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Some potential theory of processes with stationary independent increments by means of the Schwartz distribution theory

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§ 0. Introduction.

In this paper we will be primarily concerned with the most general processes with stationary independent increments on the real line $\boldsymbol{R}$. All the results are valid for the higher dimensional such processes without change. It is only for saving the notation that we restrict ourselves to the one-dimensional processes.

We will summarize the contents of the paper with the main results being picked up in (A) to (D).

§ 1 through § 3 are of quite analytic character. Let $(\mu_t)_{t\geqq 0}$ be a convolution semi-group of probability measures on $\boldsymbol{R}$. Let $(P_t)_{t\geqq 0}$ be the semi-group of Markov kernels defined by $P_t f=\int f(x+y)\mu_t(dy)$ and $(U_\lambda)_{\lambda>0}$, the resolvent of (P_t). $\mathcal{C}_0$ stands for the space of continuous functions vanishing at infinity and $(\mathcal{D}'_{L^p})$, $1\leqq p\leqq\infty$, the spaces of distributions introduced by L. Schwartz [10; Chap. VI, § 8]. It has been known that, for every $f\in\mathcal{C}_0^2=\{f\in\mathcal{C}_0\,;\,f',f''\in\mathcal{C}_0\}$, the uniform limit of $t^{-1}[P_t f-f]$ as $t\to 0$ is given by

$$(0.1)\qquad Af(x):=af'(x)+\frac{\sigma^2}{2}f''(x)+\int_{\boldsymbol{R}\setminus\{0\}}\Big[f(x+y)-f(x)-\frac{y}{1+y^2}f'(x)\Big]\nu(dy),$$

where $a\in\boldsymbol{R}$, $\sigma^2\geqq 0$ and ν is the so-called Lévy measure. L. Schwartz's basic results on the spaces $(\mathcal{D}'_{L^p})$ make it possible to extend the operators P_t, U_λ and A to those on the space of "bounded distributions" $(\mathcal{D}'_{L^\infty})$. Hence, for example, if f is a bounded function, Af is well-defined and belongs to $(\mathcal{D}'_{L^\infty})$. We see that Af is of the form $\tilde{A}^0*f$ by means of an element $\tilde{A}^0\in(\mathcal{D}'_{L^1})$, where "$*$" means the convolution. It should be noted that C. S. Herz [5] called this distribution $\tilde{A}^0$ a *generalized Laplacian* and studied its structure in a little different context from ours.

The following theorem is fundamental throughout the paper and it is proved in § 2.

(A) For every $\lambda>0$ and $f\in(\mathcal{D}'_{L^\infty})$, $u=U_\lambda f$ is the unique solution in $(\mathcal{D}'_{L^\infty})$

of $(\lambda-A)u=f$.

It is well known that (P_t) defines a strongly continuous, contraction semi-group over the Banach space C_0. The next theorem in § 3 is an immediate consequence of (A).

(B) The infinitesimal generator of (P_t) over C_0 is exactly the operator A with the domain $D(A:C_0)=\{f\in C_0;\ Af\in C_0\}$.

Similar results are valid also for L^p for every $1\leqq p<\infty$ and even for the topological vector spaces $(\mathcal{D}'_{L^p})$, $1\leqq p\leqq\infty$. This theorem has been known for some special cases (see K. Itô [8] for Brownian motion and S. Watanabe [11] for stable process), but it is new in the general form in the circle of my knowledge.

§§ 4 and 5 are devoted to potential theory of the Markov process $(X(t))_{t\geqq 0}$ having (P_t) as its transition function. Let $(\tilde{\mu}_t)_{t\geqq 0}$ be the convolution semi-group obtained from (μ_t) by the reflection at the origin and $(\tilde{P}_t)_{t\geqq 0}$, its associated semi-group of Markov kernels. $(\tilde{U}_\lambda)_{\lambda>0}$ stands for the resolvent of $(\tilde{P}_t)$ and $(\tilde{X}(t))_{t\geqq 0}$, the Markov process having $(\tilde{P}_t)$ as its transition function. The semi-groups (P_t) and $(\tilde{P}_t)$ are in duality relative to the Lebesgue measure dx in the sense that

$$(P_t f, g)=(f, \tilde{P}_t g), \qquad f, g\geqq 0, \tag{0.2}$$

where $(f, g)=\int f(x)g(x)dx$.

Combining (A) and some general results on the dual processes, we prove the following theorem in § 4.

(C) Let u be a bounded, λ-excessive function $(\lambda>0)$ and let $\mu:=(\lambda-A)u$. Then μ is a positive measure and $u\cdot dx$ is the $\tilde{U}_\lambda$-potential of μ. Moreover, for every open set G,

$$H_G^\lambda u\cdot dx=\mu\tilde{H}_G^\lambda\tilde{U}_\lambda, \tag{0.3}$$

where H_G^λ[resp. $\tilde{H}_G^\lambda$] is the kernel of λ-order hitting measure of G with respect to the process $(X(t))$[resp. $(\tilde{X}(t))$].

Let G be an open subset of $\boldsymbol{R}$. A finite, λ-excessive function u is said to be *λ-harmonic on G*, if

$$u=H_{\complement K}^\lambda u \tag{0.4}$$

for every compact subset K of G, where $\complement K=\boldsymbol{R}\setminus K$. Obviously this definition makes sense for every standard process. A typical such function is the λ-order hitting probability function $H_{\complement B}^\lambda \mathbf{1}$ of the set $\complement B$ for every B such that the interior of B contains G. For the present case we can characterize a bounded, λ-harmonic function by means of the operator A as follows.

(D) Suppose that $(U_\lambda)_{\lambda>0}$ is absolutely continuous with respect to dx. Then, except for simultaneously $\lambda=0$ and $A=0$, a bounded, $\lambda(\geqq 0)$-excessive function u is $\lambda(\geqq 0)$-harmonic on an open set G if and only if

(0.5) $(\lambda - A)u = 0$ on G (in the sense of distribution).

The proof of (D) is given in § 5. This theorem or its variant has been known for various special processes and it was applied to study the properties of hitting probabilities. (The Brownian motion case is classical. See S. Watanabe [**11**] for stable process and H. Cramér [**3**], the author [**13**] for compound Poisson process.)

GENERAL NOTATION. $\mathscr{B}(\boldsymbol{R})$ is the σ-algebra of all Borel sets in $\boldsymbol{R} = (-\infty, +\infty)$. Every function is supposed to be Borel measurable and real-valued (allowing $\pm\infty$). Otherwise stated, a measure stands for a nonnegative, completely additive set function from $\mathscr{B}(\boldsymbol{R})$ into $\bar{\boldsymbol{R}}_+ = [0, +\infty]$. A *kernel* $K(x, E)$ is a function from $\boldsymbol{R} \times \mathscr{B}(\boldsymbol{R})$ into $\bar{\boldsymbol{R}}_+$ such that, for each $x \in \boldsymbol{R}$, $K(x, \cdot)$ is a measure and, for each $E \in \mathscr{B}(\boldsymbol{R})$, $K(\cdot, E)$ is a Borel measurable function. ε_x denotes the unit measure at x. The unit measure ε_0 at the origin is also denoted by δ. For a nonnegative function f, write $f \cdot dx$ for the measure $\mu(E) = \int_E f(x)dx$. I_E, $E \in \mathscr{B}(\boldsymbol{R})$, denotes the indicator of the set E.

