname{ann} M)^{\perp\perp} \cong ((\operatorname{ann} M)^{\perp})^*.$$

Hence we have $\dim M = 1$, since $(\operatorname{ann} M)^{\perp}$ is 1-dimensional by assumption.

(1.6) REMARK. We can show that any simple right C-comodule M is isomorphic to a minimal right coideal of C, as follows. For any $\gamma \in M^*$, the map $m \mapsto (\gamma \otimes 1)\psi(m)$ is a module map of M into C, where $\psi: M \to M \otimes C$ is the comodule structure map of C. If $M\neq(0)$, there exists an element $\gamma\in M^*$ such that the map $m\mapsto(\gamma\otimes 1)\psi(m)$ is not a zero-map. Since M is simple, this map is monic.

(1.7) Let C be a coalgebra over k and let M be a right C-comodule with the structure map $\psi: M\to M\otimes C$. Let $C(M)$ denote the subspace of C spanned by

$$\{c_i\in C\mid \psi(m)=\sum m_i\otimes c_i \text{ for all } m\in M\}$$

where $\{m_i\}$ are linearly independent over k. Since $(\psi\otimes 1)\psi=(1\otimes\Delta)\psi$, we have that $C(M)$ is a subcoalgebra of C. One verifies immediately that if M is finite dimensional then $C(M)$ is so and if $M_1\subseteqq M_2$ then $C(M_1)\subseteqq C(M_2)$.

(1.8) PROPOSITION. (1) *If M is a simple right C-comodule then $C(M)$ is a simple subcoalgebra of C.*

(2) *Conversely, if K is any simple subcoalgebra of C, there exists a simple right C-comodule M such that $C(M)=K$.*

PROOF. (1) Regard M as the left rational C^*-module in natural way, we claim that $(\operatorname{ann} M)^{\perp}\supseteqq C(M)$. As we have seen in Proposition (1.5), (2), $(\operatorname{ann} M)^{\perp}$ is a simple sub-coalgebra of C. Hence we have $(\operatorname{ann} M)^{\perp}=C(M)$ and so $C(M)$ is simple.

Let $\{m_i\}_{i=1,\cdots,n}$ be a basis for M and denote $\psi(m_i)=\sum_{j=1}^{n} m_j\otimes c_{ji}$. Then $C(M)$ coincides with the space spanned by $\{c_{ji}\}$. If f is in $\operatorname{ann} M$, then $0=f\cdot m_i=\sum m_j\langle f, c_{ji}\rangle$, for all i. This implies that $\langle f, c_{ji}\rangle=0$, for all i, j, which proves what we claimed above.

(2) Let M be a minimal right coideal contained in K. Since $\Delta(M)\subseteqq M\otimes K$, we have $C(M)\subseteqq K$. Thus we conclude that we must have $C(M)=K$.

(1.9) COROLLARY. *Let C be a coalgebra. Then*

(1) *C is the sum of $C(M)$ for all finite dimensional right C-comodules M.*

(2) *The coradical $R(C)$ of C is the sum of $C(M)$ for all simple right C-comodules M.*

PROOF. (1) Let h be in C and write $\Delta(h)=\sum_{i=1}^{n} h_i\otimes g_i$, where $\{h_i\}$ are linearly independent over k. The subspace M spanned by $\{h_i\}$ is a finite dimensional right C-comodule with $\psi=\Delta\mid M$: $M\to M\otimes C$. Then $C(M)$ contains the subspace spanned by $\{g_i\}$. We have $h=(\varepsilon\otimes 1)\Delta(h)=\sum\varepsilon(h_i)g_i\in C(M)$, and this shows (1). (2) follows immediately from (1) and Proposition (1.8).

§ 2. Pro-affine algebraic groups and Hopf algebras.

Throughout this section, we assume that the base field k is algebraically closed.

(2.1) Let H be a Hopf algebra over k, and let H^0 denote the dual Hopf algebra of H. We recall that its elements are those k-linear map $f: H \to k$ which annihilate some cofinite two-sided ideal in H. We say that the Hopf algebra H is reduced if it has no nilpotent elements other than 0. If the base field k is of characteristic 0, any (commutative) Hopf algebra is reduced (cf. [3], Th. 13.1.2). It is important to note the fact that the category of reduced Hopf algebras over k is naturally anti-equivalent to the category of pro-affine algebraic groups over k (see [1]). Further, if H is a reduced Hopf algebra over k, we have that the group algebra $kG(H^0)$ is dense in H^*, that is, $(kG(H^0))^\perp = (0)$ in H. Therefore (1.5) and a remark in (1.4) imply the following:

PROPOSITION. *Let H be a reduced Hopf algebra over k.*

(1) *H is co-semi-simple if and only if every rational $kG(H^0)$-module is completely reducible.*

(2) *H is pointed if and only if all simple rational $kG(H^0)$-modules are 1-dimensional.*

(2.2) REMARK. When H is finitely generated as an algebra, $G(H^0)$ is an affine algebraic group. In this case the term "a rational $kG(H^0)$-module" coincides with the term "a rational $G(H^0)$-module" for the affine algebraic group $G(H^0)$.

