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A formula on some odd-dimensional Riemannian manifolds related to the Gauss-Bonnet formula

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§ 1. Introduction.

Let (N^{2n}, g) be a compact orientable Riemannian manifold of $2n$-dimension. The generalised Gauss-Bonnet formula is

$$\frac{(-1)^n}{2^{2n}\pi^n n!}\int_N \sum \varepsilon_{i_1\cdots i_{2n}} \Omega'_{i_1 i_2} \wedge \cdots \wedge \Omega'_{i_{2n-1} i_{2n}} = \chi(N), \tag{1.1}$$

where Ω'_{ij} denote the curvature forms and $\chi(N)$ is the Euler-Poincaré characteristic. The left hand side of (1.1) is a differential geometric or Riemannian geometric quantity and the right hand side is a topological quantity. In (1.1), even dimensionality is essential.

For a compact orientable Riemannian manifold (M^{2n+1}, g) of odd dimension, we have $\chi(M)=0$. This shows that $M=M^{2n+1}$ admits a vector field ξ with no singular points. If we try to find some formula on (M^{2n+1}, g) analogous to (1.1), some restriction on this ξ may be necessary and it might be hoped that the right hand side is a linear combination of Betti numbers.

We assume that $\xi=e_0$ is a unit vector field. Let w_0 be the 1-form dual to e_0 with respect to g. Then we have local fields of orthonormal vectors $e_0, e_1, \cdots, e_{2n}$ and the dual $w_0, w_1, \cdots, w_{2n}$. We call this frame field a ξ-frame field. By Ω_{AB} $(A, B=0, 1, \cdots, 2n)$ we denote the curvature forms with respect to the above frame field. By $\beta_r(M)$ we denote the r-th Betti number of M. In this paper we have

THEOREM A. *Let (M^{2n+1}, g) be a compact Riemannian manifold admitting a unit Killing vector ξ and let (e_0, e_i) be a ξ-frame field. Assume that*

$$\Omega_{0i} = w_i \wedge w_0, \qquad i=1, \cdots, 2n, \tag{1.2}$$

and that each trajectory of ξ is of constant length $l(\xi)$. Then

$$\frac{(-1)^n}{l(\xi)2^{2n}\pi^n n!}\int_M F(\Omega_{ij}, w_0) = \sum_{r=0}^{n}(n+1-r)(-1)^r \beta_r(M), \tag{1.3}$$

where, putting $dw_0 = \sum \varphi_{AB} w_A \wedge w_B$,

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(1.4) $$F(\Omega_{ij}, w_0)=\sum \varepsilon_{i_1\cdots i_{2n}}(\Omega_{i_1i_2}-\varphi_{i_1i_2}dw_0-\sum \varphi_{i_1k}w_k\wedge\varphi_{i_2l}w_l)\wedge \cdots \wedge(\Omega_{i_{2n-1}i_{2n}}-\varphi_{i_{2n-1}i_{2n}}dw_0-\sum\varphi_{i_{2n-1}k}w_k\wedge\varphi_{i_{2n}l}w_l)\wedge w_0 .$$

The condition (1.2) is independent of the choice of ξ-frame fields. In fact, let R be the Riemannian curvature tensor of (M^{2n+1}, g). Then (1.2) is equivalent to

(1.2)′ $$R(X, \xi)Y=g(X, Y)\xi-g(\xi, Y)X$$

for any vector fields X and Y on M. A Riemannian manifold (M^{2n+1}, g) admitting a unit Killing vector ξ satisfying (1.2) or (1.2)′ is called a Sasakian manifold (or a normal contact Riemannian manifold). In particular, Sasakian manifolds with constant $l(\xi)$ are canonically related to Hodge manifolds (i. e., Kählerian manifolds whose fundamental 2-form defines an integral cocycle). Contact manifolds are orientable.

