J. Math. Soc. Japan
Vol. 24, No. 2, 1972

Periodic maps and circle actions

Dedicated to Professor Shigeo Sasaki on his 60th birthday

By Fuichi UCHIDA

(Received Sept. 4, 1971)

§ 0. Introduction.

Fix a compact Lie group G and a family $\mathcal{F}$ of subgroups of G. We consider all (φ, M) where M is a closed oriented smooth manifold, and $\varphi: G\times M\to M$ is an orientation preserving smooth G-action so that for $x\in M$ the isotropy subgroup

$$G_x=\{g\in G,\ \varphi(g, x)=x\}$$

is conjugate to a member of $\mathcal{F}$. Then a bordism group $\mathcal{O}_n(G\,;\mathcal{F})$ of $\mathcal{F}$-free oriented G-manifolds is defined.

Let S^1 be the unit circle in the field of complex numbers and regard it a compact Lie group. Let $Z_m=\{t\in S^1,\ t^m=1\}$ be the cyclic subgroup of order m.

Given an oriented Z_m-manifold (φ, M), consider a cartesian product $S^1\times M$ and let Z_m act on $S^1\times M$ by

$$t\cdot(z, x)=(zt^{-1}, \varphi(t, x))$$

for $t\in Z_m$, $z\in S^1$ and $x\in M$. Denote by $S^1\times_{Z_m}M$ the orbit manifold and by $[z, x]$ the point of $S^1\times_{Z_m}M$ represented by (z, x) of $S^1\times M$. Then there is a circle action $\bar{\varphi}$ on $S^1\times_{Z_m}M$ given by

$$\bar{\varphi}(\lambda, [z, x])=[\lambda z, x]\,.$$

If (φ, M) is an oriented $\mathcal{F}$-free Z_m-manifold, then $(\bar{\varphi}, S^1\times_{Z_m}M)$ is $\mathcal{F}$-free S^1-manifold and this induces an extension homomorphism

$$\boldsymbol{E}:\ \mathcal{O}_n(Z_m\,;\mathcal{F})\longrightarrow\mathcal{O}_{n+1}(S^1\,;\mathcal{F})\,.$$

On the other hand, let $\mathcal{F}$ be a family of subgroups of S^1, denote by $\mathcal{F}_m$ the family of subgroups of Z_m given by

$$\mathcal{F}_m=\{Z_m\cap H,\ H\in\mathcal{F}\}\,.$$

Let (φ, M) be an oriented $\mathcal{F}$-free S^1-manifold, then the restriction $(\varphi|Z_m, M)$ is an oriented $\mathcal{F}_m$-free Z_m-manifold and this restriction induces a homomorphism

$$R: \mathcal{O}_n(S^1; \mathcal{F}) \longrightarrow \mathcal{O}_n(Z_m; \mathcal{F}_m).$$

In this paper we consider these homomorphisms for some family $\mathcal{F}$. And our main results are the following:

THEOREM 5.2. *Let m be an odd integer, then the restriction homomorphism*

$$R: \mathcal{O}_n(S^1; \{Z_m, \{1\}\}) \longrightarrow \mathcal{O}_n(Z_m; \{Z_m, \{1\}\})$$

is a null homomorphism.

THEOREM 5.4. *Let m be an odd integer, then the extension homomorphism*

$$E: \mathcal{O}_n(Z_m; \{Z_m, \{1\}\}) \longrightarrow \mathcal{O}_{n+1}(S^1; \{Z_m, \{1\}\})$$

is injective.

THEOREM 6.1. *Let m be an odd integer, then the restriction homomorphism*

$$R: \mathcal{O}_n(S^1; \{S^1, \{1\}\}) \longrightarrow \mathcal{O}_n(Z_m; \{Z_m, \{1\}\})$$

is injective.

First of all we recall some well known elementary property of the bordism group of G-manifolds (§ 1) and the bordism group of G-vector bundles (§ 2). Next we study vector bundles with particular circle actions (§ 4). Then we can prove the main theorems (§ 5—§ 6) and Ω-module structure of $\mathcal{O}_*(S^1; \{Z_m, \{1\}\})$.

Finally we consider principal circle actions on a product of spheres (§ 7) and we have the following:

THEOREM 7.3. *Let m, n be positive integers, then any principal smooth circle action on $S^{2m+1}\times S^{2n+1}$ bords as a principal smooth circle action.*

§ 1. The bordism of G-manifolds.

Fix a compact Lie group G, a compact oriented G-manifold (φ, M) consists of a compact oriented smooth manifold M and an orientation preserving smooth G-action $\varphi: G\times M\to M$ on M.

1.1. Given families $\mathcal{F}\supset\mathcal{F}'$ of subgroups of G, a compact oriented G-manifold (φ, M) is $(\mathcal{F}, \mathcal{F}')$-free if the following conditions are satisfied:

(1) if $x\in M$, then the isotropy subgroup

$$G_x=\{g\in G, \varphi(g, x)=x\}$$

is conjugate to a member of $\mathcal{F}$,

(2) if $x\in \partial M$, then G_x is conjugate to a member of $\mathcal{F}'$.

If $\mathcal{F}'$ is the empty family, then necessarily ∂M is empty and M is closed. In this case we call (φ, M) $\mathcal{F}$-free.

