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Minimal submanifolds with m-index 2 and generalized Veronese surfaces

Dedicated to Professor Kentaro Yano on his 60th birthday

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For a submanifold M in a Riemannian manifold $\bar{M}$, the *minimal index* (*m-index*) at a point of M is by definition the dimension of the linear space of all the 2nd fundamental forms with vanishing trace. The *geodesic codimension* (*g-codim*) of M in $\bar{M}$ is defined by the minimum of codimensions of M in totally geodesic submanifolds of $\bar{M}$ containing M.

In [8] and [9], the author investigated minimal submanifolds with m-index 2 everywhere in Riemannian manifolds of constant curvature and gave some typical examples of such submanifolds with g-codim 3 and g-codim 4 in the space forms of Euclidean, elliptic and hyperbolic types. Each example is the locus of points on a moving totally geodesic submanifold intersecting orthogonally a surface at a point. This surface is called the base surface. This situation is quite analogous to the case of the right helicoid in E^3 generated by a moving straight line along a base helix.

When the ambient space is Euclidean, the base surface of the example in case of g-codim 4 is a minimal surface in a 6-sphere, whose equations are analogous to those of the so-called Veronese surface which is a minimal surface in a 4-sphere with m-index 2 and g-codim 2. In [2], T. Itoh gave a minimal surface of the same sort in an 8-sphere.

In the present paper, the author will give some examples of minimal submanifolds with m-index 2 and g-codim of any integer $\geqq 2$ in the space forms of Euclidean, elliptic and hyperbolic types. The base surfaces corresponding to the minimal submanifolds with m-index 2 and even geodesic codimension in Euclidean spaces will be called generalized Veronese surfaces.

§ 1. Preliminaries

Let $M=M^n$ be an n-dimensional submanifold of an $(n+\nu)$-dimensional Riemannian manifold $\bar{M}=\bar{M}^{n+\nu}$ of constant curvature $\bar{c}$. Let $\bar{\omega}_A$, $\bar{\omega}_{AB}=-\bar{\omega}_{BA}$, $A, B=1, 2, \cdots, n+\nu$, be the basic and connection forms of $\bar{M}$ on the orthonormal frame bundle $F(\bar{M})$ which satisfy the structure equations:

(1.1) $$d\bar{\omega}_A = \sum_B \bar{\omega}_{AB} \wedge \bar{\omega}_B\,, \qquad d\bar{\omega}_{AB} = \sum_C \bar{\omega}_{AC} \wedge \bar{\omega}_{CB} - \bar{c}\bar{\omega}_A \wedge \bar{\omega}_B\,.$$

Let B be the subbundle of $F(\bar{M})$ over M composed of $b=(x, e_1, \cdots, e_{n+\nu}) \in F(\bar{M})$ such that $(x, e_1, \cdots, e_n) \in F(M)$, where $F(M)$ is the orthonormal frame bundle of M with the Riemannian metric induced from $\bar{M}$. Then deleting the bars of $\bar{\omega}_A$, $\bar{\omega}_{AB}$ on B, we have

(1.2) $$\omega_\alpha = 0\,, \qquad \omega_{i\alpha} = \sum_j A_{\alpha ij}\omega_j\,, \qquad A_{\alpha ij} = A_{\alpha ji}\,,$$
$$\alpha = n+1, \cdots, n+\nu\,; \quad i, j = 1, 2, \cdots, n\,.$$

At $x \in M$, we denote the normal space to $M_x = T_xM$ in $\bar{M}_x = T_x\bar{M}$ by N_x.

In the following, we suppose that M is minimal and of m-index 2 in $\bar{M}$ at each point. Then, N_x is decomposed as

$$N_x = N'_x + O'_x\,, \qquad N'_x \perp O'_x\,,$$

where O'_x is the linear set of normal vectors such that the corresponding 2nd fundamental forms vanish. By means of the above assumption, we have $\dim N'_x = 2$.

Let B_1 be the set of b such that $e_{n+1}, e_{n+2} \in N'_x$ which is a principal bundle over M of structure group $O(n) \times O(2) \times O(\nu-2)$. Setting $A_\alpha = (A_{\alpha ij})$, on B_1 we have

(1.3) $$A_{n+3} = A_{n+4} = \cdots = A_{n+\nu} = 0\,.$$

Since we have[1)]

$$\omega_{n+1,\beta} \equiv \omega_{n+2,\beta} \equiv 0 \pmod{\omega_1, \cdots, \omega_n} \quad \text{on } B_1$$

for $\beta > n+2$, we define a bilinear mapping $\psi' : M_x \times N'_x \to O'_x$ by

(1.4) $$\psi'(X, V) = \sum_\beta \langle V, e_{n+1}\omega_{n+1,\beta}(X) + e_{n+2}\omega_{n+2,\beta}(X) \rangle e_\beta\,.$$

ψ' is called the 1*st torsion operator* of M in $\bar{M}$.[2)]

Now, let $\mathfrak{l}_x$ be the space of relative nullity of M in $\bar{M}$ at x, which is by definition the subspace of M_x spanned by all tangent vectors that are annihilated simultaneously by all the 2nd fundamental forms of M at x. $\dim \mathfrak{l}_x$ is called the index of relative nullity of M at x. We have easily

LEMMA 1. *If M is minimal and of m-index 2 at a point, then the index of relative nullity of M is at most $n-2$ there.*

In the above mentioned case, we say that M has *maximal relative nullity* if its index of relative nullity is $n-2$ everywhere. Then, we have[3)]

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1) Lemma 1 in [8].

2) §1 in [**9**].

3) Lemmas 1, 2 and Theorem 1 in [8].

THEOREM A. *If M is minimal and of m-index 2 and $\phi' \neq 0$ everywhere, then M has maximal relative nullity and $\mathfrak{l}_x$ is the kernel of ϕ'_V for any $V \in N'_x$, $V \neq 0$, where $\phi'_V(X) = \phi'(X, V)$, and ϕ'_V has the same image.*

Supposing that M has maximal relative nullity, we decompose M_x as follows:

$$M_x = \mathfrak{w}_x + \mathfrak{l}_x\,, \qquad \mathfrak{w}_x \perp \mathfrak{l}_x\,,$$
$$\dim \mathfrak{w}_x = 2\,, \qquad \dim \mathfrak{l}_x = n-2\,,$$

and can choose $b \in B_1$ such that $e_1, e_2 \in \mathfrak{w}_x$ and $e_3, \cdots, e_n \in \mathfrak{l}_x$ and

$$(1.5) \qquad \begin{cases} \omega_{1,n+1} = \lambda\omega_1\,, & \omega_{2,n+1} = -\lambda\omega_2\,, \quad \omega_{r,n+1} = 0\,,\text{[4]} \\ \omega_{1,n+2} = \mu\omega_2\,, & \omega_{2,n+2} = \mu\omega_1\,, \quad \omega_{r,n+2} = 0\,, \end{cases}$$
$$r = 3, 4, \cdots, n\,; \qquad \lambda \neq 0\,, \quad \mu \neq 0\,.$$

We denote the set of such frames $b \in B_1$ by B'_1, which is a smooth submanifold[5] of B_1 in general. If we can choose smooth local fields e_{n+1}, e_{n+2} of $N' = \cup N'_x$ such that $\langle A_{n+1}, A_{n+2}\rangle = 0$, M is called nicely of m-index 2. If M is nicely of m-index 2, we may consider B'_1 is smooth. Then, on B'_1 we have

$$(1.6) \qquad \omega_{1r} + i\omega_{2r} = (p_r + iq_r)(\omega_1 + i\omega_2)\,, \qquad 2 < r \leqq n\,.\text{[6]}$$

The vector fields $P = \sum_r p_r e_r$ and $Q = \sum_r q_r e_r$ of M are called the *principal* and *subprincipal asymptotic vector fields*, respectively.

THEOREM B. *Let M be minimal and nicely of m-index 2 in $\bar{M}$ of constant curvature $\bar{c}$. If M has maximal relative nullity, then we have:*

(1) *The distribution $\mathfrak{l} = \cup \mathfrak{l}_x$ is completely integrable and its integral submanifolds are totally geodesic in $\bar{M}$.*
(2) *The distribution $\mathfrak{w} = \cup \mathfrak{w}_x$ is completely integrable if and only if $Q \equiv 0$.*
(3) *When $Q \equiv 0$, the integral surfaces of w are totally umbilic in M.*
(4) *When $P \neq 0$ and $Q \equiv 0$, the integral curves of the vector field P are geodesics in $\bar{M}$.*

Under the conditions of Theorem B and $Q \equiv 0$, on B'_1 we have[7]

$$(1.7) \qquad \{d \log \lambda - \langle P, dx\rangle - i(2\omega_{12} - \sigma\hat{\omega}_1)\} \wedge (\omega_1 + i\omega_2) = 0\,,$$

$$(1.8) \qquad \{d\sigma + i(1-\sigma^2)\hat{\omega}_1\} \wedge (\omega_1 + i\omega_2) = 0\,,$$

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4) Lemma 2 in [8].

5) A singular frame b of the smoothness of B'_1 is over such a point $x \in M$ that $\xi_1 A_{n+1} + \xi_2 A_{n+2}$, $\xi_1^2 + \xi_2^2 = 1$, describes a circle in the space S_n of all symmetric matrixes of order n.

6) Lemma 3 in [8].

7) Lemma 6 in [8].

(1.9) $$d\omega_{12} = -\{\|P\|^2 + \bar{c} - \lambda^2 - \mu^2\}\omega_1 \wedge \omega_2\,,$$

(1.10) $$d\hat{\omega}_1 = -\frac{1}{\lambda\mu}\{2\lambda^2\mu^2 - \|F\|^2 - \|G\|^2\}\omega_1 \wedge \omega_2\,,$$

where $\sigma = \mu/\lambda$, $\hat{\omega}_1 = \omega_{n+1,n+2}$ and $F = \sum_{\gamma > n+2} f_\gamma e_\gamma$ and $G = \sum_{\gamma > n+2} g_\gamma e_\gamma$ are defined through means of the fact[8] $\lambda\omega_{n+1,\gamma} + i\mu\omega_{n+2,\gamma}$ can be written on B_1' as

(1.11) $$\lambda\omega_{n+1,\gamma} + i\mu\omega_{n+2,\gamma} = (f_\gamma + ig_\gamma)(\omega_1 - i\omega_2)\,, \qquad \gamma > n+2\,.$$

LEMMA 2. *Under the conditions of Theorem* B *and* $Q \equiv 0$, $\bar{c} \neq 0$ *implies* $P \neq 0$.

PROOF. If $P \equiv 0$ on a neighborhood of M, then (1.6) implies

$$\omega_{ar} = 0\,, \qquad a = 1, 2\,, \qquad r = 3, \cdots, n \qquad \text{on} \quad B_1'\,,$$

from which we have

$$d\omega_{ar} = -\bar{c}\omega_a \wedge \omega_r = 0\,.$$

Hence it must be $\bar{c} = 0$. Q. E. D.

According to Lemmas 7, 8, 9, 10 and Theorem 3 in [8], we have the following theorem which plays a fundamental role in this paper.

THEOREM C. *Let M^n be an n-dimensional maximal**) *minimal submanifold in an $(n+\nu)$-dimensional space form $\bar{M}^{n+\nu}$ (of constant curvature $\bar{c}$) which is nicely of m-index* 2 *and has maximal relative nullity and $Q \equiv 0$, then it is a locus of $(n-2)$-dimensional totally geodesic subspaces $L^{n-2}(y)$ in $\bar{M}^{n+\nu}$ through points y of a base surface W^2 lying in either a Riemannian hypersphere in $\bar{M}^{n+\nu}$ with center z_0 such that*

- (i) *$L^{n-2}(y)$ intersects orthogonally with W^2 at y and contains the geodesic radius from z_0 to y,*
- (ii) *The $(n-3)$-dimensional tangent spaces to the intersection of $L^{n-2}(y)$ and the hypersphere at y are parallel along W^2 in $\bar{M}^{n+\nu}$,*

or an $(n+2)$-dimensional linear space in $\bar{M}^{n+\nu} = E^{n+\nu}$ such that M^n is $(n-2)$-dimensionally cylindrical and $L^{n-2}(y)$ are its generators intersecting orthogonally W^2 at y.

REMARK. W^2 in this theorem is an integral surface of the distribution $\mathfrak{w}$ and the geodesic radius from z_0 to y in the first case is the integral curve of P. In this case, denoting the length measured from z_0 to y along the geodesic ray starting z_0 by v, we have

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8) (1.12) in [8].

*) "maximal" means here that M^n is not a submanifold of any other n-dimensional minimal submanifold of $\bar{M}^{n+\nu}$.

$$p=\|P\|=\begin{cases}\sqrt{\bar{c}}\cot\sqrt{\bar{c}}\,v & (\bar{c}>0),\\ 1/v & (\bar{c}=0),\\ \sqrt{-\bar{c}}\coth\sqrt{-\bar{c}}\,v & (\bar{c}<0).\end{cases} \tag{1.12}$$

§ 2. Minimal submanifolds with m-index 2 in space forms.

In the following, we shall investigate M^n in $\bar{M}^{n+\nu}$ as in Theorem C and use the notations in § 1.

