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On some new birational invariants of algebraic varieties and their application to rationality of certain algebraic varieties of dimension 3

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§ 1. Introduction.

Let V be an n-dimensional algebraic variety defined over an algebraically closed field of characteristic zero. In the previous paper [7] the author introduced a birational invariant $\kappa(V)$ of V and classified algebraic varieties into $n+2$ classes according to the value of $\kappa(V)$, say $-\infty, 0, 1, \cdots, n$. Here we shall introduce some new birational invariants $\nu(V)$ and $Q_m(V)$ for $m=1, 2, \cdots$ which would be useful in the birational classification of algebraic varieties. In fact, in some cases where $\kappa(V)=-\infty$, $\nu(V)$ may be $-\infty, 0, 1, \cdots, n-1$ and so gives more information than $\kappa(V)$ alone. Using this invariant we shall obtain criteria of rationality of certain algebraic varieties of dimension 3. As an application of these criteria we can verify partially the following Hartshorne conjecture concerning ample vector bundles.

CONJECTURE H_n. *Let V be an n-dimensional non-singular projective algebraic variety whose tangent vector bundle is ample. Then V is isomorphic to a projective space $\boldsymbol{P}^n$.*

This was answered only for $n=1, 2$. Note that the proof of H_2 requires[1] the structure theorem of rational surfaces due to Castelnuovo and Andreotti. In this paper we shall show that *if V satisfies the hypothesis of H_3 then V is birationally equivalent to $\boldsymbol{P}^3$.*

The following conjecture due to Frankel [2] is well known in differential geometry.

CONJECTURE F_n. *Let M be an n-dimensional compact Kähler manifold which has a positive holomorphic sectional curvature. Then M is isomorphic to $\boldsymbol{P}^n$.*

Kobayashi and Ochiai showed that the Conjecture H_n implies F_n, see [10].

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1) Note that we can verify F_2 without using the structure theorem of rational surfaces.

The author intended to solve F_3, but he could only verify the rationality of M if M satisfies the condition of F_3. After completing this paper he heard some good news: that F_3 is true (see [**12**]). However, since H_3 has not yet been solved, the author decided to publish this. The author expresses his thanks to Dr. T. Ochiai, who communicated to him the Frankel conjecture.

For the convenience of the reader, we start with recalling the definitions of D-dimension and of Kodaira dimension of algebraic varieties.

§ 2. Notation.

Let k be an algebraically closed field of characteristic zero. Sometimes, especially in § 5, we assume $k=\boldsymbol{C}$, the field of complex numbers. We shall work in the category of schemes over k. Let V be a complete algebraic variety and D a Cartier divisor, which is by definition locally determined by one equation. With such a divisor D we can associate the invertible sheaf $\mathcal{O}(D)$ on V, and conversely a certain rational section of $\mathcal{O}(D)$ defines D. Let $\mu: V^* \to V$ be a normalization of V. Now we define $l_V(D)$ to be the dimension of the vector space $H^0(V^*, \mu^*\mathcal{O}(D))$.

Consider $l(mD)$ as a function of m. In the case in which there exists a positive integer m_0 such that $l(m_0D)>0$, we have the following estimate

$$\alpha m^{\kappa} \leqq l(mm_0D) \leqq \beta m^{\kappa}$$

for large values of m, where α, β are fixed real positive numbers and κ a non-negative integer. Since the κ is independent of the choice of m_0, α and β, we can define D-dimension of V, written $\kappa(D, V)$, to be κ. In the other case, we put $\kappa(D, V)=-\infty$. Note that $\kappa(D, V)$ takes one of the values $-\infty, 0, 1, \cdots, n$ and that $\kappa(D, V)\leqq 0$ if and only if $l(mD)\leqq 1$ for any $m>0$.

Furthermore, we indicate by $\sim$ the linear equivalence of divisors. We have three kinds of divisor groups as follows:

$\mathcal{G}(V)=$ the group of all divisors,

$\mathcal{G}_a(V)=$ the group of divisors algebraically equivalent to zero,

$\mathcal{G}_l(V)=$ the group of divisors linearly equivalent to zero.