A special case is as follows:

THEOREM B. *Let* (M^3, g) *be a compact* 3*-dimensional Riemannian manifold admitting a unit Killing vector* ξ *such that*

(1.5) $$R(X, \xi)\xi=g(X, \xi)\xi-X .$$

If each trajectory of ξ *is of constant length* $l(\xi)$, *then*

(1.6) $$\frac{1}{l(\xi)2\pi}\Big[\int_M K(\xi^\perp)dM+3\,\mathrm{Vol}(M)\Big]=2-\beta_1(M),$$

where $K(\xi^\perp)$ *means sectional curvature of the* 2*-plane orthogonal to* ξ *and* $\mathrm{Vol}(M)$ *denotes the total volume of* (M^3, g).

Two typical examples, Sasakian manifolds and Riemannian product manifolds, show a clear difference between expressions of linear combinations of Betti numbers (see (7.1) and (8.2)).

§ 2. Preliminaries.

For local fields of orthonormal vectors $(e_A, A=0, 1, \cdots, 2n)$ and the dual 1-forms (w_A) on a Riemannian manifold (M^{2n+1}, g), the structure equations are

(2.1) $$dw_A=\sum w_B\wedge w_{BA} \qquad (A, B=0, 1, \cdots, 2n),$$

(2.2) $$dw_{AB}=\sum w_{AC}\wedge w_{CB}+\Omega_{AB},$$

where w_{AB} and Ω_{AB} denote the connection forms and curvature forms, respectively; $w_{AB}+w_{BA}=0$ and $\Omega_{AB}+\Omega_{BA}=0$.

Let $({}^*e_A)$ be another frame field such that

(2.3) $${}^*e_A=\sum a_{AB}e_B, \qquad a_{AB}(x)\in O(2n+1).$$

Then the curvature forms ${}^*\Omega_{AB}$ with respect to $({}^*e_A)$ satisfy

(2.4) $$*\Omega_{AB}=\sum a_{AC}a_{BD}\Omega_{CD}\,.$$

§ 3. Sasakian structures.

Let (M^{2n+1}, g) be a Riemannian manifold admitting a unit Killing vector ξ satisfying

(3.1) $$R(X, \xi)Y=g(X, Y)\xi-g(Y, \xi)X\,,$$

where $R(X, Y)Z=\nabla_{[X,Y]}Z-[\nabla_X, \nabla_Y]Z$ and ∇ denotes the Riemannian connection. Since ξ is a Killing vector, it satisfies $\nabla_X(\nabla\xi)Y+R(X, \xi)Y=0$ (this relation is equivalent to the fact that ξ is an infinitesimal affine transformation). Hence, the left hand side of (3.1) may be replaced by $-\nabla_X(\nabla\xi)Y$. Such a Riemannian manifold is called a Sasakian manifold or normal contact Riemannian manifold (cf. Sasaki-Hatakeyama [8], Hatakeyama-Ogawa-Tanno [6], etc.) and it is denoted by (M^{2n+1}, ξ, g). For completeness we give a brief summary of relations of structure tensors (see [6], up to constant factors). We define a (1, 1)-tensor field φ by $\varphi=-\nabla\xi$, i.e., $\varphi X=-\nabla_X\xi$. By $\nabla_X(g(\xi, \xi))=0$, we have $\varphi\xi=-\nabla_\xi\xi=0$. Next, by $\nabla_X(\varphi\xi)=0$, we have $(\nabla_X\varphi)\xi+\varphi\nabla_X\xi=\nabla_X(-\nabla\xi)\xi-\varphi\varphi X=0$. The last equation and (3.1) give

(3.2) $$\varphi\varphi X=-X+g(\xi, X)\xi\,.$$

Considering the inner product of the both sides of (3.2) and Y, and noticing that $\varphi=-\nabla\xi$ is skew-symmetric with respect to g, we have

(3.3) $$g(\varphi X, \varphi Y)+g(\xi, X)g(\xi, Y)=g(X, Y)\,.$$

If w_0 is the 1-form dual to ξ with respect to g, i.e., $w_0(X)=g(\xi, X)$, by $\varphi=-\nabla\xi$ we have

(3.4) $$dw_0(X, Y)=2g(X, \varphi Y)\,.$$

w_0 satisfies $w_0\wedge(dw_0)^n\neq 0$ and is called a contact form. With respect to local coordinates (x^A), we have