(2.3) Let G be a connected affine algebraic group, and H its coordinate ring. One sees immediately from Lie-Kolchin theorem that (1) G is solvable if and only if all simple representations of G are 1-dimensional; (2) G is unipotent if and only if all simple representations of G are trivial; (3) G is a torus if and only if H is co-semi-simple and pointed. Therefore, proposition (2.1) leads to the following.

PROPOSITION. *Let G be a connected affine algebraic group and H its coordinate ring. Then in each cases* (1), (2) *and* (3), *the following three conditions* (a), (b) *and* (c) *are equivalent each other.*

(1) (a) *G is a torus,* (b) *H is co-semi-simple and pointed,* (c) $H = kG(H)$.

(2) (a) *G is unipotent,* (b) *H is irreducible,* (c) $R(H) = k \cdot 1$.

(3) (a) *G is solvable,* (b) *H is pointed,* (c) *$R(H)$ is co-commutative.*

PROOF. (3) If H is pointed, then $R(H) = kG(H)$. Hence $R(H)$ is co-commutative. Conversely if $R(H)$ is co-commutative, then $R(H)$ is pointed since k is algebraically closed ([3], Lemma 8.0.1. (c)). Hence H is pointed.

(2.4) DEFINITIONS. Let H be a reduced Hopf algebra over k. We say that the pro-affine algebraic group $G(H^0)$ is *linearly reductive* if H is co-semi-simple, and *connected* if H is an integral domain. Further a connected pro-affine algebraic group $G(H^0)$ is called a *torus* (resp. to be *unipotent*) if H is co-semi-simple and pointed (resp. irreducible).

§ 3. Coradical of a Hopf algebra.

(3.1) PROPOSITION. *Let H be a Hopf algebra over an algebraically closed field of characteristic* 0. *Then the coradical $R(H)$ is a sub-Hopf algebra of H.*

PROOF. Let K_i $(i=1, 2)$ be simple sub-coalgebras. From (1.8) there exist simple right H-comodules M_i $(i=1, 2)$ such that $C(M_i)=K_i$ $(i=1, 2)$, where $C(M_i)$ is in the sense of (1.7). Note that right H-comodule M is simple if and only if M is simple as a $G(H^0)$-module, for $G(H^0)$ is dense in H^*. Since the base field k is of characteristic 0, $M_1 \otimes M_2$ is semi-simple as a $G(H^0)$-module (see, [1], Th. 12.2). Therefore we have

$$K_1 \cdot K_2 = C(M_1)C(M_2) = C(M_1 \otimes M_2) \subseteqq R(H).$$

This means that $R(H)$ is a subalgebra of H. Now let K be any simple sub-coalgebra of H. Since the antipode S of H is anti-coalgebra map and involutive $(S \circ S = 1)$, we have that $S(K)$ is simple. This implies that $S(R(H)) \subseteqq R(H)$.

(3.2) REMARK. It is not true in general that the coradical is a sub-Hopf algebra. For instance, let k be of characteristic p $(\neq 0)$ and consider the linear algebraic group $SL_2(k)$. Let H be the coordinate ring of $SL_2(k)$, then we can denote it by

$$H = k[x_{11}, x_{12}, x_{21}, x_{22}]/(\det x_{ij} - 1) = k[\bar{x}_{11}, \bar{x}_{12}, \bar{x}_{21}, \bar{x}_{22}]$$

where $\bar{x}_{ij}$ is the class of x_{ij} modulo $(\det x_{ij} - 1)$. Let C be the subspace spanned by $\{\bar{x}_{11}, \bar{x}_{12}, \bar{x}_{21}, \bar{x}_{22}\}$. Then one verifies immediately that C is a sub-coalgebra of H and C^* is isomorphic to the algebra $M_2(k)$ of the 2×2 matrices with coefficients in k.

Thus C is simple and hence $C \subset R(H)$. Therefore $R(H)$ contains a generator system of the algebra. If $R(H)$ is a subalgebra of H, we must have $R(H) = H$, and hence H is co-semi simple. This is a contradiction, for $SL_2(k)$ is not completely reducible as an algebraic group.

(3.3) REMARK. Let H be a Hopf algebra over a field k and assume that the coradical $R(H)$ is a sub-Hopf algebra of H. We have not been able to verify that under what condition, there is a Hopf algebra map $q: H \rightarrow R(H)$ such that $q|_{R(H)} = id_{R(H)}$. Recently, M.-H. Takeuchi [4] has proved the above for a (commutative) Hopf algebra over a field of characteristic 0. His proof is based on Hopf algebra theory and this gives a semi-direct decomposition of affine algebraic groups over a field of characteristic 0 into its unipotent radical and a linearly reductive subgroup. We shall show in the following sections the corresponding theory in Hopf algebras.

