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On homotopy invariance of triangulability of certain 5-manifolds

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Kirby [4] has constructed a non-triangulable 6-manifold having the same homotopy type as $S^2\times S^4$. Extending his method, S. Ichiraku [3] proves that there is a non-triangulable manifold which is homotopy equivalent to a given PL-manifold satisfying certain conditions of dimension $\geqq 6$. Therefore, in dimensions greater than 5, it is likely that the homotopy invariance of triangulability fails in almost all cases. However, in dimension 5 there are some examples which intimate the homotopy invariance of triangulability [1], [2]. In this paper we will study the problem to what extent this invariance holds. We will state our main result in § 1, and will give a proof in §§ 2~3.

The author is indebted to helpful discussions with S. Morita.

§ 1. Our main result.

THEOREM 1. *Let M^5 be a closed orientable topological 5-manifold such that*

(i) *$\pi_1(M^5)$ is an abelian group without 2-torsions, and*

(ii) *$Sq^2: H^2(M^5; \boldsymbol{Z}_2) \longrightarrow H^4(M^5; \boldsymbol{Z}_2)$ is a zero map.*

Then for any homotopy equivalence $f: M^5 \longrightarrow L^5$ of M^5 to another 5-manifold L^5, we have

$$f^*k(L^5)=k(M^5),$$

where $k \in H^4(\ ;\boldsymbol{Z}_2)$ denotes the obstruction to PL-triangulation [5]. (We will refer this class as the Kirby-Siebenmann class.)

S. Morita [6] has proved that if M_0^5 is an orientable closed PL 5-manifold with $\pi_1(M_0^5)\cong \boldsymbol{Z}_2$, then there is a non-triangulable manifold N^5 having the same homotopy type as M_0^5. So the condition (i) is essential.

COROLLARY 1. *Replacing* (ii) *in Theorem* 1 *by the hypothesis that M^5 is a spin-manifold, we have the same conclusion.*

This is independently proved by T. Matumoto by a more geometrical argument (unpublished).

PROOF OF COROLLARY 1. Since $H_1(M^5; \boldsymbol{Z})$ has no 2-torsions, neither does $H^2(M^5; \boldsymbol{Z})$ by the universal coefficient theorem. Thus the Bockstein

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$\delta: H^1(M^5; \boldsymbol{Z}_2) \longrightarrow H^2(M^5; \boldsymbol{Z})$ is zero; so $Sq^1 = (\text{mod } 2) \circ \delta : H^1(M^5; \boldsymbol{Z}_2) \longrightarrow H^2(M^5; \boldsymbol{Z}_2)$ is a zero map.

Let u_i be the i-th Wu class. Then we have

$$w_2 = u_2 + Sq^1 u_1 + Sq^2 u_0 = u_2 .$$

Thus by the hypothesis $w_2(M^5) = 0$, we have $u_2(M^5) = 0$. So $Sq^2 : H^3(M^5; \boldsymbol{Z}_2) \longrightarrow H^5(M^5; \boldsymbol{Z}_2)$ is zero. By Cartan formula,

$$0 = Sq^2(x \cup y) = Sq^2 x \cup y + Sq^1 x \cup Sq^1 y + x \cup Sq^2 y = Sq^2 x \cup y , \tag{1}$$

where $x \in H^2(M^5; \boldsymbol{Z}_2)$, $y \in H^1(M^5; \boldsymbol{Z}_2)$. Here we used again the fact that $Sq^1 : H^1(M^5; \boldsymbol{Z}_2) \longrightarrow H^2(M^5; \boldsymbol{Z}_2)$ is zero. Since (1) holds for any y, we have $Sq^2 x = 0$ by the Poincaré duality. Corollary 1 follows by Theorem 1. Q. E. D.

EXAMPLE. *A 5-manifold which is homotopy equivalent to* $S^i \times T^{5-i}$ $(1 \leqq i \leqq 5)$ *is triangulable.*

The triangulability of homotopy tori was proved by Hsiang and Wall [2]. However, [2] includes a statement (about homotopy invariance of a certain cohomology class) which is true in their case but false in general.*) (Cf. Theorem 2, below.) The triangulability of a homotopy-$S^4 \times S^1$ is first proved by S. Fukuhara. See also [1].