The inner product (,), convolution $*$ and Fourier transform $\mathscr{F}(\cdot)$ are defined for various objects. Let f, g [resp. μ, ν] be functions [resp. measures]. Let T, S be Schwartz distributions and φ, a test function. Let K be a kernel. One defines

$$(f, g) = \int f(x)g(x)dx, \qquad (\mu, g) = \int \mu(dx)g(x),$$

$$(T, \varphi) = T(\varphi),$$

$$Kf(x) = \int K(x, dy)f(y), \qquad \mu K(E) = \int \mu(dx)K(x, E).$$

Note that (f, g) may be written as $(f \cdot dx, g)$. One also can write $\varepsilon_x K$ for $K(x, \cdot)$. Let us define

$$\tilde{f}(x) = f(-x), \qquad \tilde{\mu}(dx) = \mu(-dx), \qquad (\tilde{T}, \varphi) = (T, \tilde{\varphi}).$$

Furthermore one defines

$$f * g(x) = \int f(x-y)g(y)dy, \qquad \mu * f(x) = \int \mu(dy)f(x-y),$$

$$\mu * \nu(E) = \iint I_E(x+y)\mu(dx)\nu(dy),$$

$$T * \varphi(x) = (T_y, \varphi(x-y)), \qquad (T * S, \varphi) = (T, \tilde{S} * \varphi).$$

Finally the Fourier transform of f etc. is defined by

$$\mathscr{F}(f)(\xi) = \int_{\boldsymbol{R}} e^{ix\xi} f(x)dx \qquad \text{etc}.$$

Of course, (,), $*$ and $\mathscr{F}(\cdot)$ are defined only for reasonable objects.

§ 1. Preliminaries.

Let $(\mu_t)_{t\geqq 0}$ be a convolution semi-group of probability measures on $\boldsymbol{R}=(-\infty, +\infty)$;

$$\mu_t * \mu_s = \mu_{t+s} \quad (t, s \geqq 0), \qquad \mu_0 = \delta . \tag{1.1}$$

It is known that μ_t converges weakly* to $\mu_0=\delta$ as $t \to 0$. Let $(P_t)_{t\geqq 0}$ be the semi-group of Markov kernels defined by

$$P_t(x, B) = \varepsilon_x * \mu_t(B), \qquad B \in \mathcal{B}(\boldsymbol{R}) . \tag{1.2}$$

Note that

$$P_t f(x) = (\varepsilon_x P_t, f) = (\varepsilon_x * \mu_t, f) = (\varepsilon_x, \tilde{\mu}_t * f) = \tilde{\mu}_t * f(x) . \tag{1.3}$$

The resolvent of (P_t) is denoted by $(U_\lambda)_{\lambda>0}$;

$$\begin{aligned} U_\lambda(x, B) :&= \int_0^\infty e^{-\lambda t} P_t(x, B) dt = \varepsilon_x * \Big(\int_0^\infty e^{-\lambda t} \mu_t(B) dt \Big) \\ &= \varepsilon_x * U_\lambda^0(B), \qquad B \in \mathcal{B}(\boldsymbol{R}), \end{aligned} \tag{1.4}$$

where

$$U_\lambda^0(B) := \int_0^\infty e^{-\lambda t} \mu_t(B) dt = \varepsilon_0 U_\lambda(B) = U_\lambda(0, B) . \tag{1.5}$$

It is clear that $(\tilde{\mu}_t)_{t\geqq 0}$ defines also a convolution semi-group of probability measures. $(\tilde{P}_t)$ and $(\tilde{U}_\lambda)$ respectively denote the semi-group of Markov kernels and the resolvent associated with $(\tilde{\mu}_t)$. It follows that

$$(\tilde{U}_\lambda)^0(B) = U_\lambda^0(-B) = (U_\lambda^0)^\sim(B) . \tag{1.6}$$

As in (1.3) one sees that $U_\lambda f(x) = \tilde{U}_\lambda^0 * f(x)$. It is easy to see that $(\tilde{P}_t)$ [resp. $(\tilde{U}_\lambda)$] is the co-semi-group of (P_t) [resp. the co-resolvent of (U_λ)] with respect to the Lebesgue measure dx; for $f, g \geqq 0$,

$$(P_t f, g) = (\tilde{\mu}_t * f, g) = (f, \mu_t * g) = (f, \tilde{P}_t g), \tag{1.7}$$

$$(U_\lambda f, g) = (f, \tilde{U}_\lambda g) . \tag{1.8}$$

The Fourier transform (= characteristic function) of μ_t is of the Lévy-Khintchine canonical form;

$$\begin{aligned} \mathcal{F}(\mu_t)(\xi) &= \int_{-\infty}^\infty e^{ix\xi} \mu_t(dx) \\ &= \exp\Big[t\Big\{ ia\xi - \frac{\sigma^2}{2}\xi^2 + \int_{\boldsymbol{R}\setminus\{0\}} \Big(e^{i\xi y} - 1 - \frac{i\xi y}{1+y^2} \Big) \nu(dy) \Big\} \Big] \\ &= e^{-t\psi(\xi)}, \end{aligned} \tag{1.9}$$

$$\psi(\xi) = -ia\xi + \frac{\sigma^2}{2}\xi^2 + \int_{\boldsymbol{R}\setminus\{0\}} \Big(1 - e^{i\xi y} + \frac{i\xi y}{1+y^2} \Big) \nu(dy), \tag{1.10}$$

where $a \in \boldsymbol{R}$, $\sigma^2 \geqq 0$, and the so-called "Lévy measure" ν is a measure on $\boldsymbol{R} \setminus \{0\}$ such that

$$\int_{\boldsymbol{R}\setminus\{0\}} \frac{y^2}{1+y^2} \nu(dy) < \infty .$$

It then follows that

$$\mathcal{F}(U_\lambda^0) = \int_0^\infty e^{-\lambda t} \mathcal{F}(\mu_t) dt = \int_0^\infty e^{-\lambda t} e^{-t\psi(\xi)} dt = \frac{1}{\lambda + \psi(\xi)}, \qquad \lambda > 0 . \tag{1.11}$$

For a bounded function f with a second continuous derivative, define the operators A and $\tilde{A}$ by

$$Af(x) := af'(x) + \frac{\sigma^2}{2} f''(x) + \int_{\boldsymbol{R}\setminus\{0\}} \Big[f(x+y) - f(x) - \frac{y}{1+y^2} f'(x) \Big] \nu(dy) . \tag{1.12}$$

$$\tilde{A}f(x) := -af'(x) + \frac{\sigma^2}{2} f''(x) + \int_{\boldsymbol{R}\setminus\{0\}} \Big[f(x-y) - f(x) + \frac{y}{1+y^2} f'(x) \Big] \nu(dy) . \tag{1.13}$$

Since $\psi(\xi) \rightsquigarrow \psi(-\xi)$, $(a, \sigma^2, \nu) \rightsquigarrow (-a, \sigma^2, \tilde{\nu})$ as μ_t is transformed into $\tilde{\mu}_t$, $\tilde{A}$ corresponds to $(\tilde{\mu}_t)$ as well as A does to (μ_t). Note that $\tilde{A}$ is the formal adjoint (= co-operator) of A;

$$(Af, g) = (f, \tilde{A}g) . \tag{1.14}$$

§ 2. The equation $(\lambda - A)u = f$.

In this section we will extend the domain of A to the space $(\mathcal{D}'_{L^\infty})$ of bounded distributions and then solve the equation $(\lambda - A)u = f$ for every $f \in (\mathcal{D}'_{L^\infty})$ and $\lambda > 0$.

