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On Dirichlet series whose coefficients are class numbers of integral binary cubic forms

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Introduction

0.1. We call, after M. Sato, a pair (G, V) of a finite dimensional complex vector space V and an algebraic subgroup G in $GL(V)$ a *prehomogeneous vector space* when there exists a G-orbit in V of the same dimension as V. M. Sato constructed a systematic theory of prehomogeneous vector spaces, and as an application of his results, attached certain "distribution valued zeta-functions" to prehomogeneous vector spaces satisfying several additional conditions.[1] It was also pointed out by Sato that there would exist certain Dirichlet series with functional equations which are intimately related to them, when G is defined over an algebraic number field. In the present paper we give an example of such Dirichlet series whose coefficients have arithmetical significance.

0.2. To state the main result in an explicit form, let L denote the lattice of integral binary cubic forms:

$$L=\{F(u, v)=x_1u^3+x_2u^2v+x_3uv^2+x_4v^3\,;\ (x_1, x_2, x_3, x_4)\in \boldsymbol{Z}^4\}\,.$$

The lattice L becomes an $SL(2, \boldsymbol{Z})$-module if we put

$$\gamma\cdot F(u, v)=F((u, v)\gamma) \qquad (\gamma\in SL(2, \boldsymbol{Z}),\ F\in L)\,.$$

We call two elements x, y of L equivalent if there exists a $\gamma\in SL(2, \boldsymbol{Z})$ such that $x=\gamma\cdot y$. For every integer $m\neq 0$, we denote by L_m the set of integral binary cubic forms whose discriminants are m. It is known that there exist only finitely many equivalence classes in L_m. We denote by $h(m)$ the number of equivalence classes in L_m. We put

$$\hat{L}=\{F(u, v)=x_1u^3+x_2u^2v+x_3uv^2+x_4v^3\in L\,;\ x_2, x_3\in 3\boldsymbol{Z}\}\,.$$

Then $\hat{L}$ is an $SL(2, \boldsymbol{Z})$-submodule of L. We denote by $\hat{h}(m)$ the number of equivalence classes in L_m which are contained in $\hat{L}$. Now we define four Dirichlet series as follows:

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1) A Survey of "the theory of prehomogeneous vector spaces" is given in [8].

$$\xi_1(L, s)=\sum_{n=1}^{\infty}\frac{h(n)}{n^s}-3^{-1}\sum_{(x,y)\in \mathbf{Z}^2-\{0\}}\frac{1}{(9x^2+3xy+y^2)^{2s}},$$

$$\xi_2(L, s)=\sum_{n=1}^{\infty}\frac{h(-n)}{n^s},$$

$$\xi_1(\hat{L}, s)=\sum_{n=1}^{\infty}\frac{\hat{h}(n)}{n^s}-3^{-1-4s}\sum_{(x,y)\in \mathbf{Z}^2-\{0\}}\frac{1}{(x^2+xy+y^2)^{2s}},$$

$$\xi_2(\hat{L}, s)=\sum_{n=1}^{\infty}\frac{\hat{h}(-n)}{n^s}.$$

In the second chapter of this paper we prove the following:

THEOREM. (i) *The above four Dirichlet series converge absolutely for* $\operatorname{Re} s>1$. *They can be continued analytically to functions holomorphic in the whole plane except for simple poles at* 1 *and* 5/6. *Furthermore, they satisfy the following functional equation:*

$$\begin{pmatrix}\xi_1(L, 1-s)\\ \xi_2(L, 1-s)\end{pmatrix}$$

$$=\Gamma\left(s-\frac{1}{6}\right)\Gamma(s)^2\Gamma\left(s+\frac{1}{6}\right)2^{-1}3^{6s-2}\pi^{-4s}\begin{pmatrix}\sin 2\pi s & \sin \pi s\\ 3\sin \pi s & \sin 2\pi s\end{pmatrix}\begin{pmatrix}\xi_1(\hat{L}, s)\\ \xi_2(\hat{L}, s)\end{pmatrix}.$$

(ii) *Their residues at* $s=1$ *and at* $s=\frac{5}{6}$ *are given in the following table.*

Table of Residues

of / at	$\xi_1(L, s)$	$\xi_2(L, s)$	$\xi_1(\hat{L}, s)$	$\xi_2(\hat{L}, s)$
$s=1$	$\frac{\pi^2}{9}$	$\frac{\pi^2}{6}$	$\frac{\pi^2}{162}$	$\frac{\pi^2}{81}$
$s=\frac{5}{6}$	$\frac{\sqrt{3}}{18}r$	$\frac{1}{6}r$	$\frac{\sqrt{3}}{162}r$	$\frac{1}{54}r$

(*We put* $r=\zeta\left(\frac{2}{3}\right)\frac{\Gamma\left(\frac{1}{3}\right)(2\pi)^{1/3}}{\Gamma\left(\frac{2}{3}\right)}$.)

0.3. To make the present paper self-contained, we briefly describe some of Sato's results with his permission. The author wishes to express his most sincere gratitude to Professor M. Sato who gave Lectures entitled "The theory of prehomogeneous vector spaces and zeta functions" in 1969 at the University of Tokyo and generously permitted the author to use his results still unpublished in European language. The author also wishes to thank heartily to Professor K. Aomoto who read the manuscript and made many useful suggestions. A short summary of the present paper was given in [**9**].

Notation

We denote by $\boldsymbol{Z}, \boldsymbol{Q}, \boldsymbol{R}$ and $\boldsymbol{C}$ the ring of integers, the rational number field, the real number field and the complex number field. We use det, Sup. and Supp. as abbreviation of determinant, supremum and support, respectively.

We denote by $\Gamma(z)$, $\zeta(z)$, the usual Gamma function and the Riemann zeta function.

For a C^∞-manifold X, $C(X)$ (resp. $C^\infty(X)$) denotes the space of all complex valued continuous (resp. C^∞-) functions on X, and $C_0(X)$ (resp. $C_0^\infty(X)$) the space of all functions in $C(X)$ (resp. $C^\infty(X)$) with compact support. When X has a (prescribed) measure we denote by $L_1(X)$ the space of integrable functions on X.

Chapter 1. An introduction to the theory of prehomogeneous vector spaces

§ 1. Fundamental properties of prehomogeneous vector spaces

In this chapter, we describe a Sato's theory which is necessary for our later applications.[2)]

Let V be a complex vector space of dimension n and G be a complex algebraic subgroup of $GL(V, \boldsymbol{C})$. We write the action of G on V as follows: $g\cdot x$ ($g\in G$, $x\in V$). For any $x\in V$, we denote by G_x the isotropy subgroup of x in G, i.e., $G_x=\{g\in G\,;\, g\cdot x=x\}$. We call the pair (G, V) a *prehomogeneous vector space* when there exists an $x\in V$ such that

$$\dim G-\dim G_x=n\,. \tag{1.1}$$

We call $x\in V$ a singular point of V when the equality (1.1) does not hold. We denote by S the set of all singular points of V, $S=\{x\in V\,;\, \dim G-\dim G_x<n\}$. It is easy to see that S is a G-invariant proper algebraic subset of V and that $(V-S)$ is a single G-orbit. For every $g\in G$ and a rational function R on V, we denote by R_g the rational function on V defined as follows: $R_g(x)=R(g\cdot x)$ ($x\in V$). We call a non-zero rational function R a *relative-invariant* on V when there exists a rational character μ of G such that $R_g=\mu(g)R$ ($\forall g\in G$). We say that R is a *relative-invariant* on V *corresponding to* a rational character μ.

It is obvious that relative-invariants corresponding to the same rational character coincide with each other up to constant factors. In the following, we assume that (G, V) is a prehomogeneous vector space. Furthermore we assume that the following two additional conditions are satisfied for the pair

2) Cf. [8].

(G, V).

1. *G is a reductive algebraic group defined over the real number field and V has an $\boldsymbol{R}$-structure.*

(In the following we denote by $G_{\boldsymbol{R}}$ (resp. $V_{\boldsymbol{R}}$) the set of $\boldsymbol{R}$-rational points of G (resp. V).)

2. *The set of singular points S in V is an irreducible hypersurface of V.*

Then there exists a polynomial P which is $\boldsymbol{C}$-irreducible and $\boldsymbol{R}$-valued on $V_{\boldsymbol{R}}$ such that

$$S=\{x\in V\,;\ P(x)=0\}\,. \tag{1.2}$$

PROPOSITION 1.1. (i) *Every relative-invariant on V is of the form cP^m $(c\in \boldsymbol{C}^\times,\ m\in \boldsymbol{Z})$.*

(ii) *P is a homogeneous polynomial.*

PROOF. (i) For every $g\in G$, P_g is an irreducible polynomial whose set of zeros is $g^{-1}\cdot S=S$. Since S is an irreducible hypersurface, P_g coincides with P up to a constant factor. Hence, there exists a rational character χ of G such that $P_g=\chi(g)P$ $(\forall g\in G)$. Let R be any relative-invariant on V. Then the set of zeros and poles of R is a proper algebraic subset of V which is G-stable. Therefore it is contained in S. Consequently, any prime factor of R must coincide with P. So, R is of the form cP^m, for some $c\in \boldsymbol{C}^\times$, $m\in \boldsymbol{Z}$.

(ii) For any $t\in \boldsymbol{C}^\times$, we denote by $P^{(t)}$ the polynomial on V defined as follows: $P^{(t)}(x)=P(t\cdot x)$. Since $P_g=\chi(g)P$ for $g\in G$, we have $P_g^{(t)}=\chi(g)P^{(t)}$ $(\forall g\in G)$. Therefore, $P^{(t)}$ coincides with P up to a constant factor. Hence, P is homogeneous. q. e. d.

In the following, we denote by χ the rational character of G such that

$$P(g\cdot x)=\chi(g)P(x) \qquad (\forall g\in G,\ x\in V)\,. \tag{1.3}$$

We also denote by d the degree of the homogeneous polynomial P. Let V^* be the dual space of V. Thus G acts on V^*. We denote the action of G on V^* as follows: $g^*\cdot y$ $(g\in G,\ y\in V^*)$. We have thus

$$\langle g\cdot x, g^*\cdot y\rangle=\langle x, y\rangle \qquad (\forall g\in G,\ x\in V,\ y\in V^*)\,.$$

We put $V_{\boldsymbol{R}}^*=\{y\in V^*\,;\ \langle x, y\rangle\in \boldsymbol{R}\ (\forall x\in V_{\boldsymbol{R}})\}$.

LEMMA 1.1. *There exist a linear mapping τ of V onto V^* and an involution ι of G which satisfy the following conditions:*

(i) $\tau(V_{\boldsymbol{R}})=V_{\boldsymbol{R}}^*$,

(ii) $\tau(g\cdot x)=(g^\iota)^*\cdot\tau(x)$ $(\forall g\in G)$,

(iii) $\chi(g^\iota)=\chi^{-1}(g)$.

PROOF. Since G is a reductive algebraic group defined over $\boldsymbol{R}$, we may assume that[3] the matrix form of the group $G_{\boldsymbol{R}}$ with respect to a suitable

3) See [3].

base $\{e_1, \cdots, e_n\}$ of $V_{\boldsymbol{R}}$ has the following property: Every element g of $G_{\boldsymbol{R}}$ can be uniquely expressed as $g=k\cdot p$, where $k\in G_{\boldsymbol{R}}\cap O(n, \boldsymbol{R})$ and $p\in G_{\boldsymbol{R}}\cap S_+(n, \boldsymbol{R})$. ($O(n, \boldsymbol{R})$ is the group of real orthogonal matrices and $S_+(n, \boldsymbol{R})$ is the cone of positive definite $n\times n$ symmetric matrices.) Then it is clear that ${}^tg^{-1}\in G_{\boldsymbol{R}}$ whenever $g\in G_{\boldsymbol{R}}$. Therefore if we put $g^{\iota}={}^tg^{-1}$ $(g\in G)$, ι defines an involution of G. Let $\{f_1, \cdots, f_n\}$ be the base of $V_{\boldsymbol{R}}^*$ dual to $\{e_1, \cdots, e_n\}$. We define a linear mapping τ of V into V^* as follows: $\tau\left(\sum_{i=1}^n x_ie_i\right)=\sum_{i=1}^n x_if_i$. Then it is clear that $\tau(V_{\boldsymbol{R}})=V_{\boldsymbol{R}}^*$ and that $\tau(g\cdot x)=(g^{\iota})^*\tau(x)$. When $G_{\boldsymbol{R}}\ni g=kp$ ($k\in G_{\boldsymbol{R}}\cap O(n, \boldsymbol{R})$, $p\in G_{\boldsymbol{R}}\cap S_+(n, \boldsymbol{R})$), we have $\chi(g^{\iota})=\chi({}^tg^{-1})=\chi(kp^{-1})=\chi(k)\chi(p^{-1})$. Since P is real valued on $V_{\boldsymbol{R}}$, the equality (1.3) implies that χ is real valued on $G_{\boldsymbol{R}}$. Hence $\chi(k)=\pm 1$, and we have

$$\chi(g^{\iota})=\chi(k)^{-1}\chi(p)^{-1}=\chi(g)^{-1} \qquad (\forall g\in G_{\boldsymbol{R}}),$$

and

$$\chi(g^{\iota})=\chi(g)^{-1} \qquad (\forall g\in G).$$

q. e. d.

We note that one can choose $\{e_1, \cdots, e_n\}$ so that $P(e_1)\neq 0$. In the following we choose always such a base $\{e_i\}$.

It follows from Lemma 1.1 that (G, V^*) is also a prehomogeneous vector space. In the following we choose a linear mapping τ of V onto V^* and an involution ι of G which satisfy conditions of Lemma 1.1 and fix them once for all. We put $Q(y)=P(\tau^{-1}y)$ $(y\in V^*)$. Then Q is a homogeneous polynomial of degree d on V^* which is real valued on $V_{\boldsymbol{R}}^*$ and satisfies the following equality:

$$Q(g^*\cdot y)=\chi(g)^{-1}Q(y) \qquad (\forall g\in G,\ y\in V^*).$$

Therefore Q is a relative-invariant on V^* which corresponds to the rational character χ^{-1}. Hence, Q is, up to a real constant factor, independent of the choice of τ.

We denote by $Q(\nabla_x)$ the differential operator with constant coefficients on $V_{\boldsymbol{R}}$ such that

$$Q(\nabla_x)e^{\langle x,y\rangle}=Q(y)e^{\langle x,y\rangle} \qquad (\forall y\in V_{\boldsymbol{R}}^*).$$

It is easy to see that there exists a polynomial $b_\nu(s)$ with real coefficients of degree at most $d\nu$ such that

$$(1.4) \qquad Q(\nabla_x)^\nu P^s(x)=b_\nu(s)P^{s-\nu}(x) \qquad (\nu=1, 2, \cdots;\ s\in \boldsymbol{C})^{4)}.$$

It is clear that $b_\nu(s)$ is, up to a non-zero real constant factor, independent of the choice of τ and that $b_\nu(s)=b_1(s)b_{\nu-1}(s-1)$.

PROPOSITION 1.2. (i) *The degree of $b_1(s)$ is d, the degree of P.*

4) This is an equality between two functions on the universal covering space of $V-S$.

(ii) *If we put* $\chi_0(g)=\det(g)$, *we have* $\chi_0^2=\chi^{\frac{2n}{d}}$ (*d is a divisor of* $2n$).

PROOF. (i) We can choose τ and a base $\{e_1,\cdots,e_n\}$ of $V_{\boldsymbol{R}}$ such that $\langle e_i,\tau e_j\rangle=\delta_{ij}$ $(i,j=1,\cdots,n)$ and that $P(e_1)\neq 0$. We identify V and V^* with $\boldsymbol{C}^n$ via the bases $\{e_1,\cdots,e_n\}$ and $\{\tau e_1,\cdots,\tau e_n\}$ respectively. Then $V_{\boldsymbol{R}}$ and $V_{\boldsymbol{R}}^*$ are identified with $\boldsymbol{R}^n$. We also identify G with a subgroup of $GL(n,\boldsymbol{C})$. We put

$$P^{\nu}\Big(\sum_{i=1}^{n}x_ie_i\Big)=P^{\nu}(x_1,\cdots,x_n)=\sum_{i_1+\cdots+i_n=d\nu}a^{(\nu)}_{i_1\cdots i_n}x_1^{i_1}\cdots x_n^{i_n}$$

$(a^{(\nu)}_{i_1\cdots i_n}\in\boldsymbol{R},\ (x_1,\cdots,x_n)\in\boldsymbol{C}^n)$. Then we have

$$Q\Big(\sum_{i=1}^{n}y_i\tau e_i\Big)=P(y_1,\cdots,y_n)$$

and

$$Q(\nabla_x)^{\nu}=\sum a^{(\nu)}_{i_1\cdots i_n}\Big(\frac{\partial}{\partial x_1}\Big)^{i_1}\cdots\Big(\frac{\partial}{\partial x_n}\Big)^{i_n}.$$

Since $P(e_1)\neq 0$, $a^{(1)}_{d0\cdots0}\neq 0$. We put $a=a^{(1)}_{d0\cdots0}$. Since $a^{(\nu)}_{d\nu0\cdots0}=a^{\nu}$, it follows

$$b_{\nu}(\nu)=Q(\nabla_x)^{\nu}P(x)^{\nu}=\sum(a^{(\nu)}_{i_1\cdots i_n})^2(i_1)!\cdots(i_n)!\geqq a^{2\nu}(d\nu)!\,.$$

Let the degree of $b_1(s)$ be d'. Then there exists a constant $c>0$ such that $|b_1(s)|\leqq c|s|^{d'}$ for $|s|\geqq 1$. We get

$$c^{\nu}(\nu\,!)^{d'}\geqq|b_1(\nu)b_1(\nu-1)\cdots b_1(1)|=|b_{\nu}(\nu)|\geqq a^{2\nu}(d\nu)!\qquad(\nu=1,2,\cdots).$$

Hence we must have $d'\geqq d$. It is obvious that $d'\leqq d$. Hence we get $d'=d$.

(ii) We denote by b_0 the coefficient of s^d in $b_1(s)$. Then $b_0\neq 0$. Since $Q(\nabla_x)P^s=b_1(s)P^{s-1}$, we have

$$b_0P^{d-1}=P\Big(\frac{\partial P}{\partial x_1},\cdots,\frac{\partial P}{\partial x_n}\Big).\tag{1.5}$$

Now we define the map φ of $V-S\subset\boldsymbol{C}^n$ into $V=\boldsymbol{C}^n$ as follows:

$$\varphi(x)=\Big(\frac{1}{P}\frac{\partial P}{\partial x_1},\cdots,\frac{1}{P}\frac{\partial P}{\partial x_n}\Big).$$

Then, it follows from (1.5) that φ maps $V-S$ into $V-S$. Furthermore it is easily seen that $\varphi(g\cdot x)={}^tg^{-1}\cdot\varphi(x)$ for every $g\in G$. Since $V-S$ is a single G-orbit and the mapping $g\to{}^tg^{-1}$ gives an automorphism of G, φ maps $V-S$ onto $V-S$. Hence the Jacobian matrix

$$J(x)=\begin{pmatrix}\frac{\partial}{\partial x_1}\frac{1}{P}\frac{\partial P}{\partial x_1},&\cdots,&\frac{\partial}{\partial x_1}\frac{1}{P}\frac{\partial P}{\partial x_n}\\ \frac{\partial}{\partial x_n}\frac{1}{P}\frac{\partial P}{\partial x_1},&\cdots,&\frac{\partial}{\partial x_n}\frac{1}{P}\frac{\partial P}{\partial x_n}\end{pmatrix}$$

is non-degenerate on a Zariski open subset of $V-S$. It is easy to see that $J(g\cdot x)={}^tg^{-1}J(x)g^{-1}$ $(\forall g\in G)$. So, $\det J(x)$ is a rational function on V which does not vanish identically on V and satisfies

$$\det J(g\cdot x)=\chi_0^{-2}(g)\det J(x) \qquad (\forall g\in G)\,.$$

Thus, $\det J(x)$ is a relative-invariant on V. It follows from Proposition 1.1 that we can find $c\in \boldsymbol{C}^{\times}$ and $m\in \boldsymbol{Z}$ such that $\det J(x)=cP^m(x)$. If $t\in\boldsymbol{C}^{\times}$, we have $\det J(t\cdot x)=t^{-2n}\det J(x)$ and $P^m(t\cdot x)=t^{md}P(x)$. Hence we get $m=-\dfrac{2n}{d}$. Therefore d is a divisor of $2n$ and we have $\chi_0^2=\chi^{\frac{2n}{d}}$. q. e. d.

We put $(x, y)=\langle x, \tau y\rangle$ $(x, y\in V)$. In the following, we identify V with V^* via the bilinear form $(\ ,\ )$. We have

$$Q(\nabla_x)e^{(x,y)}=P(y)e^{(x,y)} \qquad (\forall y\in V)\,, \tag{1.6}$$

$$(g\cdot x, g^{\iota}\cdot y)=(x, y) \qquad (x, y\in V,\ g\in G)\,. \tag{1.7}$$

Furthermore we put $b_1(s)=b_0\prod_{i=1}^{d}(s-c_i)$ $(b_0\in\boldsymbol{R}^{\times},\ c_i\in\boldsymbol{C})$ and

$$\gamma(s)=\prod_{i=1}^{d}\Gamma(s-c_i+1)\,. \tag{1.8}$$

Then we get

$$b_\nu(s)=b_0^{\nu}\frac{\gamma(s)}{\gamma(s-\nu)} \qquad (\nu=1, 2, \cdots)\,. \tag{1.9}$$

§ 2. Fourier transforms of the complex powers of relative-invariants

1. Let the connected components of $(V-S)_{\boldsymbol{R}}$ be $V_1, \cdots, V_l$. Then $(V-S)_{\boldsymbol{R}}=V_1\cup\cdots\cup V_l$ and V_i $(1\leqq i\leqq l)$ is a single $(G_{\boldsymbol{R}})_0$-orbit. ($(G_{\boldsymbol{R}})_0$ is the connected component of the neutral element of the Lie group $G_{\boldsymbol{R}}$.) Since P is real valued on $V_{\boldsymbol{R}}$ and does not vanish on $(V-S)_{\boldsymbol{R}}$, the signature of P on V_i does not change. We put

$$\varepsilon_i=\operatorname*{sgn}_{x\in V_i}P(x) \qquad (i=1, \cdots, l)\,. \tag{1.10}$$

We denote by $\mathcal{S}(V_{\boldsymbol{R}})$ the space of rapidly decreasing functions on $V_{\boldsymbol{R}}$. The space $\mathcal{S}(V_{\boldsymbol{R}})$ becomes a $G_{\boldsymbol{R}}$-module if we put $(g\cdot f)(x)=f(g^{-1}\cdot x)$. We define the Fourier transform $\hat{f}$ of $f\in\mathcal{S}(V_{\boldsymbol{R}})$ as follows:

$$\hat{f}(x)=\int_{V_{\boldsymbol{R}}}f(y)e^{2\pi\sqrt{-1}(x,y)}dy\,. \tag{1.11}$$

We have, by Proposition 1.2,

$$\widehat{g\cdot f}=|\chi_0(g)|(g^{\iota})\cdot\hat{f}=|\chi^{\frac{n}{d}}(g)|(g^{\iota})\cdot\hat{f} \qquad (\forall g\in G_{\boldsymbol{R}})\,. \tag{1.12}$$

For any $f\in S(V_{\boldsymbol{R}})$ and $s\in\boldsymbol{C}$, we put

$$F_i(s, f)=\frac{1}{\gamma(s)}\int_{V_i}|P|^s f(x)dx\,. \tag{1.13}$$

Clearly, when $\operatorname{Re} s>0$, $F_i(s, f)$ is well-defined and is a holomorphic function of s.

PROPOSITION 1.3. (i) *$F_i(s, f)$ can be continued analytically as an entire function of s.*

(ii) $F_i(s, Q(\nabla_x)^\nu f)=(-1)^{d\nu}\varepsilon_i^\nu b_0^\nu F_i(s-\nu, f)$.

(iii) $F_i(s, g\cdot f)=\chi^{s+\frac{n}{d}}(g)F_i(s, f)$ $(\forall g\in(G_{\boldsymbol{R}})_0)$.

(iv) *The mapping $f\rightarrow F_i(s, f)$ defines a tempered distribution on $V_{\boldsymbol{R}}$.*

PROOF. From (1.4) and (1.10) we get

$$Q(\nabla_x)^\nu|P|^s=\varepsilon_i^\nu b_\nu(s)|P|^{s-\nu}.$$

Hence, when $\operatorname{Re} s$ is sufficiently large, we get the following equality by partial integration (cf. (1.9)).

$$\begin{aligned} F_i(s, Q(\nabla_x)^\nu f) &= \frac{(-1)^{d\nu}}{\gamma(s)}\varepsilon_i^\nu b_\nu(s)\int_{V_i}|P|^{s-\nu}f\,dx \\ &=(-1)^{d\nu}\varepsilon_i^\nu b_0^\nu F_i(s-\nu, f). \end{aligned}$$

Therefore we have

$$F_i(s, f)=(-1)^{d\nu}\varepsilon_i^\nu b_0^{-\nu}F_i(s+\nu, Q(\nabla_x)^\nu f(x)) \qquad (\nu=1, 2, \cdots),$$

when $\operatorname{Re} s$ is sufficiently large.

By this equality, we can continue analytically $F_i(s, f)$ as an entire function of s. Thus, (i) and (ii) are proved. Since V_i is a single $(G_{\boldsymbol{R}})_0$-orbit, we have, by Proposition 1.2,

$$\begin{aligned} F_i(s, g\cdot f) &= \frac{1}{\gamma(s)}\int_{V_i}|P(x)|^s f(g^{-1}\cdot x)dx \\ &=\chi(g)^s\chi_0(g)F_i(s, f) \\ &=\chi(g)^{s+\frac{n}{d}}F_i(s, f) \qquad (\forall g\in(G_{\boldsymbol{R}})_0) \end{aligned}$$

when $\operatorname{Re} s>0$. Hence we get (iii).