Consider the following homotopy commutative diagram

$$\begin{array}{ccc} & & BSPINTOP \\ & \overset{\tilde{\eta}}{\nearrow} & \downarrow \varpi \\ F/TOP & \xrightarrow[\eta]{} & BSTOP \end{array}$$

where maps η, ϖ are natural maps, $\tilde{\eta}$ the unique lift (up to homotopy) of η. Using the facts $TOP/PL \simeq K(\boldsymbol{Z}_2, 3)$ [5] and $\pi_3(STOP) \cong \boldsymbol{Z} \oplus \boldsymbol{Z}_2$, it is easily seen that $H^4(BSTOP; \boldsymbol{Z}) \cong H^4(BSO; \boldsymbol{Z})$ and $H^4(BSPINTOP; \boldsymbol{Z}) \cong H^4(BSPIN; \boldsymbol{Z})$.

Let $q_1 \in H^4(BSPINTOP; \boldsymbol{Z}) \cong \boldsymbol{Z}$ be the generator such that $\varpi^* p_1 = 2q_1$, where $p_1 \in H^4(BSTOP; \boldsymbol{Z})$ is the 1-st Pontrjagin class. Let $k \in H^4(BSTOP; \boldsymbol{Z}_2)$ be the universal Kirby-Siebenmann class. We denote by i_*, p_* the homomorphisms of cohomology groups which are induced by the coefficient homomorphism $i: \boldsymbol{Z}_2 \longrightarrow \boldsymbol{Z}_{24}$, $p: \boldsymbol{Z} \longrightarrow \boldsymbol{Z}_{24}$ (i the non-trivial map, p the projection).

The following is a key theorem to proving Theorem 1.

THEOREM 2. *In* $H^4(F/TOP; \boldsymbol{Z}_{24})$, *we have*

$$p_* \tilde{\eta}^*(q_1) + i_* \eta^*(k) = i_* k_2^2 ,$$

where $k_2 \in H^2(F/TOP; \boldsymbol{Z}_2) \cong \boldsymbol{Z}_2$ *is the generator.*

This will be proved in § 3.

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*) The author heard that this was independently pointed out by several mathematicians in 1970.

§ 2. Proof of Theorem 1.

In this section, we will prove Theorem 1 taking Theorem 2 for granted.

Let M^5, L^5, f be given as in Theorem 1. Let τ_M, τ_L be the tangent bundles of M, L, respectively. Set $\xi=\tau_M-f^*\tau_L$, then ξ has a canonical F/TOP bundle structure. By Theorem 2, we have

$$p_*(q_1(\xi))+i_*(k(\xi))=i_*k_2(\xi)^2 .$$

However by hypothesis (ii), $k_2(\xi)^2=Sq^2k_2(\xi)=0$. Thus

$$p_*(q_1(\xi))+i_*(k(\xi))=0 . \tag{2}$$

By hypothesis (i), $\pi_1(M^5)\cong$(free abelian)$\oplus$(odd torsions). Let $\tilde{M}\xrightarrow{\pi}M^5$, $\tilde{L}\xrightarrow{\pi'}L^5$ be odd-fold coverings such that $\pi_1(\tilde{M})\cong\pi_1(\tilde{L})=$ a free abelian group. Let $\tilde{f}:\tilde{M}\longrightarrow\tilde{L}$ be the induced homotopy equivalence.

LEMMA 1. *$p_1(\tilde{M})=\tilde{f}^*p_1(\tilde{L})$ where p_1 is the* 1*-st* (*integral*) *Pontrjagin class.*

Although this is an easy consequence of [5] and [7], we will give a proof for completeness. In [7], Novikov proved that L_k-class of smooth (or PL) $4k+1$-manifold is homotopy invariant. His proof is easily extended to topological manifolds by the technique of [5] and [8]. However, the proof includes a certain transversality argument, so some care is needed to the case of L_1-class of topological 5-manifolds. (Cf. [10].) Let CP_2 be a complex projective surface with fundamental class γ. Then by the higher dimensional topological analogy of Novikov's result, we have $L_1(\tilde{M})\times\gamma=L_2(\tilde{M}\times CP_2)=(\tilde{f}\times id)^*L_2(\tilde{L}\times CP_2)=\tilde{f}^*L_1(\tilde{L})\times\gamma$. Since $\times\gamma$ is an isomorphism, we have $L_1(\tilde{M})=\tilde{f}^*L_1(\tilde{L})$. This implies that the rational p_1 of topological 5-manifolds is homotopy invariant. However, in our case $H^4(\tilde{M};\boldsymbol{Z})\cong H_1(\tilde{M};\boldsymbol{Z})$ which is free abelian, and so $H^4(\tilde{M};\boldsymbol{Z})\longrightarrow H^4(\tilde{M};\boldsymbol{Q})$ is injective. Hence we have the desired invariance of the integral p_1.