We start with a review of the theory of distributions. The definition of spaces L^p $(1 \leqq p \leqq \infty)$, $(\mathcal{D})$, $(\mathcal{D}')$, $(\mathcal{E})$, $(\mathcal{E}')$ is quite familiar and so it is omitted. The L^p-norm is denoted by $\|\cdot\|_p$. $\mathcal{B}^0$ stands for the space of bounded measurable functions[1], C_0 [resp. C_u] for the space of continuous functions with $f(\pm\infty) = 0$ [resp. the space of bounded, uniformly continuous functions] and $\mathcal{M}_b$ for the space of bounded signed measures. $(\mathcal{D}_{L^p})$, $1 \leqq p \leqq \infty$, denotes the space of C^∞-functions φ such that the n-th derivative $\varphi^{(n)}$ is in L^p for every $n \geqq 0$. A sequence $(\varphi_j)_{j \geqq 1}$ converges to φ in $(\mathcal{D}_{L^p})$ if $\|\varphi_j^{(n)} - \varphi^{(n)}\|_p \to 0$ for every $n \geqq 0$. $(\mathcal{D}_{L^\infty})$ is also denoted by $\mathcal{B}$. Space $\dot{\mathcal{B}}$ is defined by the subset of $\mathcal{B}$ consisting of those functions φ such that $\varphi^{(n)} \in C_0$ for every $n \geqq 0$. One now defines the spaces of distributions $(\mathcal{D}'_{L^p})$ by

$$\begin{aligned} (\mathcal{D}'_{L^p}) &:= (\mathcal{D}_{L^{p'}})' \quad \text{with} \quad \frac{1}{p'} = 1 - \frac{1}{p} \qquad \text{for} \quad 1 < p \leqq \infty , \\ &:= (\dot{\mathcal{B}})' \qquad \text{for} \quad p = 1 . \end{aligned} \tag{2.1}$$

1) $\mathcal{B}^0 = L^\infty$ as the space of distributions. We prefer to write $\mathcal{B}^0$ rather than L^∞ in most cases.

See Chap. VI, § 8 of [10] for the basic topological properties of $(\mathcal{D}_{L^p})$ and $(\mathcal{D}'_{L^p})$.

We collect those results borrowed from the book of L. Schwartz [10], mostly, Chap. VI, § 8.

LEMMA 2.1. (a) *If* $S\in(\mathcal{D}'_{L^p})$, $T\in(\mathcal{D}'_{L^q})$ *and* $r^{-1}=p^{-1}+q^{-1}-1\geqq 0$, *then* $S*T\in(\mathcal{D}'_{L^r})$. *The mapping* $(S, T)\to S*T$ *is continuous from* $(\mathcal{D}'_{L^p})\times(\mathcal{D}'_{L^q})$ *to* $(\mathcal{D}'_{L^r})$. *In particular, if* $S\in(\mathcal{D}'_{L^1})$ *and* $T\in(\mathcal{D}'_{L^\infty})$, *then* $S*T\in(\mathcal{D}'_{L^\infty})$.

(b) *If* $T\in(\mathcal{D}'_{L^p})$, $\varphi\in(\mathcal{D}_{L^q})$ *and* $r^{-1}=p^{-1}+q^{-1}-1\geqq 0$, *then* $T*\varphi\in(\mathcal{D}_{L^r})$. *The mappings* $T\to T*\varphi$ *and* $\varphi\to T*\varphi$ *are continuous from* $(\mathcal{D}'_{L^p})$ *and* $(\mathcal{D}_{L^q})$ *to* $(\mathcal{D}_{L^r})$, *respectively.*

(c) *If* $S\in(\mathcal{D}'_{L^p})$, $T\in(\mathcal{D}'_{L^q})$ *with* $1\leqq p, q\leqq 2$, *then* $\mathcal{F}(S*T)$ *is a function and it is the product of the two functions* $\mathcal{F}(S)$ *and* $\mathcal{F}(T)$; $\mathcal{F}(S*T)=\mathcal{F}(S)\cdot\mathcal{F}(T)$.

(d) *For* $1\leqq p\leqq q\leqq\infty$,

$$(\mathcal{D}'_{L^q})\supset(\mathcal{D}'_{L^p})\supset L^p\cup\mathcal{M}_b\cup(\mathcal{E}')\,. \tag{2.2}$$

For the proofs of (a), (b) and (c), see p. 203, p. 204 and p. 270 of [10]. The proof of (d) is quite easy.

Based on (1.14) one defines

$$(Af, \varphi):=(f, \tilde{A}\varphi) \quad \text{for } f\in(\mathcal{D}'_{L^\infty}) \text{ and } \varphi\in(\mathcal{D}_{L^1})\,. \tag{2.3}$$

It follows that $Af\in(\mathcal{D}'_{L^\infty})$. In fact it is not difficult to see that the relation "$\varphi_j\to\varphi$ in $(\mathcal{D}_{L^1})$" implies that $\tilde{A}\varphi_j\to\tilde{A}\varphi$ in $(\mathcal{D}_{L^1})$. However we will give an alternative justification which is more convenient for the present situation. Define the distributions A^0 and $\tilde{A}^0$ by

$$(A^0, \varphi):=A\varphi(0)\,, \qquad (\tilde{A}^0, \varphi):=\tilde{A}\varphi(0)\,. \tag{2.4}$$

Note that $(\tilde{A})^0=(A^0)^\sim$.

LEMMA 2.2. (a) A^0 *is a sum of* $A_1^0\in(\mathcal{E}')$ *and* $A_2^0\in\mathcal{M}_b$:

$$A^0=A_1^0+A_2^0\,. \tag{2.5}$$

A_2^0 *can be chosen as the restriction of* ν *over the set* $\{x;\,|x|\geqq 1\}$. *In particular,* $A^0\in(\mathcal{D}'_{L^1})$.

(b) *For every* $\varphi\in(\mathcal{D}_{L^1})$,

$$A\varphi(x)=\tilde{A}^0*\varphi(x)\in(\mathcal{D}_{L^1})\,. \tag{2.6}$$

(c) *For every* $f\in(\mathcal{D}'_{L^\infty})$,

$$Af=\tilde{A}^0*f\in(\mathcal{D}'_{L^\infty})\,. \tag{2.7}$$

(d)

$$\mathcal{F}(A^0)=-\psi(\xi)\,, \qquad \mathcal{F}(\tilde{A}^0)=-\psi(-\xi)\,. \tag{2.8}$$

PROOF. (a) The first two assertions are immediate from definition and then the third is obvious from Lemma 2.1 (d).

(b) By virtue of Lemma 2.1 (b), $\tilde{A}^0 * \varphi(x) \in (\mathcal{D}_{L^1})$. For a fixed x, set $\zeta(y) = \varphi(x-y)$. Since $\zeta'(0) = -\varphi'(x)$ and $\zeta''(0) = \varphi''(x)$, one has

$$\begin{aligned}\tilde{A}^0 * \varphi(x) &= (\tilde{A}^0_y, \varphi(x-y)) = \tilde{A}\zeta(0) \\ &= -a\zeta'(0) + \frac{\sigma^2}{2}\zeta''(0) + \int_{R\setminus\{0\}} \Big[\zeta(-y) - \zeta(0) + \frac{y}{1+y^2}\zeta'(0)\Big]\nu(dy) \\ &= a\varphi'(x) + \frac{\sigma^2}{2}\varphi''(x) + \int_{R\setminus\{0\}} \Big[\varphi(x+y) - \varphi(x) - \frac{y}{1+y^2}\varphi'(x)\Big]\nu(dy) \\ &= A\varphi(x)\,.\end{aligned}$$

(c) By virtue of (a) and Lemma 2.1 (a), $\tilde{A}^0 * f \in (\mathcal{D}'_{L^\infty})$ whenever $f \in (\mathcal{D}'_{L^\infty})$. For $\varphi \in (\mathcal{D}_{L^1})$ one has

$$\begin{aligned}(Af, \varphi) &= (f, \tilde{A}\varphi) = (f, A^0 * \varphi) \qquad \text{by (b)} \\ &= (\tilde{A}^0 * f, \varphi)\,.\end{aligned}$$