It is well known that $\mathcal{G}(V)/\mathcal{G}_a(V)\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$ is finite dimensional, so we denote its dimension by $\rho(V)$. This is the Picard number of V. Assume V is non-singular and projective. Then the arithmetic genus $p_a(V)$ is defined to be $\dim H^n(V, \mathcal{O}_V)-\dim H^{n-1}(V, \mathcal{O}_V)+\cdots+(-1)^{n-1}\dim H^1(V, \mathcal{O}_V)$, and the irregularity $q(V)$ may be defined to be $\dim H^1(V, \mathcal{O}_V)$. Both are birational invariants, so we can define them for an arbitrary algebraic variety by making use of a non-singular projective model of V.

§ 3. Kodaira dimension.

Let V be an algebraic variety of dimension n. By Hironaka [4] we take a non-singular projective model V^* of it. We denote by $K(V^*)$ a canonical divisor of V^*, then the m-genus of V^* is defined to be $l(mK(V^*))$ for every $m \geqq 1$. Each of these is a birational invariant, so we can define the m-genus of V and write $P_m(V)$. The Kodaira dimension $\kappa(V)$ is defined to be $\kappa(K(V^*), V^*)$, so we get an estimate.

$$\alpha m^{\kappa} \leqq P_{mm_0}(V) \leqq \beta m^{\kappa},$$

for large values of m, where $\kappa = \kappa(V)$. Note that we define κ(an empty set) $= -\infty$.

We recall some geometric properties of Kodaira dimension.

I. Suppose $\kappa(V) \geqq 0$. Then we have a fibre space[2)] of non-singular projective algebraic varieties $f: V^* \to W$ satisfying

1. V^* is birationally equivalent to V,
2. $\dim W = \kappa(V)$,
3. any general fibre V_w^* of f over a general point w is irreducible,
4. $\kappa(V_w^*) = 0$.

Conversely, if a fibre space satisfies the properties above, it is birationally equivalent to $f: V^* \to W$.

II. Let $f: V \to W$ be a fibre space of algebraic varieties whose general fibre V_w is irreducible. Then we have

$$\kappa(V) \leqq \kappa(V_w) + \dim W.$$

III. Let $f: \tilde{V} \to V$ be a finite unramified covering map of non-singular projective algebraic varieties. Then $\kappa(\tilde{V}) = \kappa(V)$.

We shall make some remarks which are easily checked.

REMARK 1. $P_{m_1} > 0$ and $P_{m_2} > 0$ imply $P_{m_1+m_2} > 0$. So $P_{m_1 m_2} = 0$ yields $P_{m_1} = P_{m_2} = 0$.

REMARK 2. Let $f: V \to W$ be a generically surjective rational mapping of varieties with the same dimension. Then $P_m(V) \geqq P_m(W)$ for $m \geqq 1$.

§ 4. Definition of $Q_m(V)$ and $\nu(V)$.

Let V be an n-dimensional non-singular projective algebraic variety and D a divisor on V. We write $D = \sum n_p C_p$ where the C_p are prime divisors and $n_p \neq 0$, and define the m-genus $P_m(D)$ of D to be $\sup_p P_m(C_p)$. Now we put

$$Q_m(D, V) = \inf \{P_m(E);\ E \sim rD,\ r \neq 0\}.$$

2) Here we say that $f: V \to W$ is a fibre space if f is surjective.

Similarly, we define $\kappa(D)=\sup_p \kappa(C_p)$ and

$$\nu(D, V)=\inf\{\kappa(E);\ E\sim rD,\ r\neq 0\}.$$

First we establish

PROPOSITION 1. *$Q_m(K(V), V)$ and $\nu(K(V), V)$ are birational invariants.*

We define $Q_m(V)$ and $\nu(V)$ for an algebraic variety V as follows:

$$Q_m(V)=Q_m(K(V^*), V^*), \qquad \nu(V)=\nu(K(V^*), V^*),$$

where V^* is a non-singular projective model of V. These are well defined by Proposition 1.

In order to prove this proposition we need two lemmas concerning birational mappings.

LEMMA 1. *Any exceptional subvariety on a non-singular projective algebraic variety is a ruled*[3] *variety.*

This was first discovered by S. Abhyanker by means of the valuation theory [1]. Here we wish to prove this using Hironaka's theorem about the elimination of points of indeterminacy of birational mappings [4].