Take a positive constant $C_1>0$ such that $|P(x)|\leqq C_1(1+\|x\|^2)^{\frac{d}{2}}$ $(\forall x\in V_{\boldsymbol{R}})$, ($\|\ \|^2$ is a fixed positive definite quadratic form on $V_{\boldsymbol{R}}$). When $\operatorname{Re} s>0$, we have, for a positive constant C_2,

$$\begin{aligned} |F_i(s, f)| &\leqq \frac{1}{|\gamma(s)|}\int_{V_1}|P(x)|^{\operatorname{Re} s}|f(x)|\,dx \\ &\leqq \frac{C_1^{\operatorname{Re} s}}{|\gamma(s)|}\operatorname*{Sup.}_{x\in V_{\boldsymbol{R}}}\{(1+\|x\|^2)^{\frac{d}{2}\operatorname{Re} s+\frac{n+1}{2}}|f(x)|\}\int_{V_{\boldsymbol{R}}}(1+\|x\|^2)^{-\frac{n+1}{2}}dx \\ &=\frac{C_2C_1^{\operatorname{Re} s}}{|\gamma(s)|}\operatorname*{Sup.}_{x\in V_{\boldsymbol{R}}}\{(1+\|x\|^2)^{\frac{d}{2}\operatorname{Re} s+\frac{n+1}{2}}|f(x)|\}. \end{aligned}$$

When $-m\leqq\operatorname{Re} s<-m+1$ $(m=1, 2, \cdots)$, we get

$$|F_i(s, f)| = |b_0|^{-m} |F_i(s+m, Q(\nabla_x)^m f)|$$
$$\leqq \frac{C_2 |b_0|^{-m} C_1^{\mathrm{Re}\,(s+m)}}{|\gamma(s+m)|} \operatorname*{Sup.}_{x \in V_R} (1+\|x\|^2)^{\frac{d}{2}\mathrm{Re}\,(s+m)+\frac{n+1}{2}} |Q(\nabla_x)^m f(x)| .$$

Hence, it is obvious that the mapping $f \to F_i(s, f)$ defines a tempered distribution on V_R. q. e. d.

COROLLARY. *Let* $\| \ \|^2$ *be a positive definite quadratic form on* V_R. *Then there exist positive numbers* C_1 *and* C_2 *such that the following inequalities hold: when* $m-1 \leqq \mathrm{Re}\, s < m$ $(m = 1, 2, \cdots)$,

$$|F_i(s, f)| \leqq C_2 \frac{C_1^m}{|\gamma(s)|} \operatorname*{Sup.}_{x \in V_R} \{ (1+\|x\|^2)^{\frac{md+n+1}{2}} |f(x)| \} ;$$

and when $-m \leqq \mathrm{Re}\, s \leqq -m+1$ $(m = 1, 2, \cdots)$,

$$|F_i(s, f)| \leqq C_2 \frac{C_1^{m+\nu}}{|\gamma(s+m+\nu)|} \operatorname*{Sup.}_{x \in V_R} \{ (1+\|x\|^2)^{\frac{d(1+\nu)+n+1}{2}} |Q(\nabla_x)^{m+\nu} f(x)| \}$$
$$(\nu = 0, 1, 2, \cdots) .$$

We state three lemmas which are necessary for the proof of Theorem 1.

LEMMA 1.2. *Let* T *be a distribution on* V_i *such that*

$$T(g \cdot f) = \chi(g)^{s+\frac{n}{d}} T(f) \qquad (f \in C_0^\infty(V_i))$$

for every $g \in (G_R)_0$. *Then there exists a* $c \in \boldsymbol{C}$ *such that*

$$T(f) = c \int_{V_i} |P|^s f(x) dx .$$

PROOF. Put $\tilde{T} = |P|^{-s-\frac{n}{d}} T$. Then $\tilde{T}$ is a distribution on V_i such that $\tilde{T}(g \cdot f) = \tilde{T}(f)$ $(\forall g \in (G_R)_0)$. On the other hand, it follows from Proposition 1.2 that V_i is a homogeneous space of $(G_R)_0$ with an invariant measure $|P|^{-\frac{n}{d}} dx$. Therefore, by Theorem 3.1 of Bruhat [4], there exists a $c \in \boldsymbol{C}$ such that

$$\tilde{T}(f) = c \int_{V_i} f(x) |P|^{-\frac{n}{d}} dx \qquad (\forall f \in C_0^\infty(V_i)) .$$

Thus we get

$$T(f) = c \int_{V_i} |P|^s f(x) dx . \qquad \text{q. e. d.}$$

LEMMA 1.3. *Let* T *be a distribution on* $\boldsymbol{R}^n$ *whose order* r *is finite and whose support is contained in the set of zeros of an infinitely differentiable function* φ *on* $\boldsymbol{R}^n$. *Then* $(\varphi)^L T = 0$ *when* $L > r$.

PROOF. We take an infinitely differentiable function β on $\boldsymbol{R}$ such that

$$\beta(t) = 1 \quad \text{when} \quad |t| \leqq 1$$

and

$$\beta(t) = 0 \quad \text{when} \quad |t| \geqq 2 .$$

Put $U=\{x\in \boldsymbol{R}^n\,;\ \varphi(x)=0\}$. When $f\in C_0^\infty(\boldsymbol{R}^n)$, $f-\beta\Big(\dfrac{1}{\eta}\varphi\Big)f\in C_0^\infty(\boldsymbol{R}^n-U)$ for every $\eta>0$, where we put $\beta\Big(\dfrac{1}{\eta}\varphi\Big)(x)=\beta\Big(\dfrac{1}{\eta}\varphi(x)\Big)$. Since the support of T is contained in U, we have

$$T(f)=T\Big(\beta\Big(\frac{1}{\eta}\varphi\Big)f\Big)\qquad (\forall f\in C_0^\infty(\boldsymbol{R}^n),\ \eta>0)\,.$$

Now we assume that the support of f is contained in a relatively compact open subset W. Since the order of T is r, there exists a positive constant C_1 such that

$$|T(f)|\leqq C_1 \operatorname*{Sup.}_{x\in \boldsymbol{R}^n} \sum_{i_1+\cdots+i_n\leqq r}\left|\frac{\partial^{i_1+\cdots+i_n}}{\partial x_1^{i_1}\cdots\partial x_n^{i_n}} f\right| \qquad (\forall f\in C_0^\infty(W))\,.$$

If L is a natural number which is larger than r, we get, for every $f\in C_0^\infty(W)$,

$$\begin{aligned}|T(\varphi^L f)|&=\left|T\Big(\varphi^L\beta\Big(\frac{1}{\eta}\varphi\Big)f\Big)\right|\\ &\leqq C_1 \operatorname*{Sup.}_{x\in \boldsymbol{R}^n} \sum_{i_1+\cdots+i_n\leqq r}\left|\frac{\partial^{i_1+\cdots+i_n}}{\partial x_1^{i_1}\cdots\partial x_n^{i_n}}\varphi^L\beta\Big(\frac{1}{\eta}\varphi\Big)f\right|.\end{aligned}$$

Since $\beta\Big(\dfrac{1}{\eta}\varphi(x)\Big)=0$ when $|\varphi(x)|\geqq 2\eta$, we have $|T(\varphi^L f)|\leqq C_2(f)\eta^{L-r}$ $(\forall \eta>0)$, where $C_2(f)$ is a positive number which does not depend on η.

Whence, we get $T(\varphi^L f)=0$ $(\forall f\in C_0^\infty(W))$. Since W is an arbitrary relatively compact open subset, we have $\varphi^L T=0$. q. e. d.

LEMMA 1.4. *There exists a $q(t)\in C^\infty(\boldsymbol{R})$ which satisfies the following two conditions:*

(i) *All the derivatives of $q(t)$ are bounded functions on R and $q(t)=0$ for $t<1$.*

(ii) *Put*

$$\lambda(z)=\int_0^\infty t^{z-1}q(t)dt\qquad (\operatorname{Re} z<0)\,.$$

For every pair of positive numbers c_1, c_2 $(c_1>c_2)$, there exists a $c_3>0$ such that the following inequality holds:

$$|\lambda(z)|\geqq c_3\exp\{-\sqrt{|\operatorname{Im} z|}\}\qquad (-c_1<\operatorname{Re} z<-c_2)\,.$$

PROOF. We put

$$\varphi(z)=-\frac{1}{\pi}\int_{-\infty}^{\infty}\sqrt{|t|}\,\frac{1+tz}{i(t-z)}\,\frac{dt}{1+t^2}\qquad (z=x+iy,\ y>0)\,.$$

Then, $\varphi(z)$ is a holomorphic function on the upper half plane and

$$\begin{aligned}\operatorname{Re}\varphi(z)&=-\frac{1}{\pi}\int_{-\infty}^{\infty}\sqrt{|t|}\,\frac{y}{(x-t)^2+y^2}\,dt\\ &=-\frac{\sqrt{y}}{\pi}\int_{-\infty}^{\infty}\frac{\sqrt{\left|t+\dfrac{x}{y}\right|}}{1+t^2}\,dt\,.\end{aligned}$$

Since $\sqrt{|t+x|} \leqq \sqrt{|t|}+\sqrt{|x|}$,

$$\int_{-\infty}^{\infty} \frac{\sqrt{|t+x|}}{1+t^2} dt \leqq \pi\sqrt{|x|}+\delta_1 \qquad \left(\delta_1=\int_{-\infty}^{\infty} \frac{\sqrt{|t|}}{1+t^2} dt\right).$$

On the other hand, when $x \geqq 0$, the following inequality holds:

$$\int_{-\infty}^{\infty} \frac{\sqrt{|t+x|}}{1+t^2} dt \geqq \int_{0}^{\infty} \frac{\sqrt{|t+x|}}{1+t^2} dt \geqq \frac{\pi}{2}\sqrt{|x|}.$$

Similarly, $\int_{-\infty}^{\infty} \frac{\sqrt{|t+x|}}{1+t^2} dt \geqq \sqrt{|x|}\frac{\pi}{2}$, when $x \leqq 0$. Hence, we get

$$-\sqrt{|x|}-\delta_1'\sqrt{|y|} \leqq \operatorname{Re}\varphi(z) \leqq -\frac{1}{2}\sqrt{|x|} \qquad \left(\delta_1'=\frac{1}{\pi}\delta_1\right).$$

Now we put $\lambda(z)=\exp\varphi(-iz)$. Then $\lambda(z)$ is a holomorphic function in the left half plane $\{z;\ \operatorname{Re} z<0\}$, and satisfies the following inequality

$$\exp\{-\sqrt{|\operatorname{Im} z|}-\delta_1'\sqrt{|\operatorname{Re} z|}\} \leqq |\lambda(z)| \leqq \exp\left\{-\frac{1}{2}\sqrt{|\operatorname{Im} z|}\right\}. \tag{1.14}$$

Put $q(t)=\frac{1}{2\pi\sqrt{-1}}\int_{\operatorname{Re} z=x_0<0}\lambda(z)t^{-z}dz$ $(t>0)$. Then one sees by (1.14) that the integral converges absolutely and that $q(t)=0$ for $t<1$. It can be easily seen that $q(t)$ is infinitely differentiable and all the derivatives are bounded. It follows from Mellin's formula

$$\lambda(z)=\int_0^{\infty} t^{z-1}q(t)dt.$$

Since $\lambda(z)$ satisfies the inequality (1.14), $\lambda(z)$ has all the required properties.

q. e. d.

THEOREM 1.1. (i) *We have*

$$F_i\left(s-\frac{n}{d}, \hat{f}\right)=\gamma(-s)(2\pi)^{-ds}|b_0|^s e^{\frac{\pi\sqrt{-1}}{2}ds}\sum_{j=1}^{l}\varepsilon_{ij}(s)t_{ij}(s)F_j(-s, f),$$

where $f \in \mathcal{S}(V_{\boldsymbol{R}})$,

$$\varepsilon_{ij}(s)=\begin{cases} 1 & \textit{when} \quad \varepsilon_i\varepsilon_j b_0>0, \\ e^{-\pi\sqrt{-1}s} & \textit{when} \quad \varepsilon_i\varepsilon_j b_0<0, \end{cases}$$

and $t_{ij}(s)$ *is a polynomial in* $e^{-2\pi\sqrt{-1}s}$ *whose degree does not exceed*

$$\begin{cases} \left[\frac{d}{2}\right] & \textit{when} \quad \varepsilon_i\varepsilon_j b_0>0, \\ \left[\frac{d-1}{2}\right] & \textit{when} \quad \varepsilon_i\varepsilon_j b_0<0. \end{cases}$$

(ii) *We have* $b_1(s)=(-1)^d b_1\left(1-s-\frac{n}{d}\right)$.

PROOF.[5] (i) It follows from Proposition 1.3 that the mapping $f \to F_i\left(s-\frac{n}{d}, \hat{f}\right)$ defines a tempered distribution on $V_{\boldsymbol{R}}$. By (1.12) and Proposition 1.3, we get

$$F_i\left(s-\frac{n}{d}, \widehat{g\cdot f}\right)=\chi(g)^{-s}\chi(g)^{\frac{n}{d}}F_i\left(s-\frac{n}{d}, \hat{f}\right) \qquad (\forall g\in(G_{\boldsymbol{R}})_0,\ f\in\mathcal{S}(V_{\boldsymbol{R}}))\,.$$

Therefore it follows, from Lemma 1.2, that there exist functions $c_{i1}(s), \cdots, c_{il}(s)$ of s such that

$$F_i\left(s-\frac{n}{d}, \hat{f}\right)=\gamma(-s)\sum_{j=1}^{l}c_{ij}(s)F_j(-s, f)$$

for every $f\in C_0^\infty((V-S)_{\boldsymbol{R}})$. For any $f\in C_0^\infty((V-S)_{\boldsymbol{R}})$, $F_i\left(s-\frac{n}{d}, \hat{f}\right)$ and $\gamma(-s)F_j(-s, f)$ are entire functions of s. For any $s\in\boldsymbol{C}$ and j $(1\leqq j\leqq l)$, there exists an $f\in C_0^\infty(V_j)$ such that $\gamma(-s)F_j(-s, f)\neq 0$. Hence, $c_{ij}(s)$ is an entire function of s $(1\leqq j\leqq l)$.

Making use of (1.6) and (1.11), it can be easily proved that $\widehat{P^\nu f}=(2\pi\sqrt{-1})^{-\nu d}Q(\nabla_x)^\nu\hat{f}$. It follows from Proposition 1.3 that

$$F_i(s, \widehat{P^\nu f})=(2\pi\sqrt{-1})^{-\nu d}\varepsilon_i^\nu(-1)^{d\nu}b_0^\nu F_i(s-\nu, \hat{f}) \tag{1.15}$$

$(\nu=1, 2, \cdots;\ f\in\mathcal{S}(V_{\boldsymbol{R}}))$. Therefore, we have

$$\begin{aligned}(-2\pi\sqrt{-1})^{-\nu d}\varepsilon_i^\nu b_0^\nu\gamma(\nu-s)\sum_{j=1}^{l}c_{ij}(s-\nu)F_j(\nu-s, f)\\ =\gamma(-s)\sum_{j=1}^{l}\varepsilon_j^\nu c_{ij}(s)F_j(-s+\nu, f)\frac{\gamma(\nu-s)}{\gamma(-s)} \qquad (\forall f\in C_0^\infty((V-S)_{\boldsymbol{R}}))\,.\end{aligned}$$

Hence,

$$c_{ij}(s)=(\varepsilon_i\varepsilon_j)^\nu b_0^\nu(-2\pi\sqrt{-1})^{-\nu d}c_{ij}(s-\nu) \qquad (\nu=1, 2, \cdots)\,. \tag{1.16}$$

Put

$$c_{ij}(s)=(2\pi)^{-ds}|b_0|^s e^{\frac{-\sqrt{-1}}{2}ds}\varepsilon_{ij}(s)t_{ij}(s) \tag{1.17}$$

$(\varepsilon_{ij}(s)=e^{-\frac{\pi s}{2}\sqrt{-1}(1-\operatorname{sgn}\varepsilon_i\varepsilon_j b_0)})$. Then $t_{ij}(s)$ is an entire function of s such that

$$t_{ij}(s+1)=t_{ij}(s)\,. \tag{1.18}$$

Put

$$T_s(f)=F_i\left(s-\frac{n}{d}, \hat{f}\right)-\gamma(-s)\sum_{j=1}^{l}c_{ij}(s)F_j(-s, f) \qquad (f\in\mathcal{S}(V_{\boldsymbol{R}}))\,.$$

Then T_s is a tempered distribution on $V_{\boldsymbol{R}}$ for $\operatorname{Re}s\leqq 0$. Furthermore $T_s(f)=0$ provided $f\in C_0^\infty((V-S)_{\boldsymbol{R}})$. We take a base in $V_{\boldsymbol{R}}$ and identify $V_{\boldsymbol{R}}$ with $\boldsymbol{R}^n$. For any non negative integers N, M, we define a semi-norm $\nu(N, M)$ on $\mathcal{S}(V_{\boldsymbol{R}})$ as follows:

5) The following proof is due to the author.

$$\nu(N, M)(f)=\operatorname*{Sup.}_{x\in V_{\boldsymbol{R}}}(1+\|x\|^2)^N\sum_{i_1+\cdots+i_n\leqq M}\left|\frac{\partial^{i_1+\cdots+i_n}}{\partial x_1^{i_1}\cdots\partial x_n^{i_n}}f\right|$$

($\|x\|^2=x_1^2+x_2^2+\cdots+x_n^2$). It follows from Corollary to Proposition 1.3, that if we put $N_0=\left[\frac{n}{d}\right]+2$ and take a sufficiently large natural number M_0, there exist positive numbers C_1 and C_2 such that

$$\left|F_i\left(s-\frac{n}{d}, \hat{f}\right)\right|\leqq C_1\frac{\nu(M_0, M_0)(f)}{\left|\gamma\left(s+N_0-\frac{n}{d}\right)\right|}\qquad(0\leqq -\mathrm{Re}\, s\leqq 1) \tag{1.19}$$

and that

$$|\gamma(-s)F_j(-s, f)|\leqq C_2\nu(M_0, M_0)(f)\qquad(0\leqq -\mathrm{Re}\, s\leqq 1). \tag{1.20}$$

Therefore, $\{T_s;\ 0\leqq -\mathrm{Re}\, s\leqq 1\}$ is a family of distributions on $V_{\boldsymbol{R}}$ whose orders do not exceed M_0 and whose supports are contained in S.

It follows from Lemma 1.3 that $T_s(P^\nu f)=0$ $(0\leqq -\mathrm{Re}\, s\leqq 1,\ f\in C_0^\infty(V_{\boldsymbol{R}}))$ if $\nu>M_0$. Also we see from the equalities (1.15) and (1.16) that

$$T_s(P^\nu f)=(-2\pi\sqrt{-1})^{-\nu d}(\varepsilon_i b_0)^\nu T_{s-\nu}(f).$$

Hence, we have $T_{s-\nu}(f)=0$ $(\forall f\in C_0^\infty(V_{\boldsymbol{R}}),\ 0\leqq -\mathrm{Re}\, s\leqq 1)$. Since $T_s(f)$ is an analytic function of s, we have $T_s(f)=0$ $(\forall s\in \boldsymbol{C},\ f\in C_0^\infty(V_{\boldsymbol{R}}))$. Since T_s is a tempered distribution, we get $T_s(f)=0$ $(\forall s\in\boldsymbol{C},\ f\in\mathcal{S}(V_{\boldsymbol{R}}))$. Thus

$$F_i\left(s-\frac{n}{d}, \hat{f}\right)=\gamma(-s)\sum_{j=1}^{l}c_{ij}(s)F_j(-s, f)\qquad(\forall s\in\boldsymbol{C},\ f\in\mathcal{S}(V_{\boldsymbol{R}})). \tag{1.21}$$

It follows from the inequalities (1.19) and (1.20) that, when $0\leqq -\mathrm{Re}\, s\leqq 1$, the equality (1.21) holds for any $f\in C^\infty(V_{\boldsymbol{R}})$ such that $\nu(M_0+1, M_0)(f)<\infty$. (We note that, for any such f, there exists a sequence $\{f_j\}$ in $\mathcal{S}(V_{\boldsymbol{R}})$ such that $\lim_{j\to\infty}\nu(M_0, M_0)(f-f_j)=0$.) Put

$$\omega=\frac{1}{d}\sum_{i=1}^{n}\frac{(-1)^{i-1}}{P}x_i dx_1\wedge\cdots\wedge dx_{i-1}\wedge dx_{i+1}\wedge\cdots\wedge dx_n,$$

$$G^1=\{g\in(G_{\boldsymbol{R}})_0;\ \chi(g)=1\}$$

and

$$K_j=\{x\in V_j;\ P(x)=\varepsilon_j\}.$$

Then ω is a differential form on $(V-S)_{\boldsymbol{R}}$ such that $dx=dP\wedge\omega$. K_j is a real nonsingular hypersurface on which G^1 operates transitively. It is immediate to see that there exists an orientation on K_j such that $\omega>0$ on K_j.

Take an $h_j\in C_0^\infty(K_j)$ such that

$$\int_{K_j}h_j(x)\omega=1.$$

Furthermore we take a $q(t)\in C^\infty(\boldsymbol{R})$ which satisfies the conditions of Lemma 1.4. For every natural number L we define $f_j^{(L)}\in C^\infty(V_{\boldsymbol{R}})$ as follows:

$$f_j^{(L)}(x)=\begin{cases} 0 & \text{when } x\notin V_j\,, \\ |P|^{-L}(x)q(|P(x)|)h_j\left(\dfrac{x}{|P(x)|^{1/d}}\right) & \text{when } x\in V_j\,. \end{cases}$$

The support of $f_{(j)}^{(L)}$ is contained in the set:

$$\left\{x\in V_j\,;\ |P(x)|\geqq 1,\ \frac{x}{|P(x)|^{1/d}}\in \mathrm{Supp.}\,h_j\right\}.$$

Therefore we can take L so large that

$$\nu(M_0+1,\,M_0)(f_j^{(L)})<\infty \tag{1.22}$$

and that

$$L>1+\frac{n}{d}\,. \tag{1.23}$$

Then (1.21) yields

$$F_i\left(s-\frac{n}{d},\,\hat{f}_j^{(L)}\right)=\gamma(-s)c_{ij}(s)F_j(-s,\,f_j^{(L)}) \qquad (0\leqq -\mathrm{Re}\, s\leqq 1)\,.$$

On the other hand we get

$$\begin{aligned}\gamma(-s)F_j(-s,\,f_j^{(L)})&=\int_{V_j}|P|^{-s-L}(x)q(|P|(x))h_j\left(\frac{x}{|P|^{1/d}(x)}\right)dP\wedge\omega\\ &=\int_0^\infty t^{-s-L+\frac{n}{d}-1}q(t)dt\int_{K_j}h_j(x)\omega\\ &=\lambda\left(-s-L+\frac{n}{d}\right) \qquad \left(\lambda(z)=\int_0^\infty t^{z-1}q(t)dt\right).\end{aligned}$$

Also we see from (1.17) that

$$t_{ij}(s)=\frac{(2\pi)^{ds}|b_0|^{-s}e^{-\frac{\pi\sqrt{-1}}{2}ds}}{\lambda\left(-s-L+\frac{n}{d}\right)\varepsilon_{ij}(s)}F_i\left(s-\frac{n}{d},\,\hat{f}_j^{(L)}\right) \qquad (0\leqq -\mathrm{Re}\, s\leqq 1)\,. \tag{1.24}$$

By (1.19), we have

$$\left|F_i\left(s-\frac{d}{n},\,\hat{f}_j^{(L)}\right)\right|\leqq c_1\frac{\nu(M_0,\,M_0)(f_j^{(L)})}{\left|\gamma\left(s+N_0-\frac{n}{d}\right)\right|} \qquad (0\leqq -\mathrm{Re}\, s\leqq 1)\,. \tag{1.25}$$

From Lemma 1.4 and (1.23), it follows that there exists a $c_3>0$ such that

$$\left|\lambda\left(-s-L+\frac{n}{d}\right)\right|\geqq c_3\exp\{-\sqrt{|\mathrm{Im}\, s|}\} \qquad (0\leqq -\mathrm{Re}\, s\leqq 1)\,. \tag{1.26}$$

Making use of relations (1.24), (1.25), (1.22), (1.26) and (1.8), we can conclude that there exists a positive number c_4 such that the following inequality holds:

$$|t_{ij}(s)| \leqq c_4 \frac{|e^{-\frac{\sqrt{-1}}{2}ds}\varepsilon_{ij}^{-1}(s)| \exp\sqrt{|\mathrm{Im}\, s|}}{\prod_{k=1}^{d}\left|\Gamma\left(s+N_0-\frac{n}{d}-c_i+1\right)\right|} \qquad (0 \leqq -\mathrm{Re}\, s \leqq 1).$$

Applying Stirling's asymptotic formula for the Gamma function, we can conclude that for any $\eta > 0$, there exists a positive number c such that the following inequality holds:

$$|t_{ij}(s)| \leqq c|e^{-\frac{\pi\sqrt{-1}}{2}ds}\varepsilon_{ij}^{-1}(s)|e^{\left(\frac{\pi d}{2}+\eta\right)|\mathrm{Im}\, s|} \qquad (0 \leqq -\mathrm{Re}\, s \leqq 1).$$

We know, by (1.18), that $t_{ij}(s)$ is an entire function of s with period 1. Hence $t_{ij}(s)$ must be a polynomial in $e^{-2\pi\sqrt{-1}s}$ whose degree does not exceed

$$\begin{cases} \left[\frac{d}{2}\right] & \text{when} \quad \varepsilon_{ij}(s)=1, \\ \left[\frac{d-1}{2}\right] & \text{when} \quad \varepsilon_{ij}(s)=e^{-\pi\sqrt{-1}s}. \end{cases}$$

Thus, the first part of our theorem is proved.