§ 4. Semi-direct products of Hopf algebras.

(4.1) Let K, L be Hopf algebras over a field k (where k is not necessarily algebraically closed), σ; $L \to K\otimes L$ an algebra map. Then (σ, K) is called a *Hopf action* on L if the following diagrams commute:

(HA1)

$$\begin{CD} L @>{\sigma}>> K\otimes L \\ @V{\sigma}VV @VV{\sigma}V \\ K\otimes L @>{\Delta\otimes 1}>> K\otimes K\otimes L \end{CD} \qquad\qquad \begin{array}{ccc} K\otimes L & \xleftarrow{\ \sigma\ } & L \\ \downarrow{\scriptstyle \varepsilon\otimes 1} & & \| \\ k\otimes L & = & L \end{array}$$

(HA2)

$$\begin{CD} L @>{\sigma}>> @. @. K\otimes L \\ @V{\Delta}VV @. @. @VV{1\otimes\Delta}V \\ L\otimes L @>{\sigma\otimes\sigma}>> K\otimes L\otimes K\otimes L @>{1\otimes T\otimes 1}>> K\otimes K\otimes L\otimes L @>{M\otimes 1\otimes 1}>> K\otimes L\otimes L \end{CD}$$

$$\begin{CD} L @>{\sigma}>> K\otimes L \\ @V{\varepsilon}VV @VV{1\otimes\varepsilon}V \\ k=k\otimes k @>{u\otimes 1}>> K\otimes k \end{CD}$$

where the tensor product $\otimes$ is always understood to be 'over k', and T is the twist map $a\otimes b \mapsto b\otimes a$.

(L, σ) is a left K-comodule by the condition (HA1). As for the structure of K-comodule of L, the condition (HA2) is equivalent to say that $\Delta : L \to L\otimes L$ and $\varepsilon : L \to k$ are K-comodule maps.

(4.2) Let (σ, K) be a Hopf action on L. The *semi-direct* (*co-smash*) *product* of L with K, written $L \flat K$, is a Hopf algebra defined as follows:

(1) As an algebra $L\flat K$ is $L \otimes K$. We shall often denote $l\otimes k$ by $l \flat k$ $(l \in L,\ k \in K)$.

(2) The coalgebra structure is defined by

$$\Delta_{L\flat K} : L\otimes K \xrightarrow{\ \Delta\otimes\Delta\ } L\otimes L\otimes K\otimes K \xrightarrow{\ 1\otimes\sigma\otimes 1\otimes 1\ } L\otimes K\otimes L\otimes K\otimes K$$
$$\xrightarrow{\ 1\otimes 1\otimes T\otimes 1\ } L\otimes K\otimes K\otimes L\otimes K \xrightarrow{\ 1\otimes M\otimes 1\otimes 1\ } L\otimes K\otimes L\otimes K,$$

$$\varepsilon_{L\flat K}:\ L\otimes K \xrightarrow{\varepsilon\otimes\varepsilon} k\otimes k = k.$$

(3) The antipode is defined by

$$S_{L\flat K}:\ L\otimes K \xrightarrow{S_L\otimes S_K} L\otimes K \xrightarrow{\sigma\otimes 1} K\otimes L\otimes K$$
$$\xrightarrow{T\otimes 1} L\otimes K\otimes K \xrightarrow{1\otimes S_K\otimes 1} L\otimes K\otimes K \xrightarrow{1\otimes M_K} L\otimes K.$$

Note that for Hopf algebras which are the coordinate rings of affine algebraic groups, the above Hopf algebra $L\flat K$ is the coordinate ring of the semi-direct product of two affine algebraic groups whose coordinate rings are respectively L and K under the action corresponding to (σ, K).

(4.3) We prove that $L\flat K$ is a Hopf algebra.

Since $\Delta_{L\flat K}$ is an algebra map and $L\otimes K$ is generated by $\{l\otimes 1,\ 1\otimes k \mid l\in L,\ k\in K\}$, to show the co-associativity of $\Delta_{L\flat K}$, it suffices to show the following two equalities:

(a) $(1\otimes 1\otimes \Delta_{L\flat K})\Delta_{L\flat K}(1\otimes k)=(\Delta_{L\flat K}\otimes 1\otimes 1)\Delta_{L\flat K}(1\otimes k)$ for all $k\in K$,

(b) $(1\otimes 1\otimes \Delta_{L\flat K})\Delta_{L\flat K}(l\otimes 1)=(\Delta_{L\flat K}\otimes 1\otimes 1)\Delta_{L\flat K}(l\otimes 1)$ for all $l\in L$.

(a) is immediate.