PROOF. Here, by an exceptional subvariety on a non-singular projective algebraic variety V we mean a prime divisor S on V such that there exists a birational mapping f of V onto a non-singular projective algebraic variety V^* so that the proper transform of S by f (which we denote by $f[S]$ with Zariski [17]) has the codimension $\geqq 2$. Such an S we call an exceptional subvariety for f. First, consider the case in which $f: V\to V^*$ is a monoidal transformation with a non-singular center C. Then there exists only one exceptional subvariety $f^{-1}(C)$ for f. It is clear that $f^{-1}(C)$ is a ruled variety with the base variety C. Second, consider the case in which $f: V\to V^*$ is a composition of a finite number of successive monoidal transformations with non-singular centers $\{f_i: V_{i+1}\to V_i\}$ $0\leqq i\leqq l-1$. This means

(i) $V_0=V^*$ and $V_l=V$,

(ii) $f=f_0\circ f_1\circ\cdots\circ f_{l-1}$.

Now we proceed to prove that the exceptional subvariety S is ruled by induction with respect to l. Choose the largest number i such that the codimension of $f_i\cdots f_{l-1}[S]\geqq 2$. Suppose that $i=0$. Then $\bar{S}=f_1\cdots f_{l-1}[S]$ is a prime divisor and so $\bar{S}$ is exceptional for f_0. Hence, $\bar{S}$ is ruled. Since S is birationally equivalent to $\bar{S}$, we see that S is ruled. When $i\geqq 1$, from induction hypothesis it follows that S is ruled. Finally, consider the case in which f is a birational mapping. By Hironaka's theorem we can eliminate the point of indeterminacy of the birational mapping f^{-1}, so we have a finite succession of monoidal transformations with non-singular centers $\{g_j: V_{j+1}\to V_j\}$

3) We say V is ruled if V is birationally equivalent to the product of $\boldsymbol{P}^1$ and some algebraic variety.

$0 \leqq j \leqq l-1$ such that $V_0 = V^*$ and $h = f^{-1} \cdot g_0 \cdots g_{l-1}$ is a morphism of V_l onto V. Now, $\tilde{S} = h^{-1}[S]$ is a prime divisor because h is a morphism. Then $\tilde{S}$ is an exceptional subvariety for $g_0 \cdots g_{l-1}$. Hence, by the consideration above we see that $\tilde{S}$ is ruled and so S is ruled.

LEMMA 2. *Let V_1 and V_2 be non-singular projective algebraic varieties and f a birational mapping of V_1 onto V_2. Then it follows that*

$$K(V_2) \sim f_*(K(V_1)) + E\,,$$

where E denotes a divisor whose irreducible components are all exceptional subvarieties for f^{-1}. Such an E is called the exceptional divisor for f^{-1}.

PROOF. We begin with the case where f is a birational morphism. In the usual way we get two homomorphisms f^* and f_* so that $f_* f^* = id$ and $(f^* f_* - id)\mathcal{G}(V_1) = \mathcal{E}_f(V_1)$ by which we denote the group of exceptional divisors for f. It is easy to check

$$f^* K(V_2) \equiv K(V_1) \mod \mathcal{G}_l(V_1) + \mathcal{E}_f(V_2)\,. \tag{1}$$

Now consider the case in which f may not be a morphism. We can find a non-singular projective algebraic variety V_3 and two birational morphisms $f_1 : V_3 \to V_1$ and $f_2 : V_3 \to V_2$ such that $f \cdot f_1 = f_2$. Then we have by (1)

$$f_1^* K(V_1) \equiv K(V_3) \mod \mathcal{G}_l(V_3) + \mathcal{E}_{f_1}(V_3)\,,$$

$$f_2^* K(V_2) \equiv K(V_3) \mod \mathcal{G}_l(V_3) + \mathcal{E}_{f_2}(V_3)\,.$$

These yield

$$f_2^* K(V_2) \equiv f_1^*(K(V_1)) + E_1 \mod \mathcal{G}_l(V_3) + \mathcal{E}_{f_2}(V_3)\,,$$

where E_1 is an exceptional divisor for f_1. Hence, we obtain

$$K(V_2) = f_{2*} f_2^*(K(V_2)) \sim f_{2*} f_1^*(K(V_1)) + f_{2*}(E_1)\,. \tag{2}$$

We define $f_*(K(V))$ to be $f_{2*} f_1^*(K(V_1))$. Since $f_{2*}(E_1)$ is exceptional for f^{-1}, (2) proves Lemma 2.