(ii) We can choose τ so that $(x, y)=\langle x, \tau y\rangle=(y, x)$ (cf. Lemma 1.1). Then it follows from (1.6) and (1.11) that

$$\widehat{Q(\nabla_x)^\nu f}=(-2\pi\sqrt{-1})^{\nu d}P^\nu \hat{f}.$$

Since

$$F_i\left(s-\frac{n}{d}, \hat{f}\right)=\gamma(-s)\sum_{j=1}^{l} c_{ij}(s)F_j(-s, f),$$

we obtain

$$(-2\pi\sqrt{-1})^{\nu d}\varepsilon_{i}^{\nu}\frac{\gamma\left(s+\nu-\frac{n}{d}\right)}{\gamma\left(s-\frac{n}{d}\right)}\gamma(-s-\nu)\sum_{j=1}^{l} c_{ij}(s+\nu)F_j(-s-\nu, f)$$

$$=\gamma(-s)(-1)^{d\nu}\sum_{j=1}^{l} c_{ij}(s)F_j(-s-\nu, f)b_0^\nu\varepsilon_j^\nu \qquad (\forall f\in\mathcal{S}(V_{\boldsymbol{R}}),\ s\in\boldsymbol{C}).$$

Taking the equality (1.16) into account, we have

$$\frac{\gamma\left(s+\nu-\frac{n}{d}\right)}{\gamma\left(s-\frac{n}{d}\right)}c_{ij}(s)=(-1)^{d\nu}\frac{\gamma(-s)}{\gamma(-s-\nu)}c_{ij}(s).$$

Since there exists a j $(1\leqq j\leqq l)$ such that $c_{ij}(s)\neq 0$,

$$\frac{\gamma\left(s+\nu-\frac{n}{d}\right)}{\gamma\left(s-\frac{n}{d}\right)}=(-1)^{d\nu}\frac{\gamma(-s)}{\gamma(-s-\nu)}.$$

Therefore, by (1.9), $b_\nu\left(s+\nu-\frac{n}{d}\right)=(-1)^{d\nu}b_\nu(-s)$ and $b_1(s)=(-1)^d b_1\left(1-s-\frac{n}{d}\right)$. Since b_i is, up to a constant factor, independent of the choice of τ, the second part of our theorem is established. q. e. d.

2.[6] In the following we put

$$c_{ij}(s)=(2\pi)^{-ds}|b_0|^s e^{\frac{\pi\sqrt{-1}}{2}ds}\varepsilon_{ij}(s)t_{ij}(s) \qquad (1\leqq i,\ j\leqq l)\,,$$

and

$$\omega=\frac{1}{d}\sum_{i=1}^{n}\frac{(-1)^{i-1}x_i}{P}\,dx_1\wedge\cdots\wedge dx_{i-1}\wedge dx_{i+1}\wedge dx_n\,.$$

Then ω is a differential form on $(V-S)_{\boldsymbol{R}}$ and we have $dx=dP\wedge\omega$. We also put

$$K_j(t)=\{x\in V_j\,;\ P(x)=\varepsilon_j t\} \qquad (t>0,\ 1\leqq j\leqq l)\,.$$

Then $K_j(t)$ is a real non singular hypersurface in $V_{\boldsymbol{R}}$ and there exists an orientation on $K_j(t)$ such that $\omega>0$ on $K_j(t)$. The following lemma can be easily proved.

LEMMA 1.5.

$$\int_{V_i}|P(x)|^s f(x)dx=\int_0^\infty t^s\left\{\int_{K_j(t)}f(x)\omega\right\}dt \qquad (\operatorname{Re}s\geqq 0,\ f\in\mathcal{S}(V_{\boldsymbol{R}}))\,.$$

PROPOSITION 1.4. *We assume that* $\operatorname{Re}c_1>\operatorname{Re}c_2\geqq\cdots\geqq\operatorname{Re}c_d$ (*cf.* (1.8)). *Then we have*

$$\lim_{t\to+0}t^{c_1}\int_{K_i(t)}\hat{f}(x)\omega$$
$$=\left\{\prod_{k=2}^{d}\Gamma(c_1-c_k)\right\}\sum_{j=1}^{l}c_{ij}\left(\frac{n}{d}+c_1-1\right)\int_{V_j}|P(x)|^{1-c_1-\frac{n}{d}}f(x)dx$$
$$(\forall f\in C_0^\infty(V_{\boldsymbol{R}}-S_{\boldsymbol{R}}))\,.$$

PROOF. Take an $f\in C_0^\infty(V_{\boldsymbol{R}}-S_{\boldsymbol{R}})$. It follows from Lemma 1.5 that

$$\gamma(s-1)F_i(s-1,\hat{f})=\int_{V_i}|P(x)|^{s-1}\hat{f}(x)dx \tag{1.27}$$
$$=\int_0^\infty t^{s-1}\left\{\int_{K_i(t)}\hat{f}(x)\omega\right\}dt\,.$$

Regarded as a function of s, the left side of the equality (1.27) is rapidly decreasing on any line of the form: $\operatorname{Re}s=\sigma_0$ $(\sigma_0>1)$. Hence we have

$$\int_{K_i(t)}\hat{f}(x)\omega=\frac{1}{2\pi\sqrt{-1}}\int_{\operatorname{Re}s=\sigma_0}t^{-s}\gamma(s-1)F_i(s-1,\hat{f})ds$$

$(\sigma_0>\operatorname{Max.}(1,c_1))$. Also we see from Theorem 1.1 that

6) The following subsection is due to the author.

$$\gamma(s-1)F_i(s-1,\hat{f})=\gamma(s-1)\sum_{j=1}^{d}c_{ij}\Big(s+\frac{n}{d}-1\Big)\int_{V_j}|P|^{1-s-\frac{n}{d}}(x)f(x)dx\,.$$

It follows from Theorem 1.1 and the assumption that $(s-c_1)\gamma(s-1)\times c_{ij}\Big(s+\dfrac{n}{d}-1\Big)$ is a holomorphic function of s if $\operatorname{Re}s>c_2$ and is slowly increasing on any line parallel to the imaginary axis. We also know that

$$\int_{V_j}|P(x)|^{1-s-\frac{n}{d}}f(x)dx$$

is an entire function of s which is rapidly decreasing on any line parallel to the imaginary axis. Therefore we have

$$\begin{aligned}
&t^{c_1}\int_{K_i(t)}\hat{f}(x)\omega\\
&=\frac{1}{2\pi\sqrt{-1}}\int_{\operatorname{Re}s=\sigma_0}t^{c_1-s}\gamma(s-1)\sum_{j=1}^{l}c_{ij}\Big(s+\frac{n}{d}-1\Big)\Big\{\int_{V_j}|P(x)|^{1-s-\frac{n}{d}}f(x)dx\Big\}ds\\
&=\Big\{\prod_{k=2}^{d}\Gamma(c_1-c_k)\Big\}\sum_{j=1}^{l}c_{ij}\Big(c_1+\frac{n}{d}-1\Big)\int_{V_j}|P(x)|^{1-c_1-\frac{n}{d}}f(x)dx\\
&\quad+\frac{1}{2\pi\sqrt{-1}}\int_{\operatorname{Re}s=\sigma_2}t^{c_1-s}\gamma(s-1)\sum_{j=1}^{l}c_{ij}\Big(s+\frac{n}{d}-1\Big)\Big\{\int_{V_j}|P(x)|^{1-s-\frac{n}{d}}f(x)dx\Big\}ds
\end{aligned}$$

where $\operatorname{Re}c_2<\sigma_2<\operatorname{Re}c_1$. Therefore

$$\lim_{t\to+0}t^{c_1}\int_{K_i(t)}\hat{f}(x)\omega=\Big\{\prod_{k=2}^{d}\Gamma(c_1-c_k)\Big\}\sum_{j=1}^{l}c_{ij}\Big(c_1+\frac{n}{d}-1\Big)\int_{V_j}|P(x)|^{1-c_1-\frac{n}{d}}f(x)dx\,.$$

q. e. d.

We denote by $S'_{\boldsymbol{R}}$ the set of regular points of the irreducible hypersurface $S_{\boldsymbol{R}}$: $S'_{\boldsymbol{R}}=\{x\in S_{\boldsymbol{R}};\ (dP)_x\neq 0\}$. $S'_{\boldsymbol{R}}$ is a Zariski-open subset of $S_{\boldsymbol{R}}$.

We say that two connected components V_j and V_k of $(V_{\boldsymbol{R}}-S_{\boldsymbol{R}})$ are *mutually neighbouring* if $\bar{V}_j\cap\bar{V}_k\cap S'_{\boldsymbol{R}}\neq\emptyset$ ($\bar{V}_j$ is the closure of V_j, with respect to the Euclidean topology).

PROPOSITION 1.5. *Let V_j and V_k be mutually neighbouring. Then* $c_{ij}(1)+c_{ik}(1)=0$ $(1\leqq i\leqq l)$.

PROOF. Take a $p\in\bar{V}_j\cap\bar{V}_k\cap S'_{\boldsymbol{R}}$. There exists a neighbourhood U of p in $V_{\boldsymbol{R}}$ which satisfies the following two conditions:

1. $U=U\cap V_j\cup U\cap V_k\cup U\cap S'$.
2. We can take a positive number η and an orientation preserving C^∞-diffeomorphism φ of $I_\eta=(-\eta,\eta)^n$ onto U such that $P(\varphi((y_1,\cdots,y_n)))=y_1$ $((y_1,\cdots,y_n)\in I_\eta)$ and that $\varphi(0,\cdots,0)=p$. We put

$$J(y)=\frac{\partial(x_1,\cdots,x_n)}{\partial(y_1,\cdots,y_n)}\qquad((x_1,\cdots,x_n)=\varphi(y_1,\cdots,y_n))\,.$$

Take an $f\in C_0^\infty(U)$. We may assume that $\varepsilon_j=1$, $\varepsilon_k=-1$. Then

$$\lim_{s\to 1-0}(1-s)\int_{V_j}|P(x)|^{-s}f(x)dx$$
$$=\lim_{s\to 1-0}(1-s)\int_0^{\eta}dy_1\int_{-\eta}^{\eta}dy_2\cdots\int_{-\eta}^{\eta}dy_n y_1^{-s}J(y)f(\varphi(y))$$
$$=\int_{-\eta}^{\eta}dy_2\int_{-\eta}^{\eta}dy_3\cdots\int_{-\eta}^{\eta}dy_n J((0, y_2, \cdots, y_n))f(\varphi(0, y_2, \cdots, y_n)).$$

Similarly, it can be proved that

$$\lim_{s\to 1-0}(1-s)\int_{V_k}|P(x)|^{-s}f(x)dx$$
$$=\int_{-\eta}^{\eta}dy_2\cdots\int_{-\eta}^{\eta}dy_n J((0, y_2, \cdots, y_n))f(\varphi(0, y_2, \cdots, y_n)).$$

Since $U=U\cap V_j\cup U\cap V_k\cup U\cap(S)'_R$, we have $\int_{V_i}|P(x)|^{-s}f(x)dx=0$ unless $i=j$ or k. Since $F_i\left(s-\frac{n}{d}, \hat{f}\right)$ is an entire function of s, $\lim_{s\to 1-0}(1-s)F_i\left(s-\frac{n}{d}, \hat{f}\right)=0$. It follows from Theorem 1.1 that

$$F_i\left(s-\frac{n}{d}, \hat{f}\right)=\gamma(-s)\sum_{r=1}^{l}c_{ij}(s)F_j(-s, f)$$
$$=c_{ij}(s)\int_{V_j}|P(x)|^{-s}f(x)dx+c_{ik}(s)\int_{V_k}|P(x)|^{-s}f(x)dx.$$

Therefore

$$\lim_{s\to 1-0}(1-s)F_i\left(s-\frac{n}{d}, \hat{f}\right)$$
$$=(c_{ij}(1)+c_{ik}(1))\int_{-\eta}^{\eta}dy_2\cdots\int_{-\eta}^{\eta}dy_n J((0, y_2, \cdots, y_n))f(\varphi(0, y_2, \cdots, y_n))$$
$$=0.$$

As we can take an $f\in C_0^{\infty}(U)$ such that

$$\int_{-\eta}^{\eta}dy_2\cdots\int_{-\eta}^{\eta}dy_n J((0, y_2, \cdots, y_n))f(\varphi(0, y_2, \cdots, y_n))\neq 0,$$

we conclude that

$$c_{ij}(1)+c_{ik}(1)=0. \qquad \text{q. e. d.}$$

Chapter 2. Dirichlet series associated with the vector space of binary cubic forms

§ 1. The vector space of binary cubic forms

1. We denote by V the vector space of binary cubic forms.

$$V=\{F(u, v)=x_1u^3+x_2u^2v+x_3uv^2+x_4v^3\,;\ (x_1, x_2, x_3, x_4)\in \boldsymbol{C}^4\}.$$

We put $G=GL(2, \boldsymbol{C})$. Then V becomes a G-module if we put

$$g\cdot F((u, v))=F((u, v)g) \qquad (g\in G).$$

We define a linear isomorphism of $\boldsymbol{C}^4$ onto V as follows: $\boldsymbol{C}^4\ni x\to F_x\in V$, where

$$F_x((u, v))=x_1u^3+x_2u^2v+x_3uv^2+x_4v^3, \qquad (x=(x_1, x_2, x_3, x_4)).$$

In the following we identify V with $\boldsymbol{C}^4$ via this isomorphism. We put $g\cdot(F_x)=F_{g\cdot x}$ ($x\in\boldsymbol{C}^4$, $g\in G$). When

$$g=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in G$$

and $x=(x_1, x_2, x_3, x_4)\in\boldsymbol{C}^4$, we have

$$(2.1) \qquad (g\cdot x)'=\begin{pmatrix}\alpha^3 & \alpha^2\beta & \alpha\beta^2 & \beta^3\\ 3\alpha^2\gamma & \alpha^2\delta+2\alpha\beta\gamma & 2\alpha\beta\delta+\gamma\beta^2 & 3\beta^2\delta\\ 3\alpha\gamma^2 & 2\alpha\gamma\delta+\gamma^2\beta & \alpha\delta^2+2\beta\gamma\delta & 3\beta\delta^2\\ \gamma^3 & \gamma^2\delta & \gamma\delta^2 & \delta^3\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}.$$

Let $F\in V$, we say that $(u, v)\in P_1(\boldsymbol{C})$ is a root of F if $F((u, v))=0$. We identify $P_1(\boldsymbol{C})$ with the Riemann sphere $\boldsymbol{C}\cup\{\infty\}$ by the mapping:

$$(u, v)\to\frac{u}{v}.$$

For every $F=F_x\in V$, we denote by $P(F)=P(x)$ the discriminant of F. By the definition, we have

$$(2.2) \qquad P(x)=x_2^2x_3^2+18x_1x_2x_3x_4-4x_1x_3^3-4x_2^3x_4-27x_1^2x_4^2.$$

We say that a binary cubic form F is degenerate if $P(F)=0$. It is well-known that P is an irreducible polynomial on V and

$$(2.3) \qquad P(g\cdot x)=(\det g)^6P(x) \qquad (g\in G,\ x\in\boldsymbol{C}^4=V).$$

In the following we put

$$(2.4) \qquad \chi(g)=(\det g)^6 \qquad (g\in G).$$

The following lemma is well-known.

LEMMA 2.1. (i) *Every non zero binary cubic form F has, taking multiplicities into account, three roots; they are all distinct if and only if F is non-degenerate.*

(ii) *Let λ_1, λ_2 and λ_3 be the roots of $0\neq F\in V$. Then the roots of $g\cdot F$ ($g\in G$) are $g\cdot\lambda_1$, $g\cdot\lambda_2$ and $g\cdot\lambda_3$, where we put*

$$g\cdot\lambda=\frac{\delta\lambda-\gamma}{-\beta\lambda+\alpha} \qquad \left(g=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix},\ \lambda\in\boldsymbol{C}\cup\{\infty\}\right).$$

We denote by $\mathfrak{z}_3$ the central cyclic subgroup of order 3 in G. Then $G/\mathfrak{z}_3$ operates effectively on V and can be identified with an algebraic subgroup $\hat{G}$ in $GL(V)$.

PROPOSITION 2.1.[7] *The pair $(\hat{G}, V)$ is a prehomogeneous vector space whose set of singular points coincides with the set of degenerate binary cubic forms.*

PROOF. It follows from Lemma 2.1 that G operates transitively on the set of non-degenerate binary cubic forms. Let S be the set of degenerate binary cubic forms. Then we have

$$S=\{F_x\,;\ P(x)=0\} \tag{2.5}$$

and S is a G-invariant proper algebraic subset of V. Therefore $(\hat{G}, V)$ is a prehomogeneous vector space whose set of singular points is S. q. e. d.

In the following we denote by S the set of degenerate binary cubic forms.

We put $V_{\boldsymbol{R}}=\{F_x\,;\ x\in\boldsymbol{R}^4\}\cong\boldsymbol{R}^4$. Then $\hat{G}$ is defined over $\boldsymbol{R}$ and $\hat{G}_{\boldsymbol{R}}$ can be identified with $G_{\boldsymbol{R}}=GL(2, \boldsymbol{R})$. Since $\hat{G}$ is reductive and S is an irreducible hypersurface in V, the prehomogeneous vector space $(\hat{G}, V)$ satisfies the conditions 1 and 2 in § 1 of Chapter 1.

2. We define G_+, V_1 and V_2 as follows:

$$G_+=(G_{\boldsymbol{R}})_0=\{g\in GL(2, \boldsymbol{R})\,;\ \det g>0\}\,,$$
$$V_1=\{x\in V_{\boldsymbol{R}}\,;\ P(x)>0\}\,,\qquad V_2=\{x\in V_{\boldsymbol{R}}\,;\ P(x)<0\}\,.$$

Obviously, $(V-S)_{\boldsymbol{R}}=V_1\cup V_2$.

PROPOSITION 2.2. (i) *The group G_+ acts transitively on V_1 and the isotropy subgroup of any point of V_1 in G_+ is a cyclic group of order* 3.

(ii) *The group G_+ acts simply transitively on V_2.*

PROOF. (i) Take an $x\in V_1$, then F_x has three distinct roots in $\boldsymbol{R}\cup\{\infty\}$. Via the fractional linear action, G_+ acts transitively on the set of triples of distinct points on $R\cup\{\infty\}$. Hence it follows from Lemma 2.1 that there exists a $g\in G_+$ such that $g\cdot F_x$ has roots 0, 1 and ∞.

Then $g\cdot F_x((u, v))=c(u^2v-uv^2)$ $(c\in\boldsymbol{R}^\times)$. Putting $g'=\sqrt[3]{c}^{\,-1}g$, we have $g'\cdot F_x((u, v))=u^2v-uv^2$. Therefore if we put $x_0=(0, 1, -1, 0)\in V_1$, we have $V_1=G_+\cdot x_0$. We denote by I_{x_0} the isotropy subgroup of x_0 in G_+. Take a $g=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in I_{x_0}$, then the fractional linear transformation $z\to g\cdot z=\dfrac{\delta z-\gamma}{-\beta z+\alpha}$ induces a permutation of $\{0, 1, \infty\}$ without fixed point unless $g\neq 1$. Therefore, it induces a cyclic permutation of $\{0, 1, \infty\}$. Hence we have $g\cdot z=z$ or $g\cdot z=\dfrac{-z+1}{-z}$ or $g\cdot z=\dfrac{-1}{z-1}$ $(\forall z\in\boldsymbol{C}\cup\{\infty\})$. So, there exists a $t\in\boldsymbol{R}^\times$ such that $g=\begin{pmatrix}t & \\ & t\end{pmatrix}$ or $g=\begin{pmatrix}t & \\ & t\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & -1\end{pmatrix}$ or $g=\begin{pmatrix}t & \\ & t\end{pmatrix}\begin{pmatrix}-1 & -1\\ 1 & \end{pmatrix}$. Since $g\cdot x_0=x_0$,

7) This proposition was communicated to the author by Professor M. Sato.

we obtain, by (2.1),

$$I_{x_0}=\left\{1, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & \end{pmatrix}\right\}. \qquad \text{q. e. d.}$$

(ii) Take an $x \in V_2$, then F_x has roots z, $\bar{z}$ and λ ($z \in \boldsymbol{C}$, $\operatorname{Im} z > 0$; $\lambda \in \boldsymbol{R} \cup \{\infty\}$). We can take a $g \in G_+$ such that $g \cdot F_x$ has roots $\sqrt{-1}$, $-\sqrt{-1}$ and ∞. Then we have $g \cdot F_x((u, v)) = c(u^2 v + v^3)$ ($c \in \boldsymbol{R}^\times$). Put $g' = \sqrt[3]{c}^{-1} \cdot g$. Then $g' \cdot F_x((u, v)) = u^2 v + v^3$. Therefore if we put $x_1 = (0, 1, 0, 1)$, $V_2 = G_+ \cdot x_1$. Take a $g \in G_+$ such that $g \cdot x_1 = x_1$. Then the fractional linear transformation corresponding to g fixes $\sqrt{-1}$, $-\sqrt{-1}$ and ∞. Hence g is of the form $\begin{pmatrix} t & \\ & t \end{pmatrix}$ ($t \in \boldsymbol{R}^\times$). Since $g \cdot x_1 = x_1$, we have $g = 1$. Thus, the isotropy subgroup of x_1 in G_+ consists only of the identity element. q. e. d.

In the following we put $G^1 = SL(2, \boldsymbol{R}) \subset G_+$.

The following proposition can be similarly proved.

PROPOSITION 2.3. *We put $S_{\boldsymbol{R}} = S \cap V_{\boldsymbol{R}}$. Then $S_{\boldsymbol{R}}$ decomposes into the union of three orbits under the action of $G^1 = SL(2, \boldsymbol{R})$ as follows:*
$S_{\boldsymbol{R}} = G^1 \cdot w_1 \cup G^1 \cdot w_2 \cup \{0\}$, where we put $w_1 = (0, 0, 1, 0)$ and $w_2 = (0, 0, 0, 1)$. Further we have

$$I_{w_1} = \{g \in G^1 ;\ g w_1 = w_1\} = \{1\}$$

and

$$I_{w_2} = \{g \in G^1 ;\ g w_2 = w_2\} = \left\{\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} ;\ x \in \boldsymbol{R}\right\}.$$

3. We define the usual subgroups K, A_+, N and N' of G_+ as follows:

$$K = \left\{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} ;\ \theta \in \boldsymbol{R}\right\},$$

$$A_+ = \left\{\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} ;\ t > 0\right\},$$

$$N = \left\{\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} ;\ x \in \boldsymbol{R}\right\},$$

$$N' = \left\{\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} ;\ x \in \boldsymbol{R}\right\}.$$

Furthermore, we introduce the following notations.

$$k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad (\theta \in \boldsymbol{R}),$$

$$a_t = \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \qquad (t \in \boldsymbol{R}^\times),$$

$$n(x) = \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} \qquad (x \in \boldsymbol{R}),$$

$$\nu(x)=\begin{pmatrix}1 & x\\ & 1\end{pmatrix} \qquad (x\in \boldsymbol{R}).$$

The following lemma is well-known and can be easily proved.

LEMMA 2.2. (i) *The mapping* $K\times A_+\times N\ni(k, a, n)\rightarrow kan\in G^1$ *is an analytic diffeomorphism of* $K\times A_+\times N$ *onto* G^1.

(ii) *For any* $g\in G^1$, *there exists a unique* $(\theta(g), t(g), u(g))\in(\boldsymbol{R}/2\pi\boldsymbol{Z})\times\boldsymbol{R}_+\times\boldsymbol{R}$ *such that* $g=k_{\theta(g)}a_{t(g)}n(u(g))$. *In particular,* $t(\nu(x))=\sqrt{1+x^2}^{-1}$.