(d)

$$\begin{aligned}(\mathcal{F}A^0, \varphi) :&= (A^0, \mathcal{F}\varphi) = A(\mathcal{F}\varphi)(0) \\ &= a(\mathcal{F}\varphi)'(0) + \frac{\sigma^2}{2}(\mathcal{F}\varphi)''(0) \\ &\quad + \int_{R\setminus\{0\}} \Big[(\mathcal{F}\varphi)(y) - (\mathcal{F}\varphi)(0) - \frac{y}{1+y^2}(\mathcal{F}\varphi)'(0)\Big]\nu(dy) \\ &= a(ix, \varphi) + \frac{\sigma^2}{2}((ix)^2, \varphi) \\ &\quad + \int_{R\setminus\{0\}} \Big[(e^{ixy}, \varphi) - (1, \varphi) - \frac{y}{1+y^2}(ix, \varphi)\Big]\nu(dy) \\ &= (-\psi(x), \varphi)\,.\end{aligned}$$

In the same way as in A, one can extend the domain of P_t and U_λ to $(\mathcal{D}'_{L^\infty})$ because of (1.7) and (1.8). One gets

$$P_t f = \tilde{\mu}_t * f \in (\mathcal{D}'_{L^\infty})\,, \quad U_\lambda f = \tilde{U}^0_\lambda * f \in (\mathcal{D}'_{L^\infty}) \qquad \text{for } f \in (\mathcal{D}'_{L^\infty})\,. \tag{2.9}$$

REMARK 2.3. By virtue of Lemma 2.1, formulas (2.7) and (2.9), each of A, P_t and U_λ makes invariant each of the spaces $(\mathcal{D}_{L^p})$, $1 \leqq p < \infty$, $(\dot{\mathcal{B}})$ and $(\mathcal{D}'_{L^p})$, $1 \leqq p \leqq \infty$. Moreover, every P_t and U_λ makes invariant every L^p, $1 \leqq p \leqq \infty$ and C_0 (see Theorem 3.2).

The following theorem is fundamental throughout the paper.

THEOREM 2.4. *Let $f \in (\mathcal{D}'_{L^\infty})$. Then the equation*

$$(\lambda - A)u = f\,, \qquad \lambda > 0 \tag{2.10}$$

has the unique solution $u = U_\lambda f$ in $(\mathcal{D}'_{L^\infty})$.

PROOF. (a) Suppose first that $f \in (\mathcal{D}_{L^1})$. One claims that (2.10) has the unique solution $u = U_\lambda f$ in $(\mathcal{D}_{L^1})$. By the above remark, $U_\lambda f = \tilde{U}_\lambda^0 * f \in (\mathcal{D}_{L^1})$. By virtue of Lemma 2.1 (c), formulas (1.11) and (2.8) one gets

$$\mathcal{F}((\lambda - A)U_\lambda f) = \mathcal{F}((\lambda\delta - \tilde{A}^0) * \tilde{U}_\lambda^0 * f)$$
$$= (\lambda + \psi(-\xi)) \frac{1}{\lambda + \psi(-\xi)} \mathcal{F}(f) = \mathcal{F}(f),$$

so that $(\lambda - A)U_\lambda f = f$. If $(\lambda - A)u = 0$ for $u \in (\mathcal{D}_{L^1})$,

$$\mathcal{F}((\lambda - A)u) = (\lambda + \psi(-\xi))\mathcal{F}(u) = 0.$$

Since $\lambda + \psi(-\xi) \neq 0$ $(\lambda > 0, \operatorname{Re}\psi(-\xi) \geqq 0)$ it follows that $\mathcal{F}(u) = 0$ and, a fortiori, $u = 0$.

(b) Consider now the general case $f \in (\mathcal{D}'_{L^\infty})$. By Remark 2.3, $U_\lambda f = \tilde{U}_\lambda^0 * f \in (\mathcal{D}'_{L^\infty})$. Let $\varphi \in (\mathcal{D}_{L^1})$. Applying the result of part (a) to the " $\sim$ " system, one has

$$((\lambda - A)U_\lambda f, \varphi) = (f, \tilde{U}_\lambda(\lambda - \tilde{A})\varphi) = (f, \varphi),$$

so that $(\lambda - A)U_\lambda f = f$. Suppose that $(\lambda - A)u = 0$ for $u \in (\mathcal{D}'_{L^\infty})$. For any $\theta \in (\mathcal{D}_{L^1})$, set $\varphi = \tilde{U}_\lambda \theta \in (\mathcal{D}_{L^1})$. Then, $(\lambda - \tilde{A})\varphi = \theta$ by (a). Hence one has

$$(u, \theta) = (u, (\lambda - \tilde{A})\varphi) = ((\lambda - A)u, \varphi) = 0 \quad \text{by assumption},$$

which proves that $u = 0$.

§ 3. The infinitesimal generators.

Before giving the precise description of the infinitesimal generator of (P_t) we will prove a theorem of K. Itô [7][2] showing that the operator A is the generator in a rough sense.

THEOREM 3.1. *If f is a bounded function with a second continuous derivative, then*

(3.1) $\dfrac{P_t f - f}{t}$ *converges uniformly on every compact sets to Af as $t \to 0$.*

In particular, if $f \in C_0^2 = \{f \in C_0;\ f' \in C_0, f'' \in C_0\}$, then the convergence in (3.1) *is uniform on the whole space* $\boldsymbol{R}$.

PROOF. In Remark 3.3 we will give an operator-theoretical proof, making use of Theorem 3.2. We here present a proof based on those results in Feller's book [**4**; Chap. XVII] which are summarized below.

(a) When $t \to 0$, $t^{-1}y^2\mu_t(dy)$ converges *properly* to a canonical measure

2) See also G. Hunt [**6**], P. Courrège [**2**]. K. Yosida [**14**] had earlier proved the L_1-version of this theorem.

$M(dy)$. In other words, there is a Radon measure M such that $\int_{-\infty}^{\infty} \frac{M(dy)}{1+y^2} < \infty$, and one has

$$\lim_{t\to 0} t^{-1} \int f(y) y^2 \mu_t(dy) = \int f(y) M(dy) \tag{3.2}$$

for every continuous function f such that $y^2 f(y)$ is bounded. This implies that

$$t^{-1}\Big[\int_{|y|\geqq 1} \mu_t(dy) + \int_{|y|<1} y^2 \mu_t(dy)\Big] \tag{3.3}$$

is bounded in $t>0$, and

$$\lim_{\substack{t\to 0\\ N\to\infty}} t^{-1} \int_{|y|>N} \mu_t(dy) = 0\,. \tag{3.4}$$

(b) If $b_t = \int \frac{y}{1+y^2} \mu_t(dy)$, then $t^{-1} b_t$ converges to some constant b as $t\to 0$.

(c) The relation with the Lévy-Khintchine canonical form is the following;

$$\begin{cases} b = a\,, \qquad \sigma^2 = M(\{0\})\,, \\ \nu(dy) = y^2 M(dy) \qquad \text{for} \quad y \neq 0\,. \end{cases} \tag{3.5}$$

Let f be a bounded function such that f' and f'' are continuous. For each fixed x, set

$$\begin{aligned} g_x(y) &= \Big\{ f(x+y) - f(x) - \frac{y}{1+y^2} f'(x) \Big\} \times y^{-2} \qquad \text{for} \quad y \neq 0 \\ &= \frac{1}{2} f''(x) \qquad \text{for} \quad y = 0\,. \end{aligned}$$

It then follows that

$$\begin{aligned} t^{-1}[P_t f(x) - f(x)] &= t^{-1} \int_{-\infty}^{\infty} [f(x+y) - f(x)] \mu_t(dy) \\ &= b_t f'(x) + t^{-1}\Big[\int_{-\infty}^{\infty} g_x(y) y^2 \mu_t(dy)\Big] \\ &\to a f'(x) + \int_{-\infty}^{\infty} g_x(y) M(dy) \qquad \text{as} \quad t \to 0\,. \end{aligned} \tag{3.6}$$

We have to show that the above convergence is uniform on any compact set K. Take any $\varepsilon>0$ and choose $N>0$ and $\delta_1>0$ so that

$$t^{-1} \int_{|y|>N} \mu_t(dy) + \int_{|y|>N} \frac{M(dy)}{y^2} < \varepsilon \qquad \text{for every} \quad t < \delta_1\,. \tag{3.7}$$