By Lemma 1, $2\pi^*q_1(\xi)=\pi^*p_1(\xi)=p_1(\tilde{M})-\tilde{f}^*p_1(\tilde{L})=0$. Noting that $H^4(\tilde{M};\boldsymbol{Z})$ is torsion free, we have

$$\pi^*q_1(\xi)=0 . \tag{3}$$

Combining (2) and (3), we obtain

$$\pi^*i_*k(\xi)=0 . \tag{4}$$

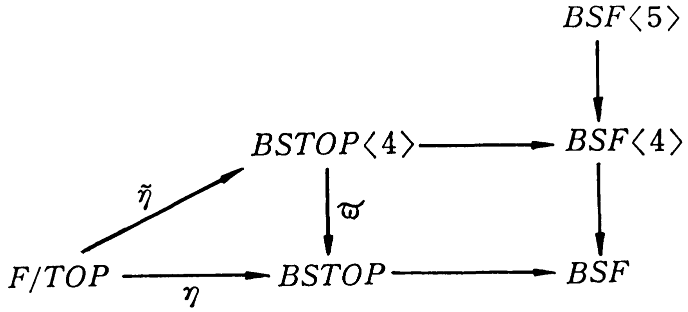
LEMMA 2. *Let $\tilde{Y}^n\xrightarrow{\nu}Y^n$ be an odd-fold covering of a closed manifold Y^n. Then $\nu^*:H^i(Y^n;\boldsymbol{Z}_2)\longrightarrow H^i(\tilde{Y}^n;\boldsymbol{Z}_2)$ is injective for any $i\geqq 0$.*

This is obvious as ν is degree 1 with respect to $\boldsymbol{Z}_2$-cohomology.

By the lemma, π^* is injective. Also $i_*:H^4(M^5;\boldsymbol{Z}_2)=Z_2\oplus\cdots\oplus Z_2\longrightarrow\boldsymbol{Z}_{24}\oplus\cdots\oplus\boldsymbol{Z}_{24}=H^4(M^5;\boldsymbol{Z}_{24})$ is clearly injective. So we have by (4) $k(\xi)=k(M)-f^*k(L)=0$. This completes the proof of Theorem 1.

§ 3. Proof of Theorem 2.

For a topological space X, denote by $X\langle n\rangle$ the $(n-1)$-connected fibre space of X. Consider the homotopy commutative diagram:



(N. B. $BSTOP\langle 4\rangle = BSPINTOP$) .

The map $\pi_4(BSTOP\langle 4\rangle) \longrightarrow \pi_4(BSF\langle 4\rangle)$ is $p+i: \boldsymbol{Z}\oplus\boldsymbol{Z}_2 \longrightarrow \boldsymbol{Z}_{24}$; so the left-hand side of the equation in Theorem 2, $p_*\tilde{\eta}^*q_1+i_*\eta^*k$, is the obstruction to lifting $F/TOP \xrightarrow{\tilde{\eta}} BSTOP\langle 4\rangle \longrightarrow BSF\langle 4\rangle$ to $F/TOP \longrightarrow BSF\langle 5\rangle$.

LEMMA 3. $p_*\tilde{\eta}^*q_1+i_*\eta^*k \neq 0$.

PROOF OF LEMMA 3. If $F/TOP \longrightarrow BSF\langle 4\rangle$ were lifted to $BSF\langle 5\rangle$, for any finite 4-complex Y^4 and any map $g: Y^4 \longrightarrow F/TOP$, the composition $Y^4 \xrightarrow{g} F/TOP \longrightarrow BSTOP\langle 4\rangle \longrightarrow BSF\langle 4\rangle$ would be null-homotopic, for $BSF\langle 5\rangle$ is 4-connected. However, the next counter-example shows that this is not the case. Thus $p_*\tilde{\eta}^*q_1+i_*\eta^*k \neq 0$. Q. E. D.