We define an invariant measure dg on G^1 as follows:

$$(2.6) \qquad \int_{G^1}f(g)dg=\frac{1}{2\pi}\int_0^{2\pi}d\theta\int_{-\infty}^{\infty}du\int_0^{\infty}d^\times t f(k_\theta n(u)a_t)$$

$$=\frac{1}{2\pi}\int_0^{2\pi}d\theta\int_{-\infty}^{\infty}du\int_0^{\infty}t^{-2}d^\times tf(k_\theta a_t n(u)) \qquad (f\in L_1(G^1)).$$

We also define an invariant measure dg on G_+ as follows:

$$(2.7) \qquad \int_{G_+}f(g)dg=\int_0^{\infty}d^\times\lambda\int_{G^1}f\left(\begin{pmatrix}\lambda & \\ & \lambda\end{pmatrix}g_1\right)dg_1 \qquad (f\in L_1(G_+)).$$

Let dx be the standard Euclidean measure on $V_{\boldsymbol{R}}$:

$$dx=dx_1dx_2dx_3dx_4 \qquad (x=(x_1, x_2, x_3, x_4)\in V_{\boldsymbol{R}}).$$

We note that

$$(2.8) \qquad \int_{G_+}f(g)dg=\frac{1}{2\pi}\int_0^{\infty}d^\times\lambda\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}du\int_{-\infty}^{\infty}d^\times tf\left(\begin{pmatrix}\lambda & \\ & \lambda\end{pmatrix}\nu(x)n(u)a_t\right)$$

$$(\forall f\in L_1(G^1)).$$

PROPOSITION 2.4. *Take an* $f\in C_0(V-S)_{\boldsymbol{R}}$, *then the following equalities hold:*

(i) (2.9) $$\int_{G_+}f(g\cdot y)dg=\frac{1}{4\pi}\int_{V_1}(P(x))^{-1}f(x)dx \qquad (y\in V_1),$$

(ii) (2.10) $$\int_{G_+}f(g\cdot y)dg=\frac{1}{12\pi}\int_{V_2}(-P(x))^{-1}f(x)dx \qquad (y\in V_2).$$

PROOF. (i) It follows from Proposition 2.2, that it is sufficient to prove the equality (2.9) for $y=(0, 1, 1, 0)$. We put

$$(z_1, z_2, z_3, z_4)=\begin{pmatrix}\lambda & \\ & \lambda\end{pmatrix}\nu(x)n(u)a_t\cdot y.$$

Then we have, by (2.1),

$$z_1=\lambda^3\{tx+(2ut+t^{-1})x^2+(u^2t+ut^{-1})x^3\}$$

$$z_2=\lambda^3\{t+2(2ut+t^{-1})x+3(u^2t+ut^{-1})x^2\}$$

$$z_3=\lambda^3\{2ut+t^{-1}+3x(u^2t+ut^{-1})\}$$

$$z_4=\lambda^3(u^2t+ut^{-1}).$$

Since $P(x)^{-1}dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$ is a unique (up to a constant factor) G_+-invariant 4-form on V_1, there exists a $c \in \boldsymbol{R}^\times$ such that

$$P(z)^{-1}dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4 = cd^\times\lambda \wedge dx \wedge du \wedge d^\times t\,.$$

By a straightforward computation one has $c=-6$.

The group G_+ can be viewed as a threefold covering space of V_1 by the mapping $g \to g\cdot y$. Hence, by (2.8),

$$\int_{G_+} f(g\cdot y)\,dg = 3\cdot\frac{1}{2\pi}\cdot\frac{1}{6}\int_{V_1} f(x)P(x)^{-1}dx = \frac{1}{4\pi}\int_{V_1} f(x)P(x)^{-1}dx\,.$$

(ii) It is sufficient to prove the equality (2.10) for $y=(1, 0, 0, 1)$. We put

$$(z_1, z_2, z_3, z_4) = \begin{pmatrix}\lambda & \\ & \lambda\end{pmatrix}\nu(x)n(u)a_t\cdot y\,.$$

Then we have

$$\begin{aligned} z_1 &= \lambda^3\{t^3(1+ux)^3+t^{-3}x^3\} \\ z_2 &= \lambda^3\{3t^3u(1+ux)^2+3x^2t^{-3}\} \\ z_3 &= \lambda^3\{3t^3u^2(1+ux)+3xt^{-3}\} \\ z_4 &= \lambda^3(t^3u^3+t^{-3})\,. \end{aligned}$$

There exists a $c \in \boldsymbol{R}^\times$ such that

$$P(z)^{-1}dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4 = cd^\times\lambda \wedge dx \wedge du \wedge d^\times t\,.$$

By easy computations, we have $c=-6$. Since the mapping $g \to g\cdot y$ gives a bijection from G_+ onto V_2,

$$\begin{aligned} \int_{G_+} f(g\cdot y)dg &= \frac{1}{2\pi}\cdot\frac{1}{6}\int_{V_2} f(x)(-P(x))^{-1}dx \\ &= \frac{1}{12\pi}\int_{V_2} f(x)(-P(x))^{-1}dx\,. \end{aligned}$$ q. e. d.

COROLLARY. *The equality* (2.9) *or* (2.10) *holds, if the integral on either side of* (2.9) *or* (2.10) *converges absolutely.*

4. We define a non-degenerate alternating form $\langle\, ,\rangle$ on V as follows:

(2.11) $$\langle x, y\rangle = x_4y_1 - \frac{1}{3}x_3y_2 + \frac{1}{3}x_2y_3 - x_1y_4\,.$$

Then

(2.12) $$\langle g\cdot x, g^\iota\cdot y\rangle = \langle x, y\rangle \qquad (\forall g \in G)\,,$$

where ι is an involution of G defined as follows:

(2.13) $$g^\iota = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}{}^t g^{-1}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

In particular, if $g\in SL(2, \boldsymbol{C})$, $\langle g\cdot x, g\cdot y\rangle=\langle x, y\rangle$. In the following, we identify V with V^* via this alternating form.

For every $f\in\mathcal{S}(V_{\boldsymbol{R}})$, we put

$$\hat{f}(x)=\int_{V_{\boldsymbol{R}}} f(y)e^{2\pi\sqrt{-1}\langle x,y\rangle}dy\,. \tag{2.14}$$

Then we have

$$f(x)=\frac{1}{9}\int_{V_{\boldsymbol{R}}}\hat{f}(y)e^{+2\pi\sqrt{-1}\langle x,y\rangle}dy\,.$$

§ 2. Fourier transforms of complex powers of the discriminant

1. In the following two subsections 1 and 2 we construct two tempered distributions Σ_1 and Σ_2 on $V_{\boldsymbol{R}}$ whose supports are contained in $S_{\boldsymbol{R}}$; we then calculate their Fourier transforms. They are "G^1-invariant integrals" on the G^1-orbits in $S_{\boldsymbol{R}}$ and play important roles in the remaining part of this chapter. We put $w_1=(0, 0, 1, 0)\in S\cap V_{\boldsymbol{R}}$. For every $f\in\mathcal{S}(V_{\boldsymbol{R}})$ and $z\in\boldsymbol{C}$ ($\operatorname{Re} z>1$) we define $\Sigma_1(f, z)$ as follows:

$$\Sigma_1(f, z)=\frac{1}{2\pi}\int_0^{2\pi}d\theta\int_{-\infty}^{\infty}du\int_0^{\infty}t^{-z}f(k_\theta n(u)a_t\cdot w_1)d^\times t\,. \tag{2.15}$$

LEMMA 2.3. (i) *The integral defining $\Sigma_1(f, z)$ converges absolutely if* $\operatorname{Re} z>1$. *Regarded as a function of z, it can be continued analytically as a meromorphic function in the whole plane whose poles are simple and are situated on the set* $\{1, -1, -3, -5, \cdots\}$.

(ii) *If we denote by $\Sigma_1(f)$ the value of $\Sigma_1(f, z)$ at $z=0$, the mapping $f\to\Sigma_1(f)$ defines a tempered distribution on $V_{\boldsymbol{R}}$, whose support is contained in $S_{\boldsymbol{R}}$.*

PROOF. We put $\tilde{f}(x)=\frac{1}{2\pi}\int_0^{2\pi}f(k_\theta\cdot x)d\theta$. Then $\tilde{f}\in\mathcal{S}(V_{\boldsymbol{R}})$.

We write $\tilde{f}(x)=\tilde{f}(x_1, x_2, x_3, x_4)$ for $x=(x_1, x_2, x_3, x_4)$. We have, by (2.1) and (2.15),

$$\begin{aligned}\Sigma_1(f, z)&=\int_{-\infty}^{\infty}du\int_0^{\infty}t^{-z}\tilde{f}(n(u)a_t\cdot w_1)d^\times t\\&=\int_{-\infty}^{\infty}du\int_0^{\infty}t^{-z}\tilde{f}(0, 0, t^{-1}, t^{-1}u)d^\times t\,.\end{aligned}$$

Hence, the above integral converges absolutely if $\operatorname{Re} z>1$ and is equal to

$$\int_{-\infty}^{\infty}\int_0^{\infty}t^{z-1}\tilde{f}(0, 0, t, u)dud^\times t\,.$$

Since $\int_{-\infty}^{\infty}\tilde{f}(0, 0, t, u)du$ is an even rapidly decreasing function of t, (i) and (ii) follow from the well-known elementary results in calculus. q. e. d.

PROPOSITION 2.5. *For every $f\in C_0^\infty((V-S)_{\boldsymbol{R}})$, one has*

$$\Sigma_1(\hat{f}) = -\pi\int_{V_1} f(x)dx - \frac{\pi}{3}\int_{V_2} f(x)dx\,.$$

PROOF. We may assume that $f(k_\theta\cdot x) = f(x)$ $(\forall\theta\in\boldsymbol{R})$. It follows from Lemma 2.2 (ii), that

$$\int_{-\infty}^{\infty} du\int_0^{\infty} t^{-z}\hat{f}(\nu(x)n(u)a_t\cdot w_1)d^\times t = (1+x^2)^{-\frac{z}{2}-1}\Sigma_1(\hat{f}, z) \qquad (\mathrm{Re}\, z>1)\,.$$

So, if $\mathrm{Re}\, z>1$, the integral

$$\int_{-\infty}^{\infty} dx\int_{-\infty}^{\infty} du\int_0^{\infty} t^{-z}\hat{f}(\nu(x)n(u)a_t\cdot w_1)d^\times t$$

converges absolutely and is equal to

$$\int_{-\infty}^{\infty}(1+x^2)^{-\frac{z}{2}-1}dx\Sigma_1(\hat{f}, z) = \sqrt{\pi}\,\frac{\Gamma\left(\frac{z+1}{2}\right)}{\Gamma\left(\frac{z}{2}+1\right)}\Sigma_1(\hat{f}, z)\,.$$

On the other hand, by (2.1),

$$\nu(x)n(u)a_t\cdot w_1 = t^{-1}(x^2(1+ux),\ x(2+3ux),\ 1+3ux,\ u)\,.$$

Therefore we have, if $\mathrm{Re}\, z>1$,

$$\begin{aligned}
\sqrt{\pi}\,\frac{\Gamma\left(\frac{z+1}{2}\right)}{\Gamma\left(\frac{z}{2}+1\right)}\Sigma_1(\hat{f}, z) &= \lim_{\varepsilon\to+0}\int_{-\infty}^{\infty} dx\int_{-\infty}^{\infty} du\int_0^{\infty} t^{-z}e^{-\varepsilon t^{-1}}\hat{f}(\nu(x)n(u)a_t\cdot w_1)d^\times t \\
&= \lim_{\varepsilon\to+0}\int_{-\infty}^{\infty} dx\int_{-\infty}^{\infty} du\int_0^{\infty} t^{-z}e^{-\varepsilon t^{-1}} \\
&\quad\times\Big\{\int \exp 2\pi\sqrt{-1}\,t^{-1}\Big[u(y_1-xy_2+x^2y_3-x^3y_4) \\
&\quad -\frac{1}{3}y_2+\frac{2}{3}xy_3-x^2y_4\Big]f(y)dy\Big\}\, d^\times t^{8)} \\
&= \lim_{\varepsilon\to+0}\int_{-\infty}^{\infty} dx\int_0^{\infty} t^{-z}e^{-\varepsilon t^{-1}} \\
&\quad\times\Big\{\int t\exp 2\pi\sqrt{-1}\,t^{-1}\Big[-\frac{1}{3}y_2+\frac{2}{3}xy_3-x^2y_4\Big] \\
&\quad\times f(xy_2-x^2y_3+x^3y_4,\ y_2,\ y_3,\ y_4)dy_2dy_3dy_4\Big\}\, d^\times t \\
&= \lim_{\varepsilon\to+0}\int_{-\infty}^{\infty} dx\,\frac{\Gamma(z-1)}{\left(\varepsilon-2\pi\sqrt{-1}\Big[-\frac{1}{3}y_2+\frac{2}{3}xy_3-x^2y_4\Big]\right)^{z-1}} \\
&\quad\times f(xy_2-x^2y_3+x^3y_4,\ y_2,\ y_3,\ y_4)dy_2dy_3dy_4\,.
\end{aligned}$$

8) See (2.11).

Since $f(y)$ is an even function of y,

$$\sqrt{\pi}\frac{\Gamma\left(\frac{z+1}{2}\right)}{\Gamma\left(\frac{z}{2}+1\right)}\Sigma_1(\hat{f}, z)$$

$$=\lim_{\varepsilon\to+0}\frac{\Gamma(z-1)}{2}\int_{-\infty}^{\infty}dx\int\Big\{\Big(\varepsilon-2\pi\sqrt{-1}\Big[-\frac{1}{3}y_2+\frac{2}{3}xy_3-x^2y_4\Big]\Big)^{1-z}$$

$$+\Big(\varepsilon+2\pi\sqrt{-1}\Big[-\frac{1}{3}y_2+\frac{2}{3}xy_3-x^2y_4\Big]\Big)^{1-z}\Big\}$$

$$\times f(xy_2-x^2y_3+x^3y_4, y_2, y_3, y_4)\,dy_2dy_3dy_4\,.$$

By the assumption, the support of f is compact and is contained in $(V-S)_{\boldsymbol{R}}$. Hence, when $(xy_2-x^2y_3+x^3y_4, y_2, y_3, y_4)$ varies in the support of f,

$$\left|-\frac{1}{3}y_2+\frac{2}{3}xy_3-x^2y_4\right|=\frac{1}{3}\left|\frac{d}{dx}(-y_1+xy_2-x^2y_3+x^3y_4)\right|$$

is always larger than a fixed positive number. (We note that (y_1, y_2, y_3, y_4) is in $V-S$ if and only if the equation $-y_1+xy_2-x^2y_3+x^3y_4=0$ has no double roots.) Hence, if $\operatorname{Re} z>1$, we have

$$\sqrt{\pi}\frac{\Gamma\left(\frac{z+1}{2}\right)}{\Gamma\left(\frac{z}{2}+1\right)}\Sigma_1(\hat{f}, z)$$

$$=\Gamma(z-1)\cos\frac{\pi}{2}(1-z)(2\pi)^{1-z}3^{z-1}\int_{-\infty}^{\infty}dx\int|-y_2+2xy_3-3x^2y_4|^{1-z}$$

$$\times f(xy_2-x^2y_3+x^3y_4, y_2, y_3, y_4)dy_2dy_3dy_4\,.$$

But the above integral converges absolutely if $\operatorname{Re} z\geqq 0$, hence, one obtains

$$\pi\Sigma_1(\hat{f})=\left(-\frac{\pi}{2}\right)(2\pi)\frac{1}{3}\int_{-\infty}^{\infty}dx\int|-y_2+2xy_3-3x^2y_4|$$

$$\times f(xy_2-x^2y_3+x^3y_4, y_2, y_3, y_4)dy_2dy_3dy_4$$

$$=-\frac{\pi^2}{3}\Big\{3\int_{V_1}f(y)dy+\int_{V_2}f(y)dy\Big\}\,. \qquad \text{q. e. d.}$$

REMARK. We can prove Proposition 2.5 for any $f\in\mathcal{S}(V_{\boldsymbol{R}})$ and that the following equality holds.

$$\Sigma_1(g\cdot f)=\Sigma_1(f) \qquad (g\in G_{\boldsymbol{R}},\ f\in\mathcal{S}(V_{\boldsymbol{R}}))\,,$$

where we put $g\cdot f(x)=f(g^{-1}\cdot x)$.

2. We put $w_2=(0, 0, 0, 1)\in S\cap V_{\boldsymbol{R}}$. For every $f\in\mathcal{S}(V_{\boldsymbol{R}})$, we define $\Sigma_2(f)$ as follows:

$$\Sigma_2(f)=\frac{1}{2\pi}\int_0^{2\pi}d\theta\int_0^{\infty}t^{-2}f(k_\theta a_t\cdot w_2)d^{\times}t\,. \tag{2.16}$$

The following lemma can be easily proved.

LEMMA 2.4. (i) *The mapping* $f\rightarrow\Sigma_2(f)$ *defines a tempered distribution on* $V_{\boldsymbol{R}}$ *whose support is contained in* $S_{\boldsymbol{R}}$.

(ii) *We have* $\Sigma_2(g\cdot f)=\chi(g)^{1/6}\Sigma_2(f)$ $(f\in\mathcal{S}(V_{\boldsymbol{R}}),\ g\in G_{\boldsymbol{R}})$, *where we put* $g\cdot f(x)=f(g^{-1}\cdot x)$.

PROPOSITION 2.6. *For any* $f\in C_0^\infty((V-S)_{\boldsymbol{R}})$, *one has*

$$\Sigma_2(\hat{f})=\frac{\Gamma\left(\frac{1}{3}\right)(2\pi)^{\frac{1}{3}}}{6\pi\Gamma\left(\frac{2}{3}\right)}\left\{\sqrt{3}\int_{V_1}(P(x))^{-\frac{1}{6}}f(x)dx+\int_{V_2}(-P(x))^{-\frac{1}{6}}f(x)dx\right\}^{9)}.$$

PROOF. We may assume that $f(k_\theta\cdot x)=f(x)$ $(\forall\theta\in\boldsymbol{R})$. Then we have, by Lemma 2.2 (ii),

$$\int_0^\infty t^{-2}\hat{f}(\nu(x)a_t\cdot w_2)d^\times t=\frac{1}{1+x^2}\int_0^\infty t^{-2}\hat{f}(a_t\cdot w_2)d^\times t\,.$$

Hence

$$\pi\Sigma_2(f)=\int_{-\infty}^\infty dx\int_0^\infty t^{-2}\hat{f}(\nu(x)a_t\cdot w_2)d^\times t\,.$$

By (2.1), we have $\nu(x)a_t\cdot w_2=t^{-3}(x^3,3x^2,3x,1)$. Therefore

$$\pi\Sigma_2(\hat{f})=\int_{-\infty}^\infty dx\int_0^\infty t^{-2}\left\{\int\exp 2\pi\sqrt{-1}\,t^{-3}(y_1-xy_2+x^2y_3-x^3y_4)f(y)dy\right\}d^\times t\,.$$

Taking into account that $f(y)$ is an even function of y,

$$\pi\Sigma_2(f)=\frac{1}{3}\Gamma\left(\frac{2}{3}\right)(2\pi)^{-\frac{2}{3}}\cos\frac{\pi}{3}\int_{-\infty}^\infty dx\int f(y)|y_1-xy_2+x^2y_3-x^3y_4|^{-\frac{2}{3}}dy\,.$$

We put $Q(x)=y_1-xy_2+x^2y_3-x^3y_4$. When $y\in V_1$, it follows from Proposition 2.2 that there exists a $g=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in G_+$ such that

$$Q\left(\frac{\alpha x+\gamma}{\beta x+\delta}\right)(\beta x+\delta)^3=x^2-x\,.$$

We have, by (2.2) and (2.3), $(\alpha\delta-\beta\gamma)^6P(y)=1$. Hence

$$\int_{-\infty}^\infty|y_1-xy_2+x^2y_3-x^3y_4|^{-\frac{2}{3}}dx=(P(y))^{-\frac{1}{6}}\int_{-\infty}^\infty|x^2-x|^{-\frac{2}{3}}dx$$

$$=(P(y))^{-\frac{1}{6}}3\frac{\Gamma\left(\frac{1}{3}\right)^2}{\Gamma\left(\frac{2}{3}\right)}\,.$$

On the other hand, when $y\in V_2$, there exists a $g=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in G_+$ such that

9) It can be proved that this equality holds for every $f\in\mathcal{S}(V_{\boldsymbol{R}})$.

$$Q\Big(\frac{\alpha x+\gamma}{\beta x+\delta}\Big)(\beta x+\delta)^3=x^2+1\,.$$

We have $(\alpha\delta-\beta\gamma)^6 P(y)=-4$ and

$$\int_{-\infty}^{\infty}|y_1-xy_2+x^2y_3-x^3y_4|^{-\frac{2}{3}}dx=2^{\frac{1}{3}}(-P(y))^{-\frac{1}{6}}\int_{-\infty}^{\infty}(x^2+1)^{-\frac{2}{3}}dx$$

$$=2^{\frac{1}{3}}(-P(y))^{-\frac{1}{6}}\sqrt{\pi}\,\frac{\Gamma\big(\frac{1}{6}\big)}{\Gamma\big(\frac{2}{3}\big)}\,.$$

Hence one obtains

$$\pi\Sigma_2(\hat{f})=\frac{\Gamma\big(\frac{2}{3}\big)}{3}(2\pi)^{-\frac{2}{3}}\cos\frac{\pi}{3}\Big\{3\frac{\Gamma\big(\frac{1}{3}\big)^2}{\Gamma\big(\frac{2}{3}\big)}\int_{V_1}(P(x))^{-\frac{1}{6}}f(x)dx$$

$$+2^{\frac{1}{3}}\sqrt{\pi}\,\frac{\Gamma\big(\frac{1}{6}\big)}{\Gamma\big(\frac{2}{3}\big)}\int_{V_2}(-P(x))^{-\frac{1}{6}}f(x)dx\Big\}$$

$$=\frac{(2\pi)^{\frac{1}{3}}}{6}\frac{\Gamma\big(\frac{1}{3}\big)}{\Gamma\big(\frac{2}{3}\big)}\Big\{\sqrt{3}\int_{V_1}(P(x))^{-\frac{1}{6}}f(x)dx$$

$$+\int_{V_2}(-P(x))^{-\frac{1}{6}}f(x)dx\Big\}$$

(note that we have used the relations $2^{\frac{2}{3}}\sqrt{\pi}\,\Gamma\big(\frac{1}{3}\big)=\Gamma\big(\frac{1}{6}\big)\Gamma\big(\frac{2}{3}\big)$, $\Gamma\big(\frac{1}{3}\big)\Gamma\big(\frac{2}{3}\big)=\frac{2}{\sqrt{3}}\pi$). q. e. d.

3. By the definition, $Q(\nabla_x)$ is a differential operator with constant coefficients on V_R such that

$$Q(\nabla_x)e^{\langle x,y\rangle}=P(y)e^{\langle x,y\rangle}\,.$$

Hence it follows from (2.2) and (2.11),

$$Q(\nabla_x)=Q\Big(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3},\frac{\partial}{\partial x_4}\Big), \tag{2.17}$$

where we put $Q(y_1,y_2,y_3,y_4)=81y_2^2y_3^2+162y_1y_2y_3y_4-108y_2^3y_4-108y_1y_3^3-27y_1^2y_4^2$. By the definition, $b_1(z)$ is a polynomial of z such that

$$P(x)Q(\nabla_x)P^z(x)=b_1(z)P^z(x)\qquad(z\in\boldsymbol{C})\,.$$

PROPOSITION 2.7.[10] *We have* $b_1(s)=2^4\cdot3^6\cdot s^2\big(s-\frac{1}{6}\big)\big(s+\frac{1}{6}\big)$.

10) This proposition is due to Sato.

PROOF. We know that $b_1(s)$ is a polynomial of degree 4. Let b_0 be its leading coefficient. We put $\partial_i P = \frac{\partial P}{\partial x_i}$ $(i=1, \cdots, 4)$. Then we have, by the definition of b_1, $Q(\partial_1 P, \partial_2 P, \partial_3 P, \partial_4 P) = b_0 P^3$. Putting $x=(0, 1, 1, 0)$ in the above equality we obtain $b_0 = 11664 = 2^4 \cdot 3^6$. By Theorem 1.1 (ii), we have $b_1(s) = b_1(-s)$. It follows easily from the definition of $b_1(s)$ that $b_1(0)=0$. So, there exists an even monic quadratic polynomial $c(s)$ such that

$$b_1(s) = 2^4 \cdot 3^6 \cdot s^2 c(s). \tag{2.18}$$

Now let $f \in C_0^\infty(V_1)$. Then it follows from Proposition 2.6,

$$\pi \Sigma_2(\hat{f}) = \frac{(2\pi)^{\frac{1}{3}}}{2\sqrt{3}} \frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} \int_{V_1} (P(x))^{-\frac{1}{6}} f(x) dx.$$

Since $\Sigma_2(P\hat{f})=0$, we have

$$\int_{V_1} (P(x))^{-\frac{1}{6}} Q(\nabla_x) f(x) dx = b_1\left(-\frac{1}{6}\right) \int_{V_1} (P(x))^{-\frac{7}{6}} f(x) dx = 0$$

for every $f \in C_0^\infty(V_1)$. Thus $b_1\left(-\frac{1}{6}\right)=0$. Hence, from (2.18), it follows $c\left(-\frac{1}{6}\right)=0$. Since $c(s)$ is an even monic quadratic polynomial, $c(s) = \left(s+\frac{1}{6}\right)\left(s-\frac{1}{6}\right)$. Therefore $b(s) = 2^4 \cdot 3^6 \cdot s^2 \left(s-\frac{1}{6}\right)\left(s+\frac{1}{6}\right)$. q. e. d.