Since the family of vector-valued y-functions $(y^2 g_x(y), g_x(y))$, $x \in K$, is uniformly bounded and equicontinuous on $|y| \leqq N$, it is totally bounded in uniform norm over the set $\{|y| \leqq N\}$. Therefore one can choose a finite subset $\{x_i\} \subset K$ such that, for each $x \in K$, there is some x_i satisfying

$$\sup_{|y|\leqq N} |y^2 g_x(y) - y^2 g_{x_i}(y)| + \sup_{|y|\leqq N} |g_x(y) - g_{x_i}(y)| < \varepsilon\,. \tag{3.8}$$

Choose $\delta_2 > 0$ such that, whenever $t < \delta_2$,

$$\left| t^{-1} \int y^2 g_{x_i}(y) \mu_t(dy) - \int g_{x_i}(y) M(dy) \right| < \varepsilon \qquad \text{for every } i. \tag{3.9}$$

Then, by virtue of (3.7), (3.8), (3.9) and the fact that

$$\sup_{\substack{x \in K \\ y \in R}} y^2 g_x(y) < \infty ,$$

there is some constant $C > 0$ (depending only on K) such that, whenever $x \in K$ and $0 < t < \delta = \min(\delta_1, \delta_2)$,

$$\left| t^{-1} \int y^2 g_x(y) \mu_t(dy) - \int g_x(y) M(dy) \right| < C\varepsilon ,$$

which proves that the convergence in (3.6) is uniform on K.

If $f \in C_0^2$, the above argument is valid for the whole line $\boldsymbol{R}$ instead of K, proving that the convergence in (3.6) is uniform on $\boldsymbol{R}$.

We now come to the precise description of the infinitesimal generator for (P_t).

THEOREM 3.2. *Let L be either of the Banach spaces L^p, $1 \leqq p < \infty$, C_0 or C_u. Then, $(P_t)_{t \geqq 0}$ defines a strongly continuous, contraction semi-group over every L. The infinitesimal generator of (P_t) over L is the operator A restricted to*

$$D(A; L) = \{ f \in L ;\ Af \in L \} , \tag{3.10}$$

that is,

$$\lim_{t \to 0} \frac{P_t f - f}{t} = Af \text{ in } L \quad \text{if and only if } f \in D(A; L). \tag{3.11}$$

PROOF. The first assertion is well known. Let us verify only the case of $L = L^p$:

$$\| P_t f \|_p^p = \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} f(x+y) \mu_t(dy) \right|^p dx$$

$$\leqq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} | f(x+y) |^p \mu_t(dy) dx = \| f \|_p^p .$$

$$\| P_t f - f \|_p^p = \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} (f(x+y) - f(x)) \mu_t(dy) \right|^p dx$$

$$\leqq \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} | f(x+y) - f(x) |^p dx \right) \mu_t(dy) .$$

Since $g(y) = \int_{-\infty}^{\infty} | f(x+y) - f(x) |^p dx$ is bounded, continuous and $g(0) = 0$, the last side of the above display converges to 0 as $t \to 0$.

The assertion on generator is obtained from Theorem 2.4 by a routine argument. Let $\dot{A}$ be the generator of (P_t) over L. By the Hille-Yosida theorem, every $u\in D(\dot{A})$ is written as $u=U_\lambda f$, $f\in L$ with

$$(\lambda-\dot{A})u=f. \tag{3.12}$$

On the other hand, by Theorem 2.4 one has

$$(\lambda-A)u=f, \tag{3.13}$$

so that $Au=\dot{A}u\in L$. Therefore, $D(\dot{A})\subset D(A\,;L)$ and A is an extension of $\dot{A}$. To show that $D(A\,;L)\subset D(\dot{A})$, take $u\in D(A\,;L)$ and let $(\lambda-A)u=f\in L$. Therefore $u=U_\lambda f$ (Theorem 2.4) and hence $u\in D(\dot{A})$ (the Hille-Yosida theorem).

REMARK 3.3. As usual, C_c denotes the space of continuous functions with compact supports. It is easy to see that $D(A\,;L)\supset C_c^2=\{f\in C_c\,;f',f''\in C_c\}$ for every L in Theorem 3.2, $D(A\,;C_0)\supset C_0^2\supset\dot{\mathscr{B}}$ and $D(A\,;L^p)\supset(\mathscr{D}_{L^p}^2)=\{f\in L^p\,;f',f''\in L^p\}\supset(\mathscr{D}_{L^p})$ for $1\leqq p<\infty$. Therefore, Theorem 3.2 implies the latter half of Theorem 3.1 and its L^p-variant.

We now show that Theorem 3.2 still implies the former half of Theorem 3.1. Let K be compact. For any $\varepsilon>0$, there is a function $g\in C_u$ such that $0\leqq g\leqq 1$, $g=0$ around K, $g=1$ near the infinity and $Ag(x)=\int g(x+y)\nu(dy)<\varepsilon$ for every $x\in K$. Then, by Theorem 3.2, $t^{-1}P_tg(x)<\varepsilon$ for every $x\in K$ if t is small enough. Let f be a bounded function with a second continuous derivative. Without loss of generality one can assume that $0\leqq f\leqq 1$. Choosing $f_1\in C_c^2$ such that $0\leqq f-f_1\leqq g$, one has for every $x\in K$,

$$|t^{-1}[P_tf-f](x)-Af(x)|\leqq|t^{-1}[P_tf_1-f_1](x)-Af_1(x)|+t^{-1}P_tg(x)+Ag(x).$$

The right hand side is uniformly small on K if t is small enough.

COROLLARY 3.4. (a) *Let $\varphi\in(\mathscr{D}_{L^p})$, $1\leqq p<\infty$ [resp. $\dot{\mathscr{B}}$]. Then*

$$\lim_{t\to 0}\frac{P_t\varphi-\varphi}{t}=A\varphi \quad \textit{in} \quad (\mathscr{D}_{L^p})\ [\textit{resp. } \dot{\mathscr{B}}]. \tag{3.14}$$

(b) *Let $f\in(\mathscr{D}'_{L^p})$, $1<p\leqq\infty$ [resp. $(\mathscr{D}'_{L^1})$]. Then*

$$\lim_{t\to 0}\frac{P_tf-f}{t}=Af \quad \textit{in} \quad \sigma(\mathscr{D}'_{L^p},\mathscr{D}_{L^{p'}}),\ \frac{1}{p'}=1-\frac{1}{p}\,[\textit{resp. } \sigma(\mathscr{D}'_{L^1},\dot{\mathscr{B}})]. \tag{3.15}$$

PROOF. (a) By Remark 3.3, if $\varphi\in(\mathscr{D}_{L^p})$, then $\varphi^{(n)}\in D(A\,;L^p)$. Hence

$$\left(\frac{P_t\varphi-\varphi}{t}\right)^{(n)}=\frac{P_t\varphi^{(n)}-\varphi^{(n)}}{t}\ \to\ A\varphi^{(n)}=(A\varphi)^{(n)} \text{ in } L^p.$$

(b) For $\varphi\in(\mathscr{D}_{L^{p'}})$,

$$\left(\frac{P_tf-f}{t},\varphi\right)=\left(f,\frac{\tilde{P}_t\varphi-\varphi}{t}\right)\ \to\ (f,\tilde{A}\varphi)=(\tilde{A}f,\varphi).$$

One can prove that the convergence in (3.15) is valid for the (strong) topology of $(\mathcal{D}'_{L^p})$. In fact we have the following theorem.