First, for an integer $m\geqq 1$, we suppose that the normal vector bundle $N=\cup N_x$ over $M=M^n$ has the following smooth orthogonal decomposition such that

$$N=N'+N''+\cdots+N^{(m)}+O^{(m)}, \tag{2.1}$$

where

$$N^{(t)}=\cup N_x^{(t)}, \qquad \dim N_x^{(t)}=2,$$

$$N_x=N_x'+\cdots+N_x^{(m)}+O_x^{(m)}$$

and

$$\bar{D}_X V\in\Gamma(N^{(t-1)})+\Gamma(N^{(t)})+\Gamma(N^{(t+1)}), \quad V\in\Gamma(N^{(t)}), \quad X\in\Gamma(T(M)), \quad t=1, 2, \cdots, m-1, \tag{2.2}$$

where $\bar{D}$ is the covariant differential operator of $\bar{M}$ and Γ denotes the set of smooth cross sections of a smooth vector bundle and

$$N^{(O)}=\cup N_x^{(O)}, \qquad N_x^{(O)}=\mathfrak{w}_x$$

and the component of $\bar{D}_X V$ on $N^{(t+1)}$ in (2.2) spans $N_x^{(t+1)}$ at each point $x\in M$.

Under the above assumption, for $t=1, 2, \cdots, m-1$, the t-th torsion operator $\psi^{(t)}$ as a tensor field on M of type $T(M)\otimes N^{(t)}\otimes N^{(t+1)}$ is defined by

$$\psi^{(t)}(X, V)=N^{(t+1)}\text{-component of } \bar{D}_X V, \quad X\in\Gamma(T(M)), \quad V\in\Gamma(N^{(t)}). \tag{2.3}$$

Now, we shall show that we can define the m-th torsion operator $\psi^{(m)}$ for M. Let B_t be the set of frames $b=(x, e_1, e_2, \cdots, e_{n+\nu})$ such that

$$e_{n+1}, e_{n+2}\in N_x'; \cdots; e_{n+2t-1}, e_{n+2t}\in N_x^{(t)},$$

which is smooth in B by the above assumption.

From now on, we use the following convention about indices: Setting $I_0=\{1, 2\}$, and $I_t=\{n+2t-1, n+2t\}, t=1, 2, \cdots, m$, if we write $\alpha_1, \alpha_2\in I_t$, then $\alpha_1<\alpha_2$.

By means of (2.2), on B_{t+1} we have

$$\omega_{\alpha\beta}=0, \quad \alpha\in I_t, \quad \beta>n+2t+2, \quad t=1, 2, \cdots, m-1. \tag{2.4}$$

Using the matrices

$$\Omega_t = \begin{pmatrix} \omega_{\alpha_1\beta_1} & \omega_{\alpha_1\beta_2} \\ \omega_{\alpha_2\beta_1} & \omega_{\alpha_2\beta_2} \end{pmatrix}, \qquad \alpha_1, \alpha_2 \in I_t\,; \quad \beta_1, \beta_2 \in I_{t+1}\,, \qquad t = 0, 1, 2, \cdots, m-1, \tag{2.5}$$

and putting

$$B'_t = B_t \cap B'_1\,, \qquad t = 1, 2, \cdots, m\,, \tag{2.6}$$

we prove the following

LEMMA 3. *There exist complex valued* (1, 2)*-matrix fields* $\Psi_t = (\Psi_{t1}, \Psi_{t2})$ *on* B'_t *satisfying the identities*

$$\Psi_t \Omega_t = \Psi_{t+1}(\omega_1 - i\omega_2) \quad \text{on } B'_{t+1}\,, \qquad t = 1, 2, \cdots, m-1\,. \tag{2.7$_t$}$$

PROOF. If we put

$$\Psi_1 = (\lambda, i\mu) \qquad \text{on } B'_1\,, \tag{2.8}$$

$$\Psi_2 = (f_{\alpha_1} + ig_{\alpha_1}, f_{\alpha_2} + ig_{\alpha_2})\,, \qquad \alpha_1, \alpha_2 \in I_2 \quad \text{on } B'_2\,, \tag{2.9}$$

then (2.7$_1$) is identical with (1.11) on B'_2.

Now, inductively we suppose (2.7$_t$) for $t \leqq s$, $s < m-1$. Since for $\alpha \in I_s$, $\gamma > n + 2(s+1)$

$$\omega_{\alpha\gamma} = 0 \qquad \text{on } B_{s+1}\,,$$

we have

$$d\omega_{\alpha\gamma} = \omega_{\alpha\beta_1} \wedge \omega_{\beta_1\gamma} + \omega_{\alpha\beta_2} \wedge \omega_{\beta_2\gamma}\,, \qquad \beta_1, \beta_2 \in I_{s+1}\,,$$

which can be written as

$$\Omega_s \wedge \begin{pmatrix} \omega_{\beta_1\gamma} \\ \omega_{\beta_2\gamma} \end{pmatrix} = 0 \qquad \text{on } B_{s+1}\,.$$

Hence, by (2.7$_s$) we have

$$\Psi_s \Omega_s \wedge \begin{pmatrix} \omega_{\beta_1\gamma} \\ \omega_{\beta_2\gamma} \end{pmatrix} = \Psi_{s+1}(\omega_1 - i\omega_2) \wedge \begin{pmatrix} \omega_{\beta_1\gamma} \\ \omega_{\beta_2\gamma} \end{pmatrix}$$

$$= -\Psi_{s+1} \begin{pmatrix} \omega_{\beta_1\gamma} \\ \omega_{\beta_2\gamma} \end{pmatrix} \wedge (\omega_1 - i\omega_2) = 0 \qquad \text{on } B'_{s+1}\,.$$

By E. Cartan's lemma, we can find $\Psi_{s+1} = (\Psi_{s+1,1}, \Psi_{s+1,2})$ on B'_{s+1} such that

$$\Psi_{s+1}\Omega_{s+1} = \Psi_{s+2}(\omega_1 - i\omega_2) \qquad \text{on } B'_{s+2}\,.$$

Thus, we have proved (2.7$_t$), $t = 1, 2, \cdots, m-1$, by induction. Q. E. D.

In the same way as in the proof of Lemma 3, for $\beta_1, \beta_2 \in I_m$, $\gamma > n + 2m$, we can put

$$\Psi_m \begin{pmatrix} \omega_{\beta_1\gamma} \\ \omega_{\beta_2\gamma} \end{pmatrix} = \Psi_{m+1,\gamma}(\omega_1 - i\omega_2) \qquad \text{on } B'_m\,, \tag{2.10}$$

where $\Psi_{m+1,\gamma}$ are functions on B'_m.

By means of these Ψ_t, we can define the following vector fields $F_t, G_t \in \Gamma(N^{(t)})$, $t=1, 2, \cdots, m$, and $F_{m+1}, G_{m+1} \in \Gamma(O^{(m)})$ by

$$\Psi_{t1}e_{\gamma_1}+\Psi_{t2}e_{\gamma_2}=F_t+iG_t\,, \qquad \gamma_1, \gamma_2 \in I_t \tag{2.11$_t$}$$

$$\sum_{\gamma>n+2m} \Psi_{m+1,\gamma}e_\gamma=F_{m+1}+iG_{m+1} \tag{2.12}$$

and call the pair (F_t, G_t) the *t-th pair of normal vector fields* associated with M^n in $\bar{M}^{n+\nu}$.

Now, we define the m-th torsion operator $\phi^{(m)}$ as a tensor field on M of type $T(M)\otimes N^{(m)}\otimes O^{(m)}$ defined by

$$\phi^{(m)}(X, V)=O^{(m)}\text{-component of } \bar{D}_X V\,, \tag{2.13}$$
$$X\in\Gamma(T(M))\,, \qquad V\in\Gamma(N^{(m)})\,.$$

LEMMA 4. *$\Psi_{t2}/\Psi_{t1}\neq$ real for $t=1, 2, \cdots, m$ and the t-th torsion operator $\phi^{(t)}(X, V)$, $X\in M_x$, $V\in N_x^{(t)}$, has the same image for $V\neq 0$ at each point x, for $t=1, 2, \cdots, m$.*

PROOF. For $t=1$, $\Psi_{12}/\Psi_{11}=i\mu/\lambda\neq$ real everywhere.

By induction, it will be sufficient to prove that supposing $\Psi_{m2}/\Psi_{m1}\neq$ real, $\phi^{(m)}$ has the property stated in the lemma.

Putting $\Psi_{m1}=a_1+ib_1$, $\Psi_{m2}=a_2+ib_2$, where a_1, b_1, a_2 and b_2 are real, $\Psi_{m2}/\Psi_{m1}\neq$ real is equivalent to

$$\Delta=a_1b_2-a_2b_1\neq 0\,.$$

By means of (2.10), (2.12), we have for $\beta_1, \beta_2\in I_m$

$$\begin{aligned}\sum_{\gamma>n+2m}\omega_{\beta_1\gamma}e_\gamma&=\frac{1}{\Delta}\{(b_2F_{m+1}-a_2G_{m+1})\omega_1+(a_2F_{m+1}+b_2G_{m+1})\omega_2\}\,,\\ \sum_{\gamma>n+2m}\omega_{\beta_2\gamma}e_\gamma&=-\frac{1}{\Delta}\{(b_1F_{m+1}-a_1G_{m+1})\omega_1+(a_1F_{m+1}+b_1G_{m+1})\omega_2\}\,,\end{aligned} \tag{2.14}$$

which shows that

$$\omega_{\beta\gamma}\equiv 0 \ (\mathrm{mod}\ \omega_1, \omega_2) \ \text{ on } B'_m\,, \qquad \text{for } \beta\in I_m\,, \ \ \gamma>n+2m\,.$$

Hence, for any $V\in N_x^{(m)}$ and $X\in M_x$, we have

$$\begin{aligned}\phi^{(m)}(X, V)&=\sum_{\beta\in I_m,\gamma>n+2m}\langle V, e_\beta\rangle\,\omega_{\beta\gamma}(X)e_\gamma\\ &=\frac{1}{\Delta}\{(b_2v_1-b_1v_2)\omega_1(X)+(a_2v_1-a_1v_2)\omega_2(X)\}F_{m+1}\\ &\quad+\frac{1}{\Delta}\{-(a_2v_1-a_1v_2)\omega_1(X)+(b_2v_1-b_1v_2)\omega_2(X)\}G_{m+1}\,,\end{aligned}$$

where $v_1=\langle V, e_{\beta_1}\rangle$, $v_2=\langle V, e_{\beta_2}\rangle$, $\beta_1, \beta_2 \in I_m$. This equality shows that for any fixed $V\in N_x^{(m)}$, $V\neq 0$, the set of images $\phi^{(m)}(X, V)$ is the space spanned by F_{m+1} and G_{m+1}.

On the other hand, from the above argument we see that supposing $\Psi_{t2}/\Psi_{t1}\neq$ real for $t<m$, the set of images $\phi^{(t)}(X, V)$ for a fixed $V\in N_x^{(t)}$, $V\neq 0$, is the space spanned by F_{t+1} and G_{t+1}. By the supposition stated at the beginning of this section, it must be $N_x^{(t+1)}$. Hence, we have

$$F_{t+1}\wedge G_{t+1}\neq 0\,,$$

which is equivalent to

$$\Psi_{t+1,2}/\Psi_{t+1,1}\neq \text{real}\,. \qquad \text{Q. E. D.}$$

By virtue of Lemma 4, if $F_{m+1}\wedge G_{m+1}\neq 0$ everywhere, then we can replace m by $m+1$ in the above argument.

LEMMA 5. *If $F_{m+1}\equiv G_{m+1}\equiv 0$, then the geodesic codimension of M^n is $2m$. If $F_{m+1}\wedge G_{m+1}\equiv 0$ and F_{m+1} or $G_{m+1}\neq 0$ everywhere, then the geodesic codimension of M^n is $2m+1$.*

PROOF. If $F_{m+1}\equiv G_{m+1}\equiv 0$, then by (2.14) we have

$$\omega_{\beta\gamma}=0 \quad \text{on } B'_m\,, \qquad \text{for} \quad \beta\in I_m\,,\ \gamma>n+2m\,,$$

we have also

$$\omega_{i\gamma}=0\,, \qquad \omega_{\alpha\gamma}=0 \qquad \text{on} \quad B'_m\,,$$

$$\text{for} \quad i=1, 2, \cdots, n\,; \quad \alpha=n+1, \cdots, n+2m-2\,, \quad \gamma>n+2m\,.$$

These equations show that there exists an $(n+2m)$-dimensional totally geodesic submanifold $\bar{M}^{n+2m}\subset \bar{M}^{n+\nu}$ containing M^n.