To verify (b), we use the notation

$$\sigma(l)=\sum l_{(k)}\otimes l_{(l)}\in K\otimes L.$$

From (HA2),

$$(1\otimes\Delta)\sigma=(M\otimes 1\otimes 1)(1\otimes T\otimes 1)(\sigma\otimes\sigma)\Delta.$$

Hence we have

$$(1\otimes 1\otimes\Delta)(1\otimes\sigma)\Delta(l)=(1\otimes M\otimes 1\otimes 1)(1\otimes 1\otimes T\otimes 1)(1\otimes\sigma\otimes\sigma)(1\otimes\Delta)\Delta(l)$$

and

$$\sum l_{(1)}\otimes l_{(2)(k)}\otimes l_{(2)(l)(1)}\otimes l_{(2)(l)(2)}=\sum l_{(1)}\otimes l_{(2)(k)}l_{(3)(k)}\otimes l_{(2)(l)}\otimes l_{(3)(l)}.$$

Applying $1\otimes 1\otimes 1\otimes\sigma$ to both sides of the above equation, we have

$$\begin{aligned}\sum l_{(1)}&\otimes l_{(2)(k)}\otimes l_{(2)(l)(1)}\otimes l_{(2)(l)(2)(k)}\otimes l_{(2)(l)(2)(l)}\\ &=\sum l_{(1)}\otimes l_{(2)(k)}\underline{l_{(3)(k)}}\otimes l_{(2)(l)}\otimes \underline{l_{(3)(l)(k)}}\otimes \underline{l_{(3)(l)(l)}}\\ &=\sum l_{(1)}\otimes l_{(2)(k)}\underline{l_{(3)(k)(1)}}\otimes l_{(2)(l)}\otimes \underline{l_{(3)(k)(2)}}\otimes \underline{l_{(3)(l)}} \qquad \text{(by (HA1))}.\end{aligned}$$

This shows that the equation (b) holds.

We can also prove the property of the antipode, as follows.

(4.4) LEMMA. $(1\otimes S_L)\sigma=\sigma S_L$

i.e. $\sum l_{(k)}\otimes S(l_{(l)})=\sum S(l)_{(k)}\otimes S(l)_{(l)}$, *for all* $l\in L$.

PROOF. We note that σ is an element of $\mathrm{Alg}(L, K\otimes L)$. For $f, g\in \mathrm{Alg}(L, K\otimes L)$, we define the convolution product $f*g=M_{K\otimes L}(f\otimes g)\Delta_L$. Thus $\mathrm{Alg}(L, K\otimes L)$ has a group structure under this product. The unit of $\mathrm{Alg}(L, K\otimes L)$ is $u_{K\otimes L}\varepsilon_L$, and for $f\in \mathrm{Alg}(L, K\otimes L)$, the inverse of f is fS_L (see [3], Th. 4.0.5). Then our task is to show $(1\otimes S_L)\sigma*\sigma=u_{K\otimes L}\varepsilon_L$. Now

$$\begin{aligned}(1\otimes S_L)\sigma*\sigma &= (M_K\otimes M_L)(1\otimes T\otimes 1)(1\otimes S_L\otimes 1\otimes 1)(\sigma\otimes\sigma)\Delta_L\\ &= (M_K\otimes M_L)(1\otimes 1\otimes S_L\otimes 1)(1\otimes T\otimes 1)(\sigma\otimes\sigma)\Delta_L\\ &= (M_K\otimes M_L(S_L\otimes 1))(1\otimes T\otimes 1)(\sigma\otimes\sigma)\Delta_L\\ &= (1\otimes M_L(S_L\otimes 1))(M_K\otimes 1\otimes 1)(1\otimes T\otimes 1)(\sigma\otimes\sigma)\Delta_L\\ &= (1\otimes M_L(S_L\otimes 1))(1\otimes \Delta_L)\sigma \qquad \text{by (HA2)}\\ &= (1\otimes u_L\varepsilon_L)\sigma \qquad S_L \text{ an antipode}\\ &= (1\otimes u_L)(u_K\otimes 1)\varepsilon_L \qquad \text{by (HA2)}\\ &= u_{K\otimes L}\varepsilon_L\,.\end{aligned}$$

Thus the Lemma is proved.