(3.4)′ $$dw_0=\sum\varphi_{AB}\,dx^A\wedge dx^B\,.$$

Sasakian manifolds (more generally contact manifolds) are orientable. Let $(\xi=e_0, e_1, \cdots, e_{2n}; w_0, w_1, \cdots, w_{2n})$ be a ξ-frame field. Then it is not difficult to see that (3.1) is equivalent to $\Omega_{0i}=w_i\wedge w_0$ $(i=1, \cdots, 2n)$, since

$$\Omega_{AB}=(1/2)\sum R_{ABCD}w_C\wedge w_D\,,$$

where we have put $R(e_C, e_D)e_B=\sum R_{ABCD}e_A$.

§ 4. Boothby-Wang's fibering.

Let (M^{2n+1}, ξ, g) be a Sasakian manifold and assume that ξ is regular (cf. Boothby-Wang [1], etc.). Then we have the fibering

$$\pi: M^{2n+1} \longrightarrow M^{2n+1}/\xi = B^{2n},$$

where (B^{2n}, J, G) is a Kählerian manifold (more precisely, Hodge manifold) with (almost) complex structure tensor J and the Kähler metric tensor G (see Hatakeyama [5], p. 181, etc.). w_0 is an infinitesimal connection form on this principal bundle. J and G satisfy

$$g(X, Y) = G(\pi X, \pi Y)\cdot\pi + w_0(X)w_0(Y), \tag{4.1}$$

$$(Ju)^* = \varphi u^*, \tag{4.2}$$

u^* denoting the horizontal lift of a vector field u on B^{2n} with respect to w_0. Conversely, every Hodge manifold (B^{2n}, J, G) gives a Sasakian manifold (M^{2n+1}, ξ, g) with regular ξ. Furthermore, we have

$$dw_0(X, Y) = 2G(\pi X, J\pi Y)\cdot\pi = 2g(X, \varphi Y).$$

Let $(f_i,\ i = 1, \cdots, 2n)$ be local fields of orthonormal vectors in B^{2n}. Then $(\xi = e_0,\ e_i = f_i^*)$ is a ξ-frame field and the Riemannian connection forms w_{AB} with respect to (e_A) are given by

$$w_{00} = 0, \tag{4.3}$$

$$w_{0i} = -w_{i0} = -\sum \varphi_{ij} w_j, \tag{4.4}$$

$$w_{ji} = \pi^*(w'_{ji}) - \varphi_{ij} w_0, \tag{4.5}$$

where $\varphi_{ij} = g(e_i, \varphi e_j)$, and w'_{ji} are the connection forms on (B^{2n}, G) with respect to (f_i) (cf. Kobayashi [7], Proposition 2). The curvature forms Ω_{AB} are given by (cf. [7], Proposition 3)

$$\Omega_{00} = 0, \tag{4.6}$$

$$\Omega_{0i} = -\Omega_{i0} = -\sum \varphi_{ik}\varphi_{kl} w_l \wedge w_0, \tag{4.7}$$

$$\Omega_{ji} = \pi^*(\Omega'_{ji}) - \sum (\varphi_{ij}\varphi_{kl} + \varphi_{ik}\varphi_{jl}) w_k \wedge w_l, \tag{4.8}$$

where Ω'_{ji} are the curvature forms on (B^{2n}, G). Hence, we have

$$\pi^*(\Omega'_{ij}) = \Omega_{ij} - \varphi_{ij} dw_0 - \sum \varphi_{ik} w_k \wedge \varphi_{jl} w_l. \tag{4.9}$$

§ 5. The Theorem A.