Now we proceed with the proof of birational invariance of $\nu(K(V), V)$. Let f be a birational mapping of V_1 onto V_2. By the definition of $\nu(K(V_1), V_1)$, there is an integer $r \neq 0$ and a divisor $D = \sum n_p C_p$ such that $rK(V_1) \sim D$ and $\kappa(C_p) \leqq \nu(K(V_1), V_1)$. Hence, from the proof of Lemma 2 it follows that

$$r f_*(K(V_1)) \sim \sum n_p f_*(C_p) + E^*\,,$$

where E^* is exceptional for f^{-1} so that $\kappa(E^*) < 0$, because E^* consists of ruled varieties by Lemma 1. In view of Lemma 2, we obtain

$$rK(V_2) \sim \sum n_p f_*(C_p) + E^* - E\,,$$

where E is exceptional for f^{-1}. If $f_*(C_p) = 0$, we have by definition $\kappa(f_*(C_p)) < 0$ and if $f_*(C_p) \neq 0$, we have $\kappa(f_*(C_p)) = \kappa(C_p) \leqq \nu(K(V_1), V_1)$. Thus we get

$\nu(K(V_2), V_2) \leqq \nu(K(V_1), V_1)$. By the same argument we have $\nu(K(V_1), V_1) \leqq \nu(K(V_2), V_2)$, so $\nu(K(V_1), V_1) = \nu(K(V_2), V_2)$. The birational invariance of $Q_m(K(V), V)$ can be proved in the same way. q. e. d.

REMARK 3. It is easy to check $Q_m(V) \leqq \beta^* m^{\nu(V)}$ for large values of m, where β^* is a positive real constant. The inequality $\alpha^* m^{\nu(V)} < Q_{m \cdot m_4}(V)$ for some positive integer m_4 and for a positive constant α^*, has only been established when $n \leqq 3$.

REMARK 4. Let $f: \tilde{V} \to V$ be a finite unramified covering map of non-singular projective algebraic varieties. Then $\nu(\tilde{V}) = \nu(V)$. But the analogue for Remark 2 does not hold. For instance, consider an elliptic curve and a 2-sheeted ramified covering map $\phi: E \to \boldsymbol{P}^1$. We have a finite ramified covering map $E \times E \to E \times \boldsymbol{P}^1$. Clearly $\nu(E \times E) = -\infty$ and $\nu(E \times \boldsymbol{P}^1) = \kappa(E) = 0$.

REMARK 5. $Q_{m_1 m_2}(V) = 0$ implies $Q_{m_1}(V) = Q_{m_2}(V) = 0$. However, the complete analogue for Remark 1 seems to be doubtful.

§ 5. Some properties of $Q_m(V)$ and $\nu(V)$.

The following proposition is in some sense an analogue of the Property II of Kodaira dimension.

PROPOSITION 2. *Let V and W be algebraic varieties of dimension n and $n-1$, respectively. Let $f: V \to W$ be a surjective morphism whose general fibre V_w is irreducible. Then we have*

$$\nu(V) \geqq \nu(V_w) + \kappa(W) .$$

PROOF. By virtue of Proposition 1 we can assume that V and W are non-singular projective. By the definition of $\nu(V)$ we have a divisor $D = \sum_{p=1}^{s} n_p C_p$ such that $D \sim rK(V)$ for an integer $r \neq 0$ and $\kappa(C_p) \leqq \nu(V)$ for every p. Suppose that $f(C_p) = W$ for some p. Then we have $\kappa(C_p) \geqq \kappa(W)$ and so $\nu(V) \geqq \kappa(W)$. Suppose that $f(C_p) \subsetneqq W$ for any $p = 1, \cdots, s$. Then we obtain $\bigcup_{p=1}^{s} f(C_p)$ is a proper Zariski closed subset of W. This implies $D \mid V_w = 0$ for a general fibre V_w. So we obtain

$$rK(V_w) \sim rK(V) \mid V_w \sim 0 .$$

From this it follows that V_w is an elliptic curve. q. e. d.

This proof shows that if V_w is not elliptic then $Q_m(V) \geqq P_m(W)$ for every $m \geqq 1$.