If $F_{m+1}\wedge G_{m+1}\equiv 0$ and F_{m+1} or $G_{m+1}\neq 0$ everywhere, then we can consider the set of frames $b=(x, e_1, \cdots, e_{n+2m+1})\in B_m$ such that

$$F_{m+1}=fe_{n+2m+1}\,, \qquad G_{m+1}=ge_{n+2m+1}\,,$$

which we denote by B_{m+1}. Then, on $B'_{m+1}=B_{m+1}\cap B'_1$ we have

$$\omega_{\alpha\gamma}=0\,, \qquad \text{for} \quad \alpha\in I_m\,, \quad \gamma>n+2m+1\,,$$

from which we get

$$d\omega_{\alpha\gamma}=\omega_{\alpha,n+2m+1}\wedge\omega_{n+2m+1,\gamma}=0\,.$$

Using the notations in the proof of Lemma 4, (2.14) implies

$$\omega_{\alpha 1,n+2m+1}=\frac{1}{\Delta}\{(b_2f-a_2g)\omega_1+(a_2f+b_2g)\omega_2\}\,,$$

$$\omega_{\alpha 2,n+2m+1}=-\frac{1}{\Delta}\{(b_1f-a_1g)\omega_1+(a_1f+b_1g)\omega_2\}\,,$$

from which we get

$$\omega_{\alpha_1,n+2m+1}\wedge\omega_{\alpha_2,n+2m+1}=-\frac{1}{\Delta}(f^2+g^2)\omega_1\wedge\omega_2\neq 0 \quad \text{on } B'_{m+1}.$$

Hence, it must be

$$\omega_{n+2m+1,\gamma}=0 \quad \text{on } B'_{m+1} \qquad \text{for} \quad \gamma>n+2m+1.$$

Therefore, by an analogous argument to the first case, we see that there exists an $(n+2m+1)$-dimensional totally geodesic submanifold $\bar{M}^{n+2m+1}$ in $\bar{M}^{n+\nu}$ containing M^n. Q. E. D.

By virtue of these lemmas, we can conclude the following

THEOREM 1. *Let M^n be an n-dimensional submanifold in an $(n+\nu)$-dimensional space form $\bar{M}^{n+\nu}$ as in Theorem C. Then, the normal vector bundle N over M^n in $\bar{M}^{n+\nu}$ can be decomposed orthogonally and smoothly in general as follows:*

(i) *If the geodesic codimension of M^n in $\bar{M}^{n+\nu}$ is even, say 2m, then*

$$N=N'+N''+\cdots+N^{(m)}+O^{(m)};$$

(ii) *If the geodesic codimension of M^n in $\bar{M}^{n+\nu}$ is odd, say 2m+1, then*

$$N=N'+N''+\cdots+N^{(m+1)}+O^{(m+1)},$$

where the fibre $N_x^{(t)}$ of $N^{(t)}$ is of 2-dimension for $t=1, 2, \cdots, m$ and 1-dimension for $t=m+1$, and for $X\in\Gamma(T(M^n))$ and $V\in\Gamma(N^{(t)})$

$$\bar{D}_X V\in\Gamma(N^{(t-1)})+\Gamma(N^{(t)})+\Gamma(N^{(t+1)})$$

for $t=1,\cdots,m-1$ in case (i) and $t=1,\cdots,m$ in case (ii) and

$$\bar{D}_X V\in\Gamma(N^{(t-1)})+\Gamma(N^{(t)})$$

for $t=m$ in case (i) and for $t=m+1$ in case (ii),

here $N^{(0)}$ means $w=\cup w_x$. Furthermore,

$$N'+N''+\cdots+N^{(m)} \quad \text{or} \quad N'+N''+\cdots+N^{(m+1)}$$

are the normal vector bundles of M^n in the totally geodesic submanifolds of the least dimensions in $\bar{M}^{n+\nu}$ containing M.

In the following, we call M in Theorem 1 *S-type* or *T-type* according to g-codim M^n = even or odd.

In the rest of this section, we shall prepare some formulas on the curvature of the normal vector bundles treated in this section. Putting

(2.15) $$\hat{\omega}_t=\omega_{\alpha_1\alpha_2}, \qquad \alpha_1, \alpha_2\in I_t,$$

the curvature form of the vector bundle $N^{(t)}$ is given by

(2.16) $$d\hat{\omega}_t=-\sum_{\beta\in I_{t-1}}\omega_{\beta\alpha_1}\wedge\omega_{\beta\alpha_2}-\sum_{\beta\in I_{t+1}}\omega_{\alpha_1\beta}\wedge\omega_{\alpha_2\beta}.$$

On the other hand, we put

$$\det\begin{pmatrix}\Psi_t\\ \bar{\Psi}_t\end{pmatrix}=\Psi_{t1}\bar{\Psi}_{t2}-\Psi_{t2}\bar{\Psi}_{t1}=-2i\Delta_t, \tag{2.17}$$

then Δ_t is real and the area with sign of the paralellogram made by the t-th pair of normal vectors (F_t, G_t). By Lemma 3, we have

$$\sum_{\beta\in I_{t+1}}\omega_{\alpha_1\beta}\wedge\omega_{\alpha_2\beta}=-\frac{1}{\Delta_t}\|\Psi_{t+1}\|^2\omega_1\wedge\omega_2, \tag{2.18}$$

where

$$\|\Psi_t\|^2=|\Psi_{t1}|^2+|\Psi_{t2}|^2=\|F_t\|^2+\|G_t\|^2 \qquad \text{for} \quad t=1, 2, \cdots, m \tag{2.19}$$

and

$$\|\Psi_{m+1}\|^2=0 \qquad \text{in case (i)},$$

$$\|\Psi_{m+1}\|^2=|\Psi_{m+1}|^2=\|F_{m+1}\|^2+\|G_{m+1}\|^2 \qquad \text{in case (ii)}$$

of Theorem 1. Analogously, we have

$$\sum_{\beta\in I_{t-1}}\omega_{\beta\alpha_1}\wedge\omega_{\beta\alpha_2}=\frac{\Delta_t}{(\Delta_{t-1})^2}\|\Psi_{t-1}\|^2\omega_1\wedge\omega_2. \tag{2.20}$$

Using (2.18), (2.20), (2.16) can be written as

$$d\hat{\omega}_t=-\left\{\frac{\Delta_t}{(\Delta_{t-1})^2}\|\Psi_{t-1}\|^2-\frac{1}{\Delta_t}\|\Psi_{t+1}\|^2\right\}\omega_1\wedge\omega_2 \quad \text{on } B'_m, \tag{2.21}$$

$$\text{for} \quad t=1, 2, \cdots, m,$$

where we put

$$\Psi_0=(1, i) \qquad \text{and} \qquad \Delta_0=1.$$

(2.21) coincides with (1.10) for $t=1$.

Finally, we compute the exterior derivative of Ω_t. By means of the structure equations and the relations above, we have easily

$$d\Omega_t=\hat{\omega}_t\wedge J\Omega_t+\Omega_t J\wedge\hat{\omega}_{t+1} \quad \text{on } B'_m, \qquad \text{for} \quad t=0, 1, \cdots, m-1, \tag{2.22}$$

where

$$J=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

and $\hat{\omega}_0=\omega_{12}$, we have also

$$d(\omega_1-i\omega_2)=(i\omega_{12}-\langle P, dx\rangle)\wedge(\omega_1-i\omega_2) \qquad \text{on } B'_m \tag{2.23}$$

by (1.6) and $Q\equiv 0$. From (2.7$_t$), we get

$$d\Psi_t\wedge\Omega_t+\Psi_t d\Omega_t=d\Psi_{t+1}\wedge(\omega_1-i\omega_2)+\Psi_{t+1}d(\omega_1-i\omega_2).$$

Substituting (2.22) and (2.23) in the above equality, we have

$$\begin{aligned}&\{d\Psi_t+\Psi_t J\hat{\omega}_t\}\wedge\Omega_t \\ &=\{d\Psi_{t+1}+\Psi_{t+1}(i\omega_{12}-\langle P, dx\rangle)+\Psi_{t+1}J\hat{\omega}_{t+1}\}\wedge(\omega_1-i\omega_2) \quad \text{on } B'_m,\\ &\qquad \text{for} \quad t=0, 1, \cdots, m-1.\end{aligned} \tag{2.24}$$

Furthermore, we have

$$\Omega_t = \frac{i}{2\Delta_t}\begin{pmatrix} \bar{\Psi}_{t,2} & -\Psi_{t,2} \\ -\bar{\Psi}_{t,1} & \Psi_{t,1} \end{pmatrix}\begin{pmatrix} \Psi_{t+1}(\omega_1 - i\omega_2) \\ \bar{\Psi}_{t+1}(\omega_1 + i\omega_2) \end{pmatrix} \tag{2.25}$$

from (2.7_t) and then

$$\Psi_t J \Omega_t = \frac{i}{2\Delta_t}(-\|\Psi_t\|^2, \Psi_{t1}^2 + \Psi_{t2}^2)\begin{pmatrix} \Psi_{t+1}(\omega_1 - i\omega_2) \\ \bar{\Psi}_{t+1}(\omega_1 + i\omega_2) \end{pmatrix}. \tag{2.26}$$

§ 3. Minimal submanifolds with m-index 2 of S-type and T-type.

In this section, we shall continue the investigation of $M = M^n$ in $\bar{M} = \bar{M}^{n+\nu}$ as in Theorem 1, supposing $\nu = 2m$ or $2m+1$ according to the case (i) or (ii).

Now, we introduce the following three conditions for a base surface W^2 of M^n which is an integral surface of the distribution $\mathfrak{w}$:

(α) $\Psi_{t2}/\Psi_{t1} = c_t$ is constant on W^2 for $t = 1, \cdots, m$.

(β) W^2 is of constant curvature c.

(γ) $\hat{\omega}_t \neq 0$ for $t = 1, 2, \cdots, m$.

THEOREM 2. *Let M^n be a submanifold of $\bar{M}^{n+\nu}$ as stated above. If a base surface W^2 of M^n satisfies conditions (α), (β) and (γ), then we can find a submanifold $B''_m \subset B'_m$ over W^2 on which*

(i) *$c_t = i$ or $-i$ for $t = 1, 2, \cdots, m$;*

(ii) *Ψ_t is constant for $t = 1, 2, \cdots, m+1$;*

(iii) *$\langle F_t, G_t \rangle = 0$ for $t = 1, 2, \cdots, m$;*

(iv) *$\hat{\omega}_t = (t+1)\sigma_t\omega_{12}$, where $\sigma_t = c_t/i$, for $t = 1, 2, \cdots, m$;*

and

(v) *$c > 0$ when M^n is of S-type and $c = 0$ when M^n is of T-type.*

PROOF. Since $\Psi_1 = (\lambda, i\mu)$, $c_1 = i\mu/\lambda = i\sigma$. Therefore $\sigma = \sigma_1$ is constant and so from (1.8) we get

$$(1 - \sigma^2)\hat{\omega}_1 \wedge (\omega_1 + i\omega_2) = 0,$$

hence

$$(1 - \sigma^2)\hat{\omega}_1 = 0 \quad \text{on } W^2. \tag{3.1}$$

By means of (γ), it must be $1 - \sigma^2 = 0$, i.e.

$$\sigma = 1 \quad \text{or} \quad -1, \tag{3.2}$$

i.e.

$$c_1 = i \quad \text{or} \quad -i.$$

Since we have

$$d\omega_{12} = -c\omega_1 \wedge \omega_2 \quad \text{on } W^2$$

by means of (β), we get from (1.9) and (3.2)

$$\lambda^2 + \mu^2 = 2\lambda^2 = \|P\|^2 + \bar{c} - c. \tag{3.3}$$

By Theorem C and its remark, $\|P\|$ is constant on W^2. Hence, by (3.2) and (3.3), λ and μ i.e. Ψ_1 must be constant. Then, (1.7) becomes

$$(2\omega_{12}-\sigma\hat{\omega}_1)\wedge(\omega_1+i\omega_2)=0 \qquad \text{on } W^2,$$

hence we get

$$\hat{\omega}_1=2\sigma_1\omega_{12}, \qquad \sigma_1=\sigma. \tag{3.4}$$

Now, we take $s\leqq m$ and suppose that (i), (ii), (iii) and (iv) are true for $t<s$. Then, putting

$$\Psi_t=a_t(1, c_t), \qquad a_t=\text{const.}, \qquad t<s, \tag{3.5}$$

we get from (2.24) the equality

$$\Psi_{s-1}J\hat{\omega}_{s-1}\wedge\Omega_{s-1}=(d\Psi_s+i\Psi_s\omega_{12}+\Psi_sJ\hat{\omega}_s)\wedge(\omega_1-i\omega_2).$$

(2.26) becomes for $t=s-1$

$$\Psi_{s-1}J\Omega_{s-1}=-i\frac{|a_{s-1}|^2}{\Delta_{s-1}}\Psi_s(\omega_1-i\omega_2).$$

By means of (3.5) and (2.17), we have

$$\sigma_{s-1}|a_{s-1}|^2=\Delta_{s-1},$$

and hence

$$\Psi_{s-1}J\Omega_{s-1}=-i\sigma_{s-1}\Psi_s(\omega_1-i\omega_2).$$

Using these, we obtain from the above equality

$$\{d\Psi_s+i\Psi_s(\omega_{12}+\sigma_{s-1}\hat{\omega}_{s-1})+\Psi_sJ\hat{\omega}_s\}\wedge(\omega_1-i\omega_2)=0.$$

Since $\hat{\omega}_{s-1}=s\sigma_{s-1}\omega_{12}$ by the assumption, we obtain

$$\{d\Psi_s+i(s+1)\Psi_s\omega_{12}+\Psi_sJ\hat{\omega}_s\}\wedge(\omega_1-i\omega_2)=0. \tag{3.6}$$