Given (φ, M), define $-(\varphi, M)=(\varphi, -M)$, with the structure precisely the same as (φ, M) except for orientation. Also define $\partial(\varphi, M)=(\varphi, \partial M)$. Note that if (φ, M) is $(\mathcal{F}, \mathcal{F}')$-free, then $(\varphi, \partial M)$ is $\mathcal{F}'$-free. Define (φ, M) and

(φ', M') to be isomorphic if there exists an equivariant orientation preserving diffeomorphism of M onto M'.

An $(\mathcal{F}, \mathcal{F}')$-free compact oriented n-dimensional G-manifold (φ, M) will be said to bord if there exists an $(\mathcal{F}, \mathcal{F}')$-free compact oriented $(n+1)$-dimensional G-manifold (ψ, W) together with a regularly embedded compact n-manifold M_1 in ∂W with M_1 invariant under the G-action ψ, such that (ψ, M_1) is isomorphic to (φ, M), and such that G_x is conjugate to a member of $\mathcal{F}'$ for $x \in \partial W - M_1$. Also M_1 is required to have orientation induced by that of W.

Also (φ_1, M_1) is bordant to (φ_2, M_2) if the disjoint union $(\varphi_1, M_1)+(\varphi_2, -M_2)$ bords. The bordism is an equivalence relation on the class of $(\mathcal{F}, \mathcal{F}')$-free compact oriented n-dimensional G-manifolds (cf. [**2**], § 21; [**3**], p. 139). Denote by $[\varphi, M]$ the equivalence class represented by (φ, M).

The bordism classes constitute an abelian group $\mathcal{O}_n(G; \mathcal{F}, \mathcal{F}')$ under the operation of disjoint union. If $\mathcal{F}'$ is empty, denote the above group by $\mathcal{O}_n(G; \mathcal{F})$. The direct sum

$$\mathcal{O}_*(G; \mathcal{F}, \mathcal{F}') = \bigoplus_n \mathcal{O}_n(G; \mathcal{F}, \mathcal{F}')$$

is naturally an Ω-module where Ω is the oriented Thom cobordism ring.

1.2. Suppose now that $\mathcal{F} \supset \mathcal{F}'$ are fixed families of subgroups of G. Every $\mathcal{F}'$-free G-manifold is also $\mathcal{F}$-free, this inclusion induces a homomorphism

$$\alpha: \ \mathcal{O}_n(G; \mathcal{F}') \longrightarrow \mathcal{O}_n(G; \mathcal{F}).$$

Similarly every $\mathcal{F}$-free G-manifold is also $(\mathcal{F}, \mathcal{F}')$-free, inducing a homomorphism

$$\beta: \ \mathcal{O}_n(G; \mathcal{F}) \longrightarrow \mathcal{O}_n(G; \mathcal{F}, \mathcal{F}').$$

Finally there is a homomorphism

$$\partial: \ \mathcal{O}_n(G; \mathcal{F}, \mathcal{F}') \longrightarrow \mathcal{O}_{n-1}(G; \mathcal{F}')$$

given by $\partial[\varphi, M] = [\varphi, \partial M]$. Then the following sequence is exact (cf. [**3**], p. 140):

$$\longrightarrow \mathcal{O}_n(G; \mathcal{F}') \xrightarrow{\alpha} \mathcal{O}_n(G; \mathcal{F}) \xrightarrow{\beta} \mathcal{O}_n(G; \mathcal{F}, \mathcal{F}') \xrightarrow{\partial} \mathcal{O}_{n-1}(G; \mathcal{F}') \xrightarrow{\alpha} .$$

1.3. In particular, set

$$\Omega_n(G) = \mathcal{O}_n(G; \{\{1\}\}),$$

$$SF_n(G) = \mathcal{O}_n(G; \{G, \{1\}\})$$

and call the bordism group of principal oriented G-manifolds ([**2**], p. 50) and of semi-free oriented G-manifolds [**7**] respectively.

§ 2. The bordism of G-vector bundles.

An oriented G-vector bundle (φ, ξ) over a compact manifold consists of a smooth vector bundle $\xi: E(\xi) \rightarrow B(\xi)$ over a compact smooth manifold $B(\xi)$ whose total space $E(\xi)$ is oriented and an orientation preserving smooth G-action $\varphi: G \times E(\xi) \rightarrow E(\xi)$ as a group of bundle maps. Such an oriented G-vector bundle (φ, ξ) is called to be of dimension (n, k) if the dimension of $B(\xi)$ is n and the fiber dimension of ξ is k.

2.1. Let H be a closed normal subgroup of G and $\mathcal{F}$ a family of subgroups of G satisfying the following condition:

(*) if $K \in \mathcal{F}$, then $H \cap K$ is a proper subgroup of H.

An oriented G-vector bundle (φ, ξ) is of type $(H, \mathcal{F})$ if the following conditions are satisfied:

(1) if $x \in E(\xi)$ is a zero vector, then the isotropy subgroup $G_x = H$,

(2) if $x \in E(\xi)$ is not a zero vector, then the isotropy subgroup G_x is conjugate to a member of $\mathcal{F}$.