Let (M^{2n+1}, ξ, g) be a compact Sasakian manifold with regular ξ. In this case regularity is equivalent to the fact that all trajectories of ξ have the common length $l(\xi)$. The Gauss-Bonnet formula (for example, see Chern [2, 3]) for a compact orientable Riemannian manifold (B^{2n}, G) is

$$\frac{(-1)^n}{2^{2n}\pi^n n!}\int_B \sum \varepsilon_{i_1\cdots i_{2n}}\Omega'_{i_1i_2}\wedge\cdots\wedge\Omega'_{i_{2n-1}i_{2n}}=\chi(B)\,, \tag{5.1}$$

where $\varepsilon_{i_1\cdots i_{2n}}$ is a symbol which is 1 or -1 according as $(i_1,\cdots,i_{2n})$ is an even or odd permutation of $(1,\cdots,2n)$, and is zero otherwise. It is not difficult to see that

$$\int_M \pi^*(\Theta)\wedge w_0 = l(\xi)\int_B \Theta \tag{5.2}$$

for any $2n$-form Θ on B^{2n}. Therefore, we get

$$\frac{(-1)^n}{l(\xi)2^{2n}\pi^n n!}\int_M \sum \varepsilon_{i_1\cdots i_{2n}}\pi^*(\Omega'_{i_1i_2})\wedge\cdots\wedge\pi^*(\Omega'_{i_{2n-1}i_{2n}})\wedge w_0=\chi(B)\,. \tag{5.3}$$

THEOREM A. *Let (M^{2n+1}, ξ, g) be a compact Sasakian manifold with regular ξ. Then*

$$\begin{aligned}&\frac{(-1)^n}{l(\xi)2^{2n}\pi^n n!}\int_M \sum \varepsilon_{i_1\cdots i_{2n}}(\Omega_{i_1i_2}-\varphi_{i_1i_2}dw_0-\sum\varphi_{i_1k}w_k\wedge\varphi_{i_2l}w_l)\wedge\\ &\qquad\cdots\wedge(\Omega_{i_{2n-1}i_{2n}}-\varphi_{i_{2n-1}i_{2n}}dw_0-\sum\varphi_{i_{2n-1}k}w_k\wedge\varphi_{i_{2n}l}w_l)\wedge w_0\\ &=\sum_{r=0}^{n}(n+1-r)(-1)^r\beta_r(M)\,,\end{aligned} \tag{5.4}$$

where $\beta_r(M)$ denotes the r-th Betti number of M^{2n+1}.

PROOF. First we notice that the integrand is independent of the choice of ξ-frame fields. By (4.9) and (5.3), it suffices to show

$$\chi(B)=\sum_{r=0}^{n}(n+1-r)(-1)^r\beta_r(M)\,. \tag{5.5}$$

The exact sequence of Gysin for $\pi: M^{2n+1}\to B^{2n}$ is

$$\begin{aligned}&0\longrightarrow H^1(B;\boldsymbol{R})\xrightarrow{\pi^*}H^1(M;\boldsymbol{R})\longrightarrow H^0(B;\boldsymbol{R})\\ &\xrightarrow{L_0}H^2(B;\boldsymbol{R})\xrightarrow{\pi^*}H^2(M;\boldsymbol{R})\longrightarrow H^1(B;\boldsymbol{R})\longrightarrow\cdots\\ &\xrightarrow{L_{p-2}}H^p(B;\boldsymbol{R})\xrightarrow{\pi^*}H^p(M;\boldsymbol{R})\longrightarrow H^{p-1}(B;\boldsymbol{R})\longrightarrow\cdots,\end{aligned}$$

where $H^p(M;\boldsymbol{R})$ (or $H^p(B;\boldsymbol{R})$) is the p-th cohomology group of M^{2n+1} (or B^{2n}) with real coefficient $\boldsymbol{R}$, and L_p sends $\lambda\in H^p(B;\boldsymbol{R})$ to $W\wedge\lambda\in H^{p+2}(B;\boldsymbol{R})$, W being the fundamental 2-form of the Kählerian manifold (B^{2n}, J, G) (cf. Chern-Spanier [**4**], Serre [**9**]). Since (B^{2n}, J, G) is Kählerian, L_p is an into isomorphism for $p\leqq(2n-2)/2$. Therefore $\beta_1(M)=\beta_1(B)$ and

$$\beta_p(M)=\beta_p(B)-\beta_{p-2}(B)\,,\qquad 2\leqq p\leqq n\,, \tag{5.6}$$

$$\beta_p(M)=\beta_{p-1}(B)-\beta_{p+1}(B)\,,\qquad n+1\leqq p\leqq 2n \tag{5.7}$$