On the other hand, from (2.21) and (β) we get

$$\begin{aligned} s\sigma_{s-1}c &= \frac{\Delta_{s-1}}{(\Delta_{s-2})^2}\|\Psi_{s-2}\|^2-\frac{1}{\Delta_{s-1}}\|\Psi_s\|^2 \\ &= \frac{\sigma_{s-1}|a_{s-1}|^2}{|a_{s-2}|^4}2|a_{s-2}|^2-\frac{1}{\sigma_{s-1}|a_{s-1}|^2}\|\Psi_s\|^2, \end{aligned}$$

and hence

$$\|\Psi_s\|^2=|a_{s-1}|^2\left\{2\frac{|a_{s-1}|^2}{|a_{s-2}|^2}-sc\right\}, \tag{3.7}$$

which shows that $\|\Psi_s\|$ is constant on W^2. Since

$$\Psi_s=(\Psi_{s1}, \Psi_{s2})=\Psi_{s1}(1, c_s),$$

(3.6) can be written in components as

$$\{d\log\Psi_{s1}+i(s+1)\omega_{12}-c_s\hat{\omega}_s\}\wedge(\omega_1-i\omega_2)=0,$$

$$\left\{d\log\Psi_{s1}+i(s+1)\omega_{12}+\frac{1}{c_s}\hat{\omega}_s\right\}\wedge(\omega_1-i\omega_2)=0,$$

from which we get

$$\left(c_s+\frac{1}{c_s}\right)\hat{\omega}_s \wedge (\omega_1-i\omega_2)=0 .$$

By the condition (γ), we get

$$c_s+\frac{1}{c_s}=0, \qquad \text{i.e.} \qquad c_s=i \quad \text{or} \quad -i ,$$

hence $\sigma_s=1$ or -1. By means of the above consideration, we may put

$$\Psi_{s,1}=a_s e^{i\theta_s}, \qquad \Psi_{s,2}=i\sigma_s\Psi_{s,1}, \tag{3.8}$$

where a_s is a positive real constant. Then (3.7) can be written as

$$a_s^2=|a_{s-1}|^2\left\{\frac{|a_{s-1}|^2}{|a_{s-2}|^2}-\frac{s}{2}c\right\} \tag{3.9}$$

and (3.6) is equivalent to

$$\{(s+1)\omega_{12}+d\theta_s-\sigma_s\hat{\omega}_s\} \wedge (\omega_1-i\omega_2)=0 ,$$

hence

$$\hat{\omega}_s=\sigma_s\{(s+1)\omega_{12}+d\theta_s\} .$$

Now, if we replace e_{β_1}, e_{β_2}, $\beta_1, \beta_2 \in I_s$, with

$$e^*_{\beta_1}=e_{\beta_1}\cos\theta_s+e_{\beta_2}\sin\theta_s , \qquad e^*_{\beta_2}=-e_{\beta_1}\sin\theta_s+e_{\beta_2}\cos\theta_s ,$$

then we get

$$\hat{\omega}^*_s=\bar{D}e^*_{\beta_1}\cdot e_{\beta_2}=\hat{\omega}_s+d\theta_s .$$

Therefore, if we use only such frames, we may assume that $\theta_s=0$, and then we have

$$\hat{\omega}_s=(s+1)\sigma_s\omega_{12} \tag{3.10}$$

and

$$\Psi_s=a_s(1, i\sigma_s) , \tag{3.11}$$

which is constant on W^2. Furthermore, we have

$$F_s=a_s e_{\beta_1} , \qquad G_s=\sigma_s a_s e_{\beta_2}$$

and so

$$\langle F_s, G_s\rangle=0 . \tag{3.12}$$

From the above argument, we see that there is a submanifold B''_m of B'_m over W^2 on which

$$\Psi_t=a_t(1, i\sigma_t) , \quad a_t>0 , \quad \sigma_t=\pm 1 \qquad \text{for} \quad t=1, 2, \cdots , m ; \tag{3.13}$$

$$a_t^2=a_{t-1}^2\left\{\left(\frac{a_{t-1}}{a_{t-2}}\right)^2-\frac{t}{2}c\right\} \qquad \text{for} \quad t=2, 3, \cdots , m \tag{3.14}$$

and (iii) and (iv) are true, where we put $a_0=1$.

Finally we consider (ii) for $t=m+1$ and (v). First, we suppose that M^n is of S-type. Then, $\Psi_{m+1}=0$. (2.21) for $t=m$ implies

$$c=\frac{2}{m+1}\left(\frac{a_m}{a_{m-1}}\right)^2 \tag{3.15}$$

hence $c>0$.

Second, we suppose that M^n is of T-type, then (2.10) becomes on B''_m

$$a_m(\omega_{\alpha_1\gamma}+i\sigma_m\omega_{\alpha_2\gamma})=\Psi_{m+1}(\omega_1-i\omega_2),$$

where $\alpha_1, \alpha_2\in I_m$, $\gamma=n+2m+1$ and $\Psi_{m+1}=\Psi_{m+1,\gamma}$, from which we get by exterior derivation

$$\{d\Psi_{m+1}+i\Psi_{m+1}(\omega_{12}+\sigma_m\hat{\omega}_m)\}\wedge(\omega_1-i\omega_2)=0. \tag{3.16}$$

On the other hand, from (2.21) for $t=m$, we have

$$|\Psi_{m+1}|^2=a_m^2\left\{2\left(\frac{a_m}{a_{m-1}}\right)^2-(m+1)c\right\}, \tag{3.17}$$

which is (3.7) for $s=m$. Therefore $|\Psi_{m+1}|$ is constant on B''_m. Hence we can put

$$\Psi_{m+1}=a_{m+1}e^{i\varphi}, \qquad a_{m+1}>0, \qquad a_{m+1}=\text{constant} \tag{3.18}$$

and then substituting this into (3.16) we have

$$d\varphi+\omega_{12}+\sigma_m\hat{\omega}_m=0.$$

Since $\hat{\omega}_m=(m+1)\sigma_m\omega_{12}$, this equality becomes

$$d\varphi+(m+2)\omega_{12}=0 \qquad \text{on } B''_m. \tag{3.19}$$

Therefore it must be $d\omega_{12}=0$, hence we get

$$c=0. \tag{3.20}$$

Q. E. D.

§4. The Frenet formulas of W^2 of S-type.

In this section, we shall investigate W^2 in Theorem 2 in detail, in the case that M^n is of S-type.

By means of Theorem 2, we may suppose that

$$c=1, \tag{4.1}$$

in the following. Then, putting

$$\left(\frac{a_t}{a_{t-1}}\right)^2=b_t, \qquad \text{for} \quad t=2, 3, \cdots, m, \tag{4.2}$$

(3.14) and (3.15) can be written as

$$b_t=b_{t-1}-\frac{t}{2}, \qquad t=2, 3, \cdots, m$$

and

$$b_m = \frac{m+1}{2}.$$

Hence we have

(4.3) $$b_t = \frac{(m-t+1)(m+t+2)}{4}, \qquad t=1, 2, \cdots, m.$$

On the other hand, from (1.9), (1.10), (3.4) and (4.1) we have

$$1 = \|P\|^2 + \bar{c} - 2\lambda^2$$

and

$$1 = \lambda^2 - \frac{a_2^2}{\lambda^2},$$

since $\|F_2\|^2 + \|G_2\|^2 = \|\Psi_2\|^2 = 2a_2^2$. Using $\lambda^2 = a_1^2$ and (4.3) for $t=2$, we get easily

(4.4) $$\lambda^2 = \frac{m(m+3)}{4}$$

and

(4.5) $$\|P\|^2 + \bar{c} = \frac{(m+1)(m+2)}{2}.$$

Since (2.7_t) can be written by (3.13) as

$$(1, i\sigma_t)\Omega_t = \sqrt{b_{t+1}}(1, i\sigma_{t+1})(\omega_1 - i\omega_2),$$

we have

(4.6) $$\begin{cases} \omega_{\alpha_1\beta_1} = \sqrt{b_{t+1}}\,\omega_1, & \omega_{\alpha_1\beta_2} = \sigma_{t+1}\sqrt{b_{t+1}}\,\omega_2, \\ \omega_{\alpha_2\beta_1} = -\sigma_t\sqrt{b_{t+1}}\,\omega_2, & \omega_{\alpha_2\beta_2} = \sigma_t\sigma_{t+1}\sqrt{b_{t+1}}\,\omega_1, \end{cases}$$

$$\alpha_1, \alpha_2 \in I_t; \quad \beta_1, \beta_2 \in I_{t+1}; \qquad t=1, 2, \cdots, m-1.$$

Now, we construct the Frenet formulas for W^2. On B_m'', we have

$$\begin{aligned} \bar{D}(e_1+ie_2) &= -i(e_1+ie_2)\omega_{12} + \sum_{r=3}^{n}(\omega_{1r}+i\omega_{2r})e_r \\ &\quad + \lambda\omega_1 e_{n+1} + \mu\omega_2 e_{n+2} - i\lambda\omega_2 e_{n+1} + i\mu\omega_1 e_{n+2} \\ &= -i(e_1+ie_2)\omega_{12} + P(\omega_1+i\omega_2) + \lambda(e_{n+1}+i\sigma e_{n+2})(\omega_1 - i\omega_2) \end{aligned}$$

by (1.5), (1.6), i.e.

(4.7) $$\bar{D}(e_1+ie_2) = -i(e_1+ie_2)\omega_{12} + P(\omega_1+i\omega_2) + \lambda(e_{n+1}+i\sigma_1 e_{n+2})(\omega_1 - i\omega_2).$$

By (4.5), $\|P\|$ is constant on W^2. If $P \neq 0$, we can use frames $b \in B_m''$ such that

(4.8) $$P = pe_3.$$

Then we have easily

(4.9) $$\bar{D}e_3 = -p(e_1\omega_1 + e_2 i\omega_2).$$

Therefore, in general we have

$$\bar{D}P=-p^2(e_1\omega_1+e_2\omega_2)\,. \tag{4.10}$$

Analogously, using (4.6) we have

$$\begin{aligned}\bar{D}(e_{\alpha_1}+i\sigma_t e_{\alpha_2})=&-\sqrt{b_t}(e_{\gamma_1}+i\sigma_{t-1}e_{\gamma_2})(\omega_1+i\omega_2)-i\sigma_t(e_{\alpha_1}+i\sigma_t e_{\alpha_2})\,\hat{\omega}_t\\&+\sqrt{b_{t+1}}(e_{\beta_1}+i\sigma_{t+1}e_{\beta_2})(\omega_1-i\omega_2)\,,\end{aligned} \tag{4.11}$$

for $\alpha_1,\alpha_2\in I_t$; $\beta_1,\beta_2\in I_{t+1}$; $\gamma_1,\gamma_2\in I_{t-1}$; $t=1,2,\cdots,m-1$

and

$$\bar{D}(e_{\alpha_1}+i\sigma_m e_{\alpha_2})=-\sqrt{b_m}(e_{\gamma_1}+i\sigma_{m-1}e_{\gamma_2})(\omega_1+i\omega_2)-i\sigma_m(e_{\alpha_1}+i\sigma_m e_{\alpha_2})\hat{\omega}_m\,, \tag{4.12}$$

for $\alpha_1,\alpha_2\in I_m$, $\gamma_1,\gamma_2\in I_{m-1}$.

Now, let S^2 be a unit sphere with the line element

$$ds^2=\frac{4dzd\bar{z}}{(1+z\bar{z})^2} \tag{4.13}$$

with respect to the complex coordinate z and ω_1^*, ω_2^* and ω_{12}^* the basic and connection forms respectively given by

$$\omega_1^*+i\omega_2^*=\frac{2dz}{1+z\bar{z}}\,, \tag{4.14}$$

$$\omega_{12}^*=i\,\frac{\bar{z}dz-zd\bar{z}}{1+z\bar{z}}\,. \tag{4.15}$$

Since W^2 is locally isometric to S^2, we may regard z as a local isothermal coordinate of W^2, and then we can put

$$\omega_1+i\omega_2=e^{-i\theta}(\omega_1^*+i\omega_2^*)\,,\qquad \omega_{12}=\omega_{12}^*+d\theta\,.$$

If we put

$$\xi_t=e^{i(t+1)\theta}(e_{\alpha_1}+i\sigma_t e_{\alpha_2})\,, \tag{4.16}$$

where $\alpha_1,\alpha_2\in I_t$; $t=0,1,\cdots,m$; $I_0=\{1,2\}$, $\sigma_0=1$, then by (iv) of Theorem 2, (4.7), (4.10), (4.11) and (4.12) can be written as follows:

$$\begin{cases}\bar{D}\xi_0=\frac{1}{h}\xi_0(\bar{z}dz-zd\bar{z})+\frac{2}{h}Pdz+\frac{2\sqrt{b_1}}{h}\xi_1 d\bar{z}\,,\\[2mm] \bar{D}\xi_1=-\frac{2}{h}\sqrt{b_1}\xi_0dz+\frac{2}{h}\xi_1(\bar{z}dz-zd\bar{z})+\frac{2\sqrt{b_2}}{h}\xi_2d\bar{z}\,,\\ \vdots\\ \bar{D}\xi_t=-\frac{2\sqrt{b_t}}{h}\xi_{t-1}dz+\frac{t+1}{h}\xi_t(\bar{z}dz-zd\bar{z})+\frac{2\sqrt{b_{t+1}}}{h}\xi_{t+1}d\bar{z}\,,\\ \vdots\\ \bar{D}\xi_m=-\frac{2\sqrt{b_m}}{h}\xi_{m-1}dz+\frac{m+1}{h}\xi_m(\bar{z}dz-zd\bar{z})\,,\\[2mm] \bar{D}P=-\frac{p^2}{h}(\bar{\xi}_0dz+\xi_0d\bar{z})\,,\end{cases} \tag{4.17}$$

where

(4.18) $$h=1+z\bar{z}.$$

We have also for the moving point x of W^2

(4.19) $$dx=\frac{1}{h}(\bar{\xi}_0dz+\xi_0d\bar{z}).$$

(4.19) and (4.17) constitute the Frenet formulas of W^2.