Given (φ, ξ), define $-(\varphi, \xi) = (\varphi, -\xi)$, with the structure precisely the same as (φ, ξ) except for orientation of $E(\xi)$. Also define $\partial(\varphi, \xi) = (\varphi, \xi \mid \partial B(\xi))$. Define (φ, ξ) and (φ', ξ') to be isomorphic if there exists an equivariant orientation preserving diffeomorphism of $E(\xi)$ onto $E(\xi')$ as a bundle map.

An oriented (n, k)-dimensional G-vector bundle (φ, ξ) over a closed manifold of type $(H, \mathcal{F})$ will be said to bord if there exists an oriented $(n+1, k)$-dimensional G-vector bundle (ψ, η) over a compact manifold of type $(H, \mathcal{F})$, such that $\partial(\psi, \eta)$ is isomorphic to (φ, ξ).

Also (φ_1, ξ_1) is bordant to (φ_2, ξ_2) if the disjoint union $(\varphi_1, \xi_1) + (\varphi_2, -\xi_2)$ bords. The bordism is an equivalence relation on the class of oriented (n, k)-dimensional G-vector bundles over a closed manifold of type $(H, \mathcal{F})$. Denote by $[\varphi, \xi]$ the bordism class represented by (φ, ξ).

The bordism classes constitute an abelian group $B_n^k(G; H, \mathcal{F})$ under the operation of disjoint union. Set

$$\mathcal{M}_s(G; H, \mathcal{F}) = \bigoplus_{k+n=s} B_n^k(G; H, \mathcal{F}).$$

The direct sum

$$B^k(G; H, \mathcal{F}) = \bigoplus_n B_n^k(G; H, \mathcal{F})$$

and

$$\mathcal{M}_*(G; H, \mathcal{F}) = \bigoplus_s \mathcal{M}_s(G; H, \mathcal{F})$$

are naturally Ω-modules.

2.2. Let (φ, M) be a compact oriented G-manifold which is $(\mathcal{F} \cup \{H\}, \mathcal{F})$-free. Then each connected component of the set of all point $x \in M$, whose isotropy subgroup is H, is a regularly embedded G-invariant submanifold without boundary in $M-\partial M$. Let $\nu(M)$ be a normal bundle of this submanifold in M, then $\nu(M)$ is an oriented G-vector bundle of type $(H, \mathcal{F})$ with respect to the induced G-action. This correspondence induces a homomorphism

$$\nu: \ \mathcal{O}_n(G\,; \mathcal{F} \cup \{H\}, \mathcal{F}) \longrightarrow \mathcal{M}_n(G\,; H, \mathcal{F})\,.$$

LEMMA 2.2. *Let H be a closed normal subgroup of G and $\mathcal{F}$ a family of subgroups of G satisfying* (*). *Then the above homomorphism ν is an isomorphism of graded Ω-modules.*

The proof is easy (cf. [3], Lemma 5.2).

§ 3. Periodic maps and circle actions.

We now begin the study of relation of periodic maps and circle actions. Let S^1 be the unit circle in the field of complex numbers and regard it a compact Lie group. Let $Z_m=\{t \in S^1 \mid t^m=1\}$ be the cyclic subgroup of order m.

3.1. Let $\mathcal{F}$ be a family of subgroups of S^1, denote by $\mathcal{F}_m$ the family of subgroups of Z_m given by

$$\mathcal{F}_m=\{Z_m \cap H \mid H \in \mathcal{F}\}\,.$$

Let (φ, M) be an oriented $(\mathcal{F}, \mathcal{F}')$-free S^1-manifold, then the restriction $(\varphi \mid Z_m, M)$ is an oriented $(\mathcal{F}_m, \mathcal{F}'_m)$-free Z_m-manifold and this restriction induces a homomorphism

$$\boldsymbol{R}: \ \mathcal{O}_n(S^1\,; \mathcal{F}, \mathcal{F}') \longrightarrow \mathcal{O}_n(Z_m\,; \mathcal{F}_m, \mathcal{F}'_m)\,.$$

Then the following diagram is commutative:

$$\begin{array}{ccccccc}
\mathcal{O}_n(S^1\,; \mathcal{F}') & \xrightarrow{\alpha} & \mathcal{O}_n(S^1\,; \mathcal{F}) & \xrightarrow{\beta} & \mathcal{O}_n(S^1\,; \mathcal{F}, \mathcal{F}') & \xrightarrow{\partial} & \mathcal{O}_{n-1}(S^1\,; \mathcal{F}') \\
\downarrow \boldsymbol{R} & & \downarrow \boldsymbol{R} & & \downarrow \boldsymbol{R} & & \downarrow \boldsymbol{R} \\
\mathcal{O}_n(Z_m\,; \mathcal{F}'_m) & \xrightarrow{\alpha} & \mathcal{O}_n(Z_m\,; \mathcal{F}_m) & \xrightarrow{\beta} & \mathcal{O}_n(Z_m\,; \mathcal{F}_m, \mathcal{F}'_m) & \xrightarrow{\partial} & \mathcal{O}_{n-1}(Z_m\,; \mathcal{F}'_m)\,.
\end{array}$$

