J. Math. Soc. Japan
Vol. 24, No. 1, 1972

Local structures of groups of diffeomorphisms

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(Received March 18, 1971)

0° Introduction

Although it is well known that the groups $\mathcal{D}$ of the smooth diffeomorphisms of compact manifolds M are so called Frechet Lie groups [2, 3, 7], the category of Frechet Lie groups seems to the author to be still too huge to treat and get some useful results. Comparing with Hilbert Lie groups, Frechet Lie groups have not so nice property. This is mainly because the implicit function theorem or the Frobenius theorem does not hold in the category of Frechet manifolds in general case. As a matter of fact, these groups $\mathcal{D}$ have a little nicer property, which is an analogue with the Sobolev chain in function spaces. In fact, $\mathcal{D}$ is an inverse limit of a series of smooth Hilbert manifolds $\mathcal{D}^s$ [2, 7]. However, group operations are not so simple as additive groups of function spaces. These have a little more complicated property (see [2, 7]. Also this will be proved again in this paper as an immediate conclusion of Theorem B and Lemma 1). Anyway since here is a chain $\mathcal{D}^s$, it is natural to think that many properties of $\mathcal{D}$ (not $\mathcal{D}^s$ itself) can be proved by using this chain, and at this time, since we want to get properties of $\mathcal{D}$ (not $\mathcal{D}^s$), inequalities which are similar to Garding's inequality are becoming needful. Garding's inequality was necessary to prove the regularity of elliptic operators, and here to get properties of $\mathcal{D}$ we need some kinds of regularity theorems.

However, to get desired inequalities in general one has to write down all formulas explicitly without using local coordinates of M. This paper is one of the efforts of doing these. All results, especially several inequalities obtained in this paper, will be needful to prove the theorem which is mentioned in the introduction in [8]. To consider the properties of $\mathcal{D}$ many concepts and quantities defined on M must be lifted up to those of $\mathcal{D}$. At this point, the Theorems A, B which are mentioned below are becoming important. The purpose of this paper is to prove these theorems and as applications of these to prove some of local properties of $\mathcal{D}$ as well as some of inequalities. (See next section to know how these theorems are applied.)

To state the Theorems A, B we have to begin with the following: Let E and F be finite dimensional vector bundles over a compact n-dimensional manifold M without boundary. On E and F we can define smooth riemannian inner products, making these riemannian vector bundles, and hence riemannian connections ∇ (cf. [6] for example). For the convenience of notations, we use the same notation ∇ for connections of E and F. Moreover, since we can define a smooth riemannian connection on M, this connection will be denoted by ∇. Connections of any finite dimensional vector bundles are denoted by a simple notation ∇ as far as there is no confusion.

Now suppose W be a relatively compact open subset of E such that $W \cap E_x$ is a non empty open subset of E_x for every $x \in M$, where E_x (as well as F_x) is the fibre at x. Let f be a smooth mapping of $\overline{W}$ (the closure of W) into F such that $f(W \cap E_x) \subset F_x$, where "smooth mapping of $\overline{W}$" means that f can be extended to a smooth mapping of an open neighbourhood of $\overline{W}$ into F. Let $\nabla' f$ and ∂f be partial covariant derivatives of f along the base manifold M and along the fibre respectively (see also 2° for more precise definition).

Let $\boldsymbol{\Gamma}(E)$ (resp. $\boldsymbol{\Gamma}(F)$) be the space of smooth sections of E (resp. F) and $\boldsymbol{\Gamma}(W)$ be the totality of $u \in \boldsymbol{\Gamma}(E)$ such that $u(x) \in W_x$ for every $x \in M$. Define an inner product $\langle\, ,\rangle_k$ on $\boldsymbol{\Gamma}(E)$ or $\boldsymbol{\Gamma}(F)$ by

$$\langle u, v\rangle_k = \sum_{t=0}^{k} \int_M \langle \nabla^t u, \nabla^t v\rangle \mu ,$$

where μ is a volume element of M and $\langle\, ,\rangle$ is the inner product of $T_M^{*t} \otimes E$ (or $T_M^{*t} \otimes F$). In fact, $\nabla^t u$ can be regarded as a section of $T_M^{*t} \otimes E$ (or $T_M^{*t} \otimes F$), where T_M^{*t} is the t-tensor product of the cotangent bundle T_M^* of M.

Denote by $\Gamma^k(E)$ (resp. $\Gamma^k(F)$) the completion of $\boldsymbol{\Gamma}(E)$ (resp. $\boldsymbol{\Gamma}(F)$) under the norm $\|\ \|_k$ defined by $\|u\|_k^2 = \langle u, u\rangle_k$.

Let $\boldsymbol{\Gamma}(W, F)$ be the linear space of smooth mappings f of $\overline{W}$ into F such that $f(W_x) \subset F_x$ for every $x \in M$.

Let $C^k(E)$ denote the space of sections of E of class C^k. This is a Banach space under the norm which induces the topology of uniform convergence of all derivatives of order $\leqq k$. The Sobolev embedding theorem states that if $r \geqq 0$ and $k \geqq \left[\frac{n}{2}\right] + 1 + r$, then $\Gamma^k(E) \subset C^r(E)$ and the inclusion is continuous. Therefore, if $k \geqq \left[\frac{n}{2}\right] + 1$, the following definition makes sense:

$$\Gamma^k(W) = \{u \in \Gamma^k(E)\,;\ u(x) \in W_x \quad \text{for every } x \in M\} .$$

Obviously, $\Gamma^k(W) = \Gamma^{k_0}(W) \cap \Gamma^k(E)$ and $\boldsymbol{\Gamma}(W) = \Gamma^{k_0}(W) \cap \boldsymbol{\Gamma}(E)$, where $k_0 = \left[\frac{n}{2}\right] + 1$.