A COUNTER-EXAMPLE. *Let $h: S^2 \longrightarrow F/TOP$ represent the non-zero element of $\pi_2(F/TOP) \cong \boldsymbol{Z}_2$, and $H: S^3 \longrightarrow S^2$ the Hopf-fibration. Since $\pi_3(F/TOP) \cong 0$, the composite map $S^3 \xrightarrow{H} S^2 \xrightarrow{h} F/TOP$ is null-homotopic. Thus h is extended to a map $g: \boldsymbol{C}P_2 \longrightarrow F/TOP$. Then the composite map $\boldsymbol{C}P_2 \xrightarrow{g} F/TOP \longrightarrow BSF\langle 4\rangle$ cannot be null-homotopic.*

PROOF. The fibre of a fibration $F/TOP \xrightarrow{\tilde{\eta}} BSTOP\langle 4\rangle$ is $SPINF$. Since $\pi_2(SPINF) \cong \pi_2(F/TOP)$, h can be considered as a composition $S^2 \xrightarrow{h'} SPINF \longrightarrow F/TOP$. Regarding $\pi_2(SPINF)$ $(\cong \pi_2(F))$ as the stable 2-stem of the homotopy groups of spheres, we know that $h'\circ H: S^3 \longrightarrow SPINF$ is not null-homotopic (See Toda [9]). Since $BSTOP\langle 4\rangle$ is 3-connected, the map $\boldsymbol{C}P_2 \xrightarrow{g} F/TOP \xrightarrow{\tilde{\eta}} BSTOP\langle 4\rangle$ represents a unique element of $\pi_4(BSTOP\langle 4\rangle)$ denoted by x. We know that $\partial x = \{h'\circ H\} \neq 0$ in the exact sequence $\pi_4(F/TOP) \xrightarrow{\tilde{\eta}_*} \pi_4(BSTOP\langle 4\rangle) \xrightarrow{\partial} \pi_3(SPINF)$; so x is not contained in $\operatorname{Im}\tilde{\eta}_*$. Consider the commutative diagram:

$$
\begin{array}{ccccc}
 & & \pi_4(BSTOP\langle 4\rangle) & \longrightarrow & \pi_4(BSF\langle 4\rangle) \\
 & \overset{\tilde{\eta}_\#}{\nearrow} & \Big\downarrow \cong \varpi_\# & & \Big\downarrow \cong \\
\pi_4(F/TOP) & \xrightarrow[\eta_\#]{} & \pi_4(BSTOP) & \longrightarrow & \pi_4(BSF).
\end{array}
$$

Since $\varpi_\#(x)$ is not in the image $\eta_\#$, it is mapped to a non-zero element in $\pi_4(BSF)$. Thus the element of $\pi_4(BSF\langle 4\rangle)$ determined by the composition $CP_2 \xrightarrow{g} F/TOP \longrightarrow BSTOP\langle 4\rangle \longrightarrow BSF\langle 4\rangle$ is not zero. This reveals that the composition is not null-homotopic.

LEMMA 4. *$p_*\tilde{\eta}^*q_1 + i_*\eta^*k$ belongs to the kernel of*

$$H^4(F/TOP; \boldsymbol{Z}_{24}) \longrightarrow H^4(F/TOP\langle 4\rangle; \boldsymbol{Z}_{24}).$$

PROOF OF LEMMA 4. We will show that the composition: $F/TOP\langle 4\rangle \longrightarrow F/TOP \longrightarrow BSF\langle 4\rangle$ is lifted to $BSF\langle 5\rangle$. The obstruction lies in $H^4(F/TOP\langle 4\rangle; \pi_4(BSF\langle 4\rangle)) = \mathrm{Hom}(\pi_4(F/TOP\langle 4\rangle), \pi_4(BSF\langle 4\rangle))$, and is represented by the homomorphism $\pi_4(F/TOP\langle 4\rangle) \longrightarrow \pi_4(BSF\langle 4\rangle)$ induced by the natural map. However, this is a zero homomorphism. This completes the proof of Lemma 4.

Consider the Serre exact sequence associated with a fibration $F/TOP\langle 4\rangle \longrightarrow F/TOP \longrightarrow K(\boldsymbol{Z}_2, 2)$:

$$0 \longrightarrow H^4(K(\boldsymbol{Z}_2, 2); \boldsymbol{Z}_{24}) \longrightarrow H^4(F/TOP; \boldsymbol{Z}_{24}) \longrightarrow H^4(F/TOP\langle 4\rangle; \boldsymbol{Z}_{24}).$$

Now $H^4(K(\boldsymbol{Z}_2, 2); \boldsymbol{Z}_{24}) \cong \boldsymbol{Z}_2$ and it is generated by $i_*k_2^2$. Therefore, the unique non-zero class in the kernel of $H^4(F/TOP; \boldsymbol{Z}_{24}) \longrightarrow H^4(F/TOP\langle 4\rangle; \boldsymbol{Z}_{24})$ is $i_*k_2^2$. Now Theorem 2 follows from Lemmas 3 and 4.

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