3.2. Given an oriented Z_m-manifold (φ, M), consider a cartesian product $S^1 \times M$ and let Z_m act on $S^1 \times M$ by

$$t \cdot (z, x)=(zt^{-1}, \varphi(t, x)) \qquad \text{for} \quad t \in Z_m\,.$$

Denote by $S^1 \times_{Z_m} M$ the orbit manifold and by $[z, x]$ the point of $S^1 \times_{Z_m} M$ represented by (z, x) of $S^1 \times M$. There is a circle action $\bar{\varphi}$ on $S^1 \times_{Z_m} M$ given by

$$\bar{\varphi}(\lambda, [z, x])=[\lambda z, x]\,.$$

If (φ, M) is $(\mathcal{F}, \mathcal{F}')$-free Z_m-manifold, then $(\bar{\varphi}, S^1 \times_{Z_m} M)$ is $(\mathcal{F}, \mathcal{F}')$-free S^1-manifold and this induces an extension homomorphism

$$E: \mathcal{O}_n(Z_m; \mathcal{F}, \mathcal{F}') \longrightarrow \mathcal{O}_{n+1}(S^1; \mathcal{F}, \mathcal{F}').$$

Then the following diagram is commutative:

$$\begin{array}{ccccccc} \mathcal{O}_n(Z_m; \mathcal{F}') & \xrightarrow{\alpha} & \mathcal{O}_n(Z_m; \mathcal{F}) & \xrightarrow{\beta} & \mathcal{O}_n(Z_m; \mathcal{F}, \mathcal{F}') & \xrightarrow{\partial} & \mathcal{O}_{n-1}(Z_m; \mathcal{F}') \\ \downarrow E & & \downarrow E & & \downarrow E & & \downarrow E \\ \mathcal{O}_{n+1}(S^1; \mathcal{F}') & \xrightarrow{\alpha} & \mathcal{O}_{n+1}(S^1; \mathcal{F}) & \xrightarrow{\beta} & \mathcal{O}_{n+1}(S^1; \mathcal{F}, \mathcal{F}') & \xrightarrow{\partial} & \mathcal{O}_n(S^1; \mathcal{F}'). \end{array}$$

3.3. Set $QF_n(m) = \mathcal{O}_n(S^1; \{Z_m, \{1\}\})$, and call the bordism group of quasi-free circle actions of type (m). Then for example the following sequences are exact:

$$\longrightarrow \Omega_n(S^1) \xrightarrow{\alpha} QF_n(m) \xrightarrow{\beta} \mathcal{M}_n(S^1; Z_m, \{\{1\}\}) \xrightarrow{\partial} \Omega_{n-1}(S^1) \xrightarrow{\alpha},$$

$$\longrightarrow \Omega_n(S^1) \xrightarrow{\alpha} SF_n(S^1) \xrightarrow{\beta} \mathcal{M}_n(S^1; S^1, \{\{1\}\}) \xrightarrow{\partial} \Omega_{n-1}(S^1) \xrightarrow{\alpha},$$

$$\longrightarrow \Omega_n(Z_m) \xrightarrow{\alpha} SF_n(Z_m) \xrightarrow{\beta} \mathcal{M}_n(Z_m; Z_m, \{\{1\}\}) \xrightarrow{\partial} \Omega_{n-1}(Z_m) \xrightarrow{\alpha}$$

by (1.2) and Lemma 2.2.

§ 4. Vector bundles with circle action.

Throughout the rest of this paper, let $m = 2k+1$ be a fixed odd integer, $\zeta = \exp(2\pi\sqrt{-1}/m)$ and $P(k)$ be the set of all positive integers mutually prime to $2k+1$ and smaller than or equal to k.

4.1. First we consider oriented S^1-vector bundles of type $(Z_m, \{\{1\}\})$. Let $\xi: E(\xi) \to X$ be a smooth vector bundle with smooth circle action $\varphi: S^1 \times E(\xi) \to E(\xi)$ as group of bundle maps such that

(1) if $v \in E(\xi)$ is a zero vector, then the isotropy subgroup at v is Z_m,

(2) if $v \in E(\xi)$ is not a zero vector, then the isotropy subgroup at v is $\{1\}$, the identity subgroup.

Let $T: E(\xi) \to E(\xi)$ be a diffeomorphism given by

$$T(v) = \varphi(\zeta, v).$$

Then,

LEMMA 4.1. *There is a unique complex vector bundle structure J on ξ and there are linear subbundles $\xi_s: E(\xi_s) \to X$ of ξ for $s \in P(k)$ such that*

(a) $J(E(\xi_s)) \subset E(\xi_s)$,

(b) $T(v) = \cos\left(\frac{2s\pi}{m}\right)v + \sin\left(\frac{2s\pi}{m}\right)J(v)$ *for* $v \in E(\xi_s)$,

(c) ξ *is the Whitney sum of* ξ_s,

(d) J *is compatible with the circle action* φ, *i.e.*

$$\varphi(z, J(v)) = J(\varphi(z, v)) \text{ for } z \in S^1 \text{ and } v \in E(\xi).$$

PROOF. By Theorems (38.3) and (38.4) of [2] and by Lemma 1.6.4 of [1], there is a unique complex vector bundle structure $\boldsymbol{J}$ on ξ and there are linear subbundles $\xi_s: E(\xi_s)\to X$ of ξ for $1\leqq s\leqq k$ satisfying the above conditions (a), (b) and (c). If $s\notin P(k)$, then ξ_s is a zero vector bundle by the above property (2). Finally, let $\boldsymbol{J}_z(v)=\varphi(z^{-1}, \boldsymbol{J}(\varphi(z, v)))$ for $z\in S^1$, then $\boldsymbol{J}_z$ is a complex vector bundle structure on ξ satisfying (a), (b) and (c). Thus $\boldsymbol{J}_z=\boldsymbol{J}$ by the uniqueness of a complex vector bundle structure and the condition (d) is satisfied. q. e. d.