(see also Tanno [11]). Then we get

$$\chi(B)=\sum_{i=0}^{2n}(-1)^i\beta_i(B)$$
$$=\sum_{p=0}^{n-1}2(-1)^p\beta_p(B)+(-1)^n\beta_n(B)$$
$$=\sum_{p=0}^{n-3}2(-1)^p\beta_p(B)+3(-1)^{n-2}\beta_{n-2}(B)$$
$$+2(-1)^{n-1}\beta_{n-1}(B)-(-1)^n\beta_{n-2}(B)+(-1)^n\beta_n(B)$$
$$=\sum_{p=0}^{n-3}2(-1)^p\beta_p(B)+3(-1)^{n-2}\beta_{n-2}(B)$$
$$+2(-1)^{n-1}\beta_{n-1}(B)+(-1)^n\beta_n(M)\,.$$

Continuing this step we have (5.5). q. e. d.

An orthonormal frame ($\xi=e_0, e_1, \varphi e_1=e_{n+1}, \cdots, e_n, \varphi e_n=e_{2n}$) is called a φ-frame. With respect to a φ-frame, we have

$$(\varphi_{AB})=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -E\\ 0 & E & 0\end{pmatrix}, \tag{5.8}$$

where E denotes an $n\times n$ unit matrix.

§ 6. The special case where $\dim M=3$.

(4.9) with $i=1$ and $j=2$ is

$$\pi^*(\Omega'_{12})=\Omega_{12}-\varphi_{12}dw_0-\sum\varphi_{1k}w_k\wedge\varphi_{2l}w_l\,. \tag{6.1}$$

With respect to a φ-frame field, we have

$$\pi^*(\Omega'_{12})=\Omega_{12}-2w_1\wedge w_2-w_1\wedge w_2\,. \tag{6.1$'$}$$

Therefore

$$l(\xi)\int_B\Omega'_{12}=\int_M\Omega_{12}\wedge w_0-3\int_M w_1\wedge w_2\wedge w_3\,. \tag{6.2}$$

That is,

$$l(\xi)\chi(B)=(-1/2\pi)\Big[\int_M\Omega_{12}\wedge w_0-3\,\mathrm{Vol}\,(M)\Big], \tag{6.3}$$

where $\mathrm{Vol}\,(M)$ denotes the total volume of (M^{2n+1}, g). By (1.2), we have $R_{A012}=0$, etc., and hence we have

$$\Omega_{12}=(1/2)\sum R_{12AB}\,w_A\wedge w_B=R_{1212}\,w_1\wedge w_2 \tag{6.4}$$

where $-R_{1212}=K(e_1, e_2)$ is the sectional curvature for the (e_1, e_2)-plane. Consequently, we get

THEOREM B. *Let* (M^3, ξ, g) *be a 3-dimensional compact Sasakian manifold with regular* ξ. *Then*

(6.5) $$\frac{1}{l(\xi)2\pi}\Big[\int_M K(e_1, \varphi e_1)dM + 3\,\mathrm{Vol}(M)\Big] = 2\beta_0(M) - \beta_1(M).$$

A Riemannian manifold (M^{2n+1}, g) admitting a unit Killing vector ξ satisfying (1.5) is called a K-contact Riemannian manifold. Every K-contact Riemannian manifold of 3-dimension is Sasakian (Tanno [**10**]), and so Theorem B in the introduction is equivalent to the above one.

EXAMPLE. A unit sphere S^{2n+1} admits the standard Sasakian structure ξ (Sasaki-Hatakeyama [8]). For S^3, we have $l(\xi)=2\pi$, $K(e_1, \varphi e_1)=1$ and $\mathrm{Vol}(S^3)=2\pi^2$. On the other hand, $\beta_1(S^3)=\beta_2(S^3)=0$ and $\beta_0(S^3)=\beta_3(S^3)=1$.

§ 7. Special case where $\dim M = 5$.