4. We put

$$\omega = \frac{1}{4P} \{x_1 dx_2 \wedge dx_3 \wedge dx_4 - x_2 dx_1 \wedge dx_3 \wedge dx_4 + x_3 dx_1 \wedge dx_2 \wedge dx_4 - x_4 dx_1 \wedge dx_2 \wedge dx_3\}.$$

Then ω is a differential form on $(V-S)_{\boldsymbol{R}}$ and $dx = dP \wedge \omega$.[11]

PROPOSITION 2.8. *Let $f \in \mathcal{S}(V_{\boldsymbol{R}})$, then*

$$\lim_{\mu \to +0} \mu^{\frac{1}{6}} \int_{P=\mu} f(x)\omega = \pi \frac{\Gamma\left(\frac{1}{3}\right)^2}{\Gamma\left(\frac{2}{3}\right)} \Sigma_2(f)$$

and

$$\lim_{\mu \to +0} \mu^{\frac{1}{6}} \int_{P=-\mu} f(x)\omega = \pi\sqrt{3} \frac{\Gamma\left(\frac{1}{3}\right)^2}{\Gamma\left(\frac{2}{3}\right)} \Sigma_2(f).$$

PROOF. We put $y_1=(0, 1, 1, 0)$ and $y_2=(1, 0, 0, 1)$. It follows from Proposition 2.4 that

11) Cf. Chapter 1, §2. 2.

$$\int_0^\infty d^\times\lambda\int_{G^1}f(g_1\cdot(\lambda^3 y_1))dg_1=\frac{1}{4\pi}\int_0^\infty d^\times\mu\int_{P=\mu}f(x)\omega$$

and

$$\int_0^\infty d^\times\lambda\int_{G^1}f(g_1\cdot(\lambda^3 y_2))dg_1=\frac{1}{12\pi}\int_0^\infty d^\times\mu\int_{P=-\mu}f(x)\omega$$

for every $f\in C_0((V-S)_{\boldsymbol{R}})$ $(G^1=SL(2,\boldsymbol{R}))$. Since $P(\lambda^3 y_1)=\lambda^{12}$ and $P(\lambda^3 y_2)=-27\lambda^{12}$, we have

$$\frac{1}{4\pi}\int_{P=\mu}f(x)\omega=\frac{1}{12}\int_{G^1}f(g_1\cdot((\mu)^{\frac{1}{4}}y_1))dg_1$$

and

$$\frac{1}{12\pi}\int_{P=-\mu}f(x)\omega=\frac{1}{12}\int_{G^1}f\Big(g_1\cdot\Big(\Big(\frac{\mu}{27}\Big)^{\frac{1}{4}}y_2\Big)\Big)dg_1$$

$(f\in C_0^\infty((V-S)_{\boldsymbol{R}}),\ \mu>0)$. Obviously, the above equality holds for every $f\in\mathcal{S}(V_{\boldsymbol{R}})$. Hence, when we put

$$\tilde{f}(x)=\frac{1}{2\pi}\int_0^{2\pi}f(k_\theta\cdot x)d\theta\,,$$

we get, by (2.1) and (2.6),

$$\begin{aligned}\mu^{\frac{1}{6}}\int_{P=\mu}f(x)\omega&=\frac{\pi}{3}\mu^{\frac{1}{6}}\int_{-\infty}^\infty du\int_0^\infty\tilde{f}(n(u)a_t\cdot(\mu^{\frac{1}{4}}y_1))d^\times t\\&=\frac{\pi}{3}\mu^{\frac{1}{6}}\int_{-\infty}^\infty du\int_0^\infty\tilde{f}(\mu^{\frac{1}{4}}(0,\,t,\,2ut+t^{-1},\,u^2t+ut^{-1}))d^\times t\ {}^{12)}\\&=\frac{\pi}{3}\int_{-\infty}^\infty du\int_0^\infty\tilde{f}(0,\,t\mu^{\frac{1}{3}},\,2ut\mu^{\frac{1}{6}}+\mu^{\frac{1}{6}}t^{-1},\,u^2t+ut^{-1})d^\times t\end{aligned}$$

(replace u by $\mu^{-\frac{1}{6}}u$ and t by $\mu^{\frac{1}{12}}t$). Hence it follows

$$\begin{aligned}\lim_{\mu\to+0}\mu^{\frac{1}{6}}\int_{P=\mu}f(x)\omega&=\frac{\pi}{3}\int_{-\infty}^\infty du\int_0^\infty\tilde{f}(0,\,0,\,0,\,u^2t+ut^{-1})d^\times t\\&=\frac{\pi}{3}\int_0^\infty du\int_0^\infty\tilde{f}(0,\,0,\,0,\,u^{\frac{3}{2}}(t+t^{-1}))d^\times t\\&\quad+\frac{\pi}{3}\int_{-\infty}^\infty du\int_0^\infty\tilde{f}(0,\,0,\,0,\,(-u)^{\frac{3}{2}}(t-t^{-1}))d^\times t\\&=\frac{2}{3}\pi\int_0^\infty\tilde{f}(0,\,0,\,0,\,u^3)udu\\&\quad\times\Big\{\int_0^\infty(t+t^{-1})^{-\frac{2}{3}}d^\times t+\int_0^\infty|t-t^{-1}|^{-\frac{2}{3}}d^\times t\Big\}\ {}^{13)}\end{aligned}$$

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12) We put $\tilde{f}(x)=\tilde{f}(x_1,\,x_2,\,x_3,\,x_4)$, when $x=(x_1,\,x_2,\,x_3,\,x_4)$.
13) We note that f is an even function.

$$=\pi\frac{\Gamma\left(\frac{1}{3}\right)^2}{\Gamma\left(\frac{2}{3}\right)}\Sigma_2(f).$$

Similarly, we have

$$\mu^{\frac{1}{6}}\int_{P=-\mu}f(x)\omega=\pi\mu^{\frac{1}{6}}\int_{-\infty}^{\infty}du\int_0^{\infty}\tilde{f}\Big(\Big(\frac{\mu}{27}\Big)^{\frac{1}{4}}(t^3,\,3t^3u,\,3t^3u^2,\,t^3u^3+t^{-3})\Big)d^\times t$$

$$=\pi\int_{-\infty}^{\infty}du\int_0^{\infty}\tilde{f}\Big(\frac{1}{(27)^{\frac{1}{4}}}(\mu^{\frac{1}{2}}t^3,\,3t^3u\mu^{\frac{1}{3}},\,3t^3u^2\mu^{\frac{1}{6}},\,t^3u^3+t^{-3})\Big)d^\times t.$$

Hence we get

$$\lim_{\mu\to+0}\mu^{\frac{1}{6}}\int_{P=-\mu}f(x)\omega=(27)^{\frac{1}{6}}\pi\int_{-\infty}^{\infty}du\int_0^{\infty}\tilde{f}(0,\,0,\,0,\,t^3u^3+t^{-3})d^\times t$$

$$=\sqrt{3}\,\pi\frac{\Gamma\left(\frac{1}{3}\right)^2}{\Gamma\left(\frac{2}{3}\right)}\Sigma_2(f).\qquad\text{q. e. d.}$$

5. We put $\gamma(s)=\Gamma\left(s+\frac{5}{6}\right)\Gamma(s+1)^2\Gamma\left(s+\frac{7}{6}\right)$, then, by Theorem 1.1, we know that the following assertions are true. For every $f\in\mathcal{S}(V_R)$,

$$F_1(s,f)=\frac{1}{\gamma(s)}\int_{V_1}f(x)\,|P(x)|^s dx$$

and

$$F_2(s,f)=\frac{1}{\gamma(s)}\int_{V_2}f(x)\,|P(x)|^s dx$$

are entire analytic functions of s which satisfy the following equality:

$$\begin{pmatrix}F_1(s-1,\hat{f})\\ F_2(s-1,\hat{f})\end{pmatrix}=\gamma(-s)(2\pi)^{-4s}|b_0|^s\begin{pmatrix}a(s) & b(s)\\ c(s) & d(s)\end{pmatrix}\begin{pmatrix}F_1(-s,f)\\ F_2(-s,f)\end{pmatrix},$$ [14]

where $a(s)$, $b(s)$, $c(s)$, $d(s)$ are of the following forms;

$$\begin{aligned}a(s)&=a_{-1}e^{-2\pi\sqrt{-1}s}+a_0+a_1e^{2\pi\sqrt{-1}s}\\ b(s)&=b_{-1}e^{-\pi\sqrt{-1}s}+b_1e^{\pi\sqrt{-1}s}\\ c(s)&=c_{-1}e^{-\pi\sqrt{-1}s}+c_1e^{\pi\sqrt{-1}s}\\ d(s)&=d_{-1}e^{-2\pi\sqrt{-1}s}+d_0+d_1e^{2\pi\sqrt{-1}s}\end{aligned}\tag{2.19}$$

(a_0, d_0, $a_{\pm1}$, $b_{\pm1}$, $c_{\pm1}$, $d_{\pm1}\in\boldsymbol{C}$). Furthermore, the equality

$$|b_0|(2\pi)^{-4}\begin{pmatrix}a(s) & b(s)\\ c(s) & d(s)\end{pmatrix}\begin{pmatrix}a(1-s) & b(1-s)\\ c(1-s) & d(1-s)\end{pmatrix}\tag{2.20}$$

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14) We put $b_0=2^4\cdot3^6$.

$$=9\frac{1}{\gamma(s-1)\gamma(-s)}\begin{pmatrix}1 & \\ & 1\end{pmatrix}$$

$$=9\pi^{-4}\sin^2\pi s\sin\pi\left(s-\frac{1}{6}\right)\sin\pi\left(s+\frac{1}{6}\right)\begin{pmatrix}1 & \\ & 1\end{pmatrix}$$

holds, since $\hat{\hat{f}}=9f$.

The connected components V_1 and V_2 of V_R-S_R are obviously mutually neighbouring (see Chap. 1, § 2. 2). Therefore we get the following lemma by Proposition 1.5.

LEMMA 2.5. *We have*

$$\left\{\begin{array}{l} a(1)+b(1)=0, \\ c(1)+d(1)=0. \end{array}\right. \tag{2.21}$$

LEMMA 2.6. *We have*

$$\begin{pmatrix} a\left(\frac{1}{6}\right) & b\left(\frac{1}{6}\right) \\ c\left(\frac{1}{6}\right) & d\left(\frac{1}{6}\right)\end{pmatrix}=\frac{1}{18}\begin{pmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{3}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}. \tag{2.22}$$

PROOF. Take an $f\in C_0^\infty((V-S)_R)$ and define the C^2-valued function $q(t)$ $(t>0)$ as follows:

$$q(t)=\begin{pmatrix}\int_{p=t}\hat{f}(x)\omega \\ \int_{p=-t}\hat{f}(x)\omega\end{pmatrix}.$$

It follows from Proposition 1.4 that

$$\lim_{t\to+0} t^{\frac{1}{6}}q(t)$$

$$=\Gamma\left(\frac{1}{6}\right)^2\Gamma\left(\frac{1}{3}\right)(2\pi)^{-\frac{2}{3}}\cdot 2^{\frac{2}{3}}\cdot 3\begin{pmatrix} a\left(\frac{1}{6}\right) & b\left(\frac{1}{6}\right) \\ c\left(\frac{1}{6}\right) & d\left(\frac{1}{6}\right)\end{pmatrix}\begin{pmatrix}\int_{V_1}|P(x)|^{-\frac{1}{6}}f(x)dx \\ \int_{V_2}|P(x)|^{-\frac{1}{6}}f(x)dx\end{pmatrix}.$$

On the other hand, it follows from Proposition 2.8 and Proposition 2.6 that

$$\lim_{t\to+0} t^{\frac{1}{6}}q(t)=\pi\frac{\Gamma\left(\frac{1}{3}\right)^2}{\Gamma\left(\frac{2}{3}\right)}\begin{pmatrix}\Sigma_2(\hat{f}) \\ \sqrt{3}\,\Sigma_2(\hat{f})\end{pmatrix}$$

$$=\pi\frac{\Gamma\left(\frac{1}{3}\right)^2}{\Gamma\left(\frac{2}{3}\right)}\cdot\frac{(2\pi)^{\frac{1}{3}}\Gamma\left(\frac{1}{3}\right)}{6\pi\Gamma\left(\frac{2}{3}\right)}\begin{pmatrix}\sqrt{3} & 1 \\ 3 & \sqrt{3}\end{pmatrix}\begin{pmatrix}\int_{V_1}|P|^{-\frac{1}{6}}f(x)dx \\ \int_{V_2}|P|^{-\frac{1}{6}}f(x)dx\end{pmatrix}.$$

Hence, we get

$$\begin{pmatrix} a\left(\frac{1}{6}\right) & b\left(\frac{1}{6}\right) \\ c\left(\frac{1}{6}\right) & d\left(\frac{1}{6}\right) \end{pmatrix} = \frac{1}{18} \frac{2^{\frac{1}{3}}\pi\Gamma\left(\frac{1}{3}\right)^2}{\Gamma\left(\frac{2}{3}\right)^2\Gamma\left(\frac{1}{6}\right)^2} \begin{pmatrix} \sqrt{3} & 1 \\ 3 & \sqrt{3} \end{pmatrix} = \frac{1}{18} \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{3}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}.$$

q. e. d.

Now we are ready to prove the following.

PROPOSITION 2.9. *We have*

$$\begin{pmatrix} a(s) & b(s) \\ c(s) & d(s) \end{pmatrix} = \frac{1}{18} \begin{pmatrix} \sin 2\pi s & \sin \pi s \\ 3 \sin \pi s & \sin 2\pi s \end{pmatrix}.$$

In other words

$$\begin{pmatrix} \int_{V_1} \hat{f}(x)|P|^{s-1}dx \\ \int_{V_2} \hat{f}(x)|P|^{s-1}dx \end{pmatrix} = \Gamma\left(s-\frac{1}{6}\right)\Gamma(s)^2\Gamma\left(s+\frac{1}{6}\right)\pi^{-4s}3^{6s} \times \frac{1}{18} \begin{pmatrix} \sin 2\pi s & \sin \pi s \\ 3 \sin \pi s & \sin 2\pi s \end{pmatrix} \begin{pmatrix} \int_{V_1} f(x)|P|^{-s}dx \\ \int_{V_2} f(x)|P|^{-s}dx \end{pmatrix} \tag{2.23}$$

for every $f \in \mathcal{S}(V_{\mathbf{R}})$.

PROOF. We already know that $a(s)$, $b(s)$, $c(s)$ and $d(s)$ are of the forms described in (2.19) and satisfy relations (2.20), (2.21) and (2.22). We put $\Delta(s) = a(s)d(s)-b(s)c(s)$. Then (2.19) and (2.20) yield the following equalities:

$$\begin{aligned} &\Delta(s+1) = \Delta(s), \\ &\Delta(s)\Delta(1-s) = 3^{-8} \sin^4\pi s \cdot \sin^2\pi\left(s-\frac{1}{6}\right)\sin^2\pi\left(s+\frac{1}{6}\right), \\ &\Delta(s)\begin{pmatrix} a(1-s) & b(1-s) \\ c(1-s) & d(1-s) \end{pmatrix} \\ &\quad = 3^{-4}\sin^2\pi s \sin \pi\left(s-\frac{1}{6}\right)\sin \pi\left(s+\frac{1}{6}\right)\begin{pmatrix} d(s) & -b(s) \\ -c(s) & a(s) \end{pmatrix}. \end{aligned} \tag{2.24}$$

Taking (2.22) into account, we can conclude that $\Delta(s)$ must be equal to $\pm 3^{-4}\sin^2\pi s \cdot \sin \pi\left(s-\frac{1}{6}\right)\cdot \sin \pi\left(s+\frac{1}{6}\right)$. Assume $\Delta(s) = 3^{-4}\sin^2\pi s \sin \pi\left(s-\frac{1}{6}\right) \sin \pi\left(s+\frac{1}{6}\right)$, then it follows from (2.19), (2.22) and (2.24)

$$\begin{pmatrix} a(s) & b(s) \\ c(s) & d(s) \end{pmatrix} = \begin{pmatrix} a(s) & \frac{1}{18\sqrt{3}} \cos \pi s \\ \frac{1}{6\sqrt{3}} \cos \pi s & a(1-s) \end{pmatrix}.$$

Hence,

$$0=\Delta(1)=a(1)a(0)-\frac{1}{324}=a(1)^2-\frac{1}{324}.$$

Thus $a(1)^2=\frac{1}{324}\neq b(1)^2$. This is impossible by virtue of (2.21). Therefore $\Delta(s)=-3^{-4}\sin^2\pi s\sin\pi\left(s-\frac{1}{6}\right)\sin\pi\left(s+\frac{1}{6}\right)$ and

$$\begin{pmatrix} a(s) & b(s) \\ c(s) & d(s) \end{pmatrix}=\begin{pmatrix} a(s) & \frac{1}{18}\sin\pi s \\ \frac{1}{6}\sin\pi s & -a(1-s) \end{pmatrix}.$$

Hence

$$\begin{aligned}-a(s)a(1-s)&=\frac{1}{108}\sin^2\pi s-3^{-4}\sin^2\pi s\sin\pi\left(s-\frac{1}{6}\right)\sin\pi\left(s+\frac{1}{6}\right)\\&=\frac{1}{324}\sin^2 2\pi s.\end{aligned}$$

Since $a\left(\frac{1}{6}\right)=\frac{1}{36}\sqrt{3}$, we can conclude $a(s)=\frac{1}{18}\sin 2\pi s$. Thus

$$\begin{pmatrix} a(s) & b(s) \\ c(s) & d(s) \end{pmatrix}=\frac{1}{18}\begin{pmatrix} \sin 2\pi s & \sin\pi s \\ 3\sin\pi s & \sin 2\pi s \end{pmatrix}.$$

q. e. d.

§ 3. Dirichlet series associated with the lattice of integral binary cubic forms

1. We denote by L the lattice of integral binary cubic forms and by $\hat{L}$ the dual lattice of L. We have

$$L=\{(x_1, x_2, x_3, x_4)\in V_{\boldsymbol{R}}\,;\ x_1, x_2, x_3, x_4\in\boldsymbol{Z}\}$$

and

$$\begin{aligned}\hat{L}&=\{x\in V_{\boldsymbol{R}}\,;\ \langle x, y\rangle\in\boldsymbol{Z}\ (\forall y\in L)\}\\&=\{(x_1, x_2, x_3, x_4)\in L\,;\ 3\,|\,x_2, x_3\}.\end{aligned}$$

We put $\Gamma=SL(2,\boldsymbol{Z})\subset SL(2,\boldsymbol{R})=G^1$. Then L and $\hat{L}$ are stable under the action of Γ.

For every integer m we put $L_m=\{x\in L\,;\ P(x)=m\}$, $\hat{L}_m=L_m\cap\hat{L}$. It is obvious that L_m is a Γ-invariant subset of L.

PROPOSITION 2.10. *The set L_0 decomposes into a union of a countable number of Γ-orbits as follows:*

$$L_0=\{0\}\cup\bigcup_{m=1}^{\infty}\ \bigcup_{\gamma\in\Gamma/\Gamma\cap N}\gamma\cdot(0,0,0,m)\cup\bigcup_{m=1}^{\infty}\bigcup_{n=0}^{m-1}\bigcup_{\gamma\in\Gamma}\gamma\cdot(0,0,m,n).$$

PROOF. It is obvious that $L_0 \subset S_{\boldsymbol{R}}$. It follows from Proposition 2.3 that $S_{\boldsymbol{R}} = S_1 \cup S_2 \cup \{0\}$, where $S_1 = G^1 \cdot (0, 0, 1, 0)$ and $S_2 = G^1 \cdot (0, 0, 0, 1)$. Take an $x \in L_0 \cap S_2$, then, F_x has a rational triple root.

Hence there exists a $\gamma \in \Gamma$ such that ∞ is a triple root of $\gamma \cdot F_x$. Then we have $\gamma \cdot x = (0, 0, 0, m)$ $(m \in \boldsymbol{Z} - \{0\})$. Since $\begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \in \Gamma$, we may assume $m > 0$. It is obvious that $(0, 0, 0, m_1)$ and $(0, 0, 0, m_2)(m_1, m_2 > 0;\ m_1, m_2 \in \boldsymbol{Z})$ lie on the same Γ-orbit if and only if $m_1 = m_2$. We see from Corollary to Proposition 2.3 that the isotropy subgroup of $(0, 0, 0, m)$ in G^1 is N. Hence, we get

$$L_0 \cap S_2 = \bigcup_{m=1}^{\infty} \bigcup_{\gamma \in \Gamma / \Gamma \cap N} \gamma \cdot (0, 0, 0, m) \qquad \text{(disjoint union)}.$$

Take an $x \in L_0 \cap S_1$, then F_x has a rational double root and a rational simple root. We can take a $\gamma \in \Gamma$ such that $\gamma \cdot F_x$ has ∞ as a double root and a rational simple root. Then, we have

$$\gamma \cdot x = (0, 0, m, n) \qquad (m \in \boldsymbol{Z} - \{0\},\ n \in \boldsymbol{Z}).$$

Since $\begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \in \gamma$, we may assume $m > 0$. It can be easily proved that $(0, 0, m_1, n_1)$ and $(0, 0, m_2, n_2)(m_1, n_1;\ m_2, n_2 \in \boldsymbol{Z}$ and $m_1, m_2 > 0)$ lie on the same Γ-orbit if and only if $m_1 = m_2$ and n_1 is congruent to n_2 modulo m_1. Since G^1 acts simply transitively on S_1,

$$L_0 \cap S_1 = \bigcup_{m=1}^{\infty} \bigcup_{n=0}^{m-1} \bigcup_{\gamma \in \Gamma} \gamma \cdot (0, 0, m, n) \qquad \text{(disjoint union)},$$

and

$$L_0 = \{0\} \cup \bigcup_{m=1}^{\infty} \bigcup_{\gamma \in \Gamma / \Gamma \cap N} \gamma \cdot (0, 0, 0, m) \cup \bigcup_{m=1}^{\infty} \bigcup_{n=0}^{m-1} \bigcup_{\gamma \in \Gamma} \gamma \cdot (0, 0, m, n)$$

(disjoint union). q. e. d.

COROLLARY.

$$\hat{L}_0 = \{0\} \cup \bigcup_{m=1}^{\infty} \bigcup_{\gamma \in \Gamma / \Gamma \cap N} \gamma \cdot (0, 0, 0, m) \cup \bigcup_{m=1}^{\infty} \bigcup_{n=0}^{3m-1} \bigcup_{\gamma \in \Gamma} \gamma \cdot (0, 0, 3m, n)$$

(disjoint union).

The following proposition is classical (see e. g. [1]). We give a proof for the sake of completeness.

PROPOSITION 2.11. *The set L_m $(0 \neq m \in \boldsymbol{Z})$ decomposes into a union of a finite number of Γ-orbits. When we denote by $h(m)$ the number of Γ-orbits in L_m,*[15] *there exists a positive constant c such that the following inequality holds:*

$$h(m) < c|m|^2 \qquad (0 \neq m \in \boldsymbol{Z}).$$

PROOF. We define a subset $\mathfrak{S}'$ of $G' = SL(2, \boldsymbol{R})$ as follows:

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15) In other words, $h(m)$ is the class number of integral binary cubic forms with discriminant m.

$$\mathfrak{S}'=\left\{n(u)a_t k_\theta\,;\ \theta\in\boldsymbol{R},\ 0<t\leqq 2,\ |u|\leqq\frac{1}{2}\right\}.$$

It is well known that $G^1=SL(2,\boldsymbol{R})=\Gamma\mathfrak{S}'=\{\gamma g\,;\ \gamma\in\Gamma,\ g\in\mathfrak{S}'\}$ (see e.g. [2]). We put $y_1=(0,1,1,0)$ and $y_2=\left(\sqrt[4]{\frac{1}{27}},0,0,\sqrt[4]{\frac{1}{27}}\right)$. Furthermore we put

$$x_m=\sqrt[4]{|m|}\,y_1 \qquad \text{when}\quad m>0$$

and

$$x_m=\sqrt[4]{|m|}\,y_2 \qquad \text{when}\quad m<0.$$

Then, it follows from Proposition 2.2, that $L_m\subset G^1\cdot x_m=\Gamma\mathfrak{S}'\cdot x_m$. Hence, it is sufficient to prove that there exists a positive constant c such that the cardinality $\#\{L\cap\mathfrak{S}'\cdot x_m\}$ is smaller than $c|m|^2$. Since the set

$$E=\left\{a_t^{-1}n(u)a_t k_\theta\,;\ \theta\in\boldsymbol{R},\ 0<t\leqq 2,\ |u|\leqq\frac{1}{2}\right\}$$

is a compact subset of G^1, $E\cdot y_1$ and $E\cdot y_2$ are compact subsets of $V_{\boldsymbol{R}}$.

Hence, there exists a positive number $N>1$ such that $\operatorname*{Max}_{1\leqq i\leqq 4}|x_i|<N$ when $x\in E\cdot y_1$ or $E\cdot y_2$. Take an $x=(x_1,x_2,x_3,x_4)\in L\cap\mathfrak{S}'\cdot x_m$. Then there exists a t such that $0<t\leqq 2$ and that

$$a_t^{-1}\cdot\frac{x}{\sqrt[4]{|m|}}\in E\cdot y_1\cup E\cdot y_2.$$

Hence one gets

$$\text{Max.}\left\{t^{-3}\frac{|x_1|}{\sqrt[4]{|m|}},\ t^{-1}\frac{|x_2|}{\sqrt[4]{|m|}}\right\}<N.$$

Since $x\in\boldsymbol{L_m}$ $(m\neq 0)$, it follows $\max.\{|x_1|,|x_2|\}\geqq 1$. Therefore we have $t>\{N\sqrt[4]{|m|}\}^{-1}$. Hence

$$L\cap\mathfrak{S}'\cdot x_m\subset\left\{a_t\cdot x\,;\ \frac{1}{N\sqrt[4]{|m|}}<t\leqq 2,\ \text{Max.}\,|x_i|<\sqrt[4]{|m|}\,N\right\}$$

$$\subset\left\{x\,;\ |x_1|,\,|x_2|\leqq 8N\sqrt[4]{|m|},\ |x_3|<N^2\sqrt{|m|},\ |x_4|<N^4|m|\right\}.$$

Hence,

$$\#(L\cap\mathfrak{S}'\cdot x_m)\leqq 2^4(8N\sqrt[4]{m}+1)^2(N^2\sqrt{|m|}+1)(N^4|m|+1).$$

Thus it is proved that if we take a C sufficiently large, we have $\#(L\cap\mathfrak{S}'\cdot x_m)<C|m|^2$ $(0\neq m\in\boldsymbol{Z})$. q. e. d.

For any integer $m\neq 0$, we write $L_m=O_1(m)\cup\cdots\cup O_{h(m)}(m)$, where $O_1(m),\cdots,O_{h(m)}(m)$ are mutually disjoint Γ-orbits in L_m.