REMARK. Now, we shall give a remark on the conditions in Theorem 2. As is easily seen, the condition (γ) out of them is a little artificial from the differential geometrical point of view, because $\hat{\omega}_t$ is the connection form of the vector bundle $N^{(t)}$ over M^n and so $\hat{\omega}_t\neq 0$ has, in general, no intrinsic meaning. But, in the case of S-type, we can replace (γ) by the intrinsic condition:

(γ') $$d\hat{\omega}_t\neq 0 \qquad \text{for } t=1, 2, \cdots, m,$$

in terms of the curvature forms $d\hat{\omega}_t$ of $N^{(t)}$.

THEOREM 3. *Let M^n be a minimal submanifold of $\bar{M}^{n+2m}$ with g-codim $2m$ as in Theorem C. If a base surface W^2 of M^n satisfies the conditions:*

(α) $$\Psi_{t2}/\Psi_{t1}=c_t \text{ is constant on } W^2 \text{ for } t=1, \cdots, m,$$

(β') $$W^2 \text{ is of constant curvature } c\neq 0,$$

then $c>0$ and supposing $c=1$, there exist m complex normal vector fields ξ_1, $\cdots$, ξ_m defined on W^2 such that

(I) $$\begin{cases} \xi_t\cdot\xi_s=\xi_t\cdot\bar{\xi}_s=0, & t\neq s, \\ \xi_t\cdot\xi_t=0, \quad \xi_t\cdot\bar{\xi}_t=2, & t, s=1, 2, \cdots, m \end{cases}$$

and

$$dx=\frac{1}{h}(\bar{\xi}_0dz+\xi_0d\bar{z}),$$

(II) $$\begin{cases} \bar{D}\xi_0=\frac{1}{h}\xi_0(\bar{z}dz-zd\bar{z})+\frac{2}{h}Pdz+\frac{2\sqrt{b_1}}{h}\xi_1d\bar{z}, \\ \bar{D}\xi_1=-\frac{2\sqrt{b_1}}{h}\xi_0dz+\frac{2}{h}\xi_1(\bar{z}dz-zd\bar{z})+\frac{2\sqrt{b_2}}{h}\xi_2d\bar{z}, \\ \vdots \\ \bar{D}\xi_t=-\frac{2\sqrt{b_t}}{h}\xi_{t-1}dz+\frac{t+1}{h}\xi_t(\bar{z}dz-zd\bar{z})+\frac{2\sqrt{b_{t+1}}}{h}\xi_{t+1}d\bar{z}, \\ \vdots \\ \bar{D}\xi_m=-\frac{2\sqrt{b_m}}{h}\xi_{m-1}dz+\frac{m+1}{h}\xi_m(\bar{z}dz-zd\bar{z}), \\ \bar{D}P=-\frac{\|P\|^2}{h}(\bar{\xi}_0dz+\xi_0d\bar{z}), \end{cases}$$

where z is an isothermal complex coordinate of W^2 and $h=1+z\bar{z}$, $\xi_0=e_1+ie_2$.

PROOF. It will be sufficient to prove that condition (γ) is satisfied, by the argument in this section. We shall give a brief proof, referring to the proof of Theorem 2.

From (β') and (1.9), we have

$$\lambda^2+\mu^2=\lambda^2(1+\sigma^2)=\|P\|^2+\bar{c}-c$$

must be constant on W^2, hence λ and μ are also constant. From (1.7), we get

$$\hat{\omega}_1=\frac{2}{\sigma}\omega_{12}.$$

Since $c\neq 0$, we have $d\hat{\omega}_1\neq 0$ and so $\hat{\omega}_1\neq 0$ locally. Hence, we get (3.2) from (3.1) and also (3.4) from (1.7).

Now, take $s\leqq m$ and suppose that (i), (ii), (iii) and (iv) in Theorem 2 are true for $t<s$. Then, we have (3.7) and so $\|\Psi_s\|$ is constant on W^2. Since $\|\Psi_s\|^2=|\Psi_{s1}|^2(1+|c_s|^2)$, $|\Psi_{s1}|$ is non-zero constant. Putting $\Psi_{s1}=a_s e^{i\theta_s}$, (3.6) becomes

$$(i\{(s+1)\omega_{12}+d\theta_s\}-c_s\hat{\omega}_s)\wedge(\omega_1-i\omega_2)=0,$$

$$\left(i\{(s+1)\omega_{12}+d\theta_s\}+\frac{1}{c_s}\hat{\omega}_s\right)\wedge(\omega_1-i\omega_2)=0.$$

Putting $c_s=re^{i\alpha}$ ($r>0$), from these equalities we get

$$2r\sin\alpha\hat{\omega}_s=(1+r^2)\{(s+1)\omega_{12}+d\theta_s\}$$

and so

$$d\hat{\omega}_s=\frac{1+r^2}{2r\sin\alpha}(s+1)d\omega_{12}\neq 0,$$

since $\sin\alpha\neq 0$ by Lemma 4. Hence it must be $\hat{\omega}_s\neq 0$ locally. On the other hand, we have

$$\left(c_s+\frac{1}{c_s}\right)\hat{\omega}_s\wedge(\omega_1-i\omega_2)=0$$

from (3.6) and so it must be $c_s+\frac{1}{c_s}=0$. From the above equality, we get

$$\hat{\omega}_s=\sigma_s\{(s+1)\omega_{12}+d\theta_s\}.$$

The remaining argument is completely analogous to the proof of Theorem 2. Q.E.D.

§ 5. Determination of W^2 of S-type in E^{n+2m}.

In this section, we shall give the solutions of the equations (II) in Theorem 3 in the case $\bar{M}^{n+2m}=E^{n+2m}$. Then, we have

(5.1) $$p=\|P\|=\sqrt{\frac{(m+1)(m+2)}{2}}$$

by (4.5). Since the operator $\bar{D}$ become the ordinary differential operator in E^{n+2m}, we get

$$\frac{\partial\xi_m}{\partial\bar{z}}=-\frac{m+1}{h}z\xi_m,$$

hence we can put

(5.2) $$\xi_m=\frac{1}{h^{m+1}}F(z),$$

where $F(z)$ is a holomorphic vector function of z. Substituting (5.2) in (II) and using (4.3) we obtain

(5.3) $$\xi_{m-1}=\frac{\sqrt{2(m+1)}\bar{z}}{h^{m+1}}F(z)-\frac{1}{\sqrt{2(m+1)}h^m}F'(z).$$

Substituting (5.2) and (5.3) in (II), we obtain analogously

(5.4) $$\xi_{m-2}=\frac{\sqrt{(m+1)(2m+1)}\bar{z}^2}{h^{m+1}}F(z)-\sqrt{\frac{2m+1}{m+1}}\frac{\bar{z}}{h^m}F'(z)$$
$$+\frac{1}{2\sqrt{(2m+1)(m+1)}h^{m-1}}F''(z).$$

Now, in general, from (II), we have

(5.5) $$\xi_{t-1}=-\frac{h^{t+2}}{2\sqrt{b_t}}\frac{\partial}{\partial z}\left(\frac{\xi_t}{h^{t+1}}\right)\qquad\text{for } t=m,\ m-1,\cdots,1,$$

and

(5.6) $$e_3=\frac{P}{p}=\frac{h^2}{2p}\frac{\partial}{\partial z}\left(\frac{\xi_0}{h}\right).$$

Comparing the forms of the right hand sides of (5.2), (5.3) and (5.4) and using the equalities (5.5), we see

$$\xi_t=\Big[\frac{\bar{z}^{m-t}}{h^{m+1}}F,\ -\frac{\bar{z}^{m-t-1}}{h^m}F',\ \cdots,\ (-1)^{m+1-s}\frac{\bar{z}^{s-t-1}}{h^t}F^{(m+1-s)},$$
$$\cdots,\ (-1)^{m-t}\frac{1}{h^{t+1}}F^{(m-t)}\Big]_{R^+},\qquad t=m,\ m-1,\cdots,0,$$

by induction, where $[\ \ ]_{R^+}$ means a linear combination of the vector functions in the brace with positive constants. Therefore, we have also

(5.7) $$-e_3=\Big[\frac{\bar{z}^{m+1}}{h^{m+1}}F,\ -\frac{\bar{z}^m}{h^m}F',\ \cdots,\ (-1)^m\frac{\bar{z}}{h}F^{(m)},\ (-1)^{m+1}F^{(m+1)}\Big]_{R^+}.$$

Since e_3 is a real vector field and $h=1+z\bar{z}$ is real, $h^{m+1}e_3$ is also a real vector field and its components must be polynomials in $\bar{z}$ and also z of order at most $m+1$ respectively. Using the fact that h^t is a polynomial in z and $\bar{z}$ and of order t with respect to z and $\bar{z}$ respectively and its constant term is 1, for $t=1, \cdots, m+1$, we have from (5.7)

$$-h^{m+1}e_3 \equiv [(-1)^{m+1}F^{(m+1)}(z)]_{R^+} \ (\text{mod}\ \bar{z}, \bar{z}^2, \cdots, \bar{z}^{m+1}).$$

Hence, $F^{(m+1)}(z)$ and so $F(z)$ must be polynomial vector functions in z. According to this fact, we can put

$$F(z) = A_0 + A_1 z + \cdots + A_l z^l, \tag{5.8}$$

where $A_0, A_1, \cdots, A_l$ are constant complex vectors in C^{m+2}.

Now, for simplicity, setting

$$C_t = 2^{m-t}\sqrt{b_m b_{m-1} \cdots b_{t+1}}\,, \qquad t=0, 1, \cdots, m-1 \tag{5.9}$$

and

$$C_m = 1,$$

define the following vector functions:

$$H_t(z, \bar{z}) = C_t h^{m+1}\xi_t. \tag{5.10}$$

Then, by (5.5) we have

$$\begin{aligned} H_{t-1} &= \bar{z}\left\{(m+t+2)H_t - z\frac{\partial H_t}{\partial z}\right\} - \frac{\partial H_t}{\partial z} \\ &= -h^{t+m+3}\frac{\partial}{\partial z}\left(\frac{H_t}{h^{t+m+2}}\right), \qquad \text{for } t=m, m-1, \cdots, 2, 1. \end{aligned} \tag{5.11}$$

Since

$$H_m = F(z) = \sum_{j=0}^{l} A_j z^j,$$

we have from (5.11) for $t=m$

$$H_{m-1} = \sum_{j=0}^{l} \{(2m+2-j)\bar{z}z^j - jz^{j-1}\}A_j. \tag{5.12}$$

Analogously, from (5.11) for $t=m-1$ and (5.12) we get

$$\begin{aligned} H_{m-2} = \sum_{j=0}^{l} \{&(2m+2-j)(2m+1-j)\bar{z}^2 z^j \\ &-2j(2m+2-j)\bar{z}z^{j-1} + j(j-1)z^{j-2}\}A_j. \end{aligned}$$

Noticing the forms of these equalities, we get by induction

$$\begin{aligned} H_t = (m-t)!\sum_{j=0}^{l}\Big\{&\binom{2m+2-j}{m-t}\bar{z}^{m-t}z^j - \binom{2m+2-j}{m-t-1}\binom{j}{1}\bar{z}^{m-t-1}z^{j-1} + \\ &\cdots + (-1)^s\binom{2m+2-j}{m-t-s}\binom{j}{s}\bar{z}^{m-t-s}z^{j-s} + \cdots + (-1)^{m-t}\binom{j}{m-t}z^{j-m+t}\Big\}A_j \\ &\text{for } t=m, m-1, \cdots, 0. \end{aligned} \tag{5.13}$$

In particular, we have

$$H_0 = m!\sum_{j=0}^{l}\left\{\sum_{s=0}^{m}(-1)^s\binom{2m+2-j}{m-s}\binom{j}{s}\bar{z}^{m-s}z^{j-s}\right\}A_j \tag{5.14}$$

and

$$\xi_0 = \frac{1}{C_0 h^{m+1}} H_0(z, \bar{z}). \tag{5.15}$$

In these formulas (5.13), (5.14) and so on, we have used the convention $\binom{j}{s}=0$ for $s>j$. From (5.9) and (4.3), we have easily

$$C_0 = \sqrt{\frac{(2m+2)!}{(m+1)(m+2)}}. \tag{5.16}$$

Finally, from (5.6) and (5.14), we get

$$P = \frac{h^2}{2}\frac{\partial}{\partial z}\left(\frac{H_0}{C_0 h^{m+2}}\right) = \frac{-1}{2C_0 h^{m+1}}\left\{-h^{m+3}\frac{\partial}{\partial z}\left(\frac{H_0}{h^{m+2}}\right)\right\}.$$