If $\dim M=5$, we have

(7.1) $$\frac{1}{l(\xi)2^4\pi^2 2}\int_M 8[\pi^*(\Omega'_{12})\wedge\pi^*(\Omega'_{34}) + \pi^*(\Omega'_{13})\wedge\pi^*(\Omega'_{42}) + \pi^*(\Omega'_{14})\wedge\pi^*(\Omega'_{23})]\wedge w_0 = 3\beta_0(M) - 2\beta_1(M) + \beta_2(M).$$

If we take a φ-frame field, we have $\pi^*(\Omega'_{12}) = \Omega_{12} - w_3\wedge w_4$, $\pi^*(\Omega'_{14}) = \Omega_{14} - w_2\wedge w_3$, $\pi^*(\Omega'_{23}) = \Omega_{23} - w_1\wedge w_4$, $\pi^*(\Omega'_{34}) = \Omega_{34} - w_1\wedge w_2$, $\pi^*(\Omega'_{13}) = \Omega_{13} - 2w_2\wedge w_4 - 3w_1\wedge w_3$, and $\pi^*(\Omega'_{24}) = \Omega_{24} - 2w_1\wedge w_3 - 3w_2\wedge w_4$. Hence, we have

(7.2) $$\begin{aligned}\frac{1}{4\pi^2 l(\xi)}\int_M [\Omega_{12}\wedge\Omega_{34} &+ \Omega_{13}\wedge\Omega_{42} + \Omega_{14}\wedge\Omega_{23} + 3w_1\wedge w_3\wedge\Omega_{24} \\ &+ 3w_2\wedge w_4\wedge\Omega_{13} + 15w_1\wedge w_2\wedge w_3\wedge w_4 - w_1\wedge w_2\wedge\Omega_{12} \\ &+ 2w_1\wedge w_3\wedge\Omega_{13} - w_1\wedge w_4\wedge\Omega_{14} - w_2\wedge w_3\wedge\Omega_{23} \\ &+ 2w_2\wedge w_4\wedge\Omega_{24} - w_3\wedge w_4\wedge\Omega_{34}]\wedge w_0 \\ &= 3 - 2\beta_1(M) + \beta_2(M).\end{aligned}$$

EXAMPLE. For S^5, we have $l(\xi)=2\pi$, $\Omega_{ij} = -w_i\wedge w_j$ $(i, j=1, \cdots, 4)$ and $\mathrm{Vol}(S^5)=\pi^3$. On the other hand, $\beta_i(M)=0$ $(i=1, \cdots, 4)$ and $\beta_0(M)=\beta_5(M)=1$.

§ 8. Remarks.

(i) If (M, g) is of constant curvature k, we have

(8.1) $$R(X, Y)Z = k[g(X, Z)Y - g(Y, Z)X].$$

If a Killing vector ξ of non-zero constant length satisfies

$$R(X, \xi)Z = k[g(X, Z)\xi - g(\xi, Z)X], \qquad k > 0,$$

then we can assume the length of ξ is 1 and we can change the Riemannian metric g by $g^* = (1/k)g$ and ξ by $\xi^* = \sqrt{k}\,\xi$, so that (M, ξ^*, g^*) is a Sasakian manifold.

Every complete Riemannian manifold of constant curvature 1 and odd dimension admits a Sasakian structure (Wolf [**13**], Tanno [**12**]).

(ii) Let N be a 4-dimensional compact orientable Riemannian manifold with Betti numbers $\beta_p(N)$. Let S be a circle of length l and let $N\times S$ be the Riemannian product of N and S. A unit tangent vector field on S defines a unit Killing vector ξ on $M^5 = N\times S$ in the natural way. Its dual 1-form w_0 is parallel. Then

$$\frac{1}{l\,2^4\pi^2 2}\int_M \sum \varepsilon_{i_1\cdots i_4}\Omega_{i_1i_2}\wedge\Omega_{i_3i_4}\wedge w_0 = \chi(N) = 5\beta_0(M) - 3\beta_1(M) + \beta_2(M), \tag{8.2}$$

where $\beta_r(M)$ denotes the r-th Betti number of $M = N\times S$ and we have used $\beta_r(M) = \sum_{p+q=r}\beta_p(N)\beta_q(S)$. One sees the difference between the right hand sides of (7.1) and (8.2).

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