Now we assume $m>0$ and take an $O_i(m)$. It follows from Proposition 2.2 that the isotropy subgroup of any point of $O_i(m)$ in Γ is either the unit group or a cyclic group of order 3.

In the first case we call $O_i(m)$ an orbit of the first kind and in the second case we call $O_i(m)$ an orbit of the second kind. We denote by $h_1(m)$ the number of orbits of the first kind in L_m and denote by $h_2(m)$ the number of orbits of the second kind in L_m.

When $m<0$, Γ operates simply transitively on every orbit in L_m.

Furthermore we denote by $\hat{h}(m)$ the number of Γ-orbits in L_m which are contained in $\hat{L}$. We define $\hat{h}_1(m)$, $\hat{h}_2(m)$ $(m>0)$ in the similar manner.

PROPOSITION 2.12. *We have*

$$2h_2(m)=\sharp\{(x, y)\in \boldsymbol{Z}^2\,;\ (9x^2+3xy+y^2)^2=m\} \qquad (m=1, 2, \cdots).$$

PROOF. We assume that the isotropy subgroup of $x\in L_m$ in Γ is a cyclic group of order 3. It is well-known that any cyclic subgroup of order 3 in Γ is conjugate in Γ to the group $\left\{1, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & \end{pmatrix}\right\}$. Therefore (replacing x by a suitable $\gamma\cdot x$ $(\gamma\in\Gamma)$ if necessary) we may assume that it is $\left\{1, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}\right\}$. We put $x=(x_1, x_2, x_3, x_4)$. We assume $x_1\neq 0$ and take a root ω of the equation $x_1u^3+x_2u^2+x_3u+x_4=0$. Then the set of the roots of this equation is $\left\{\omega, \frac{-1}{\omega-1}, \frac{-\omega+1}{-\omega}\right\}$. Hence

$$-\frac{x_2}{x_1}=\omega+\frac{1}{1-\omega}+\frac{\omega-1}{\omega}=\frac{\omega^3-3\omega+1}{\omega(\omega-1)}$$

$$\frac{x_3}{x_1}=\frac{\omega}{1-\omega}-\frac{1}{\omega}+\omega-1=\frac{\omega^3-3\omega^2+1}{\omega(\omega-1)}$$

$$-\frac{x_4}{x_1}=-1.$$

Therefore

$$x_4=x_1 \quad \text{and} \quad x_2=-x_3-3x_1. \tag{2.25}$$

Hence $P(x)=(9x_1^2+3x_1x_3+x_3^2)^2$. If $x_1=0$, (2.25) must be satisfied also.

Conversely take an $x\in L_m$ which satisfies (2.25), then we can see, by (2.1) and Proposition 2.2 (i), that the isotropy subgroup of x in Γ is $\left\{1, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}\right\}$. Now we assume that both x and $\gamma\cdot x$ $(x\in L_m, \gamma\in\Gamma)$ satisfy (2.25). Then γ is a normalizer in Γ of the subgroup $\left\{1, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}\right\}$. Hence we must have $\gamma=\pm 1$, or $\pm\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ or $\pm\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$. Therefore $\gamma\cdot x=\pm x$.

Thus it is proved that if we put

$$M_m=\{(x_1, -x_3-3x_1, x_3, x_1)\,;\ x_1, x_3\in \boldsymbol{Z},\ m=(9x_1^2+3x_1x_3+x_3^2)^2\},$$

$\{\Gamma\cdot x;\ x\in M_m\}$ coincides with the set of Γ-orbits of the second kind in L_m and that two elements x_1 and x_2 in M_m lie on the same Γ-orbit if and only if $x_1=\pm x_2$. Thus

$$2h_2(m)=\#\{(x, y)\in \boldsymbol{Z}^2;\ (9x^2+3xy+y^2)^2=m\}\,.$$

COROLLARY.

$$2\hat{h}_2(m)=\#\{(x, y)\in \boldsymbol{Z}^2;\ 81(x^2+xy+y^2)^2=m\}\,.$$

2. Now we define four Dirichlet series $\xi_1(L, s)$, $\xi_2(L, s)$, $\xi_1(\hat{L}, s)$ and $\xi_2(\hat{L}, s)$ as follows:

$$\xi_1(L, s)=\sum_{n=1}^{\infty}\frac{h_1(n)+3^{-1}h_2(n)}{n^s}$$

$$=\sum_{n=1}^{\infty}\frac{h(n)}{n^s}-3^{-1}\sum_{(x,y)\in \boldsymbol{Z}^2-\{0\}}\frac{1}{(9x^2+3xy+y^2)^{2s}}\,,$$

$$\xi_2(L, s)=\sum_{n=1}^{\infty}\frac{h(-n)}{n^s}$$

$$\xi_1(\hat{L}, s)=\sum_{n=1}^{\infty}\frac{\hat{h}_1(n)+3^{-1}\hat{h}_2(n)}{n^s}$$

$$=\sum_{n=1}^{\infty}\frac{\hat{h}(n)}{n^s}-3^{-1-4s}\sum_{(x,y)\in \boldsymbol{Z}^2-\{0\}}\frac{1}{(x^2+xy+y^2)^{2s}}\,,$$

$$\xi_2(\hat{L}, s)=\sum_{n=1}^{\infty}\frac{\hat{h}(-n)}{n^s}\,.$$

Then, by Proposition 2.11, these series converge absolutely for $\mathrm{Re}\, s>3$ and represent there holomorphic functions of s.

3. We put $L'=L-L_0$ and $\hat{L}'=\hat{L}-\hat{L}_0$. We define $Z(f, L; s)$ and $Z(f, \hat{L}, s)$ $(f\in \mathcal{S}(V_{\boldsymbol{R}}),\ s\in \boldsymbol{C})$ as follows:

$$Z(f, L; s)=\int_{G_+/\Gamma}\chi(g)^s\sum_{x\in L'}f(g\cdot x)dg\,,$$

$$Z(f, \hat{L}; s)=\int_{G_+/\Gamma}\chi(g)^s\sum_{x\in \hat{L}'}f(g\cdot x)dg\,,$$

where we put $\chi(g)=(\det g)^6$.

PROPOSITION 2.13. *For* $\mathrm{Re}\, s>3$, *the above integrals both converge absolutely and*

$$Z(f, L; s)=\frac{1}{4\pi}\xi_1(L, s)\int_{V_1}|P(x)|^{s-1}f(x)dx+\frac{1}{12\pi}\xi_2(L, s)\int_{V_2}|P(x)|^{s-1}f(x)dx$$

and

$$Z(f, \hat{L}; s)=\frac{1}{4\pi}\xi_1(\hat{L}, s)\int_{V_1}|P(x)|^{s-1}f(x)dx+\frac{1}{12\pi}\xi_2(\hat{L}, s)\int_{V_2}|P(x)|^{s-1}f(x)dx\,.$$

PROOF. We call two elements of L equivalent if they lie on the same

Γ-orbit in L. Let $x_1(m), \cdots, x_{h(m)}(m)$ be a complete set of representatives of Γ-equivalence classes in L_m. We denote by $I(i, m)$ the isotropy subgroup of $x_i(m)$ in Γ and by $\nu(i, m)$ the number of elements of $I(i, m)$.

We have $\nu(i, m)=1$ if $m<0$ or if $m>0$ and $x_i(m)$ is on an orbit of the first kind. We have $\nu(i, m)=3$ if $m>0$ and $x_i(m)$ is on an orbit of the second kind. Thus

$$\sum_{x\in L'} f(g\cdot x)=\sum_{m\neq 0}\sum_{i=1}^{h(m)}\sum_{\gamma\in\Gamma/I(i,m)} f(g\gamma\cdot x_i(m)).$$

It follows from Corollary to Proposition 2.4 that the series

$$\sum_{i=1}^{h(m)}\sum_{\gamma\in\Gamma/I(i,m)}\int_{G_+/\Gamma}\chi(g)^s f(g\gamma\cdot x_i(m))dg$$

converges absolutely when $\operatorname{Re} s>1$ and is equal to

$$\begin{aligned}
&\sum_{i=1}^{h(m)}\frac{1}{\nu(i, m)}\int_{G_+}\chi(g)^s f(g\cdot x_i(m))dg\\
&=\sum_{i=1}^{h(m)}\frac{1}{\nu(i, m)}\frac{1}{|m|^s}\int_{G_+}|P(g\cdot x_i(m))|^s f(g\cdot x_i(m))dg\\
&=\begin{cases}\dfrac{1}{4\pi}\{h_1(m)+3^{-1}h_2(m)\}\dfrac{1}{|m|^s}\displaystyle\int_{V_1} f(x)|P(x)|^{s-1}dx & \text{when } m>0,\\[2ex] \dfrac{1}{12\pi}h(m)\dfrac{1}{|m|^s}\displaystyle\int_{V_2} f(x)|P(x)|^{s-1}dx & \text{when } m<0.\end{cases}
\end{aligned}$$

It follows from Proposition 2.11 that the series $\sum_{m=1}^{\infty}\frac{h_1(m)+3^{-1}h_2(m)}{m^s}$ and $\sum_{m=1}^{\infty}\frac{h(-m)}{m^s}$ are absolutely convergent for $\operatorname{Re} s>3$.

Therefore the integral $\int_{G_+/\Gamma}\chi(g)^s\sum_{x\in L'} f(g\cdot x)dg$ converges absolutely when $\operatorname{Re} s>3$ and is equal to

$$\frac{1}{4\pi}\xi_1(L, s)\int_{V_1}|P(x)|^{s-1}f(x)dx+\frac{1}{12\pi}\xi_2(L, s)\int_{V_2}|P(x)|^{s-1}f(x)dx.$$

Thus we get the first equality. The second equality can be obtained similarly. q. e. d.

COROLLARY. *We put*

$$Z^+(f, L; s)=\int_{G^+/\Gamma,\chi(g)\geqq 1}\chi(g)^s\sum_{x\in L'} f(g\cdot x)dg$$

and put

$$Z^+(f, \hat{L}; s)=\int_{G^+/\Gamma,\chi(g)\geqq 1}\chi(g)^s\sum_{x\in\hat{L}'} f(g\cdot x)dg$$

($f\in\mathcal{S}(V_{\boldsymbol{R}})$, $s\in\boldsymbol{C}$). Then these integrals converge absolutely for any $s\in\boldsymbol{C}$ and, regarded as functions of s, are entire functions of s.

The following Proposition is an immediate consequence of the Poisson summation formula.

PROPOSITION 2.14. (i) *If* $\mathrm{Re}\, s>3$,

$$Z(f, L; s)=Z^{+}(f, L; s)+Z^{+}(\hat{f}, \hat{L}; 1-s)$$
$$-\int_{G_{+}/\Gamma, \chi(g)\leqq 1}\chi(g)^{s}\Big\{\sum_{x\in L_{0}}f(g\cdot x)-\chi^{-1}(g)\sum_{x\in\hat{L}_{0}}\hat{f}(g^{\iota}\cdot x)\Big\}dg.$$[16]

(ii) *If* $\mathrm{Re}\, s>3$,

$$Z(f, \hat{L}; s)=Z^{+}(f, \hat{L}; s)+\frac{1}{9}Z^{+}(\hat{f}, L; 1-s)$$
$$-\int_{G_{+}/\Gamma, \chi(g)\leqq 1}\chi(g)^{s}\Big\{\sum_{x\in\hat{L}_{0}}f(g\cdot x)-\frac{1}{9}\chi^{-1}(g)\sum_{x\in L_{0}}\hat{f}(g^{\iota}\cdot x)\Big\}dg.$$

In the following two subsections we prove several propositions which are necessary for the computation of the integral

$$\int_{G_{+}/\Gamma, \chi(g)\leqq 1}\chi(g)^{s}\Big\{\sum_{x\in L_{0}}f(g\cdot x)-\chi^{-1}(g)\sum_{x\in\hat{L}_{0}}\hat{f}(g^{\iota}\cdot x)\Big\}dg.$$

4. By Lemma 2.2, every element of $g\in G^{1}=SL(2, \boldsymbol{R})$ can be uniquely expressed as follows: $g=k_{\theta(g)}a_{t(g)}n(u(g))$. We denote by B the group of lower triangular matrices in G^{1}. We define the Eisenstein series $E(z, g)$ ($z\in\boldsymbol{C}$, $\mathrm{Re}\, z>1$; $g\in G^{1}$) as follows:

$$E(z, g)=\sum_{\gamma\in\Gamma/\Gamma\cap B}t(g\gamma)^{z+1}. \tag{2.26}$$

The following Proposition is well-known, (see e.g. [**5**] and [**10**]).

PROPOSITION 2.15. (i) *When* $\mathrm{Re}\, z>1$, *the series* $E(z, g)$ *converges absolutely and locally uniformly with respect to* z *and* g.

(ii) *The series* $E(z, g)$ *can be continued analytically as a meromorphic function of* z *in the whole plane and satisfies the following functional equation:*

$$\xi(z+1)E(z, g)=\xi(1-z)E(-z, g),$$

where we put $\xi(z)=\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)$.

(iii) *The function:* $E(z, g)-\dfrac{1}{\xi(2)(z-1)}$, *is holomorphic in the right half plane* $\{z;\ \mathrm{Re}\, z>0\}$.

(iv) *When* $\mathrm{Re}\, z>0$, *we have the following Fourier expansion of* $E(z, k_{\theta}a_{t}\, n(u))$ *with respect to* u*:*

$$E(z, k_{\theta}a_{t}n(u))=t^{z+1}+t^{1-z}\frac{\xi(z)}{\xi(z+1)}+2t^{1-z}\sum_{m=1}^{\infty}a_{m}(z, t)\cos 2\pi mu,$$

16) For the definition of g^{ι}, see (2.13).

where $a_m(z, t)$ *is defined as follows:*

$$a_m(z, t)=\frac{1}{\zeta(1+z)}\left\{\sum_{0<d\mid m} d^{-z}\right\}\int_{-\infty}^{\infty}\frac{e^{2\pi\sqrt{-1}\,mt^2w}}{(1+w^2)^{(1+z)/2}}\,dw\,.$$

The following lemma can be easily proved using partial integration.

LEMMA 2.7. (i) *The function* $a_m(z, t)$ $(m=1, 2, \cdots)$ *is a holomorphic function of* z *in the half plane* $\{z\in \boldsymbol{C};\ \mathrm{Re}\, z>0\}$.

(ii) *For any natural number* k, *there exists a positive number* C_k *such that:*

$$\left|\int_{-\infty}^{\infty}\frac{\exp(2\pi\sqrt{-1}\,mt^2w)}{(1+w^2)^{\frac{1+z}{2}}}\,dw\right|\leqq C_k(|m|t^2)^{-k}(1+|z|)^k \qquad (\mathrm{Re}\, z>0)\,.$$

5. For any positive number C, we define a subset $\mathfrak{S}_C$ of G^1 as follows:

$$\mathfrak{S}_C=\left\{k_\theta a_t n(u)\,;\ \theta\in \boldsymbol{R},\ t\geqq C,\ |u|\leqq\frac{1}{2}\right\}.$$

When $C^2\leqq\frac{\sqrt{3}}{2}$, we have $G^1=\mathfrak{S}_C\cdot\Gamma$ and we can take a fundamental domain F in G with respect to Γ such that $F\subset\mathfrak{S}_C$ (see e.g. [2]).

For any $r\in \boldsymbol{R}$, we define a semi-norm $\mu(r)$ on $C(G^1/\Gamma)$ as follows:

$$\mu(r)(f)=\underset{g\in\mathfrak{S}_{1/2}}{\mathrm{Sup.}}\ t(g)^r|f(g)|.^{17)}$$

We put $C(G^1/\Gamma, r)=\{f\in C(G^1/\Gamma)\,;\ \mu(r)(f)<\infty\}$.

LEMMA 2.8. *When* $r>-2$, $C(G^1/\Gamma, r)\subset L_1(G^1/\Gamma)$.

PROOF. There exists a fundamental domain F of G^1 with respect to Γ such that $F\subset\mathfrak{S}_{1/2}$. Take an $f\in C(G^1/\Gamma, r)$ $(r>-2)$, we have, by (2.6),

$$\int_{G^1/\Gamma}|f(g)|\,dg=\int_F|f(g)|\,dg\leqq\int_{\mathfrak{S}_{1/2}}|f(g)|\,dg$$
$$\leqq\mu(r)(f)\int_{\mathfrak{S}_{1/2}}t(g)^{-r}dg^1=\mu(r)(f)\int_{1/2}^{\infty}t^{-r-2}d^\times t<\infty\,.$$

Therefore, $f\in L_1(G^1/\Gamma)$. q. e. d.

We denote by Ψ the space of entire functions which satisfy the following inequalities:

$$\underset{C_1<\mathrm{Re}\, w<C_2}{\mathrm{Sup.}}\{1+(\mathrm{Im}\, w)^2\}^N|\psi(w)|<\infty\quad(\forall N>0,\ -\infty<\forall C_1<\forall C_2<\infty)\,.^{18)}$$

LEMMA 2.9. *We put*

$$\mathcal{E}(\psi, w\,;g)=\frac{1}{2\pi i}\int_{\substack{\mathrm{Re}\, z=x_0\\ 1<x_0<\mathrm{Re}\, w}}\psi(z)\frac{E(z, g)}{w-z}\,dz$$

where $\psi\in\Psi$ *and* $w\in \boldsymbol{C}$ $(\mathrm{Re}\, w>1)$. *Then we have the following:*

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17) For the definition of $t(g)$, see Lemma 2.2 (ii).

18) It is obvious that $\exp(z^2)\in\Psi$.

(i) $\mathcal{E}(\phi, w; g) \in C(G^1/\Gamma, \operatorname{Re} w-1)$.

(ii) *For a fixed* ϕ,

$$\operatorname*{Sup.}_{\substack{1\leqq w\leqq M \\ g\in\mathfrak{S}_{1/2}}} |(w-1)\mathcal{E}(\phi, w; g)| < \infty, \qquad (M>1).$$

(iii) $\displaystyle\lim_{w\to 1+0} (w-1)\mathcal{E}(\phi, w; g) = \frac{\phi(1)}{\xi(2)}$.

PROOF. We put

$$E(z, g) = t(g)^{1+z} + \frac{\xi(z)}{\xi(z+1)} t(g)^{1-z} + E'(z, g).$$

(i) It follows from Proposition 2.15 (iv) and Lemma 2.7 that there exists a constant c such that the following inequality holds:

$$|E'(z, g)| \leqq ct(g)^{(1-\operatorname{Re} z)-6}(1+|z|)^3 \qquad (\operatorname{Re} z \geqq 1).$$

By the Cauchy's integral theorem, we have

$$\begin{aligned}
&\mathcal{E}(\phi, w; g) \\
&= \frac{1}{2\pi i}\int_{\operatorname{Re} z=-\operatorname{Re} w} \frac{t(g)^{1+z}}{w-z}\phi(z)dz + \left\{\frac{\xi(w)}{\xi(w+1)} t(g)^{1-w} + E'(w, g)\right\}\phi(w) \\
&\quad + \frac{1}{2\pi i}\int_{\operatorname{Re} z=\operatorname{Re} w+1}\left\{\frac{\xi(z)}{\xi(z+1)} t(g)^{1-z} + E'(z, g)\right\}\frac{\phi(z)}{w-z}dz.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\operatorname*{Sup.}_{g\in\mathfrak{S}_{1/2}} |t(g)^{\operatorname{Re} w-1}\mathcal{E}(\phi, w; g)| \\
&\leqq \frac{1}{2\pi}\int_{\operatorname{Re} z=-\operatorname{Re} w} \frac{|\phi(z)|}{|w-z|}|dz| + \left|\frac{\xi(w)}{\xi(w+1)}\phi(w)\right| + 2^6\cdot c(1+|w|)^3\phi(w) \\
&\quad + \frac{1}{2\pi}\int_{\operatorname{Re} z=\operatorname{Re} w+1}\left\{\left|2\frac{\xi(z)}{\xi(z+1)}\right| + 2^7\cdot c(1+|z|)^3\right\}|\phi(z)||dz| < \infty.
\end{aligned}$$

Hence $\mathcal{E}(\phi, w; g) \in C(G^1/\Gamma, \operatorname{Re} w-1)$.

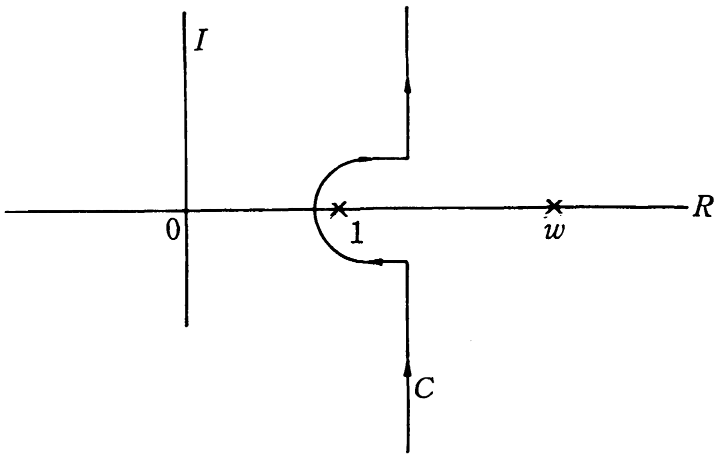
(ii) It follows from the above equality that

$$\begin{aligned}
&\operatorname*{Sup.}_{\substack{1\leqq w\leqq M \\ g\in\mathfrak{S}_{1/2}}} |(w-1)\mathcal{E}(\phi, w; g)| \\
&\leqq \frac{2^{M-1}}{4\pi}(M-1)\operatorname*{Sup.}_{1\leqq w\leqq M}\left\{\int_{\operatorname{Re} z=-w}|\phi(z)||dz|\right\} + 2^{M-1}\operatorname*{Sup.}_{1\leqq w\leqq M}\left|(w-1)\frac{\xi(w)}{\xi(w+1)}\phi(w)\right| \\
&\quad + 2^{5+M}c(M-1)\operatorname*{Sup.}_{1\leqq w\leqq M}(1+|w|)^3|\phi(w)| \\
&\quad + \frac{(M-1)}{2\pi}\operatorname*{Sup.}_{1\leqq w\leqq M}\int_{\operatorname{Re} z=w+1}\left\{2^M\frac{|\xi(z)|}{|\xi(z+1)|} + 2^{6+M}\cdot c(1+|z|)^3\right\}|\phi(z)||dz| \\
&< \infty.
\end{aligned}$$

(iii) We have, by Proposition 2.15 (iii) and the Cauchy's integral theorem,

$$\mathcal{E}(\psi, w\,;\, g) = \frac{\psi(1)}{\xi(2)}\frac{1}{w-1} + \frac{1}{2\pi i}\int_C \frac{E(z, g)}{w-z}\psi(z)dz\,,$$

where C is an integral path indicated below:



Since

$$\lim_{w\to 1+0}\frac{(w-1)}{2\pi i}\int_C \frac{E(z, g)}{w-z}\,\psi(z)dz = 0\,,$$

we get

$$\lim_{w\to 1+0}(w-1)\mathcal{E}(\psi, w\,;\, g) = \frac{\psi(1)}{\xi(2)}\,. \qquad \text{q. e. d.}$$

COROLLARY (Cf. [7]). *For every* $f \in L_1(G^1/\Gamma, dg)$,

$$\lim_{w\to 1+0}(w-1)\int_{G^1/\Gamma} f(g)\mathcal{E}(\psi, w\,;\, g)dg = \frac{\psi(1)}{\xi(2)}\int_{G^1/\Gamma} f(g)dg\,.$$

PROOF. Take a fundamental domain F in G^1 with respect to Γ such that $F \subset \mathfrak{S}_{1/2}$. Then

$$(w-1)\int_{G^1/\Gamma} f(g)\mathcal{E}(\psi, w\,;\, g)dg = (w-1)\int_F f(g)\mathcal{E}(\psi, w\,;\, g)dg$$

$(w > 1)$. By Lemma 2.9 (ii), Lemma 2.7 (iii) and the bounded convergence theorem,

$$\lim_{w\to 1+0}(w-1)\int_F f(g)\mathcal{E}(\psi, w\,;\, g)dg = \frac{\psi(1)}{\xi(2)}\int_F f(g)dg = \frac{\psi(1)}{\xi(2)}\int_{G^1/\Gamma} f(g)dg\,.$$

q. e. d.

LEMMA 2.10. *For every* $f \in \mathcal{S}(V_R)$, *define* $J_L(f)$ *and* $J_{\hat{L}}(f) \in C(G^1/\Gamma)$ *as follows:*

$$\begin{cases} J_L(f)(g) = \sum_{x\in L_0} f(g\cdot x)\,, \\ J_{\hat{L}}(f)(g) = \sum_{x\in \hat{L}_0} f(g\cdot x) \qquad (g \in G^1)\,. \end{cases}$$

(L_0 *is the set of degenerate integral binary cubic forms and* $\hat{L}_0 = L_0 \cap \hat{L}$.) *Then*

(i) $J_L(f),\ J_{\hat{L}}(f) \in C(G^1/\Gamma, -4)$

(ii) $J_L(f) - J_{\hat{L}}(\hat{f}) \in C(G^1/\Gamma, N)$ $(\forall N > 0)$.