(4.5) LEMMA. *For* $l\in L$,

$$S(l_{(1)(l)})l_{(2)(l)}\otimes S(l_{(1)(k)})S(l_{(2)(k)})=\varepsilon(l)(1\otimes 1)\,.$$

PROOF. By (HA2),

$$\sum l_{(k)}\otimes l_{(k)(1)}\otimes l_{(l)(2)}=\sum l_{(1)(k)}l_{(2)(k)}\otimes l_{(1)(l)}\otimes l_{(2)(l)}\,.$$

To both sides of the above equation, apply $(1\otimes M_L)(S_K\otimes S_L\otimes 1)$ and get

$$\sum S(l_{(k)})\otimes S(l_{(l)(1)})\,l_{(l)(2)}=\sum S(l_{(1)(k)})S(l_{(2)(k)})\otimes S(l_{(1)(l)})\,l_{(2)(l)}\,.$$

Now, we have

$$\sum S(l_{(k)})\otimes S(l_{(l)(1)})\,l_{(l)(2)}=\sum S(l_{(k)})\otimes \varepsilon(l_{(l)})1=\varepsilon(l)(1\otimes 1)\,.$$

(4.6) PROPOSITION.

$$M_{L\flat K}(S_{L\flat K}\otimes 1_{L\flat K})\Delta_{L\flat K}=u_{L\flat K}\varepsilon_{L\flat K}\,.$$

PROOF. For $l\in L$, $k\in K$,

$$\begin{aligned}&M_{L\flat K}(S_{L\flat K}\otimes 1_{L\flat K})\Delta_{L\flat K}(l\flat k)\\ &\quad=\sum S(l_{(1)(l)})\,l_{(2)(l)}\flat S(l_{(1)(k)})S(l_{(2)(k)})S(k_{(1)})k_{(2)} \qquad \text{by (4.4)}\\ &\quad=\varepsilon_{L\flat K}(l\flat k)\cdot 1_{L\flat K} \qquad \text{by (4.5)}\,.\end{aligned}$$

This means that $S_{L\flat K}$ is the antipode of the bialgebra $L\flat K$.

§ 5. Decomposition theorem for Hopf algebras.

Let H, K be Hopf algebras over a field k, $i: K \to H$ be an injective Hopf algebra map, and $p: H \to K$ be a Hopf algebra map such that $p \circ i = 1_K$. Then we shall prove that H is isomorphic to $L \flat K$ as Hopf algebras for a Hopf algebra L and a Hopf action (σ, K) on L which will be defined in (5.7) and (5.8).

(5.1) Let $\rho: H \to H \otimes K$ be the composite $(1 \otimes p)\Delta$. Then (H, ρ) is a right K-comodule.

PROOF. It is obvious since p is a coalgebra map.

(5.2) Let $\omega: H \otimes K \to H$ be the composite $M(1 \otimes i)$. Then (H, ω) is a right K-module.

(5.3) $(H; \omega, \rho)$ is a right K-Hopf module.

PROOF. We must show that $(\omega \otimes M)(1 \otimes T \otimes 1)(\rho \otimes \Delta) = \rho\omega$.

$$\begin{aligned}(\omega \otimes M)&(1 \otimes T \otimes 1)(\rho \otimes \Delta)(h \otimes k)\\ &= \sum (\omega \otimes M)(h_{(1)} \otimes k_{(1)} \otimes p(h_{(2)}) \otimes k_{(2)})\\ &= \sum h_{(1)} i(k_{(1)}) \otimes p(h_{(2)}) k_{(2)}\\ &= \rho\omega(h \otimes k) \qquad \text{for all} \quad h \in H,\ k \in K.\end{aligned}$$

(5.4) Let $L = \{h \in H \mid \rho(h) = h \otimes 1\}$, then L is a subalgebra of H. Further, the map $\varphi: L \otimes K \to H$ $(l \otimes k \mapsto l \cdot i(k))$ is an algebra isomorphism.

PROOF. Immediately from the structure theorem of Hopf modules ([3], Theorem 4.1.1).

(5.5) L is a left coideal of H. That is, $\Delta(L) \subseteqq H \otimes L$.

PROOF. See [3], Lemma 16.1.1.

(5.6) REMARKS. (a) If M is any left coideal of H and $p|M = \varepsilon|M$, then $M \subseteqq L$.

(b) The inverse map ψ of φ is given by the composite $(P \otimes 1)\rho$, where P is the map defined by the composite $\omega\ (1 \otimes S)\rho$, which maps H onto L. It follows immediately from the definition of P that $P|L = 1_L$ and $P(i(k)) = \varepsilon(k) 1_H$ for all $k \in K$.

(c) If H is co-commutative, then $S(L) \subseteqq L$. Hence L is a sub-Hopf algebra of H and so φ is a Hopf algebra isomorphism.

PROOF of (c). Let l be in L. Then

$$(S \otimes 1)(1 \otimes p)\Delta(l) = S(l) \otimes 1.$$

On the other hand,

$$\begin{aligned}(S \otimes 1)(1 \otimes p)\Delta(l) &= (S \otimes p)\Delta(l)\\ &= (S \otimes pSS)\Delta(l) \qquad (SS = 1 \text{ by commutativity of } H)\end{aligned}$$

$$= (1 \otimes pS)\Delta(S(l)) \qquad (S \text{ is a coalgebra map})$$

$$= (1 \otimes S_K p)\Delta(S(l)) \qquad (p \text{ is a Hopf algebra map}).$$

Hence we have $(1 \otimes S_K p)\Delta(S(l)) = S(l) \otimes 1$. Applying $1 \otimes S_K$ to the both sides of the equation,

$$(1 \otimes p)\Delta(S(l)) = S(l) \otimes 1 .$$

Hence we have $S(l) \in L$.