COROLLARY. *Suppose that V is birationally equivalent to $\boldsymbol{P}^1 \times W$. Then we have $Q_m(V) = P_m(W)$ and $\nu(V) = \kappa(W)$.*

PROOF. Note the formula $K(\boldsymbol{P}^1 \times W) = -2W + \boldsymbol{P}^1 \times K(W)$. This implies $Q_m(\boldsymbol{P}^1 \times W) \leqq P_m(W)$ and $\nu(\boldsymbol{P}^1 \times W) \leqq \kappa(W)$. Combining these with Propositions

1 and 2 we establish the corollary.

PROPOSITION 3. *Let V be a non-singular projective algebraic variety with a non-empty anticanonical system. Suppose that $\kappa(K(V) \mid A, A) = -\infty$ for every prime divisor A which is an irreducible component of the fixed part of $|-K(V)|$. Then $\nu(V) \leqq 0$ and so $Q_m(V) \leqq 1$ by Remark 3.*

PROOF. Let L be any irreducible component of a general member D of $|-K(V)|$. It suffices to prove $\kappa(L) \leqq 0$. Before proving this we shall deduce from the hypothesis of Proposition 3 the following

PROPERTY A. *Let A, B and C be effective divisors on V such that* (i) *A is prime,* (ii) *$B = lA + C$ for $l \geqq 0$,* (iii) *each component of C is not equal to A and* (iv) *$B + A$ is a general member of $|-K(V)|$. Then $\kappa(-B \mid A, A) \leqq 0$.*

We shall prove this by deriving $\kappa(-B \mid A, A) \leqq 0$ under the assumption $\kappa(-B \mid A, A) \geqq 0$. So, there exist an integer $m \geqq 1$ and an effective divisor X such that

$$X \sim -mB \mid A. \tag{1}$$

Hence, combined with (ii), the relation (1) yields

$$X + mC \mid A \sim -mlA \mid A. \tag{2}$$

On the other hand, by (iv) we have

$$K \mid A \sim (-B - A) \mid A,$$

and so by (2)

$$mK \mid A \sim X - mA \mid A. \tag{3}$$

Multiplying the formula (3) by l, we get from (2)

$$mlK \mid A \sim (l+1)X + mC \mid A. \tag{4}$$

Since $C \mid A$ is effective, the formula (4) implies $\kappa(K \mid A, A) \geqq 0$ so that A is an irreducible component of the moving part of $|-K(V)|$ by the hypothesis of Proposition 3. Hence, according to the Bertini theorem we have $l = 0$ and so $B = C$. This implies $\kappa(-B \mid A, A) \leqq 0$. Thus the proof of Property A is completed.

Now we write D in the form $\sum n_p C_p + lL$, where the C_p denote the irreducible components which are different from L. We use Property A in the case in which $A = L$, $C = \sum n_p C_p$ and $B = (l-1)L + \sum n_p C_p$. Hence, we conclude that

$$\kappa((-D + L) \mid L, L) \leqq 0. \tag{5}$$

We need the following lemma which may be regarded as a generalization of adjunction formula.

LEMMA 3. *Let V be a non-singular projective algebraic variety and S a* *prime divisor on V. Then it follows that*

$$P_m(S) \leqq l(m(K(V)+S) \mid S)$$

for every integer $m \geqq 1$.

PROOF. Let $f: V^* \to V$ be a monoidal transformation with a non-singular center C which is contained in the singular locus of S. We denote by S^* and E, respectively, the strict transform of S and the full inverse image of C by f. Then we have

$$K(V^*) = f^*K(V) + (\nu-1)E, \tag{6}$$

where $\nu = n - \dim C$ and

$$f^*(S) = S^* + m_1 E, \qquad \text{for some integer } m_1 \geqq 2.$$

Combining this with (6), we obtain

$$K(V^*) + S^* = f^*K(V) + f^*(S) - (m_1+1-\nu)E.$$

Restricting these divisors to S^*, we have

$$(K(V^*)+S^*) \mid S^* = f^*(K(V)+S) \mid S^* - (m_1+1-\nu)E \mid S^*, \tag{7}$$

and

$$f^*(K(V)+S) \mid S^* = \bar{f}^*((K(V)+S) \mid S)$$

where $\bar{f}$ is the restriction of f to S^*. Since E and S^* are both irreducible and $E \neq S^*$, it follows that $E \mid S^*$ is effective. First, we consider the case of $\nu = 2$. Then $m_1+1-\nu \geqq 1$. Hence, we have

$$(K(V^*)+S^*) \mid S^* \leqq \bar{f}^*((K(V)+S) \mid S).$$

Therefore, for every integer $m \geqq 1$, we obtain

$$l(m(K(V^*)+S^*) \mid S^*) \leqq l(m\bar{f}^*(K(V)+S) \mid S) = l(m(K(V)+S) \mid S). \tag{8}$$