By this lemma, the base manifold X is canonically oriented so that the orientation of the total space $E(\xi)$ is given by the complex vector bundle structure of ξ and the orientation of X.

4.2. Let $\sigma: S^1\times E(\xi)\to E(\xi)$ be a smooth circle action given by

$$\sigma(\exp\sqrt{-1}\,\theta, v)=(\cos s\theta)v+(\sin s\theta)\boldsymbol{J}(v)$$

for $v\in E(\xi_s)$. Then the circle actions φ and σ are commutative, and let $\Psi: S^1\times E(\xi)\to E(\xi)$ be a smooth map given by

$$\Psi(\exp\sqrt{-1}\,\theta, v)=\sigma(\exp(-\sqrt{-1}\,\theta/m), \varphi(\exp(\sqrt{-1}\,\theta/m), v))$$

for $0\leqq\theta\leqq 2\pi$ and $v\in E(\xi)$. Then Ψ is a principal circle action on $E(\xi)$ by definition of σ and Ψ.

Ψ is compatible with the complex vector bundle structure $\boldsymbol{J}$ on ξ and $\Psi(S^1\times E(\xi_s))\subset E(\xi_s)$. Moreover Ψ induces a principal circle action ψ on the base space X. Considering Ψ-orbit space of $E(\xi)$, we have a complex vector bundle $\bar{\xi}$ over ψ-orbit space $\bar{X}$ of X. Consequently we have the following result by the usual way (cf. [2], § 38; [3], 7.4).

LEMMA 4.2.

(a) $B_n^{2t}(S^1; Z_m, \{\{1\}\})\cong\bigoplus\Omega_{n-1}(\boldsymbol{B}S^1\times\boldsymbol{B}U(r_1)\times\boldsymbol{B}U(r_2)\times\cdots\times\boldsymbol{B}U(r_k))$, *where the sum is taken over all sequences* $(r_1, r_2, \cdots, r_k)$ *with sum* t *and* $r_s=0$ *if* $s\notin P(k)$,

(b) $B_n^{2t+1}(S^1; Z_m, \{\{1\}\})=0$,

(c) $B_n^{2t}(Z_m; Z_m, \{\{1\}\})\cong\bigoplus\Omega_n(\boldsymbol{B}U(r_1)\times\boldsymbol{B}U(r_2)\times\cdots\times\boldsymbol{B}U(r_k))$, *where the sum is taken over all sequences* $(r_1, r_2, \cdots, r_k)$ *with sum* t *and* $r_s=0$ *if* $s\notin P(k)$,

(d) $B_n^{2t+1}(Z_m; Z_m, \{\{1\}\})=0$.

4.3. By the isomorphisms of Lemma 2.2 and Lemma 4.2, we have isomorphisms of Ω-modules

$$\theta: \Omega_*(\boldsymbol{B}S^1)\otimes_{\Omega}(\bigoplus\Omega_*(\boldsymbol{B}U(r_1)\times\cdots\times\boldsymbol{B}U(r_k)))\cong\mathcal{O}_*(S^1; \{Z_m, \{1\}\}, \{\{1\}\}),$$

$$\theta: \bigoplus\Omega_*(\boldsymbol{B}U(r_1)\times\cdots\times\boldsymbol{B}U(r_k))\cong\mathcal{O}_*(Z_m; \{Z_m, \{1\}\}, \{\{1\}\}),$$

where the sums are taken over all sequences $(r_1, r_2, \cdots, r_k)$ with $r_s=0$ if $s\notin P(k)$.

We now interpret these isomorphisms for the later use. Let $\varphi: S^1\times M\to M$ be a principal smooth circle action on an oriented closed manifold M and $\xi: E(\xi)\to X$ be a smooth complex vector bundle over an oriented closed manifold X. For $s\in P(k)$ let $\xi_s: E(\xi_s)\to X$ be a complex vector subbundle of ξ and assume $\xi=\oplus\xi_s$. Let $D(\xi)$ be a total space of associated disk bundle of ξ and $\psi: S^1\times D(\xi)\to D(\xi)$ be a smooth circle action given by

$$\psi(z, v)=z^s v \qquad \text{for} \quad z\in S^1 \text{ and } v\in E(\xi_s)\cap D(\xi).$$

Now we obtain a smooth circle action Φ on $M\times D(\xi)$ given by

$$\Phi(z, (x, v))=(\varphi(z^m, x), \psi(z, v)) \qquad \text{for} \quad z\in S^1,\ x\in M \text{ and } v\in D(\xi).$$

Then the first isomorphism θ takes $[\varphi, M]\otimes[\xi]$ into $[\Phi, M\times D(\xi)]$ and the second isomorphism θ takes $[\xi]$ into $[\psi| Z_m, D(\xi)]$.