(5.7) Let Δ_L, ε_L and S_L be the maps $(P \otimes 1)\Delta_H$, $\varepsilon | L$ and $P \circ S_H$ respectively. Then $(L; \Delta_L, \varepsilon_L, S_L)$ is a Hopf algebra and it is isomorphic to $H/Hi(K^+)$ as Hopf algebras, where $K^+ = \operatorname{Ker} \varepsilon_K$.

PROOF. By (b) of (5.6), we have $\operatorname{Ker} P = Hi(K^+)$. Hence $L \cong H/Hi(K^+)$ as algebras. Moreover, the Hopf algebra structure of L coincides with the natural quotient Hopf algebra structure of $H/Hi(K^+)$.

(5.8) Let $\sigma : L \to K \otimes L$ be the composite $(p \otimes 1)\Delta_H$. Then (σ, K) is a Hopf action on L.

PROOF. (HA1) is obvious. We now prove (HA2).

$$\begin{aligned}
&(M \otimes 1 \otimes 1)(1 \otimes T \otimes 1)(\sigma \otimes \sigma)\Delta_L(l) \\
&\quad = \sum (M \otimes 1 \otimes 1)(1 \otimes T \otimes L)(\sigma \otimes \sigma)(l_{(1)} i Sp(l_{(2)}) \otimes l_{(3)}) \\
&\quad = \sum (M \otimes 1 \otimes 1)(1 \otimes T \otimes 1)(p(l_{(1)}) p i Sp(l_{(4)}) \otimes l_{(2)} i Sp(l_{(3)}) \otimes p(l_{(5)}) \otimes l_{(6)}) \\
&\quad = \sum p(l_{(1)}) Sp(l_{(4)}) p(l_{(5)}) \otimes l_{(2)} i Sp(l_{(3)}) \otimes l_{(6)} \\
&\quad = \sum p(l_{(1)}) \varepsilon(l_{(4)}) \otimes l_{(2)} i Sp(l_{(3)}) \otimes l_{(5)} \\
&\quad = \sum p(l_{(1)}) \otimes l_{(2)} i Sp(l_{(3)}) \otimes l_{(4)} \\
&\quad = \sum p(l_{(1)}) \otimes \Delta_L(l_{(2)}) \\
&\quad = (1 \otimes \Delta_L)\sigma(l) \qquad \text{for all } \ l \in L .
\end{aligned}$$

Further

$$(1 \otimes \varepsilon)\sigma(l) = \sum p(l_{(1)}) \varepsilon(l_{(2)}) = p(l) = \varepsilon(l) 1_K .$$

(5.9) THEOREM. *Under the above situation, the algebra isomorphism φ of $L \otimes K$ onto H is a Hopf algebra isomorphism of $L \flat K$ onto H.*

PROOF. It remains to show that φ is a coalgebra map.

$$\begin{aligned}
(\varphi \otimes \varphi)\Delta_{L \flat K}(l \flat k) &= \sum (\varphi \otimes \varphi)(l_{(1)} i Sp(l_{(2)}) \flat p(l_{(3)}) k_{(1)} \otimes l_{(4)} \flat k_{(2)}) \\
&= \sum l_{(1)} i Sp(l_{(2)}) i S(l_{(3)}) i(k_{(1)}) \otimes l_{(4)} i(k_{(2)}) \\
&= \sum l_{(1)} \varepsilon(l_{(2)}) i(k_{(1)}) \otimes l_{(3)} i(k_{(2)}) \\
&= \sum l_{(1)} i(k_{(1)}) \otimes l_{(2)} i(k_{(2)}) \\
&= \Delta_H \varphi(l \flat k) .
\end{aligned}$$

(5.10) EXAMPLE. Let X, Y be indeterminates over a field k. Let $H=k[X, X^{-1}, Y]$. H has a Hopf algebra structure determined by $\Delta(X)=X\otimes X$, $\Delta(Y)=X\otimes Y+Y\otimes X^{-1}$, $\varepsilon(X)=1$, $\varepsilon(Y)=0$, $S(X)=X^{-1}$, $S(Y)=-Y$. Then $K=k[X, X^{-1}]$ is a sub-Hopf algebra of H. Let $p: H\rightarrow K$ be a Hopf algebra map defined by $p(X)=X$, $p(Y)=0$. Clearly we have $p\circ i=1_K$.

By easy computations, $L=k[Z]$ where $Z=XY$. Here the Hopf algebra structure of L is given by $\Delta_L(Z)=1\otimes Z+Z\otimes 1$, $\varepsilon_L(Z)=0$, $S_L(Z)=-Z$. And the Hopf action (σ, K) on L is given by $\sigma(Z)=X^2\otimes Z$.