Thus, noticing (5.11) and (5.13), we have

$$P = -\frac{\sqrt{(m+1)(m+2)}\,(m+1)!}{2\sqrt{(2m+2)!}\;h^{m+1}} \times \sum_{j=0}^{l}\left\{\sum_{s=0}^{m+1}(-1)^s\binom{2m+2-j}{m-s+1}\binom{j}{s}\bar{z}^{m-s+1}z^{j-s}\right\}A_j. \tag{5.17}$$

Now, we search for the condition that P is real. Looking (5.17), we consider the vector function

$$G_m(z, \bar{z}) = \sum_{j=0}^{l}\left\{\sum_{s=0}^{m+1}(-1)^s\binom{2m+2-j}{m-s+1}\binom{j}{s}\bar{z}^{m-s+1}z^{j-s}\right\}A_j$$

and find the condition that G_m is real. Since

$$G_m(z, \bar{z}) = \sum_{s=0}^{m+1}(-1)^s\bar{z}^{m-s+1}\sum_{j=s}^{s+m+1}\binom{2m+2-j}{m-s+1}\binom{j}{s}z^{j-s}A_j, \tag{5.18}$$

we have

$$\begin{aligned}\overline{G_m(z, \bar{z})} &= \sum_{s=0}^{m+1}(-1)^s z^{m-s+1}\sum_{j=s}^{s+m+1}\binom{2m+2-j}{m-s+1}\binom{j}{s}\bar{z}^{j-s}\bar{A}_j \\ &= \sum_{s=0}^{m+1}(-1)^s\bar{z}^{m-s+1}\sum_{j=s}^{s+m+1}(-1)^{m+1-j}\binom{2m+2-j}{m-s+1}\binom{j}{s}z^{j-s}\bar{A}_{2m+2-j}\end{aligned}$$

by suitable changes of indices. Hence, from the condition $\overline{G_m(z, \bar{z})} = G_m(z, \bar{z})$, we obtain the conditions

$$A_j = (-1)^{m+1-j}\bar{A}_{2m+2-j}, \qquad j = 0, 1, \cdots, 2m+2 \tag{5.19}$$

and in particular

$$A_{m+1} = \bar{A}_{m+1}.$$

And, we get also $l=2m+2$. Substituting (5.19) into (5.18), and changing the indices, we have

$$G_m(z,\bar z)=\sum_{j=0}^{m}(-1)^j\Big\{\sum_{s=0}^{j}(-1)^s\binom{2m+2-j}{m+1-s}\binom{j}{s}(z\bar z)^s\Big\} \times(\bar z^{m+1-j}A_j+z^{m+1-j}\bar A_j)+(-1)^{m+1}\sum_{s=0}^{m+1}(-1)^s\binom{m+1}{s}^2(z\bar z)^s A_{m+1}. \tag{5.20}$$

At last, we have to investigate the condition (I) in Theorem 3. Looking into the right hand sides of (II) in Theorem 3, we see that the condition (I) will be satisfied identically, if it is true at $z=0$, as in the case of the curve theory.

From (5.10), (5.13) and (4.3), we have

$$\xi_t|_{z=0}=\frac{1}{C_t}H_t(0,0)=\frac{(-1)^{m-t}\sqrt{(m-t)!}}{\sqrt{(2m+2)(2m+1)\cdots(m+t+3)}}A_{m-t}, \tag{5.21}$$
$$\text{for } t=0,1,\cdots,m-1,$$

and

$$\xi_m|_{z=0}=H_m(0,0)=A_0 \tag{5.22}$$

and from (5.17) and (5.20) we get

$$P|_{z=0}=\frac{(-1)^m(m+1)\sqrt{m!}}{2\sqrt{(2m+2)(2m+1)\cdots(m+3)}}A_{m+1}. \tag{5.23}$$

Therefore, the condition (I) can be replaced by the following (III) for solutions of (II) in the present case:

$$\text{(III)}\quad\begin{cases} A_t\cdot A_t=0,\ t=0,1,\cdots,m;\ A_{m+1}=\bar A_{m+1};\\ A_t\cdot A_s=A_t\cdot\bar A_s=0,\qquad t\neq s,\ t,s=0,1,\cdots,m+1;\\ A_0\cdot\bar A_0=2,\ A_t\cdot\bar A_t=2\cdot\dfrac{(2m+2)(2m+1)\cdots(2m-t+3)}{t!}=2\dbinom{2m+2}{t}, \end{cases}$$
$$t=1,2,\cdots,m,m+1.$$

Thus, through these arguments, we reach the stage to give the exact form of W^2. From (II), we get easily

$$x+\frac{1}{p^2}P=\text{a fixed point in } E^{2m+3}=C^{m+1}\times R.$$

We may suppose that this fixed point for W^2 is the origin of E^{2m+3}. Then, from (5.17), (5.20) and (5.1) we get

$$x=\frac{\sqrt{m!}}{(m+2)\sqrt{(2m+2)(2m+1)\cdots(m+3)}\,(1+z\bar z)^{m+1}} \times\Big[\sum_{j=0}^{m}(-1)^j\{\sum_{s=0}^{j}(-1)^s\binom{2m+2-j}{m+1-s}\binom{j}{s}(z\bar z)^s\}(\bar z^{m+1-j}A_j+z^{m+1-j}\bar A_j) \tag{5.24}$$

$$+(-1)^{m+1}\sum_{s=0}^{m+1}(-1)^s\binom{m+1}{s}^2(z\bar{z})^s A_{m+1}\Big].$$

We denote this isometrically immersed surface from the unit sphere into E^{2m+3} by W_m^2. According to Theorem C, we can construct an n-dimensional minimal submanifold M^n in E^{n+2m}. That is, first of all, we make the following identification:

$$E^{n+2m}=R^{n-4}\times R^{2m+4},\ R^{2m+4}\cong C^{m+2}$$

and

$$R^{2m+4}=E^{2m+3}\times R\,.$$

We take a W_m^2 in E^{2m+3} as stated above. For any point $y\in W_m^2$, let $L^{n-2}(y)$ be a linear subspace through y such that

$$L^{n-2}(y)\| R^{n-4}\times R \quad\text{and}\quad L^{n-2}(y)\| P(y)\,,\qquad y=x(z)\,.$$

Then, the locus of points on the moving $L^{n-2}(y)$ is an n-dimensional submanifold in E^{n+2m} as in Theorem. We denote this submanifold by $M_m^n(S, E)$.

Thus, by virtue of the argument in this section, we have

THEOREM 4. *Let M^n be a minimal submanifold in E^{n+2m} with g-codim $2m$ as in Theorem C. If a base surface W^2 of M^n satisfies the conditions (α), (β') in Theorem 3 and $c=1$, then M^n is locally congruent with $M_m^n(S, E)$ under the motions of E^{n+2m}.*

REMARK 1. W_m^2 given by (5.24) in E^{2m+3} clearly lies in the $(2m+2)$-sphere with radius $\dfrac{1}{p}=\sqrt{\dfrac{2}{(m+1)(m+2)}}$ and it is minimal and of g-codim $2m$ in the sphere. Enlarging these figures by the similarity of magnification p, we get a surface with analogous properties in the unit sphere S^{2m+2}, which we denote V_m^2. We shall give the equations of V_m^2 in the canonical coordinates $x_0, x_1, \cdots, x_{2m+2}$ of E^{2m+3}. By means of (III), we put

$$A_{m+1}=(-1)^{m+1}\sqrt{2\cdot\binom{2m+2}{m+1}}\,\partial/\partial x_0\,,$$

$$A_m=(-1)^m\sqrt{\binom{2m+2}{m}}\,(\partial/\partial x_1+i\partial/\partial x_2)\,,$$

$$\vdots$$

$$A_t=(-1)^t\sqrt{\binom{2m+2}{t}}\left(\frac{\partial}{\partial x_{2m-2t+1}}+i\frac{\partial}{\partial x_{2m-2t+2}}\right),$$

$$\vdots$$

$$A_1=-\sqrt{2m+2}\left(\frac{\partial}{\partial x_{2m-1}}+i\frac{\partial}{\partial x_{2m}}\right),$$

$$A_0=\left(\frac{\partial}{\partial x_{2m+1}}+i\frac{\partial}{\partial x_{2m+2}}\right),$$

then V_m^2 is given by the following equations:

$$
(5.25)\quad \begin{cases}
x_0 = \dfrac{1}{(1+z\bar{z})^{m+1}} \displaystyle\sum_{s=0}^{m+1} (-1)^s \binom{m+1}{s}^2 (z\bar{z})^s , \\
x_{2t-1} = \dfrac{1}{(1+z\bar{z})^{m+1}} \sqrt{\dfrac{(m+1)m \cdots (m-t+2)}{2\cdot(m+t+1)(m+t)\cdots(m+2)}} \\
\qquad \times \displaystyle\sum_{s=0}^{m-t+1} (-1)^s \binom{m+t+1}{s+t}\binom{m-t+1}{s} (z\bar{z})^s (z^t+\bar{z}^t) , \\
x_{2t} = \dfrac{-i}{(1+z\bar{z})^{m+1}} \sqrt{\dfrac{(m+1)m \cdots (m-t+2)}{2\cdot(m+t+1)(m+t)\cdots(m+2)}} \\
\qquad \times \displaystyle\sum_{s=0}^{m-t+1} (-1)^s \binom{m+t+1}{s+t}\binom{m-t+1}{s} (z\bar{z})^s (z^t-\bar{z}^t) ,
\end{cases}
$$

$$
t = 1, 2, \cdots, m+1 .
$$

If we consider especially the case $m=1$, then V_1^2 in S^4 is given by the equations:

$$
x_0 = \frac{1}{(1+z\bar{z})^2} (1-4z\bar{z}+z^2\bar{z}^2) ,
$$

$$
x_1 = \frac{\sqrt{3}}{(1+z\bar{z})^2} (1-z\bar{z})(z+\bar{z}) ,
$$

$$
x_2 = \frac{-i\sqrt{3}}{(1+z\bar{z})^2} (1-z\bar{z})(z-\bar{z}) ,
$$

$$
x_3 = \frac{\sqrt{3}}{(1+z\bar{z})^2} (z^2+\bar{z}^2) ,
$$

$$
x_4 = \frac{-i\sqrt{3}}{(1+z\bar{z})^2} (z^2-\bar{z}^2) .
$$

These equations show that our V_1^2 coincides with the so called Veronese surface. Therefore, we may call the minimal surface V_m^2 in S^{2m+2} the *generalized Veronese surface* of index m.

REMARK 2. The generalized Veronese surface V_m^2 can be considered as isometric immersion of P^2 with the standard metric into S^{2m+2} for odd m and into P^{2m+2} with the standard metric for even m. In fact, if we replace z by $-\dfrac{1}{\bar{z}}$ and $\bar{z}$ by $-\dfrac{1}{z}$ in the vector functions in the right hand side of (5.24):

$$
\frac{1}{(1+z\bar{z})^{m+1}} \sum_{s=0}^{j} (-1)^s \binom{2m+2-j}{m+1-s}\binom{j}{s} (z\bar{z})^s (\bar{z}^{m+1-j}A_j + z^{m+1-j}\bar{A}_j) ,
$$

we get

$$
\frac{(z\bar{z})^{m+1}}{(1+z\bar{z})^{m+1}} \sum_{s=0}^{j} (-1)^s \binom{2m+2-j}{m+1-s}\binom{j}{s} \frac{(-1)^{m+1-j}}{(z\bar{z})^s} \left(\frac{1}{z^{m+1-j}} A_j + \frac{1}{\bar{z}^{m+1-j}} \bar{A}_j \right)
$$

$$
= \frac{(-1)^{m+1}}{(1+z\bar{z})^{m+1}} \sum_{s=0}^{j} (-1)^{j-s} \binom{2m+2-j}{m+1-j+s}\binom{j}{j-s} (z\bar{z})^{j-s} (\bar{z}^{m+1-j}A_j + z^{m+1-j}\bar{A}_j) ,
$$

which guarantees the above assertion.

§ 6. Determination of W^2 of S-type in S^{n+2m} and H^{n+2m}.

Continuing to § 5, we shall try to get the solutions of the equations (II) in Theorem 3 for the case $\bar{M}^{n+2m}$ is a sphere or hyperbolic space form.

CASE: $\bar{M}^{n+2m}=S^{n+2m}(R)$.

$S^{n+2m}(R)$ denotes the $(n+2m)$-sphere of radius R and we suppose $\frac{1}{R^2}=\bar{c}$.