4.4. Next we consider the following diagram:

$$\begin{array}{ccc}
\oplus\Omega_*(BU(r_1)\times\cdots\times BU(r_k)) & \xrightarrow{\theta} & \mathcal{O}_*(Z_m;\{Z_m,\{1\}\},\{\{1\}\}) \\
 & & \downarrow E_1 \\
\Omega_*(BS^1)\otimes_{\Omega}(\oplus\Omega_*(BU(r_1)\times\cdots\times BU(r_k))) & \xrightarrow{\theta} & \mathcal{O}_*(S^1;\ \{Z_m,\{1\}\},\{\{1\}\}) \\
 & & \downarrow R_1 \\
\oplus\Omega_*(BU(r_1)\times\cdots\times BU(r_k)) & \xrightarrow{\theta} & \mathcal{O}_*(Z_m;\{Z_m,\{1\}\},\{\{1\}\}).
\end{array}$$

By definition of the restriction homomorphism R_1 and the extension homomorphism E_1 and by the interpretation of θ,

$$\theta^{-1}R_1\theta([\varphi, M]\otimes[\xi])=[M]\cdot[\xi],$$
$$\theta^{-1}E_1\theta([\xi])=[\mu, S^1]\otimes[\xi]$$

where $[M]$ is the oriented bordism class in Ω represented by M and $[\mu, S^1]$ is the bordism class in $\Omega_*(BS^1)$ represented by the left translation $\mu: S^1\times S^1\to S^1$.

Since M admits a principal circle action φ, $[M]=0$ in Ω and therefore R_1 is a null-homomorphism. On the other hand, $\Omega_*(BS^1)$ is a free Ω-module and $[\mu, S^1]$ is a member of standard Ω-basis of $\Omega_*(BS^1)$, thus E_1 is a monomorphism.

§5. Periodic maps and quasi-free circle actions.

Let $\tilde{\Omega}_n(Z_m)$ be a subgroup of $\Omega_n(Z_m)$ generated by the all bordism classes of principal Z_m-action (φ, M) such that $[M]=0$ in Ω_n, then

$$\Omega_n(Z_m)=\Omega_n\cdot[\mu, Z_m]\oplus\tilde{\Omega}_n(Z_m)$$

where $\mu: Z_m\times Z_m\to Z_m$ is a left translation (cf. [2], p. 90).

Let $\mu: S^1\times S^{2n+1}\to S^{2n+1}$ be a principal circle action given by

$$\mu(\lambda, (z_0, z_1, \cdots, z_n)) = (\lambda z_0, \lambda z_1, \cdots, \lambda z_n)$$

in complex coordinates. Then $\{[\mu, S^{2n+1}], n=0, 1, 2, \cdots\}$ is an Ω-base of the free Ω-module $\Omega_*(S^1)$ and $\{[\mu | Z_m, S^{2n+1}], n=0, 1, 2, \cdots\}$ generates the Ω-module $\tilde{\Omega}_*(Z_m)$.

5.1. By the same argument as in ([2], § 34), we have the following result.

LEMMA 5.1. *Let m be an odd integer, then*

(1) *the abelian group* $\tilde{\Omega}_n(Z_m)$ *is* 0 *for n even, and of order* m^t *for n odd where* $t = \text{rank}\,\Omega_0 + \cdots + \text{rank}\,\Omega_n$,

(2) *the sequence*

$$0 \longrightarrow \Omega_n \xrightarrow{\times[\mu, Z_m]} SF_n(Z_m) \longrightarrow \mathcal{O}_n(Z_m\,;\{Z_m, \{1\}\}, \{\{1\}\}) \longrightarrow \tilde{\Omega}_{n-1}(Z_m) \longrightarrow 0$$

is exact.

5.2. We now consider the restriction homomorphism

$$\boldsymbol{R}:\quad QF_n(m) \longrightarrow SF_n(Z_m)\,.$$

LEMMA. *Let* $[\varphi, M] \in QF_n(m)$, *then*

(1) $[M] \in \text{Tor}\,\Omega_n$, *the torsion subgroup of* Ω_n,

(2) $\boldsymbol{R}([\varphi, M]) = [M]\cdot[\mu, Z_m]$.

PROOF. Let $p > m$ be any odd prime, then the restriction $\varphi | Z_p$ gives a principal Z_p-action on M. Therefore $[M] \in p\cdot\Omega_n$ by Theorem 19.4 [2]. Consequently $[M] \in \text{Tor}\,\Omega_n$. On the other hand the image of $\boldsymbol{R}$ is contained in the kernel of β in the following diagram:

$$\begin{array}{ccccc} & & QF_n(m) & \longrightarrow & \mathcal{O}_n(S^1\,;\{Z_m, \{1\}\}, \{\{1\}\}) \\ & & \downarrow \boldsymbol{R} & & \downarrow \boldsymbol{R}_1 \\ \Omega_n & \xrightarrow{\times[\mu, Z_m]} & SF_n(Z_m) & \xrightarrow{\beta} & \mathcal{O}_n(Z_m\,;\{Z_m, \{1\}\}, \{\{1\}\})\,, \end{array}$$

since $\boldsymbol{R}_1$ is a null-homomorphism by 4.4. Thus there exists an element $[M_1] \in \Omega_n$ such that

$$[M_1]\cdot[\mu, Z_m] = \boldsymbol{R}([\varphi, M])\,.$$

Forgetting the Z_m-actions, we have

$$m\cdot[M_1] = [M] \in \text{Tor}\,\Omega_n\,.$$

Therefore $[M_1] = [M]$, since m is odd and each element of $\text{Tor}\,\Omega_n$ is of order 2. q. e. d.