PROOF. (i) We put $g = k(\theta(g))a_{t(g)}n(u(g))$ and $c(g) = k(\theta(g))a_{t(g)}n(u(g))a_{t(g)}^{-1}$. Then we have

$$f(g\cdot x) = f(c(g)(t(g)^3 x_1,\ t(g)x_2,\ t(g)^{-1}x_3,\ t(g)^{-3}x_4)) \qquad (x = (x_1, x_2, x_3, x_4)).$$

When g varies in $\mathfrak{S}_{1/2}$, $c(g)$ remains in a compact subset of G^1. By Lemma 5 in [11], we can take an $f_0 \in \mathcal{S}(V_{\boldsymbol{R}})$ such that $|f(c(g)\cdot x)| \leqq f_0(x)$ $(\forall g \in \mathfrak{S}_{1/2}, \forall x \in V_{\boldsymbol{R}})$. Hence there exists a constant C_1 such that

$$|J_L(f)(g)| \leqq \sum_{x\in L_0} f_0(t(g)^3 x_1,\ t(g)x_2,\ t(g)^{-1}x_3,\ t(g)^{-3}x_4) \leqq C_1 t(g)^4$$

$(g \in \mathfrak{S}_{1/2})$. Therefore, we have $\mu(-4)(J_L(f)) < \infty$ and $J_L(f) \in C(G^1/\Gamma, -4)$. Similarly, we can prove that $J_{\hat{L}}(f) \in C(G^1/\Gamma, -4)$.

(ii) It follows from the Poisson summation formula that

$$J_L(f)(g) - J_{\hat{L}}(\hat{f})(g) = \sum_{x\in\hat{L}'} \hat{f}(g\cdot x) - \sum_{x\in L'} f(g\cdot x),$$

$$= \sum_{x\in\hat{L}'} \hat{f}(c(g)t(g)\cdot x) - \sum_{x\in L'} f(c(g)t(g)\cdot x),$$

where we put $L' = L - L_0$ and $\hat{L}' = \hat{L} - \hat{L}_0$. We can take $f_1, f_2 \in \mathcal{S}(V_{\boldsymbol{R}})$ such that

$$|\hat{f}(c(g)\cdot x)| \leqq f_1(x) \qquad (\forall g \in \mathfrak{S}_{1/2}, \forall x \in V_{\boldsymbol{R}}),$$

$$|f(c(g)\cdot x)| \leqq f_2(x) \qquad (\forall g \in \mathfrak{S}_{1/2}, \forall x \in V_{\boldsymbol{R}}).$$

Therefore

$$|\{J_L(f) - J_{\hat{L}}(\hat{f})\}(g)| \leqq \sum_{x\in\hat{L}'} f_1(t(g)\cdot x) + \sum_{x\in L'} f_2(t(g)\cdot x) \qquad (\forall g \in \mathfrak{S}_{1/2}).$$

Hence, for every natural number m, there exists a constant C_m such that

$$|\{J_L(f) - J_{\hat{L}}(\hat{f})\}(g)|$$
$$\leqq C_m \sum_{x\in L'} \frac{1}{\{(1+t(g)^6 x_1^2)(1+t(g)^2 x_2^2)\}^m (1+t(g)^{-2}x_3^2)(1+t(g)^{-6}x_4^2)}$$

$(\forall g \in \mathfrak{S}_{1/2})$. Since $\operatorname{Max}_{x\in L'}\{|x_1|, |x_2|\} \geqq 1$, there exists a constant C'_m such that $|\{J_L(f) - J_{\hat{L}}(\hat{f})\}(g)| \leqq C'_m t(g)^{8-2m}$ $(\forall g \in \mathfrak{S}_{1/2})$. Thus we get $J_L(f) - J_{\hat{L}}(\hat{f}) \in C(G^1/\Gamma, 2m-8)$ $(m = 1, 2, \cdots)$ and $J_L(f) - J_{\hat{L}}(\hat{f}) \in C(G^1/\Gamma, N)$ $(\forall N \in \boldsymbol{R})$. q. e. d.

6. PROPOSITION 2.16. *We have*

$$\int_{G^1/\Gamma} \{\sum_{x\in L_0} f(g\cdot x) - \sum_{x\in\hat{L}_0} \hat{f}(g\cdot x)\} dg$$
$$= \frac{\zeta(2)}{2\pi}(f(0) - \hat{f}(0)) + \zeta\left(\frac{2}{3}\right)\{\Sigma_2(f) - \Sigma_2(\hat{f})\} + \zeta(-1)\{\Sigma_1(f) - 3\Sigma_1(\hat{f})\}.$$

PROOF. We put $\tilde{f}(x)=\frac{1}{2\pi}\int_0^{2\pi} f(k_\theta\cdot x)d\theta$, $J_L(f)(g)=\sum_{x\in L_0} f(g\cdot x)$, and $J_{\hat{L}}(\hat{f})(g) = \sum_{x\in\hat{L}_0}\hat{f}(g\cdot x)$. It follows from Lemma 2.8 and Lemma 2.10 that $J_L(f)-J_{\hat{L}}(\hat{f}) \in L_1(G^1/\Gamma, dg^1)$. By Corollary to Lemma 2.9, we get

$$\lim_{w\to 1+0}(w-1)\int_{G^1/\Gamma}\{J_L(f)-J_{\hat{L}}(\hat{f})\}(g)\mathcal{E}(\psi, w\,;g)dg = \frac{\psi(1)}{\xi(2)}\int_{G^1/\Gamma}\{J_L(f)-J_{\hat{L}}(\hat{f})\}(g)dg\,. \tag{2.27}$$

By Lemma 2.10 (i), Lemma 2.9 (i) and Lemma 2.8, we can see that $J_L(f)\mathcal{E}(\psi, w, g)\in L_1(G^1/\Gamma)$ and $J_{\hat{L}}(\hat{f})\mathcal{E}(\psi, w, g)\in L_1(G^1/\Gamma)$ for $\operatorname{Re} w>3$. Hence

$$\int_{G^1/\Gamma}\{J_L(f)-J_{\hat{L}}(\hat{f})\}(g)\mathcal{E}(\psi, w\,;g)dg = \int_{G^1/\Gamma} J_L(f)(g)\mathcal{E}(\psi, w\,;g)dg-\int_{G^1/\Gamma} J_{\hat{L}}(\hat{f})(g)\mathcal{E}(\psi, w\,;g)dg$$

for $\operatorname{Re} w>3$. We put

$$L_0(I) = \bigcup_{m=1}^{\infty}\ \bigcup_{\gamma\in\Gamma/\Gamma\cap N}\gamma\cdot(0, 0, 0, m)\,,$$

$$L_0(II) = \bigcup_{m=1}^{\infty}\bigcup_{n=0}^{m-1}\bigcup_{\gamma\in\Gamma}\gamma\cdot(0, 0, m, n)\,,$$

and

$$\hat{L}_0(II) = \bigcup_{m=1}^{\infty}\bigcup_{n=0}^{3m-1}\bigcup_{\gamma\in\Gamma}\gamma\cdot(0, 0, 3m, n)\,.$$

Then it follows from Proposition 2.10 that $L_0=\{0\}\cup L_0(I)\cup L_0(II)$ and from its corollary that $\hat{L}_0=\{0\}\cup L_0(I)\cup\hat{L}_0(II)$. Hence

$$\begin{aligned}\int_{G^1/\Gamma} J_L(f)(g)\mathcal{E}(\psi, w\,;g)dg &= \int_{G^1/\Gamma}\mathcal{E}(\psi, w\,;g)dg f(0)+\int_{G^1/\Gamma}\sum_{x\in L_0(I)} f(g\cdot x)\mathcal{E}(\psi, w\,;g)dg\\ &\quad+\int_{G^1/\Gamma}\mathcal{E}(\psi, w\,;g)\sum_{x\in L_0(II)} f(g\cdot x)dg \qquad (\operatorname{Re} w>3)\end{aligned} \tag{2.28}$$

and

$$\begin{aligned}\int_{G^1/\Gamma} J_{\hat{L}}(\hat{f})(g)\mathcal{E}(\psi, w\,;g)dg &= \hat{f}(0)\int_{G^1/\Gamma}\mathcal{E}(\psi, w\,;g)dg+\int_{G^1/\Gamma}\sum_{x\in L_0(I)}\hat{f}(g\cdot x)\mathcal{E}(\psi, w\,;g)dg\\ &\quad+\int_{G^1/\Gamma}\sum_{x\in\hat{L}_0(II)}\hat{f}(g\cdot x)\mathcal{E}(\psi, w\,;g)dg \qquad (\operatorname{Re} w>3)\,.\end{aligned} \tag{2.29}$$

We emphasize that all the terms in the equalities (2.28) and (2.29) can be regarded as functions of w, holomorphic in the half plane $\{w\,;\ \mathrm{Re}\,w>3\}$. In the following, two holomorphic functions $\varphi_1(w)$ and $\varphi_2(w)$ in the half plane $\{w\,;\ \mathrm{Re}\,w>3\}$ are said to be equivalent when the function $(\varphi_1-\varphi_2)(w)$ can be continued as a holomorphic function in the half plane $\{w\,;\ \mathrm{Re}\,w>0\}$. In this case, we write $\varphi_1(w)\sim\varphi_2(w)$.

Now we establish the following sublemma.

SUBLEMMA. *For any $\phi\in\Psi$ and $f\in\mathcal{S}(V_{\boldsymbol{R}})$, we have the following:*

(i) $$\int_{G^1/\Gamma}\mathcal{E}(\phi,w\,;g)dg=\frac{\phi(1)}{2(w-1)}\qquad(\mathrm{Re}\,w>1)\,.$$

(ii) $$\int_{G^1/\Gamma}\mathcal{E}(\phi,w\,;g)\sum_{x\in L_0(I)}f(g\cdot x)dg^1\sim\frac{a_0(2)}{w-2}\phi(2)\int_0^\infty\tilde{f}(0,0,0,t)dt+\frac{1}{\xi(2)}\frac{\phi(1)}{w-1}\zeta\Big(\frac{2}{3}\Big)\Sigma_2(f)\,,$$

where we put

$$\tilde{f}(x)=\frac{1}{2\pi}\int_0^{2\pi}f(k_\theta\cdot x)d\theta,\ a_0(z)=\frac{\xi(z)}{\xi(z+1)}$$

$\Big(\xi(z)=\pi^{-\frac{z}{2}}\Gamma\Big(\frac{z}{2}\Big)\zeta(z)\Big)$.

(iii) $$\int_{G^1/\Gamma}\mathcal{E}(\phi,w\,;g)\sum_{x\in L_0(II)}f(g\cdot x)dg\sim\frac{\phi(3)a_0(3)}{w-3}\Sigma_1(f,2)^{19)}+\frac{\phi(2)\zeta(0)}{w-2}a_0(2)\int_{-\infty}^\infty\tilde{f}(0,0,0,u)du+\frac{\phi(1)\zeta(-1)}{w-1}\frac{1}{\xi(2)}\Sigma_1(f)\,.$$

(iv) $$\int_{G^1/\Gamma}\mathcal{E}(\phi,w\,;g)\sum_{x\in\hat{L}_0(II)}f(g\cdot x)dg\sim\frac{1}{3}\frac{\phi(3)a_0(3)}{w-3}\Sigma_1(f,2)+\frac{\phi(2)\zeta(0)}{w-2}a_0(2)\int_{-\infty}^\infty\tilde{f}(0,0,0,u)du+\frac{3\phi(1)\zeta(-1)}{w-1}\frac{1}{\xi(2)}\Sigma_1(f)\,.$$

PROOF OF SUBLEMMA. (i) When $\mathrm{Re}\,w>1$, it follows from (2.26) and (2.6),

$$\int_{G^1/\Gamma}\mathcal{E}(\phi,w\,;g)dg=\frac{1}{2\pi i}\int_{G^1/\Gamma}\Big(\int_{\substack{\mathrm{Re}\,z=x_0\\1<x_0<\mathrm{Re}\,w}}\frac{E(z,g)}{w-z}\phi(z)dz\Big)dg$$

19) See (2.15).

$$= \frac{1}{2\pi i}\int_{G^1/\Gamma}\Bigl(\int_{\mathrm{Re}\, z=x_0}\frac{\sum_{\gamma\in\Gamma/\Gamma\cap B} t(g\gamma)^{1+z}}{w-z}\phi(z)dz\Bigr)dg$$

$$= \frac{1}{4\pi i}\int_{G^1/\Gamma\cap N}\Bigl(\int_{\mathrm{Re}\, z=x_0}\frac{t(g)^{1+z}}{w-z}\phi(z)dz\Bigr)dg$$

$$= \frac{1}{4\pi i}\frac{1}{2\pi}\int_0^{2\pi}d\theta\int_0^\infty t^{-2}d^\times t\int_0^1 du\Bigl(\int_{\mathrm{Re}\, z=x_0}\frac{t^{1+z}}{w-z}\phi(z)dz\Bigr)$$

$$= \frac{1}{4\pi i}\int_0^\infty\Bigl(\int_{\mathrm{Re}\, z=x_0}\frac{t^{z-1}}{w-z}\phi(z)dz\Bigr)d^\times t = \frac{1}{2}\frac{\phi(1)}{w-1}.$$

(ii) We put

$$\Theta_\phi^{(1)}(w) = \int_{G^1/\Gamma}\sum_{x\in L_0(I)} f(g\cdot x)\mathcal{E}(\phi, w\,; g)dg \qquad (\mathrm{Re}\, w > 3)$$

Then it follows from (2.6) and Proposition 2.15 (iv) that

$$\Theta_\phi^{(1)}(w) = \int_{G^1/\Gamma}\sum_{m=1}^\infty\sum_{\gamma\in\Gamma/\Gamma\cap N} f(g\gamma\cdot(0, 0, 0, m))\mathcal{E}(\phi, w\,; g)dg$$

$$= \sum_{m=1}^\infty\int_{G^1/\Gamma\cap N} f(g\cdot(0, 0, 0, m))\mathcal{E}(\phi, w\,; g)dg$$

$$= \sum_{m=1}^\infty\frac{1}{4\pi^2 i}\int_0^{2\pi}d\theta\int_0^\infty t^{-2}d^\times t\int_0^1 du f(k_\theta a_t n(u)\cdot(0, 0, 0, m))$$

$$\times\Bigl(\int_{\substack{\mathrm{Re}\, z=x_0\\ 3<x_0<\mathrm{Re}\, w}}\frac{E(z, k_\theta a_t n(u))}{w-z}\phi(z)dz\Bigr)$$

$$= \frac{1}{2\pi i}\sum_{m=1}^\infty\int_0^\infty\tilde{f}((0, 0, 0, t^{-3}m))\Bigl(\int_{\mathrm{Re}\, z=x_0}\frac{t^{1+z}+a_0(z)t^{1-z}}{w-z}\phi(z)dz\Bigr)t^{-2}d^\times t$$

$$= \frac{1}{2\pi i}\int_0^\infty\tilde{f}((0, 0, 0, t^{-3}))\Bigl(\int_{\mathrm{Re}\, z=x_1<-2}\frac{t^{z-1}\zeta\bigl(\frac{1-z}{3}\bigr)}{w-z}\phi(z)dz\Bigr)d^\times t$$

$$+\frac{1}{2\pi i}\int_0^\infty\tilde{f}((0, 0, 0, t^{-3}))\Bigl(\int_{\mathrm{Re}\, z=x_0}\frac{a_0(z)t^{-1-z}\zeta\bigl(\frac{1+z}{3}\bigr)}{w-z}\phi(z)dz\Bigr)d^\times t.$$

Since the integral

$$\frac{1}{2\pi i}\int_0^\infty\tilde{f}((0, 0, 0, t^{-3}))\Bigl(\int_{\mathrm{Re}\, z=x_1<-2}\frac{t^{z-1}\zeta\bigl(\frac{1-z}{3}\bigr)}{w-z}\phi(z)dz\Bigr)d^\times t$$

is an entire function of w, we have

$$\Theta_\phi^{(1)}(w)\sim\frac{1}{2\pi i}\int_0^\infty\tilde{f}((0, 0, 0, t^{-3}))\Bigl(\int_{\mathrm{Re}\, z=x_0}\frac{t^{-1-z}\zeta\bigl(\frac{1+z}{3}\bigr)a_0(z)}{w-z}\phi(z)dz\Bigr)d^\times t$$

$$=\frac{1}{2\pi i}\int_0^\infty \tilde{f}((0,0,0,t^{-3}))\Big(\int_{\substack{\mathrm{Re}\, z=x_2\\ 0<x_2<1}} \frac{t^{-1-z}\zeta\big(\frac{1+z}{3}\big)a_0(z)}{w-z}\phi(z)dz\Big)\,d^\times t$$

$$+3\frac{a_0(2)\phi(2)}{w-2}\int_0^\infty \tilde{f}((0,0,0,t^{-3}))t^{-3}d^\times t$$

$$+\frac{\zeta\big(\frac{2}{3}\big)\phi(1)}{(w-1)\xi(2)}\int_0^\infty \tilde{f}((0,0,0,t^{-3}))t^{-2}d^\times t$$

(we note that $\operatorname*{res}_{z=2}\zeta\big(\frac{1+z}{3}\big)=3$ and that $\operatorname*{res}_{z=1} a_0(z)=\frac{1}{\xi(2)}$). The integral

$$\frac{1}{2\pi i}\int_0^\infty \tilde{f}((0,0,0,t^{-3}))\Big(\int_{\substack{\mathrm{Re}\, z=x_2\\ 0<x_2<1}} \frac{t^{-1-z}\zeta\big(\frac{1+z}{3}\big)a_0(z)}{w-z}\phi(z)dz\Big)\,d^\times t$$

is a holomorphic function of w in the half plane $\{w\,;\,\mathrm{Re}\,w>0\}$. Therefore we get, by (2.16),

$$\Theta_\phi^{(1)}(w)\sim\frac{a_0(2)\phi(2)}{w-2}\int_0^\infty \tilde{f}((0,0,0,t))dt+\frac{\zeta\big(\frac{2}{3}\big)\phi(1)}{(w-1)\xi(2)}\Sigma_2(f)\,.$$

(iii) We put

$$\Theta_\phi^{(2)}(w)=\int_{G^1/\Gamma}\mathcal{E}(\phi,w\,;g)\sum_{x\in L_0(II)}f(g\cdot x)dg\qquad(\mathrm{Re}\,w>3)\,.$$

Then it follows from (2.6)

$$\Theta_\phi^{(2)}(w)=\int_{G^1/\Gamma}\mathcal{E}(\phi,w\,;g)\sum_{m=1}^\infty\sum_{n=-\infty}^\infty\sum_{\gamma\in\Gamma/\Gamma\cap N}f(g\gamma\cdot(0,0,m,n))dg$$

$$=\frac{1}{2\pi i}\int_{G^1/\Gamma\cap N}\Big\{\sum_{m=1}^\infty\sum_{n=-\infty}^\infty f(g\cdot(0,0,m,n))\Big\}\Big(\int_{\substack{\mathrm{Re}\, z=x_0\\ 3<x_0<\mathrm{Re}\, w}}\frac{E(z,g)\phi(z)}{w-z}dz\Big)dg$$

$$=\frac{1}{2\pi i}\int_0^\infty t^{-2}d^\times t\int_0^1 du\sum_{m=1}^\infty\sum_{n=-\infty}^\infty \tilde{f}(a_t\cdot(0,0,m,n+mu))\int_{\mathrm{Re}\, z=x_0}\frac{E(z,a_tn(u))\phi(z)}{w-z}dz.$$

Applying Parseval's identity for the Fourier expansions with respect to u, we have, by Proposition 2.15 (iv), the following.

$$\int_0^1\Big\{\sum_{m=1}^\infty\sum_{n=-\infty}^\infty \tilde{f}(a_t\cdot(0,0,m,n+mu))\Big\}\Big(\int_{\mathrm{Re}\, z=x_0}\frac{E(z,a_tn(u))}{w-z}\phi(z)dz\Big)du$$

$$=\int_{-\infty}^\infty\sum_{m=1}^\infty \tilde{f}((0,0,t^{-1}m,t^{-3}u))du\Big(\int_{\mathrm{Re}\, z=x_0}\frac{t^{1+z}+a_0(z)t^{1-z}}{w-z}\phi(z)dz\Big)$$

$$+\sum_{Z\ni l\neq 0}\Big\{\sum_{m=1}^\infty\frac{1}{m}\Big(\sum_{n=0}^{m-1}e^{-\frac{2\pi i l n}{m}}\Big)\int_{-\infty}^\infty \tilde{f}((0,0,t^{-1}m,t^{-3}u))e^{2\pi i\frac{lu}{m}}du\Big\}$$

$$\times\Big(\int_{\mathrm{Re}\,z=x_0}\frac{t^{1-z}a_{l0}(z,t)}{w-z}\phi(z)dz\Big)$$

$$=\int_{-\infty}^{\infty}\sum_{m=1}^{\infty}\tilde{f}((0,0,t^{-1}m,t^{-3}u))du\Big(\int_{\mathrm{Re}\,z=x_0}\frac{t^{1+z}+a_0(z)t^{1-z}}{w-z}\phi(z)dz\Big)$$

$$+2\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\Big\{\int_{-\infty}^{\infty}\tilde{f}((0,0,t^{-1}m,u))\cos(2\pi lt^3u)du\Big\}$$

$$\times\Big(\int_{\mathrm{Re}\,z=x_0}\frac{t^{4-z}a_{lm}(z,t)}{w-z}\phi(z)dz\Big).$$

Making use of Lemma 2.7, we can prove that the series

$$\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\int_0^{\infty}\Big\{\int_{-\infty}^{\infty}\tilde{f}((0,0,t^{-1}m,u))\cos(2\pi lt^3u)du\Big\}$$

$$\times\Big(\int_{\mathrm{Re}\,z=x_0}\frac{t^{4-z}a_{lm}(z,t)}{w-z}\phi(z)dz\Big)t^{-2}d^\times t$$

can be, as a function of w, continued analytically to a holomorphic function in the half plane: $\{w\in\boldsymbol{C};\ \mathrm{Re}\,w>0\}$. Therefore,

$$\Theta_\phi^{(2)}(w)\sim\frac{1}{2\pi i}\int_0^{\infty}\Big(\int_{-\infty}^{\infty}\sum_{m=1}^{\infty}\tilde{f}((0,0,t^{-1}m,t^{-3}u))du\Big)$$

$$\times\Big(\int_{\mathrm{Re}\,z=x_0}\frac{t^{1+z}+a_0(z)t^{1-z}}{w-z}\phi(z)dz\Big)t^{-2}d^\times t$$

$$=\frac{1}{2\pi i}\int_0^{\infty}\Big(\int_{-\infty}^{\infty}\tilde{f}((0,0,t^{-1},u))du\Big)\Big(\int_{\mathrm{Re}\,z=x_1<-3}\frac{t^{2+z}\zeta(-2-z)}{w-z}\phi(z)dz\Big)d^\times t$$

$$+\frac{1}{2\pi i}\int_0^{\infty}\Big(\int_{-\infty}^{\infty}\tilde{f}((0,0,t^{-1},u))du\Big)\Big(\int_{\mathrm{Re}\,z=x_0}\frac{a_0(z)\zeta(z-2)t^{2-z}}{w-z}\phi(z)dz\Big)d^\times t$$

$$\sim\frac{1}{2\pi i}\int_{\mathrm{Re}\,z=x_0}\frac{a_0(z)\zeta(z-2)}{w-z}\Sigma_1(f,z-1)\phi(z)dz.$$

It follows from Lemma 2.3 and its proof

$$\frac{1}{2\pi i}\int_{\mathrm{Re}\,z=x_0}\frac{a_0(z)\zeta(z-2)}{w-z}\Sigma_1(f,z-1)\phi(z)dz$$

$$=\frac{a_0(3)}{w-3}\Sigma_1(f,2)\phi(3)+\frac{a_0(2)\zeta(0)\phi(2)}{w-2}\operatorname*{res.}_{z=1}\Sigma_1(f,z)$$

$$+\frac{\zeta(-1)\phi(1)}{\xi(2)(w-1)}\Sigma_1(f)+\frac{1}{2\pi i}\int_{\substack{\mathrm{Re}\,z=x_2\\0<x_2<1}}\frac{a_0(z)\zeta(z-2)}{w-z}\Sigma_1(f,z-1)\phi(z)dz.$$

Hence,

$$\Theta_\phi^{(2)}(w)\sim\frac{a_0(3)}{w-3}\Sigma_1(f,2)\phi(3)+\frac{a_0(2)\zeta(0)\phi(2)}{w-2}\int_{-\infty}^{\infty}\tilde{f}((0,0,0,u))du$$

$$+\frac{\zeta(-1)\phi(1)}{\xi(2)(w-1)}\Sigma_1(f).$$

(iv) Proof of (iv) is quite similar to that of (iii). q. e. d.

Now we return to the proof of Proposition 2.16. It follows from (2.28) and Sublemma that

$$\int_{G^1/\Gamma} J_L(f)\mathcal{E}(\phi, w\,;g)dg \sim \frac{\phi(3)a_0(3)}{w-3}\Sigma_1(f, 2)$$

$$+\frac{\phi(2)a_0(2)}{w-2}\left\{\int_0^\infty \tilde{f}(0, 0, 0, t)dt+\zeta(0)\int_{-\infty}^\infty \tilde{f}(0, 0, 0, u)du\right\}$$

$$+\frac{\phi(1)}{w-1}\left(\frac{1}{2}f(0)+\frac{\zeta\left(\frac{2}{3}\right)}{\xi(2)}\Sigma_2(f)+\frac{\zeta(-1)}{\xi(2)}\Sigma_1(f)\right)$$

$$=\frac{\phi(3)a_0(3)}{w-3}\Sigma_1(f, 2)+\frac{\phi(1)}{w-1}\left(\frac{1}{2}f(0)+\frac{\zeta\left(\frac{2}{3}\right)}{\xi(2)}\Sigma_2(f)+\frac{\zeta(-1)}{\xi(2)}\Sigma_1(f)\right)$$

(we note that $\tilde{f}$ is an even function and that $\zeta(0)=-\frac{1}{2}$).