THEOREM 3.5. *Let M be either of the (complete) topological vector spaces*

(a) $$(\mathcal{D}_{L^p}),\quad 1\leqq p<\infty,\quad \dot{\mathcal{B}} \quad \text{or}$$

(b) $$(\mathcal{D}'_{L^p}),\quad 1\leqq p\leqq\infty.$$

Then (P_t) defines an equicontinuous semi-group of class (C_0) on each M[3]. *In this case the infinitesimal generator of (P_t) over M is the operator A which turns out to be a bounded operator on each M.*

Case (a) is obvious from the above corollary (a). The proof of (b) is due to S. Sugitani (private communication) and it is omitted here.

§ 4. Some results on the dual processes.

Let

$$X=(\Omega,\ \mathcal{F},\ \mathcal{F}_t,\ X(t),\ \boldsymbol{P}^x) \tag{4.1}$$

be a standard realization of (P_t) defined by (1.2). For each $B\in\mathcal{B}(\boldsymbol{R})$, define

$$T_B:=\inf\{t>0\,;\ X(t)\in B\} \qquad \text{(hitting time)} \tag{4.2}$$

$$W_B:=\inf\left\{t>0\,;\ \int_0^t I_B\circ X(s)ds>0\right\} \qquad \text{(penetration time)}, \tag{4.3}$$

$$H_B^\lambda f(x):=\boldsymbol{E}^x(e^{-\lambda T_B}f\circ X(T_B)),\qquad \lambda\geqq 0, \tag{4.4}$$

$$\underline{H}_B^\lambda f(x):=\boldsymbol{E}^x(e^{-\lambda W_B}f\circ X(W_B)),\qquad \lambda\geqq 0. \tag{4.5}$$

We often omit λ when $\lambda=0$.

We omit the well-known definition of λ $(\geqq 0)$-excessive function for (P_t) or X [**1**; p. 70]. A λ $(\geqq 0)$-*excessive measure* ν for (P_t) is a σ-finite measure such that

$$\nu\geqq\nu P_t^\lambda \qquad \text{for every}\quad t\geqq 0\ (P_t^\lambda=e^{-\lambda t}P_t), \tag{4.6}$$

$$\nu=\lim_{t\to 0}\uparrow \nu P_t^\lambda. \tag{4.7}$$

But since (P_t) is a standard semi-group, condition (4.7) is superfluous; it is a consequence of condition (4.6) (see [**1**; p. 257]). Eventually it follows that a σ-finite measure ν is λ-excessive for (P_t) if and only if it is λ-*supermedian* for the resolvent $(U_{\lambda'})$, i.e.,

$$\nu\geqq\nu(\lambda' U_{\lambda+\lambda'}) \qquad \text{for every}\quad \lambda'>0, \tag{4.8}$$

and that (4.8) implies

$$\nu=\lim_{\lambda'\to\infty}\uparrow \nu(\lambda' U_{\lambda+\lambda'}). \tag{4.9}$$

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3) For the definition of such semi-group, see K. Yosida [**15**; p. 234].

Let ν be a σ-finite measure dominated by a measure which is λ-supermedian for $(U_{\lambda'})$. For $B\in\mathscr{B}(\boldsymbol{R})$, the *balayage operator* L_B^λ for ν is defined by

$$\nu L_B^\lambda := \inf\{\nu' ; [\nu'\geqq\nu]_B,\ \nu' \text{ is } \lambda\text{-supermedian}\}\,, \tag{4.10}$$

where $[\nu'\geqq\nu]_B$ means that ν' dominates ν over B. The *balayée* νL_B^λ is λ-supermedian for $(U_{\lambda'})$ and hence λ-excessive for (P_t). A measure μ is said to *belong to the domain of* U_λ if μU_λ is a σ-finite measure. In this case, μU_λ is said to be the U_λ-*potential of* μ. Every U_λ-potential is λ-excessive for (P_t).

We need a general lemma which is valid for every standard process.

LEMMA 4.1. *Suppose that* μ *belongs to the domain of* U_λ. *For every* $B\in\mathscr{B}(\boldsymbol{R})$,

$$(\mu U_\lambda)L_B^\lambda=\mu \underline{H}_B^\lambda U_\lambda\,. \tag{4.11}$$

In particular, if $B=G$ *is open,* $\underline{H}_G^\lambda=H_G^\lambda$. *Hence*

$$(\mu U_\lambda)L_G^\lambda=\mu H_G^\lambda U_\lambda\,. \tag{4.12}$$

For the proof see Theorem 16 of [**12**].

All objects with respect to $(\tilde{P}_t)_{t\geqq 0}$ are denoted with "$\sim$" over the associated letters. The processes X and $\tilde{X}$ (=the standard realization of $(\tilde{P}_t)$) are *in duality relative to the Lebesgue measure dx* in the sense that the relations (1.7) and (1.8) hold. Note, however, that our definition of duality is weaker than that of Blumenthal and Getoor [**1**; p. 253], for we will not impose the hypothesis of absolute continuity for the resolvents (U_λ) and $(\tilde{U}_\lambda)$ in general. In the present situation the hypothesis of absolute continuity for the resolvents (U_λ) and $(\tilde{U}_\lambda)$ reduces to the simple condition that

$$U_\lambda^0 \text{ is absolutely continuous with respect to the Lebesgue measure } dx\,. \tag{4.13}$$

We sometimes assume this condition to obtain stronger results. When (4.13) is being assumed to hold we will explicitly say so.

The next lemma is valid for every pair $(X,\tilde{X})$ of standard processes which is in duality (in our sense) relative to some measure.

LEMMA 4.2. (a) *If* u *is a* λ*-excessive function for* (P_t) *such that* $\tilde{\nu}(dx)=u\cdot dx$ *is a* σ*-finite measure. Then,* $\tilde{\nu}$ *is a* λ*-excessive measure for* $(\tilde{P}_t)$. *If* (4.13) *holds, the converse is true. More precisely, for every* λ*-excessive measure* $\tilde{\nu}$ *for* $(\tilde{P}_t)$, *there is a unique* λ*-excessive function* u *for* (P_t) *such that* $\tilde{\nu}=u\cdot dx$.

(b) *Let* G *be open and* $\lambda\geqq 0$. *For every* $f, g\geqq 0$,

$$(H_G^\lambda U_\lambda f, g)=(f, \tilde{H}_G^\lambda\tilde{U}_\lambda g)\,. \tag{4.14}$$

(c) *If* u *is a* λ*-excessive function such that* $u\cdot dx$ *is* σ*-finite, then*

$$(u\cdot dx)\tilde{L}_G^\lambda=H_G^\lambda u\cdot dx\,. \tag{4.15}$$

PROOF. (a) It is enough to show when $\lambda>0$. If $u=U_\lambda f$, $f\geqq 0$, then $\tilde{\nu}=u\cdot dx=(f\cdot dy)\tilde{U}_\lambda$. Hence $\tilde{\nu}$ is a $\tilde{U}_\lambda$-potential and, a fortiori, λ-excessive for $(\tilde{P}_t)$. Then the general case is easily proved by the approximation of excessive functions by potentials.

If (4.13) holds, there exists the λ-Green function $u_\lambda(x, y)$ of $(U_\lambda, \tilde{U}_\lambda)$ relative to dx [1; p. 254]. If $\tilde{\nu}=\mu\tilde{U}_\lambda$, then

$$\tilde{\nu}(dx)=\Big(\int\mu(dy)u_\lambda(x, y)\Big)\cdot dx\,.$$

But $u=\int\mu(dy)u_\lambda(x, y)$ is a λ-excessive function for (P_t). For the general case, let $\tilde{\nu}=\lim_n\uparrow\mu_n\tilde{U}_\lambda=\lim_n\uparrow u_n\cdot dx$, where $u_n(x)=\int\mu_n(dy)u_\lambda(x, y)$. Since dx is a reference measure for the process X with (4.13) being satisfied [1; p. 259], the relation "$u_n\leqq u_{n+1}$ dx-almost surely" implies that $u_n\leqq u_{n+1}$ everywhere. Hence $u:=\lim_n\uparrow u_n$ is λ-excessive and $\tilde{\nu}=u\cdot dx$.