THEOREM 5.2. *Let m be an odd integer, then the restriction homomorphism*

$$\boldsymbol{R}:\quad QF_n(m) \longrightarrow SF_n(Z_m)$$

is a null-homomorphism.

PROOF. This follows from the above lemma (2) and the fact that if a closed oriented smooth manifold M admits a stationary point free circle action with no isotropy subgroups of even order then $[M]=0$ in Ω ([5], p. 48).

q. e. d.

5.3. We now consider the forgetting homomorphism

$$\alpha: \quad \Omega_n(S^1) \longrightarrow QF_n(m).$$

LEMMA 5.3. *Let m be an odd integer, then the forgetting homomorphism*

$$\alpha: \quad \Omega_n(S^1) \longrightarrow QF_n(m)$$

is injective and its image is a direct summand as Ω-module.

PROOF. On his notation ([5], Lemma 2), there is a commutative diagram:

$$\begin{array}{ccc} & \Omega_n(S^1) & \\ {}^{\alpha}\swarrow & & \searrow^{i} \\ QF_n(m) & \longrightarrow & \bar{O}_n(m) \end{array}$$

where i is injective and its image is a direct summand as Ω-module. This assures the result.

q. e. d.

COROLLARY. *Let $m=2k+1$ be an odd integer, then there is an isomorphism of Ω-modules*

$$QF_*(m) \cong \Omega_*(BS^1) \oplus (\oplus\, \Omega_*(BS^1 \times BU(r_1) \times BU(r_2) \times \cdots \times BU(r_k))),$$

where the sum is taken over all sequences $(r_1, r_2, \cdots, r_k)$ with $r_s=0$ if $s \notin P(k)$.

5.4. Next we consider the extension homomorphism

$$\boldsymbol{E}: \quad SF_n(Z_m) \longrightarrow QF_{n+1}(m).$$

THEOREM 5.4. *Let m be an odd integer, then the extension homomorphism*

$$\boldsymbol{E}: \quad SF_n(Z_m) \longrightarrow QF_{n+1}(m)$$

is injective.

PROOF. In the following commutative diagram:

$$\begin{array}{ccccc} \Omega_n & \xrightarrow{\times[\mu,\, Z_m]} & SF_n(Z_m) & \longrightarrow & \mathcal{O}_n(Z_m\,;\{Z_m, \{1\}\}, \{\{1\}\}) \\ \downarrow{\scriptstyle \times[\mu,\, S^1]} & & \downarrow{\scriptstyle \boldsymbol{E}} & & \downarrow{\scriptstyle \boldsymbol{E}_1} \\ \Omega_{n+1}(S^1) & \xrightarrow{\alpha} & QF_{n+1}(m) & \longrightarrow & \mathcal{O}_{n+1}(S^1\,;\{Z_m, \{1\}\}, \{\{1\}\}), \end{array}$$

the homomorphisms $\times[\mu, Z_m]$ and $\times[\mu, S^1]$ are injective and $\boldsymbol{E}_1$ is injective by 4.4 and α is injective by Lemma 5.3. Moreover the horizontal lines are exact. Therefore $\boldsymbol{E}$ is injective.

q. e. d.

COROLLARY. *$\boldsymbol{E}$: $SF_n(Z_m) \rightarrow QF_{n+1}(m)$ is an isomorphism if and only if $n=0, 1, 3$ and 5.*

PROOF. If $\boldsymbol{E}$ is an isomorphism, then $\times[\mu, S^1]$ is an isomorphism in the above diagram. Therefore $\Omega_{n-2i}=0$ for any $i>0$, from the Ω-module structure of $\Omega_*(S^1)$, and this is only the case for $n=0, 1, 3$ and 5. On the other hand $\boldsymbol{E}$ is an isomorphism for $n=0, 1, 3$ and 5 by direct calculation. We leave it to the reader. q. e. d.

§ 6. Periodic maps and semi-free circle actions.

We have considered the bordism group of semi-free circle actions in [7] and obtained the following result.

LEMMA. *The sequence*

$$0 \longrightarrow SF_n(S^1) \xrightarrow{\beta} \mathcal{O}_n(S^1; \{S^1, \{1\}\}, \{\{1\}\}) \xrightarrow{\partial} \Omega_{n-1}(S^1) \longrightarrow 0$$

is exact and there is an isomorphism of Ω-modules

$$\theta: \bigoplus_r \Omega_*(\boldsymbol{BU}(r)) \longrightarrow \mathcal{O}_n(S^1; \{S^1, \{1\}\}, \{\{1\}\})$$

which takes $[\xi]$ into $[\phi, D(\xi)]$ the bordism class of scalar multiplication on the associated disk bundle.

THEOREM 6.1. *Let m be an odd integer, then the restriction homomorphism*

$$\boldsymbol{R}: SF_n(S^1) \longrightarrow SF_n(Z_m)$$

is injective.