We know that the Hopf algebra H is the coordinate ring of affine algebraic group $G=\left\{\begin{pmatrix} x & y \\ 0 & x^{-1}\end{pmatrix} \middle| x\in k-\{0\},\ y\in k\right\}$. Further, K is the coordinate ring of the multiplicative group $\boldsymbol{G}_m$, L is that of the additive group $\boldsymbol{G}_a$ and G is a semi-direct product of $\boldsymbol{G}_m$ and $\boldsymbol{G}_a$.

(5.11) THEOREM. *Let H be a Hopf algebra over an algebraically closed field k of characteristic* 0, *then there is an irreducible Hopf algebra L such that H is the co-smash product of L with $R(H)$.*

PROOF. In view of remark (3.3) and theorem (5.9), it remains to show that $L=H/Hi(K^+)$ is irreducible where $K=R(H)$. Let $q: H\rightarrow R(H)$ (resp. $\gamma: L\rightarrow R(L)$) be a Hopf algebra projection of H (resp. L) onto $R(H)$ (resp. $R(L)$). Let p be the natural quotient map from H onto L. It follows that there is a surjective Hopf algebra map $s: R(H)\rightarrow R(L)$ such that $\gamma\circ p=s\circ q$. Since $K=k\cdot 1+K^+$ and $\operatorname{Ker} p=Hi(K^+)$, $R(L)$ must coincide with $k\cdot 1$. This means that L is irreducible.

§ 6. Exact sequences of reduced Hopf algebras.

In this section we consider the relations between algebraic subgroups (or quotient groups) of a pro-affine algebraic group and their Hopf algebras. Throughout this section we assume that the base field k is algebraically closed.

(6.1) Let H be a reduced Hopf algebra and K be a sub-Hopf algebra. Let $i: K\rightarrow H$ be the injective Hopf algebra map. We put $L=H/Hi(K^+)$, which is a Hopf algebra since $Hi(K^+)$ is a Hopf ideal of H. Here, L is not necessarily reduced in general. We say that K is *admissible* if L is reduced. In this case L is called the *cokernel* of i, and the sequence

$$0 \longrightarrow K \xrightarrow{\ i\ } H \xrightarrow{\ p\ } L \longrightarrow 0$$

is called an exact sequence of reduced Hopf algebras, where p is the quotient map. Then we have an exact sequence of pro-affine algebraic groups;

$$1 \longrightarrow G(L^0) \xrightarrow{\ p^0\ } G(H^0) \xrightarrow{\ i^0\ } G(K^0) \longrightarrow 1\,.$$

(As for affine case, see, for example [1] Th. 6.4.)

(6.2) Let H, L be reduced Hopf algebras and $p: H \to L$, be a surjective Hopf algebra map. We shall identify $G(L^0)$ with the subgroup of $G(H^0)$ by p^0. Let ρ (resp. σ) be the composite $(1 \otimes p)\Delta_H$ (resp. $(p \otimes 1)\Delta_H$). Then (H, ρ) (resp. (H, σ)) is a right (resp. left) L-comodule. The rational $G(L^0)$-module structure determined by ρ, resp. σ, is given by

$$\langle y, xf \rangle = \langle yx, f \rangle, \qquad \text{resp. } \langle y, fx \rangle = \langle xy, f \rangle$$

$$\text{for all} \quad x \in G(L^0), \qquad f \in H \quad \text{and} \quad y \in G(H^0).$$

Now we put $H^L = \{h \in H \mid \rho(h) = h \otimes 1\}$ and ${}^L H = \{h \in H \mid \sigma(h) = 1 \otimes h\}$. Then, it is easy to see that $H^L = \{f \in H \mid xf = f \text{ for all } x \in G(L^0)\}$ and ${}^L H = \{f \in H \mid fx = f \text{ for all } x \in G(L^0)\}$.

(6.3) PROPOSITION. *The followings are equivalent:*

(a) *$G(L^0)$ is normal in $G(H^0)$,*

(b) *$H^L = {}^L H$.*

PROOF. (a) $\Rightarrow$ (b): If $x \in G(L^0)$ and $y \in G(H^0)$ we have $yxy^{-1} = x' \in G(L^0)$. Therefore,

$$\langle y, xf \rangle = \langle yx, f \rangle = \langle x'y, f \rangle = \langle y, fx' \rangle.$$

Hence $f \in {}^L H \Leftrightarrow f \in H^L$.

(b) $\Rightarrow$ (a): We first show that H^L $(= {}^L H)$ is a sub-Hopf algebra of H. Since the antipode S of H is an anti-coalgebra map from H onto itself, we know from the proof of (5.6), (c) that $S(H^L) = {}^L H$. Further, H^L $(= {}^L H)$ is a sub-coalgebra, for from (5.5), H^L is a left coideal and ${}^L H$ is a right coideal. Thus we have that H^L is a sub-Hopf algebra of H.