(b) It is enough to show when $\lambda>0$ and f, g are bounded. Recall that, if f, g are functions and μ is a measure, then

$$(f, g)=\int f(x)g(x)dx\,,\qquad (\mu, g)=\int g(x)\mu(dx)\,,$$

so that (f, g) can be written as $(f\cdot dx, g)$.

Since $[U_\lambda f=H_G^\lambda U_\lambda f]_G$ and so $[U_\lambda f\cdot dx=(H_G^\lambda U_\lambda f)\cdot dx]_G$, it follows by definition that

$$(U_\lambda f\cdot dx)\tilde{L}_G^\lambda\leqq(H_G^\lambda U_\lambda f)\cdot dx\,. \tag{4.16}$$

Therefore one sees that

$$\begin{aligned}(f, \tilde{H}_G^\lambda\tilde{U}_\lambda g)&=([f\cdot dy]\tilde{H}_G^\lambda\tilde{U}_\lambda, g)\\ &=([(f\cdot dy)\tilde{U}_\lambda]\tilde{L}_G^\lambda, g)\qquad\text{by (4.12)}\\ &=([U_\lambda f\cdot dx]\tilde{L}_G^\lambda, g)\\ &\leqq([H_G^\lambda U_\lambda f]\cdot dx, g)\\ &=(H_G^\lambda U_\lambda f, g)\,.\end{aligned} \tag{4.17}$$

By the symmetry of the argument, $(H_G^\lambda U_\lambda f, g)\leqq(f, \tilde{H}_G^\lambda\tilde{U}_\lambda g)$.

(c) The proof of (b) implies that equality holds in (4.16), which proves (4.15) in case of $u=U_\lambda f$. For the general case, use the approximation by potentials.

Under hypothesis (4.13) one can refine (b) and (c) of the preceding lemma. (This is not used in the rest of the paper.)

LEMMA 4.3. *Suppose that* (4.13) *holds. For every* $B\in\mathcal{B}(\boldsymbol{R})$, $f, g\geqq 0$ *and every λ-excessive function u such that $u\cdot dx$ is σ-finite,* (4.18) $$(H_B^\lambda U_\lambda f, g)=(f, \tilde{H}_B^\lambda \tilde{U}_\lambda g),$$

(4.19) $$(\underline{H}_B^\lambda U_\lambda f, g)=(f, \underline{\tilde{H}}_B^\lambda \tilde{U}_\lambda g),$$

(4.20) $$(u\cdot dx)\tilde{L}_B^\lambda=\underline{H}_B^\lambda u\cdot dx .$$

PROOF. Equation (4.18) follows from (4.14) by virtue of the argument of [**1**; p. 262].

Theorem 9 and 16 of [**12**] tell us that there exists a set N of potential zero such that $[U_\lambda f=\underline{H}_B^\lambda U_\lambda f]_{B\setminus N}$. Since dx is a reference measure under hypothesis (4.13) by virtue of Remark (1.13) of [**1**; p. 259], one sees that

$$[U_\lambda f\cdot dx=\underline{H}_B^\lambda U_\lambda f\cdot dx]_B .$$

Then the rest of the proof is the same as in (b) of the preceding lemma.

THEOREM 4.4. *Let $\lambda>0$ and u, a bounded λ-excessive function. Then*

(4.21) $$\mu:=(\lambda-A)u\in(\mathcal{D}'_{L^\infty})$$

is a positive measure and $u\cdot dx$ is the $\tilde{U}_\lambda$-potential of μ;

(4.22) $$u\cdot dx=\mu\tilde{U}_\lambda .$$

Moreover, for every open set G,

(4.23) $$H_G^\lambda u\cdot dx=\mu\tilde{H}_G^\lambda\tilde{U}_\lambda ,$$

or equivalently,

(4.24) $$(\lambda-A)H_G^\lambda u=\mu\tilde{H}_G^\lambda .$$

Note that the theorem breaks down for $\lambda=0$. Take $u=\mathbf{1}$. Then since dx is invariant for $(\tilde{P}_t)$, it is impossible that dx is a $\tilde{U}_\lambda$-potential.

PROOF. Choose $g_n\in\mathcal{B}_+^0$ such that $u=\lim_n\uparrow U_\lambda g_n$. By Theorem 2.4,

$$(\lambda-A)U_\lambda g_n=g_n\geqq 0 .$$

By the next lemma, $g_n\to\mu$ in $(\mathcal{D}')$. Hence μ is a positive measure. But, by Theorem 2.4,

$$u=U_\lambda\mu \text{ in the distribution sense},$$

which is identical (in the usual notation) with

(4.25) $$u\cdot dx=U_\lambda\mu=\tilde{U}_\lambda^0*\mu=\mu*\tilde{U}_\lambda^0=\mu\tilde{U}_\lambda .$$

If G is open,

(4.26) $$H_G^\lambda u\cdot dx=(u\cdot dx)\tilde{L}_G^\lambda=(\mu\tilde{U}_\lambda)\tilde{L}_G^\lambda=\mu\tilde{H}_G^\lambda\cdot\tilde{U}_\lambda ,$$

using (4.15), (4.22) and (4.12).

LEMMA 4.5. *Let u_n be a sequence in $\mathcal{B}^0$ such that $\sup_n\|u_n\|_\infty<\infty$ and $u_n\to u$ pointwise. Then*

$$(4.27) \qquad Au_n \to Au \quad in \quad \sigma(\mathcal{D}'_{L^\infty}, \mathcal{D}_{L^1}).$$

PROOF. Let $\varphi \in (\mathcal{D}_{L^1})$. Since $\tilde{A}\varphi = A^0 * \varphi \in (\mathcal{D}_{L^1})$, it follows from the Lebesgue dominated convergence theorem that

$$(Au_n, \varphi) = (u_n, \tilde{A}\varphi) \rightarrow (u, \tilde{A}\varphi) = (Au, \varphi).$$

§ 5. The characterization of harmonic functions on an open set in terms of the infinitesimal generator.

Let X be the process defined by (4.1). Let G be an open subset of $\boldsymbol{R}$. A finite, $\lambda(\geqq 0)$-excessive function u is said to be *λ-harmonic on G* (*harmonic on G* if $\lambda = 0$), if, for every $B \in \mathcal{B}(\boldsymbol{R})$ which is relatively compact in G,

$$(5.1) \qquad u(x) = H^\lambda_{\complement B} u(x), \qquad x \in \boldsymbol{R},$$

where $\complement B = \boldsymbol{R} \setminus B$. If (5.1) holds dx-almost surely[4] for each B as above, we say that u is *almost λ-harmonic on G*. If (4.13) is satisfied, the almost λ-harmonicity is equivalent to the λ-harmonicity.

EXAMPLES. (a) Let $B \in \mathcal{B}(\boldsymbol{R})$. Let u be a finite, λ-excessive function. Then the functions

$$(5.2) \qquad H^\lambda_{\complement B} u \quad \text{and} \quad \underline{H}^\lambda_{\complement B} u$$

are λ-harmonic on $\operatorname{int} B$ (= the interior of B).

(b) Let $f \in \mathcal{B}^0_+$ be supported in $\complement B$. Then

$$(5.3) \qquad u = U_\lambda f$$

is λ-harmonic on $\operatorname{int} B$.