PROOF. In the following commutative diagram:

$$\begin{array}{ccccc}
SF_n(S^1) & \xrightarrow{\beta} & \mathcal{O}_n(S^1; \{S^1, \{1\}\}, \{\{1\}\}) & \xleftarrow[\cong]{\theta} & \bigoplus \Omega_*(\boldsymbol{BU}(r)) \\
\downarrow \boldsymbol{R} & & \downarrow \boldsymbol{R}_1 & & \\
SF_n(Z_m) & \xrightarrow{\beta} & \mathcal{O}_n(Z_m; \{Z_m, \{1\}\}, \{\{1\}\}) & \xleftarrow[\cong]{\theta} & \bigoplus \Omega_*(\boldsymbol{BU}(r_1)\times \cdots \times \boldsymbol{BU}(r_k))
\end{array}$$

where $m=2k+1$, the composition $\theta^{-1}\boldsymbol{R}_1\theta$ is injective by the interpretations of θ and $\boldsymbol{R}_1$. In fact $\theta^{-1}\boldsymbol{R}_1\theta$ is the induced homomorphism by the inclusion of $\boldsymbol{BU}(r)$ into the first factor of $\boldsymbol{BU}(r_1)\times \cdots \times \boldsymbol{BU}(r_k)$. So $\beta\boldsymbol{R}=\boldsymbol{R}_1\beta$ is injective by the above lemma. Therefore $\boldsymbol{R}$ is injective. q. e. d.

§ 7. Principal circle action on a product of spheres.

Let $\mu: S^1\times S^{2m+1}\times S^{2n+1} \rightarrow S^{2m+1}\times S^{2n+1}$ be a smooth circle action on $S^{2m+1}\times S^{2n+1}$ given by

$$\mu(\lambda, (u_0, \cdots, u_m), (v_0, \cdots, v_n))=((\lambda^a u_0, \cdots, \lambda^a u_m), (\lambda^b v_0, \cdots, \lambda^b v_n))$$

in complex coordinates where a, b are integers. In particular, if a is prime to b, then this is a principal circle action. And this represents a null class in $\Omega_*(S^1)$ by ([5], Lemma 2).

In this section we will state more general result.

LEMMA 7.1. *Let m, n be positive integers and let*

$$\pi:\ S^{2m+1}\times S^{2n+1}\longrightarrow M$$

be a principal S^1-bundle over a closed smooth manifold M. Then

$$H^*(M;\boldsymbol{Z}_2)\cong\wedge(x)\otimes\boldsymbol{Z}_2[c]/(c^{k+1})$$

as a ring over $\boldsymbol{Z}_2$, where c is a modulo 2 reduction of the first Chern class of the principal S^1-bundle π, $k=m$ or $k=n$, $\deg x=2(m+n-k)+1$ *and* $S_q^1c=0$.

PROOF. This follows, by direct calculation, from Poincaré duality of M and Thom-Gysin sequence ([4], Theorem 21). We leave it to the reader. q. e. d.

LEMMA 7.2. *Let M be the same as in Lemma 7.1. Then each odd dimensional Stiefel-Whitney class of M vanishes.*

PROOF. Let $V_i\in H^i(M;\boldsymbol{Z}_2)$ be a class characterized by equation

$$S_q^i\alpha=V_i\cup\alpha\qquad\text{for all}\quad\alpha\in H^{\dim M-i}(M;\boldsymbol{Z}_2),$$

and let $V=V_0+V_1+\cdots+V_i+\cdots$, then $S_qV=W(M)$, the total Stiefel-Whitney class of M by Wu's formula ([4], Theorem 17).

Since $S_q^1c=0$, we have $S_q^{2s+1}(c^{m+n-s})=0$ by the property of Steenrod operations ([6], Lemma 2.5 of Ch. I). Therefore

$$V_{2s+1}=0\,,\quad V_{2s}=a_sc^s\qquad(a_s=0\ \text{or}\ a_s=1)$$

by the ring structure of $H^*(M;\boldsymbol{Z}_2)$. Consequently,

$$\begin{aligned}W(M)&=\sum_{s,t}a_sS_q^t(c^s)\\&=\sum_{s,t}a_sS_q^{2t}(c^s)\\&=\sum_{s,t}a_s\binom{s}{t}c^{s+t}\,.\end{aligned}$$

This shows that $W_{2s+1}(M)=0$ for any s. q. e. d.

THEOREM 7.3. *Let m, n be positive integers, then any principal smooth circle action on $S^{2m+1}\times S^{2n+1}$ bords in $\Omega_*(S^1)$.*

PROOF. The orbit manifold M is odd dimensional and each odd dimensional Stiefel-Whitney class of M vanishes by Lemma 7.2. Hence all bordism Stiefel-Whitney numbers of this principal circle action vanish. And all bordism Pontrjagin numbers also vanish. Therefore this principal circle action bords by Theorem 17.5 of [2]. q. e. d.

Department of Mathematics
Osaka University

References

[1] M. F. Atiyah, *K*-theory, Benjamin, 1967.
[2] P. E. Conner and E. E. Floyd, Differentiable Periodic Maps, Springer-Verlag, 1964.
[3] P. E. Conner and E. E. Floyd, Maps of odd period, Ann. of Math., **84** (1966), 132–156.
[4] J. Milnor, Lectures on Characteristic Classes, 1957.
[5] E. Ossa, Fixpunktfreie S^1-Aktionen, Math. Ann., **186** (1970), 45–52.
[6] N. E. Steenrod, Cohomology Operations, Princeton, 1962.
[7] F. Uchida, Cobordism groups of semi-free S^1- and S^3-actions, Osaka J. Math., **7** (1970), 345–351.