According to the method in [**9**], we regard as $S^{n+2m}(R)\subset E^{n+2m+1}$. Putting

$$\frac{x}{R}=e_{n+2m+1}, \tag{6.1}$$

we have by (4.19)

$$dx=Rd\,e_{n+2m+1}=\frac{1}{h}(\bar{\xi}_0dz+\xi_0d\bar{z}).$$

On the base surface W^2 in Theorem 3 in the present case, we have easily

$$\begin{aligned}
d\xi_0 &= \frac{1}{h}\xi_0(\bar{z}dz-zd\bar{z})+\frac{2}{h}Pdz+\frac{2\sqrt{b_1}}{h}\xi_1d\bar{z}-\frac{2}{Rh}e_{n+2m+1}dz,\\
d\xi_1 &= -\frac{2\sqrt{b_1}}{h}\xi_0dz+\frac{2}{h}\xi_1(\bar{z}dz-zd\bar{z})+\frac{2\sqrt{b_2}}{h}\xi_2d\bar{z},\\
&\vdots\\
d\xi_t &= -\frac{2\sqrt{b_t}}{h}\xi_{t-1}dz+\frac{t+1}{h}\xi_t(\bar{z}dz-zd\bar{z})+\frac{2\sqrt{b_{t+1}}}{h}\xi_{t+1}d\bar{z},\\
&\vdots\\
d\xi_m &= -\frac{2\sqrt{b_m}}{h}\xi_{m-1}dz+\frac{m+1}{h}\xi_m(\bar{z}dz-zd\bar{z}),\\
dP &= -\frac{\|P\|^2}{h}(\bar{\xi}_0dz+\xi_0d\bar{z}),
\end{aligned} \tag{6.2}$$

from (II) and the above relation, where d denotes the ordinary differential operator in E^{n+2m+1}.

On the other hand, we have

$$p=\sqrt{\bar{c}}\cot\sqrt{\bar{c}}\,v=\frac{1}{R}\cot\frac{v}{R}. \tag{6.3}$$

We see easily that the point $x+\frac{1}{p^2}P$ and the unit vector

$$e_0=e_{n+2m+1}\cos\frac{v}{R}+e_3\sin\frac{v}{R},\qquad e_3=\frac{P}{p}$$

are fixed on W^2. Hence W^2 lies in the linear space E_1^{n+2m} through the point $O_1=e_0R\cos\frac{v}{R}$ and orthogonal to e_0. Now, setting

$$e_3^* = e_3 \cos \frac{v}{R} - e_{n+2m+1} \sin \frac{v}{R}, \tag{6.4}$$

we have

$$\overrightarrow{O_1 x} = -e_3^* R \sin \frac{v}{R}, \qquad P - \frac{1}{R} e_{n+2m+1} = \frac{1}{R \sin \frac{v}{R}} e_3^*$$

and

$$\|P\|^2 + \bar{c} = \frac{1}{R^2 \sin \frac{v}{R}} = \frac{(m+1)(m+2)}{2}$$

by (4.5) and (6.3). Hence the first equality of (6.2) can be written as

$$d\xi_0 = \frac{1}{h} \xi_0(\bar{z}dz - zd\bar{z}) + \frac{2}{h} P^* dz + \frac{2\sqrt{b_1}}{h} \xi_1 d\bar{z}, \tag{6.5}$$

where

$$P^* = \sqrt{\frac{(m+1)(m+2)}{2}}\, e_3^*$$

and we get easily

$$dP^* = -\frac{\|P^*\|^2}{h}(\bar{\xi}_0 dz + \xi_0 d\bar{z}). \tag{6.6}$$

Thus, we see that (6.2) turns out to be the system of equations (II) in E_1^{n+2m}, if we replace the first and last ones by (6.5) and (6.6) respectively.

In conclusion, we can construct a minimal submanifold M^n in $S^{n+2m}(R)$ in Theorem 3 as follows: We take a W_m^2 in E_1^{n+2m}. Let $L_1^{n-2}(y)$ be the $(n-2)$-dimensional linear subspace in E_1^{n+2m} as described in § 5. Then, the image $L^{n-2}(y)$ of $L_1^{n-2}(y)$ under the projection $E^{n+2m+1} \to S^{n+2m}(R)$ from the origin of E^{n+2m+1} is an $(n-2)$-dimensional great sphere of $S^{n+2m}(R)$. The locus of points on the moving $L^{n-2}(y)$ as y varies is the n-dimensional submanifold in $S^{n+2m}(R)$ to be constructed in this case. We denote it by $M_m^n(S, S)$.

CASE: $\bar{M}^{n+2m} = H^{n+2m}(\bar{c})$.

In this case, $H^{n+2m}(\bar{c})$ is the $(n+2m)$-dimensional hyperbolic space of curvature $\bar{c}$, (4.5) and (1.12) imply

$$-\bar{c} = \frac{(m+1)(m+2)}{2} \sinh^2 \sqrt{-\bar{c}}\, v. \tag{6.7}$$

Now, we use the Poincaré representation of $H^{n+2m}(\bar{c})$ in the unit disk in E^{n+2m} with the canonical coordinates $x_1, x_2, \cdots, x_{n+2m}$ and the metric

$$ds^2 = \frac{4R^2 dx \cdot dx}{(1 - x \cdot x)^2}, \qquad R = \sqrt{\frac{1}{-\bar{c}}}, \tag{6.8}$$

where $x \cdot x = \sum_j x_j x_j$. For any tangent vector X and Y, we have

$$\langle X, Y \rangle = \frac{4R^2}{L^2} X \cdot Y, \qquad L = 1 - x \cdot x,$$

where $\langle X, Y\rangle$ and $X\cdot Y$ denote the inner products of X and Y in $H^{n+2m}(\bar{c})$ and E^{n+2m}. For any tangent vector field X, we have

$$\bar{D}X=\frac{L}{2R}\Big[d\Big(\frac{2R}{L}X\Big)+\frac{2}{L}\Big\{\Big(x\cdot\frac{2R}{L}X\Big)dx-x\Big(\frac{2R}{L}X\cdot dx\Big)\Big\}\Big]. \tag{6.9}$$

Putting

$$e_3^*=\frac{2R}{L}e_3\,,\qquad \xi_t^*=\frac{2R}{L}\xi_t\,,\qquad t=0, 1, \cdots, m\,, \tag{6.10}$$

we rewrite (II) in Theorem 3 in these terms, by using (6.9). Then, we get

$$dx=\frac{L}{2Rh}(\bar{\xi}_0^*dz+\xi_0^*d\bar{z})\,, \tag{6.11}$$

$$d\xi_0^*=\frac{1}{h}\xi_0^*(\bar{z}dz-zd\bar{z})+\frac{2}{h}\Big(pe_3^*+\frac{1}{R}x\Big)dz+\frac{2\sqrt{b_1}}{h}\xi_1^*d\bar{z}\,, \tag{6.12}$$
$$-\frac{1}{Rh}(x\cdot\xi_0^*)(\bar{\xi}_0^*dz+\xi_0^*d\bar{z})\,,$$

$$d\xi_t^*=-\frac{2\sqrt{b_t}}{h}\xi_{t-1}^*dz+\frac{t+1}{h}\xi_t^*(\bar{z}dz-zd\bar{z})+\frac{2\sqrt{b_{t+1}}}{h}\xi_{t+1}^*d\bar{z} \tag{6.13}$$
$$-\frac{1}{Rh}(x\cdot\xi_t^*)(\bar{\xi}_0^*dz+\xi_0^*d\bar{z})\,,\qquad t=1, 2, \cdots, m-1\,,$$

$$d\xi_m^*=-\frac{2\sqrt{b_m}}{h}\xi_{m-1}^*dz+\frac{m+1}{h}\xi_m^*(\bar{z}dz-zd\bar{z}) \tag{6.14}$$
$$-\frac{1}{Rh}(x\cdot\xi_m^*)(\bar{\xi}_0^*dz+\xi_0^*d\bar{z})\,,$$

$$de_3^*=-\Big\{p+\frac{1}{R}(x\cdot e_3^*)\Big\}\frac{1}{h}(\bar{\xi}_0^*dz+\xi_0^*d\bar{z})\,. \tag{6.15}$$

On the other hand, any geodesic starting from the origin $O(0, \cdots, 0)$ in $H^{n+2m}(\bar{c})$ is a Euclidean straight line segment in the unit disk. The arc lengths v and r in $H^{n+2m}(\bar{c})$ and E^{n+2m} have the relations:

$$v=R\log\frac{1+r}{1-r}\,,\qquad r=\tanh\frac{v}{2R}\,.$$

Since any W^2 is congruent to others under the hyperbolic motions, we may suppose the focal point z_0 in Theorem C is the origin O. Then, we have

$$x=-e_3^*r=-e_3^*\tanh\frac{v}{2R}\,,$$

$$x\cdot\xi_t^*=0\qquad\text{for } t=0, 1, \cdots, m$$

and

$$L = 1 - x \cdot x = \frac{1}{\cosh^2 \frac{v}{2R}},$$

$$p + \frac{1}{R}(x \cdot e_3^*) = \frac{1}{R \sinh \frac{v}{R}} = \sqrt{\frac{(m+1)(m+2)}{2}}$$

by (1.12) and (6.7). Thus, the system of equation (6.11)$\sim$(6.15) can be written as

$$\begin{cases} dx = \dfrac{1}{\cosh \frac{v}{R} + 1} \cdot \dfrac{1}{h}(\bar{\xi}_0^* dz + \xi_0^* d\bar{z}), \\ d\xi_0^* = \dfrac{1}{h}\xi_0^*(\bar{z}dz - zd\bar{z}) + \dfrac{2}{h}P^* dz + \dfrac{2\sqrt{b_1}}{h}\xi_1^* d\bar{z}, \\ \vdots \\ d\xi_t^* = -\dfrac{2\sqrt{b_t}}{h}\xi_{t-1}^* + \dfrac{t+1}{h}\xi_t^*(\bar{z}dz - zd\bar{z}) + \dfrac{2\sqrt{b_{t+1}}}{h}\xi_{t+1}^*, \\ \vdots \\ d\xi_m^* = -\dfrac{2\sqrt{b_m}}{h}\xi_{m-1}^* + \dfrac{m+1}{h}\xi_m^*(\bar{z}dz - zd\bar{z}), \\ dP^* = -\dfrac{\|P^*\|}{h}(\bar{\xi}_0^* dz + \xi_0^* d\bar{z}), \end{cases}$$

where $P^* = \sqrt{\frac{(m+1)(m+2)}{2}}\, e_3^*$. This system of equalities is completely identical with the one for W^2 in Case $\bar{M}^{n+2m} = E^{n+2m}$ except the first one. Thus we see that W^2 is obtained from V_m^2 in E^{n+2m} by the similarity of magnification $1/\left(\cosh \frac{v}{R} + 1\right)$.

As in the previous case, we can construct a minimal submanifold M^n in $H^{n+2m}(\bar{c})$ in Theorem 3 as follows: First we take a base surface W^2 as mentioned above in the unit disk of E^{n+2m}. Then, we take the $(n-2)$-dimensional linear subspace $L_2^{n-2}(y)$ through each $y \in W^2$ which is parallel to P^* and orthogonal to $\xi_0^*, \cdots, \xi_m^*$. And let $L^{n-2}(y)$ be the totally geodesic subspace of $H^{n+2m}(\bar{c})$ tangent to $L_2^{n-2}(y)$ at y. The locus of points on the moving $L^{n-2}(y)$ as y varies is the n-dimensional submanifold in $H^{n+2m}(\bar{c})$ to be constructed in this case. We denote it by $M_m^n(S, H)$.

By virtue of the argument in this section, we have

THEOREM 5. *Let M^n be a minimal submanifold in $S^{n+2m}(R)$ or $H^{n+2m}\left(-\frac{1}{R^2}\right)$ with g-codim $2m$ as in Theorem C. If a base surface W^2 of M^n satisfies conditions (α), (β') in Theorem 3 and $c=1$, then M^n is locally congruent with $M_m^n(S, S)$ or $M_m^n(S, H)$ under the motions of $S^{n+2m}(R)$ or $H^{n+2m}\left(-\frac{1}{R^2}\right)$ respectively.*

§ 7. The Frenet formulas of W^2 of T-type.

In this section, we shall investigate W^2 in Theorem 2 in the case that M^n is of T-type, in detail. Using the notations in § 3, let M^n and $\bar{M}^{n+2m+1}$ be the manifolds as in Theorem 2.