Next, consider the case of $\nu \geqq 3$. In this case, $\bar{f}$ maps $E \mid S^*$ onto C where $n-2 = \dim E \mid S^* > \dim C$. For simplicity, we write $E^* = E \mid S^*$, $D = m(K(V)+S) \mid S$, $D^* = m(K(V^*)+S^*) \mid S^*$ and $e = (m_1+1-\nu)m$. Hence, we have

$$D^* = \bar{f}^*(D) - eE^*. \tag{9}$$

If $e \geqq 0$, then $D^* \leqq \bar{f}^*(D)$. This implies the formula (8). If $e < 0$, in order to establish the formula (8), it suffices to consider the case of $l(D^*) \geqq 1$. We choose $D_1^* \in |D^*|$. Applying $\bar{f}_*$ to (9), we obtain $\bar{f}_*(D_1^*) \sim D$. We write D_1 instead of $\bar{f}_*(D_1^*)$, which is effective. Thus we have

$$D_1^* \sim \bar{f}^*(D_1) + (-e)E^*.$$

In view of the inequality $\dim E^* > \dim C$, we obtain

$$l(D_1^*) = l(\bar{f}^*(D_1) + (-e)E^*) = l(\bar{f}^*(D_1)) = l(D_1).$$

This yields (8).

On the other hand, we can find a finite succession of monoidal transformations with non-singular centers $\{f_i : V_{i+1} \to V_i\}$ $0 \leqq i \leqq l-1$ such that

(i) $V_0 = V$,

(ii) let S_0 be S and S_{i+1} the strict transform by f_i for $i = 0, 1, \cdots, l-1$,

(iii) let C_i denote the center of f_i, then C_i is contained in the singular locus of S_i for $0 \leqq i \leqq l-1$,

(iv) S_l is a submanifold of V_l.

Applying the formula (8), we obtain

$$(10) \qquad l(m(K(V_l)+S_l) \mid S_l) \leqq l(m(K(V_{l-1})+S_{l-1}) \mid S_{l-1}) \leqq \cdots \leqq l(m(K(V)+S) \mid S) .$$

The usual adjunction formula is written as

$$(K(V_l)+S_l) \mid S_l = K(S_l) .$$

Combining this with (10), we have

$$P_m(S) = P_m(S_l) = l(mK(S_l)) = l(m(K(V_l)+S_l) \mid S_l) \leqq \cdots \leqq l(m(K(V)+S) \mid S) ,$$

as required. q. e. d.

From this proof when $n=3$ we get

$$(11) \qquad -K(S_l) = -(K(V_l)+S_l) \mid S_l \geqq \cdots \geqq -(K(V)+S) \mid S .$$

Now we continue to prove Proposition 3. By Lemma 3 we have

$$P_m(L) \leqq l(m(K(V)+L) \mid L) = l(m(-D+L) \mid L)$$

for any integer $m \geqq 1$, and so

$$\kappa(L) \leqq \kappa((-D+L) \mid L) .$$

Hence, from (5) we infer that $\kappa(L) \leqq 0$. q. e. d.

PROPOSITION 4. *Let V be a non-singular projective algebraic variety of $\rho(V)=1$. Suppose that there exists a generically surjective rational mapping f of V onto W whose general fibre V_w is an irreducible rational curve. Then we have $q(W)=0$ and $\nu(V)=-\infty$ so $\kappa(W)=-\infty$ by Proposition 2.*