Suppose that (5.1) holds for fixed x and B. Let T be a stopping time such that $T \leqq T_{\complement B}$ $\boldsymbol{P}^x$-almost surely. Since u is supposed to be λ-excessive,

$$(5.4) \qquad u(x) = P^\lambda_T u(x) := \boldsymbol{E}^x(e^{-\lambda T} u \circ X(T)),$$

for $u(x) \geqq P^\lambda_T u(x) \geqq H^\lambda_{\complement B} u(x) = u(x)$. Hence one has

LEMMA 5.1. *Let u be λ-excessive. Let $A, B \in \mathcal{B}(\boldsymbol{R})$ and $A \subset B$. If $u = H^\lambda_{\complement B} u$ everywhere (resp. dx-almost surely), then $u = H^\lambda_{\complement A} u$ everywhere (resp. dx-almost surely).*

THEOREM 5.2. *Let u be a bounded, $\lambda(\geqq 0)$-excessive function and G, an open set. Then the following three conditions are equivalent with each other, except if simultaneously $\lambda = 0$ and A is the zero operator.*[5]

(a) *u is almost λ-harmonic on G.*

(b) *Let $\boldsymbol{C}$ be a subfamily of $\mathcal{B}(\boldsymbol{R})$ such that each $C \in \boldsymbol{C}$ is relatively compact*

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4) Exceptional sets may depend on B.

5) $A = 0$ if and only if (P_t) is the trivial semi-group: $\mu_t = \delta$ for every $t \geqq 0$.

in G *and* $G=\bigcup_{C\in \boldsymbol{C}}(\mathrm{int}\,C)$. *For every* $C\in \boldsymbol{C}$, $u=H^{\lambda}_{\complement C}u$ *dx-almost surely.*

(c)

(5.5) $$(\lambda-A)u=0 \text{ on } G \text{ (in the distribution sense)}.$$

PROOF. (a)$\Rightarrow$(b). Obvious.

(b)$\Rightarrow$(c). By the preceding lemma one can assume that $\boldsymbol{C}$ contains a subfamily $\boldsymbol{K}$ of compact sets $K\subset G$ such that the family $\{\mathrm{int}\,K\}_{K\in \boldsymbol{K}}$ covers G. If $K\in \boldsymbol{K}$ and $\lambda>0$, by Theorem 4.4,

$$\mu:=(\lambda-A)u=(\lambda-A)H^{\lambda}_{\complement K}u=\mu\widetilde{H}^{\lambda}_{\complement K}.$$

Since $[\mu\widetilde{H}^{\lambda}_{\complement K}=0]_{\mathrm{int}\,K}$, $[\mu=0]_{\mathrm{int}\,K}$ for every $K\in \boldsymbol{K}$. Hence $[\mu=0]_G$.

If $\lambda=0$, define $u_\lambda:=H^{\lambda}_{\complement K}u$ ($\lambda>0$) for $K\in \boldsymbol{K}$. Since u_λ is λ-harmonic on $\mathrm{int}\,K$ (Example (a)), it follows from the above result that

(5.6) $$(\lambda-A)u_\lambda=0 \quad \text{on} \quad \mathrm{int}\,K.$$

Letting $\lambda\to 0$ and applying Lemma 4.5, one sees that

$$(\lambda-A)u_\lambda \longrightarrow -A(H_{\complement K}u)=-Au \quad \text{in} \quad (\mathcal{D}').$$

By (5.6), $-Au=0$ on $\mathrm{int}\,K$ for every $K\in \boldsymbol{K}$ and hence on G.

(c)$\Rightarrow$(a). By Lemma 4.5 it is enough to show that, for every compact $K\subset G$, $u=H^{\lambda}_{\complement K}u$ dx-almost surely.

Let $\lambda>0$ and $\mu:=(\lambda-A)u$. Since $[\mu=0]_K$ by the assumption, it is easy to see that $\mu=\mu\widetilde{H}^{\lambda}_{\complement K}$. By Theorem 4.4,

$$u\cdot dx=\mu\widetilde{U}_\lambda=\mu\widetilde{H}^{\lambda}_{\complement K}\cdot\widetilde{U}_\lambda=H^{\lambda}_{\complement K}u\cdot dx,$$

so that $u=H^{\lambda}_{\complement K}u$ dx-almost surely.

Next consider the case $\lambda=0$. Let $\mu=-Au$. Since $(\lambda-A)u=\mu+\lambda u$ and $\mu=\mu\widetilde{H}^{\lambda}_{\complement K}$ for $\lambda>0$, it follows that

$$u\cdot dx=(\mu+\lambda u\cdot dy)\widetilde{U}_\lambda,$$
$$H^{\lambda}_{\complement K}u\cdot dx=(\mu+\lambda u\cdot dy)\widetilde{H}^{\lambda}_{\complement K}\cdot\widetilde{U}_\lambda=\mu\widetilde{U}_\lambda+(\lambda u\cdot dy)\widetilde{H}^{\lambda}_{\complement K}\cdot\widetilde{U}_\lambda,$$

so that, for $\varphi\in C_c^+$,

(5.7) $$([u-H^{\lambda}_{\complement K}u]\cdot dx,\varphi)=([\lambda u\cdot dy-(\lambda u\cdot dz)\widetilde{H}^{\lambda}_{\complement K}]\widetilde{U}_\lambda,\varphi).$$

The left hand side converges to $([u-H_{\complement K}u]\cdot dx,\varphi)$ as $\lambda\to 0$.

On the other hand, one has for $f\geqq 0$,

(5.8) $$\begin{aligned}(\lambda u\cdot dx-(\lambda u\cdot dy)\widetilde{H}^{\lambda}_{\complement K}, f)&=(\lambda u, f-\widetilde{H}^{\lambda}_{\complement K}f)\\ &\leqq(\lambda u, f\cdot I_K)\\ &=(\lambda u I_K\cdot dx, f).\end{aligned}$$

Therefore

$$0 \leqq ([\lambda u\cdot dy-(\lambda u\cdot dz)\tilde{H}^{\lambda}_{\complement K}]\tilde{U}_{\lambda}, \varphi)$$
$$\leqq ([\lambda u I_K\cdot dy]\tilde{U}_{\lambda}, \varphi)$$
$$=(\lambda U_{\lambda}(uI_K), \varphi) \longrightarrow 0\,, \quad \lambda \longrightarrow 0\,,$$

by virtue of a result of K. Sato [**9**].[6] Hence

$$([u-H_{\complement K}u]\cdot dx, \varphi)=0\,, \qquad \varphi \in C_c^+\,,$$

which proves that $u=H_{\complement K}u$ dx-almost surely.

THEOREM 5.3. *Let u be a bounded, almost λ-harmonic function on $\boldsymbol{R}$.*

(a) *If $\lambda>0$, $u=0$ dx-almost surely.*

(b) *If $\lambda=0$, there is a constant $C \geqq 0$ such that $u=C$ dx-almost surely except for the following case: the Lévy measure ν is arithmetic (i.e. ν is supported in an arithmetic progression containing the origin) and*

$$\sigma^2=a-\int_{\boldsymbol{R}\setminus\{0\}}\frac{y}{1+y^2}\,\nu(dy)=0\,.$$

PROOF. (a) $u\cdot dx=\mu\tilde{U}_{\lambda}=0$.

(b) Since $(\lambda-A)u=\lambda u$, $u\cdot dx=(\lambda u\cdot dy)\tilde{U}_{\lambda}=\lambda U_{\lambda}u\cdot dx$. Hence, for each $\varphi \in C_c^+$, $v=u*\varphi$ is a bounded, continuous solution of the convolution equation $v=\lambda U_{\lambda}v=(\lambda\tilde{U}^0_{\lambda})*v$. It is easy to see that the probability measure $\lambda\tilde{U}^0_{\lambda}$ is nonarithmetic except for the above-described case. Therefore v must be a constant by virtue of a result of Choquet-Deny (see Feller [**4**; p. 351] for an elementary proof). Since $\varphi \in C_c^+$ is arbitrary, u is a constant dx-almost surely.

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6) For every $f \in C_0$, $\lambda U_{\lambda}f \rightarrow 0$ in C_0 as $\lambda \rightarrow 0$. This follows from Theorem 2.2 and 4.1 of [**9**].

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