Similarly, it follows from (2.29) and Sublemma that

$$\int_{G^1/\Gamma} J_{\hat{L}}(\hat{f})\mathcal{E}(\phi, w\,;g)dg^1 \sim \frac{\phi(3)a_0(3)}{3(w-3)}\Sigma_1(\hat{f}, 2)$$

$$+\frac{\phi(1)}{w-1}\left(\frac{1}{2}\hat{f}(0)+\frac{\zeta\left(\frac{2}{3}\right)}{\xi(2)}\Sigma_2(\hat{f})+\frac{3\zeta(-1)}{\xi(2)}\Sigma_1(\hat{f})\right).$$

On the other hand we have, by (2.15) and (2.14) that

$$\frac{1}{3}\Sigma_1(\hat{f}, 2)=\frac{1}{3}\int_0^\infty dt\int_{-\infty}^\infty \tilde{\hat{f}}(0, 0, t, u)du=\frac{1}{2}\int_{-\infty}^\infty\int_{-\infty}^\infty \tilde{f}(0, 0, t, u)dtdu=\Sigma_1(f, 2).$$

Hence,

$$\int_{G^1/\Gamma}\{J_L(f)-J_{\hat{L}}(\hat{f})\}\mathcal{E}(\phi, w\,;g)dg^1$$

$$\sim\frac{\phi(1)}{w-1}\left\{\frac{1}{2}(f(0)-\hat{f}(0))+\frac{\zeta\left(\frac{2}{3}\right)}{\xi(2)}(\Sigma_2(f)-\Sigma_2(\hat{f}))+\frac{\zeta(-1)}{\xi(2)}(\Sigma_1(f)-3\Sigma_1(\hat{f}))\right\}.$$

Therefore, one can see

$$\lim_{w\to 1+0}(w-1)\int_{G^1/\Gamma}\{J_L(f)-J_{\hat{L}}(\hat{f})\}\mathcal{E}(\phi, w\,;g)dg^1$$

$$=\phi(1)\left\{\frac{1}{2}(f(0)-\hat{f}(0))+\frac{\zeta\left(\frac{2}{3}\right)}{\xi(2)}(\Sigma_2(f)-\Sigma_2(\hat{f}))+\frac{\zeta(-1)}{\xi(2)}(\Sigma_1(f)-3\Sigma_1(\hat{f}))\right\}.$$

Since there exists a $\phi\in\Psi$ such that $\phi(1)\neq 0$, it follows from (2.27) that

$$\int_{G^1/\Gamma}\{J_L(f)-J_{\hat L}(\hat f)\}(g)dg^1$$
$$=\frac{\zeta(2)}{2\pi}(f(0)-\hat f(0))+\zeta\Big(\frac{2}{3}\Big)(\Sigma_2(f)-\Sigma_2(\hat f))+\zeta(-1)(\Sigma_1(f)-3\Sigma_1(\hat f))$$

$\Big($we note that $\xi(2)=\frac{\zeta(2)}{\pi}\Big)$. q. e. d.

COROLLARY TO PROPOSITION 2.16. *When* $\mathrm{Re}\, s>1$,

$$\int_{G^+/\Gamma,\chi(g)\leq 1}\chi(g)^s\Big\{\sum_{x\in L_0}f(g\cdot x)-\chi(g)^{-1}\sum_{x\in\hat L_0}\hat f(g^\iota\cdot x)\Big\}dg$$
$$=\frac{\zeta(2)}{2\pi}\Big(\frac{f(0)}{12s}-\frac{\hat f(0)}{12(s-1)}\Big)+\zeta\Big(\frac{2}{3}\Big)\Big(\frac{\Sigma_2(f)}{12s-2}-\frac{\Sigma_2(\hat f)}{12s-10}\Big)$$
$$+\zeta(-1)\Big(\frac{\Sigma_1(f)}{12s}-\frac{3}{12(s-1)}\Sigma_1(\hat f)\Big)$$

and

$$\int_{G^1/\Gamma,\chi(g)\leq 1}\chi(g)^s\Big\{\sum_{x\in\hat L_0}f(g\cdot x)-\chi(g)^{-1}\frac{1}{9}\sum_{x\in L_0}\hat f(g^\iota\cdot x)\Big\}dg$$
$$=\frac{\zeta(2)}{2\pi}\Big(\frac{f(0)}{12s}-\frac{\hat f(0)}{9\cdot 12(s-1)}\Big)+\zeta\Big(\frac{2}{3}\Big)\Big(\frac{\Sigma_2(f)}{12s}-\frac{\Sigma_2(\hat f)}{9(12s-10)}\Big)$$
$$+\zeta(-1)\Big(\frac{3\Sigma_1(f)}{12s}-\frac{\Sigma_1(\hat f)}{9\cdot 12(s-1)}\Big)\qquad(f\in\mathcal{S}(V_{\boldsymbol{R}})).$$

PROOF. We put $f_t(x)=f(tx)$ $(t>0)$. Then we have, by (2.14), $(\hat f)_t=t^{-4}(\hat f)_{t^{-1}}$. Making use of Lemma 2.3 and Lemma 2.4 we can see that $\Sigma_1(f_t)=\Sigma_1(f)$ and that $\Sigma_2(f_t)=t^{-\frac{2}{3}}\Sigma_2(f)$. We get, by Proposition 2.16 and (2.7),

$$\int_{G^+/\Gamma,\chi(g)\leq 1}\chi(g)^s\Big\{\sum_{x\in L_0}f(g\cdot x)-\chi(g)^{-1}\sum_{x\in\hat L_0}\hat f(g^\iota\cdot x)\Big\}dg$$
$$=\int_0^1 t^{12s}d^\times t\int_{G^1/\Gamma}\Big\{\sum_{x\in L_0}f_{t^3}(g_1\cdot x)-\sum_{x\in\hat L_0}(\widehat{f_{t^3}})(g_1\cdot x)\Big\}dg_1$$
$$=\int_0^1 t^{12s}\Big\{\frac{\zeta(2)}{2\pi}(f_{t^3}(0)-\widehat{f_{t^3}}(0))+\zeta\Big(\frac{2}{3}\Big)(\Sigma_2(f_{t^3})-\Sigma_2(\widehat{f_{t^3}}))$$
$$+\zeta(-1)(\Sigma_1(f_{t^3})-3\Sigma_1(\widehat{f_{t^3}}))\Big\}d^\times t$$
$$=\frac{\zeta(2)}{2\pi}\Big(\frac{f(0)}{12s}-\frac{\hat f(0)}{12(s-1)}\Big)+\zeta\Big(\frac{2}{3}\Big)\Big(\frac{\Sigma_2(f)}{12s-2}-\frac{\Sigma_2(\hat f)}{12s-10}\Big)$$
$$+\zeta(-1)\Big(\frac{\Sigma_1(f)}{12s}-\frac{3\Sigma_1(\hat f)}{12(s-1)}\Big).$$

Thus, we get the first equality. The second equality can be similarly obtained. q. e. d.

7. Now we are ready to prove the main result of this chapter.

THEOREM 2.1. (i) *Four Dirichlet series* $\xi_1(L, s)$, $\xi_2(L, s)$, $\xi_1(\hat{L}, s)$ *and* $\xi_2(\hat{L}, s)$ *can be continued analytically as meromorphic functions in the whole plane which satisfy the following functional equation:*

$$\begin{pmatrix}\xi_1(L, 1-s)\\ \xi_2(L, 1-s)\end{pmatrix}=\Gamma\left(s-\frac{1}{6}\right)\Gamma(s)^2\Gamma\left(s+\frac{1}{6}\right)\pi^{-4s}3^{6s}$$
$$\times\frac{1}{18}\begin{pmatrix}\sin 2\pi s & \sin \pi s\\ 3\sin\pi s & \sin 2\pi s\end{pmatrix}\begin{pmatrix}\xi_1(\hat{L}, s)\\ \xi_2(\hat{L}, s)\end{pmatrix}.$$

(ii) *Dirichlet series* $\xi_1(L, s)$, $\xi_2(L, s)$, $\xi_1(\hat{L}, s)$ *and* $\xi_2(\hat{L}, s)$ *have simple poles at* $s=1$ *and at* $s=\frac{5}{6}$ *and are holomorphic everywhere else. Their residues are given in the following table.*

	$\xi_1(L, s)$	$\xi_2(L, s)$	$\xi_1(\hat{L}, s)$	$\xi_2(\hat{L}, s)$
$s=1$	$\frac{\pi^2}{9}$	$\frac{\pi^2}{6}$	$\frac{\pi^2}{162}$	$\frac{\pi^2}{81}$
$s=\frac{5}{6}$	$\frac{\sqrt{3}}{18}r$	$\frac{1}{6}r$	$\frac{\sqrt{3}}{162}r$	$\frac{1}{54}r$
	$\left(We\ put\ r=\zeta\left(\frac{2}{3}\right)\frac{\Gamma(1/3)(2\pi)^{1/3}}{\Gamma(2/3)}.\right)$			

(iii) $(s-1)\left(s-\frac{5}{6}\right)\xi_1(L, s)$, $(s-1)\left(s-\frac{5}{6}\right)\xi_2(L, s)$, $(s-1)\left(s-\frac{5}{6}\right)\xi_1(\hat{L}, s)$ *and* $(s-1)\left(s-\frac{5}{6}\right)\xi_2(\hat{L}, s)$ *are entire functions whose orders are all equal to* 1.

PROOF. (i) It follows from Proposition 2.14 and Corollary to Proposition 2.16 that

$$Z(f, L; s)=Z^+(f, L; s)+Z^+(\hat{f}, \hat{L}, 1-s)$$
$$-\frac{\zeta(2)}{2\pi}\left(\frac{f(0)}{12s}-\frac{\hat{f}(0)}{12(s-1)}\right)-\zeta\left(\frac{2}{3}\right)\left(\frac{\Sigma_2(f)}{12s-2}-\frac{\Sigma_2(\hat{f})}{12s-10}\right)$$
$$-\zeta(-1)\left(\frac{\Sigma_1(f)}{12s}-\frac{3\Sigma_1(\hat{f})}{12(s-1)}\right) \tag{2.30}$$

and that

$$Z(f, \hat{L}; s)=Z^+(f, \hat{L}; s)+\frac{1}{9}Z^+(\hat{f}, L, 1-s)$$
$$-\frac{\zeta(2)}{2\pi}\left(\frac{f(0)}{12s}-\frac{\hat{f}(0)}{9\cdot 12(s-1)}\right)-\zeta\left(\frac{2}{3}\right)\left(\frac{\Sigma_2(f)}{12s-2}-\frac{\Sigma_2(\hat{f})}{9(12s-10)}\right)$$
$$-\zeta(-1)\left(\frac{3\Sigma_1(f)}{12s}-\frac{\Sigma_1(\hat{f})}{9\cdot 12(s-1)}\right)\quad (s\in \boldsymbol{C},\ \mathrm{Re}\, s>3,\ f\in \mathcal{S}(V_{\boldsymbol{R}})). \tag{2.31}$$

Since $Z^+(f, L; s)$, $Z^+(\hat{f}, \hat{L}; 1-s)$, $Z^+(f, \hat{L}; s)$ and $Z^+(\hat{f}, L; 1-s)$ are entire functions of s,[20] we can conclude that $Z(f, L; s)$ and $Z(f, \hat{L}; s)$ can be continued analytically as meromorphic functions in the whole plane which satisfy the following functional equation:

$$Z(f, L; s) = Z(\hat{f}, \hat{L}; 1-s). \tag{2.32}$$

For any $f \in \mathcal{S}(V_{\boldsymbol{R}})$, we put

$$\Phi_i(f, s) = \int_{V_i} |P(x)|^s f(x) dx \qquad (\mathrm{Re}\, s > 0) \ (i=1, 2).$$

When $\mathrm{Re}\, s > 3$, it follows from Proposition 2.13 that

$$Z(f, L; s) = \left(\frac{1}{4\pi}\xi_1(L, s)\ \frac{1}{12\pi}\xi_2(L, s)\right)\begin{pmatrix}\Phi_1(f, s-1)\\ \Phi_2(f, s-1)\end{pmatrix} \tag{2.33}$$

and that

$$Z(f, \hat{L}; s) = \left(\frac{1}{4\pi}\xi_1(\hat{L}, s)\ \frac{1}{12\pi}\xi_2(L, s)\right)\begin{pmatrix}\Phi_1(f, s-1)\\ \Phi_2(f, s-1)\end{pmatrix}. \tag{2.34}$$

Take an $f \in C_0^\infty(V_1)$, then $\Phi_1(f, s)$ is an entire function of s and we have

$$\xi_1(L, s)\Phi_1(f, s-1) = 4\pi Z(f, L; s)$$

and

$$\xi_1(\hat{L}, s)\Phi_1(f, s-1) = 4\pi Z(f, \hat{L}; s) \qquad (\mathrm{Re}\, s > 3).$$

Since there exists an $f \in C_0^\infty(V_1)$ such that $\Phi_1(f, s-1)$ does not vanish identically, we can conclude that $\xi_1(L, s)$ and $\xi_1(\hat{L}, s)$ can be continued analytically to meromorphic functions in the whole plane. Similarly, $\xi_2(L, s)$ and $\xi_2(\hat{L}, s)$ can be continued analytically to meromorphic functions in the whole plane. We can see, by Proposition 2.9, that $\Phi_1(f, s)$ and $\Phi_2(f, s)$ can be continued analytically to meromorphic functions in the whole plane which satisfy the following functional equation:

$$\begin{pmatrix}\Phi_1(\hat{f}, s-1)\\ \Phi_2(\hat{f}, s-1)\end{pmatrix} = \pi^{-4s}3^{6s}\Gamma\left(s-\frac{1}{6}\right)\Gamma(s)^2\Gamma\left(s+\frac{1}{6}\right) \times \frac{1}{18}\begin{pmatrix}\sin 2\pi s & \sin \pi s\\ 3\sin \pi s & \sin 2\pi s\end{pmatrix}\begin{pmatrix}\Phi_1(f, -s)\\ \Phi_2(f, -s)\end{pmatrix}. \tag{2.35}$$

Hence we get, by the equalities (2.32), (2.33), (2.34) and (2.35),

$$\begin{pmatrix}\xi_1(L, 1-s)\\ \frac{1}{3}\xi_2(L, 1-s)\end{pmatrix} = \Gamma\left(s-\frac{1}{6}\right)\Gamma(s)^2\Gamma\left(s+\frac{1}{6}\right)\pi^{-4s}3^{6s} \times \frac{1}{18}\begin{pmatrix}\sin 2\pi s & 3\sin \pi s\\ \sin \pi s & \sin 2\pi s\end{pmatrix}\begin{pmatrix}\xi_1(\hat{L}, s)\\ \frac{1}{3}\xi_2(\hat{L}, s)\end{pmatrix}.$$

20) See Corollary to Proposition 2.13.

(ii) Take an $f\in C_0^\infty(V_{\mathbf{R}}-S_{\mathbf{R}})$, then it follows from (2.30), (2.33), Proposition 2.5 and Proposition 2.6 that

$$(2.36)\qquad \frac{1}{4\pi}\Phi_1(f,s-1)\xi_1(L,s)+\frac{1}{12\pi}\Phi_2(f,s-1)\xi_2(L,s)$$
$$=Z^+(f,L;s)+Z^+(\hat{f},\hat{L};1-s)+\frac{\zeta(2)}{2\pi}\frac{1}{12(s-1)}(\Phi_1(f,0)+\Phi_2(f,0))$$
$$+\frac{\zeta\left(\frac{2}{3}\right)}{12\left(s-\frac{5}{6}\right)}\frac{\Gamma\left(\frac{1}{3}\right)(2\pi)^{\frac{1}{3}}}{6\pi\Gamma\left(\frac{2}{3}\right)}\left\{\sqrt{3}\,\Phi_1\left(f,-\frac{1}{6}\right)+\Phi_2\left(f,-\frac{1}{6}\right)\right\}$$
$$+\frac{3\zeta(-1)}{12(s-1)}\left(-\pi\Phi_1(f,0)-\frac{\pi}{3}\Phi_2(f,0)\right).$$

For any $s\in\mathbf{C}$, there exists an $f\in C_0^\infty(V_i)$ such that $\Phi_i(f,s)\neq 0$ $(i=1,2)$. Hence $\xi_1(L,s)$ and $\xi_2(L,s)$ have simple poles at $s=1$ and at $s=\frac{5}{6}$ and are holomorphic elsewhere. It follows from (2.36) that

$$\frac{1}{4\pi}\Phi_1(f,0)\operatorname*{res}_{s=1}\xi_1(L,s)=\frac{\Phi_1(f,0)}{12}\left(\frac{\zeta(2)}{2\pi}-3\zeta(-1)\pi\right)$$

and that

$$\frac{1}{4\pi}\Phi_1\left(f,-\frac{1}{6}\right)\operatorname*{res}_{s=\frac{5}{6}}\xi_1(L,s)=\frac{\Phi_1\left(f,-\frac{1}{6}\right)}{12}\frac{\sqrt{3}\,\Gamma\left(\frac{1}{3}\right)(2\pi)^{\frac{1}{3}}}{6\pi\Gamma\left(\frac{2}{3}\right)}\zeta\left(\frac{2}{3}\right)$$

$(\forall f\in C_0^\infty(V_1))$. Therefore,

$$\operatorname*{res}_{s=1}\xi_1(L,s)=\frac{\pi}{3}\left(\frac{\zeta(2)}{2\pi}-3\zeta(-1)\pi\right)=\frac{\pi^2}{9}\ ^{21)}$$

and

$$\operatorname*{res}_{s=\frac{5}{6}}\xi_1(L,s)=\frac{\sqrt{3}\,\Gamma\left(\frac{1}{3}\right)(2\pi)^{\frac{1}{3}}}{18\Gamma\left(\frac{2}{3}\right)}\zeta\left(\frac{2}{3}\right).$$

Similarly,

$$\operatorname*{res}_{s=1}\xi_2(L,s)=\pi\left(\frac{\zeta(2)}{2\pi}-\zeta(-1)\pi\right)=\frac{\pi^2}{6}$$

and

$$\operatorname*{res}_{s=\frac{5}{6}}\xi_2(L,s)=\frac{\Gamma\left(\frac{1}{3}\right)(2\pi)^{\frac{1}{3}}}{6\Gamma\left(\frac{2}{3}\right)}\zeta\left(\frac{2}{3}\right).$$

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21) As is well-known, $\zeta(-1)=-\frac{1}{12}$ and $\zeta(2)=\frac{\pi^2}{6}$.

Similarly, we can prove that $\xi_1(\hat{L}, s)$ and $\xi_2(\hat{L}, s)$ have simple poles at $s=1$ and at $s=\frac{5}{6}$ and are holomorphic elsewhere. The residues of $\xi_1(\hat{L}, s)$ and $\xi_2(\hat{L}, s)$ are calculated in the same manner.

(iii) There exists an $f \in C_0^\infty(V_1)$ which does not vanish identically and satisfies the following conditions (α) and (β):

$$(\alpha) \qquad f(x) \geqq 0 \qquad (\forall x \in V_R).$$

$$(\beta) \qquad \underset{x \in V_R}{\text{Sup.}} \left| \frac{\partial^{m_1+m_2+m_3+m_4}}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3} \partial x_4^{m_4}} f(x) \right| \leqq C^{\sum_{k=1}^4 m_k} \prod_{k=1}^4 (m_k)^{2m_k}$$

($\exists C>0$, $m_k=0, 1, 2, \cdots$). Then, making use of Proposition 2.13 and Corollary to Proposition 1.3, we can prove without difficulty that $Z^+(f, L; s)+Z^+(\hat{f}, \hat{L}, 1-s)$ is an entire function whose order is at most 1. Hence, it follows from (2.36) that $\Phi_1(f, s-1)(s-1)\left(s-\frac{5}{6}\right)\xi_1(L, s)$ is an entire function whose order is at most 1. Since $\Phi_1(f, s-1)$ is an entire function of exponential type of s which does not vanish identically and $(s-1)\left(s-\frac{5}{6}\right)\xi_1(L, s)$ is an entire function, we can conclude that $(s-1)\left(s-\frac{5}{6}\right)\xi_1(L, s)$ is an entire function whose order is at most 1.

Making use of the functional equation of $\xi_1(L, s)$ and Stirling's formula for the Gamma function, we can evaluate the asymptotic behaviour of $\xi_1(L, s)$ on the negative real axis and we can see that the order of $(s-1)\left(s-\frac{5}{6}\right)\xi_1(L, s)$ is not smaller than 1. Therefore the order of $(s-1)\left(s-\frac{5}{6}\right)\xi_1(L, s)$ is equal to 1. Similarly we can prove that $(s-1)\left(s-\frac{5}{6}\right)\xi_2(L, s)$, $(s-1)\left(s-\frac{5}{6}\right)\xi_1(\hat{L}, s)$ and $(s-1)\left(s-\frac{5}{6}\right)\xi_2(\hat{L}, s)$ are all entire functions whose orders are 1. q. e. d.

8. Applying theorem 2, we can get some informations about the asymptotic distributions of class numbers of integral binary cubic forms.

The first part of the following proposition follows immediately from Ikehara's Tauberian theorem (see e. g. [6]) and Theorem 2.1, while the second part can be proved combining a familiar argument in analytic number theory with Theorem 2.1.

PROPOSITION 2.17. *We put* $A_1(t)=\sum_{0<n\leqq t} h(n)$ *and* $A_2(t)=\sum_{0<n\leqq t} h(-n)$.

(i) $$\lim_{t\to+\infty} \frac{A_1(t)}{t} = \frac{\pi^2}{9} \quad \textit{and} \quad \lim_{t\to+\infty} \frac{A_2(t)}{t} = \frac{\pi^2}{6}.$$

(ii) $$\int_0^t A_1(t)dt = \frac{\pi^2}{18}t^2 + \frac{2\sqrt{3}}{55}\zeta\left(\frac{2}{3}\right)\frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}(2\pi)^{\frac{1}{3}}t^{\frac{11}{6}} + O(t^{\frac{3}{2}+\varepsilon})$$

and

$$\int_0^t A_2(t)dt = \frac{\pi^2}{12}t^2 + \frac{6}{55}\zeta\left(\frac{2}{3}\right)\frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}(2\pi)^{\frac{1}{3}}t^{\frac{11}{6}} + O(t^{\frac{3}{2}+\varepsilon})$$

for any $\varepsilon > 0$.

REMARK. For every $t \geqq 1$, we put

$$V_1(t) = \{x \in V;\ 1 \leqq P(x) \leqq t\}$$

and

$$V_2(t) = \{x \in V;\ -t \leqq P(x) \leqq -1\}.$$

We denote by $\Gamma \backslash V_i(t)$ a fundamental domain in V_i with respect to the action of Γ and denote by $\mathrm{vol}(\Gamma \backslash V_i(t))$ the Euclidean volume of $\Gamma \backslash V_i(t)$. It follows from Proposition 2.4 that

$$\mathrm{vol}(\Gamma \backslash V_1(t)) = 4\pi \int_1^{t^{1/12}} t^{12} d^\times t \int_{G'/\Gamma} dg_1 = \frac{\pi^2}{36}(t-1)$$

and that

$$\mathrm{vol}(\Gamma \backslash V_2(t)) = 12\pi \int_1^{t^{1/12}} t^{12} d^\times t \int_{G'/\Gamma} dg_1 = \frac{\pi^2}{12}(t-1).$$

On the other hand, $A_i(t)$ is the number of lattice points in $\Gamma \backslash V_i(t)$. Hence, by Proposition 2.17: (i),

$$\lim_{t \to \infty} \frac{A_1(t)}{\mathrm{vol}(\Gamma \backslash V_1(t))} = 4$$

and

$$\lim_{t \to \infty} \frac{A_2(t)}{\mathrm{vol}(\Gamma \backslash V_2(t))} = 2.$$

REMARK 2. Let $h_i(n)$ (resp. $h_r(n)$) denote the class number of irreducible (resp. reducible) integral binary cubic forms with discriminant $n (n \in \boldsymbol{Z} - \{0\})$. We have then $h(n) = h_i(n) + h_r(n)$. It is easy to see that $\sum_{n=1}^{\infty} h_r(n) n^{-s} = \sum x^{-2s}(y^2 - 4xz)^{-s}$ (resp. $\sum_{n=1}^{\infty} h_r(-n) n^{-s} = \sum x^{-2s}(4xz - y^2)^{-s}$), where summations on the right side are taken over the set of all triples of integers (x, y, z) satisfying the conditions: $x > 0$, $0 \leqq y < 2x$ and $y^2 - 4xz > 0$ (resp. $x > 0$, $0 \leqq y < 2x$ and $4xz - y^2 > 0$).

We can prove (the proof will appear elsewhere) that these Dirichlet series, absolutely convergent for $\mathrm{Re}\, s > 1$, can be continued analytically to meromorphic functions in the whole plane. Further we can show that they are holomorphic in the half plane $\{s;\ \mathrm{Re}\, s > 1/2\}$ except at $s = 1$, where each of them has a simple pole with residue $\pi^2/12$. Thus, we get

$$\sum_{0<n\leqq x} h_r(n)=(\pi^2/12)x+o(x)$$

and

$$\sum_{0<n\leqq x} h_r(-n)=(\pi^2/12)x+o(x)\,.$$

Hence it follows from Proposition 2.17 that

$$\sum_{0<n\leqq x} h_i(n)=(\pi^2/36)x+o(x)$$

and

$$\sum_{0<n\leqq x} h_i(-n)=(\pi^2/12)x+o(x)\,.$$

On the other hand, Davenport established the following results:

$$\sum_{0<n\leqq x} h_i(n)=(\pi^2/36)x+O(x^{15/16})\,,$$

$$\sum_{0<n\leqq x} h_i(-n)=(\pi^2/12)x+O(x^{15/16})$$

(see Davenport [**12**]).

Thus our estimates are consistent with his results.

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