PROOF. We first show that $q(V)$ vanishes. Assume the contrary, namely, $q(V)=0$. Then we have the non-trivial Albanese variety $\mathrm{Alb}(V)$ and the Albanese morphism α_V of V into $\mathrm{Alb}(V)$. Denote by B the image $\alpha_V(V)$. Now let C_w be a general fibre of f, then it is clear that $\alpha_V(C_w)$ reduces to a point a_w on B. We choose an effective divisor H on B such that $a_w \in B-H$. Hence, $\alpha_V^*(H) \cap C_w = \emptyset$. On the other hand $\rho(V)=1$ implies that $\alpha_V^*(H)$ is ample, so $\alpha_V^*(H) \cap C_w \neq \emptyset$. Thus we have arrived at a contradiction. From $q(W) \leqq q(V)$ we get $q(W)=0$. Moreover, the hypothesis $\rho(V)=1$ leads to the following

PROPERTY B. *Let D_1 and D_2 be divisors, then $aD_1 \sim bD_2$ where $a \neq 0$ and* *b are integers.*

Let Γ be a proper inverse image of a general hyperplane section of W by f. Then Γ is ruled. We can find $r \neq 0$ and s such that $rK(V) \sim s\Gamma$. This implies $\nu(V) = -\infty$. q. e. d.

REMARK 6. Combined with the corollary, Proposition 3 may be regarded as an extension of a theorem of Severi in the theory of algebraic surfaces. Indeed, F. Severi introduced the notion of antigenus as follows:

In case V is non-singular projective, the antigenus $p^{-1}(V)$ is defined to be the dimension of $H^0(V, \mathcal{O}(-K(V)))$. And the absolute antigenus of an algebraic variety V is defined to be the supremum of $p^{-1}(V^*)$ where V^* is any non-singular projective model of V.

Severi stated the theorem to the effect that any ruled surface with $p^{-1}(S) > 0$ has the irregularity 0 or 1 (see [**14**], [**15**]). This stimulated our discovery of Proposition 3 but is not valid in general.

EXAMPLE 1. Consider $\boldsymbol{P}^3$ and its anticanonical system $|-K(\boldsymbol{P}^3)|$. Let z_0, z_1, z_2 and z_3 form the homogeneous coordinate of $\boldsymbol{P}^3$. Then any homogeneous polynomial of degree 4 defines an element of the anticanonical system. Let S be a surface $\in |-K(\boldsymbol{P}^3)|$ defined by $z_0^4 + z_1^4 + z_2^4$. Clearly S is a ruled surface whose irregularity is equal to 3. By virtue of (11), we conclude that the absolute antigenus of S is positive.

EXAMPLE 2. (T. Oda). Let C be a non-singular projective algebraic curve of genus g and F a locally free sheaf of rank 2 on C. The projective line bundle defined by F is a ruled surface of irregularity g, which we write S_F. Then we have

$$l(-K(S_F)) = \dim H^0(C, \mathcal{O}(-K(C)) \otimes \boldsymbol{S}^2(F) \otimes (\det F)^{-1}),$$

where by $\boldsymbol{S}^2 F$ and $\det F$ we denote the symmetric product of F and the determinant of F, respectively. Choose a point p on C and put $F = \mathcal{O}(rp) \oplus \mathcal{O}(-rp)$ for $r \geqq 1$. It is clear that $\boldsymbol{S}^2(F) = \mathcal{O}(2rp) \oplus \mathcal{O} \oplus \mathcal{O}(-2rp)$ and $\det F = \mathcal{O}$. In this case we write S_r instead of S_F. Then, when r grows to infinity, $p^{-1}(S_r)$ is asymptotically equal to $2r$. This means that the absolute antigenus of a ruled surface is equal to ∞.

REMARK 7. There are many problems concerning the computation of ν. For instance,

(1) Let V be a unirational variety of dimension $\geqq 2$. Then $\nu(V) = -\infty$?

(2) Let V be an algebraic variety whose Kodaira dimension $\geqq 0$. Then $\kappa(V) - 1 \geqq \nu(V)$?

REMARK 8. In the definition of ν, we can replace linear equivalence by algebraic equivalence. That means

$$\nu^*(V) = \inf \{\kappa(D) ;\ D \text{ is alg. eq. to } rK(V)\}\ .$$

In general $\nu^*(V)\leqq\nu(V)$ and we can check that under the assumption $n=2$, $\kappa\leqq 1$, this ν^* is a deformation invariant.

§ 6. Criteria of rationality.