Since H is the union of the family of its finitely generated sub-Hopf algebras, we may assume without loss of generalities that H is finitely generated. Then we have H^L is finitely generated, because of the known fact that every sub-Hopf algebra of any finitely generated reduced Hopf algebra is finitely generated (see for example [1], Th. 6.4). Thus the set of right cosets $G(H^0)/G(L^0)$ is an affine variety, with H^L as its coordinate ring. If $x \in G(L^0)$, $y \in G(H^0)$ and $f \in H^L = {}^L H$, we have $\langle y, f \rangle = \langle xy, f \rangle = \langle yx, f \rangle$. Since H^L separates the elements of $G(H^0)/G(L^0)$, we have $yx \in G(L^0)xy = G(L^0)y$, so (b) $\Rightarrow$ (a) is proved.

We say that L is a *co-normal quotient* if it satisfies (a) or (b) of the above Proposition.

(6.4) PROPOSITION. (1) *Let* $0 \longrightarrow K \xrightarrow{i} H \xrightarrow{p} L \longrightarrow 0$ *be an exact sequence of reduced Hopf algebras. Then*

$$K \cong {}^L H = H^L.$$

(2) *Let H and L be reduced Hopf algebras, and $p: H \to L$ be a co-normal*

quotient. We put ${}^{L}H=K$. *Then*

$$0\longrightarrow K\overset{i}{\longrightarrow}H\overset{p}{\longrightarrow}L\longrightarrow 0$$

is an exact sequence of reduced Hopf algebras, where $i: K\to H$ *is the inclusion map.*

PROOF. (1) Let f be an element of K. Then we have

$$(p\otimes 1)\Delta_H i(f)=(p\otimes 1)(i\otimes i)\Delta_K(f)=(1\otimes i)(\varepsilon_K\otimes 1)\Delta_K(f)=1\otimes i(f),$$

hence $i(f)\in {}^{L}H$. This means that $i(K)\subseteq {}^{L}H$. Similarly we have $i(K)\subseteq H^{L}$.

Conversely, let f be an element of ${}^{L}H$. This means that $(p\otimes 1)\Delta(f)=1\otimes f$. Applying $1\otimes\varepsilon$ to the both sides of the equation, we have $\varepsilon(f)=p(f)$, and it follows that $({}^{L}H)^{+}\subseteq \operatorname{Ker} p$. Hence we have that $H({}^{L}H)^{+}=Hi(K^{+})$ and $G(({}^{L}H)^{0})=G(K^{0})$. This means that $i(K)={}^{L}H$.

(2) If $f\in({}^{L}H)^{+}=K^{+}$, then we have $f\in\operatorname{Ker} p$, and also $Hi(K)\subseteq\operatorname{Ker} p$. Thus the map p induces a natural epimorphism φ of $L'=H/Hi(K^{+})$ into L. Further it induces a monomorphism of $G(L^{0})$ into $G(L'^{0})$. On the other hand, it follows from (1) that ${}^{L'}H=K={}^{L}H$. Hence we have $G(L^{0})=G(L'^{0})$. Since L is reduced, it follows that $L=L'$.

The proposition shows that any exact sequence of pro-affine algebraic groups arises from that of reduced Hopf-algebras defined in (6.1).

(6.5) Let $0\longrightarrow K\overset{i}{\longrightarrow}H\overset{p}{\longrightarrow}L\longrightarrow 0$ be an exact sequence of reduced Hopf algebras. The sequence is said to be split if there exists a Hopf algebra map $q: H\to K$ such that $q\circ i=id_K$. In this case, the corresponding exact sequence of pro-affine algebraic groups splits, i. e.

$$1\longrightarrow G(L^{0})\overset{p^{0}}{\longrightarrow}G(H^{0})\underset{q^{0}}{\overset{i^{0}}{\rightleftarrows}}G(K^{0})\longrightarrow 1\,.$$

(6.6) Let $0\longrightarrow K\underset{q}{\overset{i}{\rightleftarrows}}H\overset{p}{\longrightarrow}L\longrightarrow 0$ be a split exact sequence of reduced Hopf-algebras. Then, from (5.4) and (6.4), by means of the K-comodule structure on H defined by q, L is isomorphic as algebras to H^{K} and H is isomorphic as algebras to $H^{K}\otimes {}^{L}H$. Further, from (5.9), we can define canonically a Hopf-action of K on L so that H is isomorphic to $L\,\flat\,K$ as Hopf algebras.

(6.7) REMARK. M.-H. Takeuchi [5] has also discussed the correspondence between sub-Hopf algebras of a commutative (not necessarily reduced) Hopf algebras over a field and its normal Hopf ideals.

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