Since $c=0$, (3.14) becomes

$$a_t^2=\frac{a_{t-1}^4}{a_{t-2}^2} \qquad \text{for } t=2, 3, \cdots, m,$$

and (3.17) and (3.18) imply

$$a_{m+1}^2=2\frac{a_m^4}{a_{m-1}^2}.$$

From these equalities, we have

$$\text{(7.1)} \qquad \begin{aligned} &a_t=\lambda^t, \qquad \text{for } t=1, \cdots, m, \\ &a_{m+1}=\sqrt{2}\,\lambda^{m+1}. \end{aligned}$$

Therefore we have

$$\Psi_t=\lambda^t(1, i\sigma_t), \qquad t=1, 2, \cdots, m$$

and

$$\Psi_{m+1}=\sqrt{2}\,\lambda^{m+1}e^{i\varphi}.$$

Hence, from (2.7) and (2.17), we have

$$\text{(7.2)} \qquad \begin{cases} \omega_{\alpha_1\beta_1}+i\sigma_t\omega_{\alpha_2\beta_1}=\lambda(\omega_1-i\omega_2) \\ \omega_{\alpha_1\beta_2}+i\sigma_t\omega_{\alpha_2\beta_2}=i\sigma_{t+1}\lambda(\omega_1-i\omega_2) \end{cases} \quad \text{on } B_m''$$

$$\alpha_1, \alpha_2\in I_t;\ \beta_1, \beta_2\in I_{t+1}, \qquad \text{for } t=1, 2, \cdots, m$$

and

$$\text{(7.3)} \qquad \omega_{\alpha_1\gamma}+i\sigma_m\omega_{\alpha_2\gamma}=\sqrt{2}\,\lambda e^{i\varphi}(\omega_1-i\omega_2),$$

$$\alpha_1, \alpha_2\in I_m; \qquad \gamma=n+2m+1.$$

Since $d\omega_{12}=0$, we can put locally

$$\text{(7.4)} \qquad \omega_{12}=d\theta,$$

then (3.19) becomes

$$\text{(7.5)} \qquad d\varphi=-(m+2)d\theta.$$

From (3.3), we have

$$\text{(7.6)} \qquad p^2=2\lambda^2-\bar{c}$$

By means of (7.2) and (7.3), on B_m'' we get

$$dx=e_1\omega_1+e_2\omega_2,$$

$$\bar{D}(e_1+ie_2)=-i(e_1+ie_2)\omega_{12}+P(\omega_1+i\omega_2)+\lambda(e_{n+1}+i\sigma e_{n+2})(\omega_1-i\omega_2)\,,$$
$$\bar{D}(e_{\alpha_1}+i\sigma_t e_{\alpha_2})=-\lambda(e_{\gamma_1}+i\sigma_{t-1}e_{\gamma_2})(\omega_1+i\omega_2)$$
$$-i\sigma_t(e_{\alpha_1}+i\sigma_t e_{\alpha_2})\hat{\omega}_t+\lambda(e_{\beta_1}+i\sigma_{t+1}e_{\beta_2})(\omega_1-i\omega_2)\,,$$
$$\alpha_1,\,\alpha_2\in I_t\,;\ \beta_1,\,\beta_2\in I_{t+1}\,;\ \gamma_1,\,\gamma_2\in I_{t-1}\,,\qquad t=1,2,\cdots,m-1\,,$$
$$\bar{D}(e_{\alpha_1}+i\sigma_m e_{\alpha_2})=-\lambda(e_{\gamma_1}+i\sigma_m e_{\gamma_2})(\omega_1+i\omega_2)$$
$$-i\sigma_m(e_{\alpha_1}+i\sigma_m e_{\alpha_2})\hat{\omega}_m+\sqrt{2}\,\lambda e^{i\varphi}e_\beta(\omega_1-i\omega_2)\,,$$
$$\alpha_1,\,\alpha_2\in I_m\,;\ \gamma_1,\,\gamma_2\in I_{m-1}\,;\qquad \beta=n+2m+1\,,$$
$$\bar{D}e_\beta=-\sqrt{2}\,\lambda R((e_{\alpha_1}+i\sigma_m e_{\alpha_2})e^{-i\varphi}(\omega_1+i\omega_2))\,,$$
$$\bar{D}P=-\|P\|^2(e_1\omega_1+e_2\omega_2)$$

in the same way as we obtained (4.7), (4.10)$\sim$(4.12) for M^n of S-type.

Since W^2 is flat, we can take a complex coordinate z such that $\omega_1+i\omega_2=e^{-i\theta}dz$ by (7.4). If we set

$$\xi_t=e^{i(t+1)\theta}(e_{\alpha_1}+i\sigma_t e_{\alpha_2})\,,\qquad \alpha_1,\,\alpha_2\in I_t\,,\qquad t=0,1,\cdots,m\,, \tag{7.7}$$

and

$$\varphi=-(m+2)(\theta+\theta_0)$$

by (7.5), then by (iv) of Theorem 2, the above equalities can be written as follows:

$$dx=\frac{1}{2}(\bar{\xi}_0dz+\xi_0d\bar{z})\,,$$
$$\bar{D}\xi_0=Pdz+\lambda\xi_1d\bar{z}\,,$$
$$\bar{D}\xi_t=-\lambda\xi_{t-1}dz+\lambda\xi_{t+1}d\bar{z}\,,\qquad t=1,2,\cdots,m-1\,,$$
$$\bar{D}\xi_m=-\lambda\xi_{m-1}dz+\sqrt{2}\,\lambda e^{-i(m+2)\theta_0}e_\beta d\bar{z}\,,$$
$$\bar{D}e_\beta=-\sqrt{2}\,\lambda R(\xi_m e^{i(m+2)\theta_0}dz)\,,\qquad \beta=n+2m+1\,,$$
$$\bar{D}P=-\frac{\|P\|^2}{2}(\bar{\xi}_0dz+\xi_0d\bar{z})\,,$$

but if we change the complex coordinate z to $ze^{i\theta_0}$, we may assume $\theta_0=0$.

Now, if we impose the following geometrical condition:

$$\|F_t\|=\|G_t\|\,,\qquad \langle F_t,G_t\rangle=0\qquad \text{for } t=1,\cdots,m\,,$$

from the beginning of the proof of Theorem 2, we get easily the same relations when M^n is of T-type. Thus, we have the following theorem in a little more arranged expression.

THEOREM 6. *Let M^n be a minimal submanifold of $\bar{M}^{n+2m+1}$ with g-codim $2m+1$ as in Theorem C. If a base surface W^2 of M^n is flat and the associated normal vector fields F_t, G_t of M^n satisfy*

$$\|F_t\| = \|G_t\|, \quad \langle F_t, G_t\rangle = 0 \quad on\ W^2 \qquad for\ t=1, \cdots, m,$$

then there exist m complex normal vector fields $\xi_1, \xi_2, \cdots, \xi_m$ *and a real normal vector field* η *on* W^2 *such that*

$$\text{(I)} \qquad \begin{cases} \xi_t\cdot\xi_s = \xi_t\cdot\bar{\xi}_s = \xi_t\cdot\eta = 0, & t\neq s, \\ \xi_t\cdot\xi_t = 0, \quad \xi_t\cdot\bar{\xi}_t = 2, \quad \eta\cdot\eta = 1, & t, s = 1, 2, \cdots, m \end{cases}$$

and

$$\text{(II)} \qquad \begin{cases} dx = \frac{1}{2}(\bar{\xi}_0 dz + \xi_0 d\bar{z}), \\ \bar{D}\xi_0 = P dz + \lambda\xi_1 d\bar{z}, \\ \bar{D}\xi_t = -\lambda\xi_{t-1}dz + \lambda\xi_{t+1}d\bar{z}, \qquad t=1, \cdots, m-1, \\ \bar{D}\xi_m = -\lambda\xi_{m-1}dz + \sqrt{2}\,\lambda\eta d\bar{z}, \\ \bar{D}\eta = -\frac{1}{\sqrt{2}}\lambda(\xi_m dz + \bar{\xi}_m d\bar{z}), \\ \bar{D}P = -\frac{1}{2}\|P\|^2(\bar{\xi}_0 dz + \xi_0 d\bar{z}), \end{cases}$$

where z is a complex coordinate of W^2 *and* $\xi_0 = e_1 + ie_2$.

§ 8. Examples of W^2 of T-type.

In this section, we shall give a solution of the equations (II) in Theorem 6 when $\bar{M}^{n+2m+1}$ is a Euclidean space. When $\bar{M}^{n+2m+1}$ is an elliptic or hyperbolic space form, we can get the examples of W^2 by the same method used for S-type as shown in [8] in the case $m=1$.

Supposing $\bar{M}^{n+2m+1} = E^{n+2m+1}$, from (7.6) and (1.12) we have

$$p = \|P\| = \sqrt{2}\,\lambda = \frac{1}{v} \tag{8.1}$$

and from (II) of Theorem 6 we see that $x + \frac{1}{p^2}P$ is a fixed point. We may consider this point as the origin of E^{n+2m+1}, then

$$x = -\frac{1}{p^2}P. \tag{8.2}$$

Since $\bar{D} = d$ in E^{n+2m+1}, we can easily see that the vector fields $\xi_0, \xi_1, \cdots, \xi_m, \eta, P$ satisfy the equation

$$\frac{\partial^2 X}{\partial z \partial \bar{z}} = -\lambda^2 X. \tag{8.3}$$

Supposing the completeness of M^n, we can put

$$(8.4)\qquad \frac{1}{p}P=U=\sum_{j=1}^{m+2}\{A_j\exp\lambda(z\exp(i\alpha_j)-\bar z\exp(-i\alpha_j))$$
$$+\bar A_j\exp\lambda(-z\exp(i\alpha_j)+\bar z\exp(-i\alpha_j))\}\,,$$

where A_j, $j=1,\cdots,m+2$, are constant complex vectors in C^{m+2} such that

$$(8.5)\qquad \begin{cases} A_j\cdot A_j=A_j\cdot A_k=A_j\cdot\bar A_k=0\,, & j\neq k\\ \sum_{j=1}^{m+2}A_j\cdot\bar A_j=\frac{1}{2} \end{cases}$$

and α_j, $j=1,\cdots,m+2$, are real constants. By (II) of Theorem 6, if we put

$$\xi_0=-\frac{\sqrt{2}}{\lambda}\frac{\partial U}{\partial\bar z}=\sqrt{2}\sum_j\exp(-i\alpha_j)\{A_j\exp\lambda(z\exp(i\alpha_j)-\bar z\exp(-i\alpha_j))$$
$$-\bar A_j\exp\lambda(-z\exp(i\alpha_j)+\bar z\exp(-i\alpha_j))\}\,,$$

then $\xi_0\cdot\bar\xi_0=2$, $U\cdot\xi_0=0$ and

$$\xi_0\cdot\xi_0=-4\sum_j A_j\cdot\bar A_j(\cos2\alpha_j-i\sin2\alpha_j)\,.$$

Now, by means of (II), we have inductively

$$(8.6)\qquad \xi_t=\sqrt{2}\sum_j\exp(-i(t+1)\alpha_j)\{(-1)^tA_j\exp\lambda(z\exp(i\alpha_j)-\bar z\exp(-i\alpha_j))$$
$$-\bar A_j\exp\lambda(-z\exp(i\alpha_j)+\bar z\exp(-i\alpha_j))\}\,,\qquad \text{for } t=0,1,\cdots,m\,.$$

Furthermore, if we set

$$(8.7)\qquad \xi_{-1}=U\,,\qquad \xi_{m+1}=\sqrt{2}\,\eta\,,$$

then (8.6) is also true for $t=-1$ and $m+1$. Using (8.5) and (8.6), we get

$$(8.8)\qquad \xi_t\cdot\xi_s=-2((-1)^t+(-1)^s)\sum_j A_j\bar A_j\{\cos(t+s+2)\alpha_j-i\sin(t+s+2)\alpha_j\}$$

and

$$(8.9)\qquad \xi_t\cdot\bar\xi_s=2(1+(-1)^{t+1})\sum_j A_j\cdot\bar A_j\{\cos(t-s)\alpha_j-i\sin(t-s)\alpha_j\}\,.$$

Hence, in particular we have

$$\xi_t\cdot\xi_t=-(-1)^t4\sum_j A_j\cdot\bar A_j\{\cos(2t+2)\alpha_j-i\sin(2t+2)\alpha_j\}\,,$$
$$\xi_t\cdot\bar\xi_t=4\sum_j A_j\cdot\bar A_j=2$$

and

$$\xi_t\cdot\xi_s=\xi_t\cdot\bar\xi_s=0\,,\qquad \text{if } t-s=\text{odd}\,.$$

Taking note of the fact that ξ_{m+1} is real, ξ_t given by (8.6) for $t=-1,0,\cdots$, $m+1$, are admissible as the solution of (I) and (II) of Theorem 6 in the present case, if and only if

(8.10) $$\sum_j A_j \cdot \bar{A}_j \{\cos(2t\alpha_j) - i\sin(2t\alpha_j)\} = 0, \quad t = 1, 2, \cdots, m+1,$$

and

(8.11) $$(m+2)\alpha_j \equiv \begin{cases} \frac{\pi}{2} \pmod{\pi}, & \text{when } m = \text{odd}, \\ 0 \pmod{\pi}, & \text{when } m = \text{even}. \end{cases}$$

We have a solution of (8.10) and (8.11) as follows:

(8.12) $$A_j \cdot \bar{A}_j = \frac{1}{2(m+2)}$$

and

(8.13) $$\alpha_j = \begin{cases} \frac{(2j-1)\pi}{2(m+2)}, & \text{when } m = \text{odd}, \\ \frac{j\pi}{m+2}, & \text{when } m = \text{even}, \end{cases}$$

for $j = 1, 2, \cdots, m+2$.

By virtue of (8.1), (8.2) and (8.4), we get an example of base surfaces W^2 in Theorem 6 in case $\bar{M}^{n+2m+1} = E^{n+2m+1}$ given by

$$x = -v\Big[\sum_j \Big\{A_j \exp \frac{i\sqrt{2}}{v}\Big(u_1 \sin \frac{(2j-1+\varepsilon)\pi}{2(m+2)} + u_2 \cos \frac{(2j-1+\varepsilon)\pi}{2(m+2)}\Big)$$
$$+ \bar{A}_j \exp \frac{-i\sqrt{2}}{v}\Big(u_1 \sin \frac{(2j-1+\varepsilon)\pi}{2(m+2)} + u_2 \cos \frac{(2j-1+\varepsilon)\pi}{2(m+2)}\Big)\Big\}\Big],$$

where $\varepsilon = 0$ or 1 according as m is odd or even.

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