THEOREM 1. *Let V be a non-singular projective algebraic variety of dimension* 3 *which is birationally equivalent to $\boldsymbol{P}^1\times W$, W being a non-singular projective surface.*

CRITERION I. *Suppose that V satisfies*

(1) $p_a(V)=0$,

(2) $\pi_1(V)\neq \boldsymbol{Z}/(2)$,

(3) $|-K(V)|\neq\emptyset$ *and each irreducible component A of the fixed part of* $|-K(V)|$ *satisfies* $\kappa(K(V)\mid A, A)=-\infty$.

Then V is rational.

CRITERION II. *Suppose that V satisfies*

$$\rho(V)=1. \tag{4}$$

Then V is rational.

PROOF. Since $\dim H^3(V, \mathcal{O}_V)=0$ and $\dim H^i(V, \mathcal{O}_V)=\dim H^i(W, \mathcal{O}_W)$ for $i=1,2$, we have by (1) $p_g(W)=q(W)$. From (2) it follows that $\pi_1(W)\cong\pi_1(\boldsymbol{P}^1\times W)\cong\pi_1(V)\neq\boldsymbol{Z}/(2)$. Moreover, by Propositions 2 and 3 we obtain $\kappa(W)=0$ or $-\infty$. Note that any W whose Kodaira dimension vanishes can be classified into the table below (see [12]):

TABLE

Class	p_g	q	π_1	the structure of their minimal models
I	1	2	$\boldsymbol{Z}^4$	abelian varieties of dimension 2
II	0	1		hyperelliptic surfaces
III	1	0	0	$K3$ surfaces
IV	0	0	$\boldsymbol{Z}/(2)$	Enriques surfaces

Hence, recalling the numerical invariants, we infer that $\kappa(W)=-\infty$, so $q(W)=p_g(W)=0$. On the other hand, from (4) we can deduce $q(W)=0$, $\kappa(W)=-\infty$ using Proposition 4. By the Castelnuovo criterion about rationality, we have established that W is rational, and so V is rational. q. e. d.

THEOREM 2. *Let V be a 3-dimensional non-singular projective algebraic variety whose tangent vector bundle T_V is ample. Then V is rational.*

PROOF. First we notice that

(5) $$T_V \text{ is ample}$$

implies

(6) $$-K(V) \text{ is ample by Hartshorne [3].}$$

From this

(7) $$H^p(V, \mathcal{O}_V)=0 \qquad \text{for } p=1, 2, 3$$

follows by virtue of the Kodaira vanishing theorem.

We apply Hirzebruch's formula of Riemann Roch to a 3-dimensional algebraic variety V and a divisor D. Then we have two formulas:

(8) $$-24(1-p_a(V))=KC,$$

(9) $$\sum_{j=0}^{3}(-1)^j \dim H^j(V, \mathcal{O}(D))=\frac{1}{6}D^3-\frac{1}{4}KD^2+\frac{1}{12}(K^2+C)D-\frac{1}{24}KC,$$

where K and C are a canonical divisor and a canonical curve of V, respectively. In the case in which $D=-K(V)$ and $-K(V)$ is ample, these yield

(10) $$\dim|-K(V)| \geqq -K(V)^3+2 \geqq 3.$$

Moreover, V satisfies the condition (3), because $-K(V)$ is ample.

On the other hand, referring to [10] we get

(11) $$\dim H^0(V, \mathcal{O}(T_V)) \geqq 6.$$

By Matsumura [13] we know in general that if $\dim H^0(V, \mathcal{O}(T_V)) > \dim V$, then V is ruled. Combined with this, (11) leads to the fact that V is ruled. Now, assume that V is not rational. By Theorem 1, we have $\pi_1(V)=\boldsymbol{Z}/(2)$, so V is birationally equivalent to the product of $\boldsymbol{P}^1$ and an Enriques surface. We can construct the finite unramified covering manifold $\tilde{V}$ of V which is birationally equivalent to $\boldsymbol{P}^1\times\tilde{W}$, $\tilde{W}$ being a $K3$ surface. This implies

(12) $$\dim H^2(\tilde{V}, \mathcal{O})=p_g(\tilde{W})=1.$$

However, $T_{\tilde{V}}$ is also ample by Hartshorne (see Proposition 4.3 in [3]). Hence we have $\dim H^2(\tilde{V}, \mathcal{O})=0$. This contradicts (12).

Note that Hironaka constructed a 3-dimensional non Kähler compact complex manifold which is rational and whose Picard number is 1.

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