Let V^{k_1} be a bounded open subset of $\Gamma^{k_1}(W)$ for $k_1 \geqq k_0$, where "bounded" means that there is $u \in V^{k_1}$ such that $\|v - u\|_{k_1}$ is bounded for every $v \in V^{k_1}$.

Put $V^s=\Gamma^s(W)\cap V^{k_1}$, $\boldsymbol{V}=\boldsymbol{\Gamma}(W)\cap V^{k_1}$.

Let α denote the triple $(k, l, \boldsymbol{V})$. We define a semi-norm $|\ |_\alpha$ on $\boldsymbol{\Gamma}(W, F)$ by

$$|f|_\alpha=\sum_{\substack{0\leqq p+q\leqq k\\ 0\leqq l'\leqq l}}\sup\left\{\int|(\nabla^p\partial^{q+l'}f)(v(x))|^2\mu\,;\ v\in\boldsymbol{V}\right\}.$$

Of course $|f|_\alpha=0$ implies $f(v(x))\equiv 0$ for any $v\in\boldsymbol{V}$. Let $\boldsymbol{N}$ be the totality of $f\in\boldsymbol{\Gamma}(W, F)$ such that $f(v(x))\equiv 0$ for every $v\in\boldsymbol{V}$. $\boldsymbol{N}$ contains the null space of $|\ |_\alpha$, $\alpha=(k, l, \boldsymbol{V})$ for any k and l.

Define a norm $\|\ \|_\alpha$ on $\boldsymbol{\Gamma}(W, F)/\boldsymbol{N}$ by the following:

$$\|\bar f\|_\alpha=\inf\{|f+f'|_\alpha\,;\ f'\in\boldsymbol{N}\},$$

where $\bar f$ is the equivalence class of f. Obviously $\|\ \|_\alpha$ is a norm. Let $\Gamma^{k,l}(W, F, V^{k_1})$ denote the completion of $\boldsymbol{\Gamma}(W, F)/\boldsymbol{N}$ by $\|\ \|_\alpha$.

For the simplicity, we assume $k_1\leqq n+5$ throughout this paper.

Now, we can state the theorems:

THEOREM A. *Suppose* $s\geqq n+5$. *Let* $\bar f\in\Gamma^{s,0}(W, E^{*m}\otimes F, V^{k_1})$, $w\in V^s$ *and* $v_j\in\Gamma^s(E)$, $j=1, 2, \cdots, m$. *Then the following inequality holds:*

$$\begin{aligned}\|\bar f(w)(v_1, \cdots, v_m)\|_s^2\leqq{}&\|\bar f\|_{\alpha_1}^2P(\|w\|_{2k_0}^2)\|w\|_s^2\|v_1\|_{k_0}^2\cdots\|v_m\|_{k_0}^2\\ &+\|\bar f\|_{\alpha_2}^2Q(\|w\|_{2k_0}^2)\Big(\sum_{j=1}^m\|v_1\|_{k_0}^2\cdots\|v_{j-1}\|_{k_0}^2\|v_j\|_s^2\|v_{j+1}\|_{k_0}^2\cdots\|v_m\|_{k_0}^2\Big)\\ &+\|\bar f\|_\alpha^2R_s(\|w\|_{k_0+1}^2)\|v_1\|_{k_0}^2\cdots\|v_m\|_{k_0}^2\\ &+\|\bar f\|_{\alpha'}^2S_s(\|w\|_{s-1}^2)\|v_1\|_{s-1}^2\cdots\|v_m\|_{s-1}^2,\end{aligned}$$

$\alpha_1=(k_0, 1, \boldsymbol{V})$, $\alpha_2=(k_0, 0, \boldsymbol{V})$, $\alpha=(s, 0, \boldsymbol{V})$, $\alpha'=(s-1, 0, \boldsymbol{V})$, *where* $k_0=\left[\frac{n}{2}\right]+1$, P, Q, R_s *and* S_s *are polynomials with positive coefficients such that* P, Q *do not depend on* s *and* R_s, S_s *depend on* s.

Essential point of this inequality is that the polynomials P, Q do not depend on s and the highest term $\|\ \|_s$ comes in like linear mappings.

THEOREM B. *Define a mapping* Φ: $\boldsymbol{\Gamma}(W, F)/\boldsymbol{N}\times\boldsymbol{V}\to\boldsymbol{\Gamma}(F)$ *by* $\Phi(\bar f, w)(x)=f(w(x))$. *Then* Φ *can be extended to the* C^l-*mapping of* $\Gamma^{s,l}(W, F, V^{k_1})\times V^s$ *into* $\Gamma^s(F)$ *for* $s\geqq n+5$.

One may find a similar result in 2° of [2] with respect to the differentiability of composition of maps and in fact this is an immediate conclusion of the above theorem and Lemma 1 in the next section. These Theorems A, B are proved in 3°.

1° Local properties of groups of diffeomorphisms

Let M be a compact n-dimensional manifold without boundary. Suppose M has a smooth riemannian structure. Let Exp denote the exponential mapping defined by this smooth riemannian structure. So there is a relatively compact and open tubular neighbourhood U of the zero section of T_M (the tangent bundle of M) such that Exp_x (the exponential mapping restricted to the tangent space T_xM at x) is a diffeomorphism of $U\cap T_xM$ (sometimes this is denoted by U_x) onto an open neighbourhood of x in M. There exists a tubular neighbourhood V of the zero section of T_M such that $\mathrm{Exp}\,(d\ \mathrm{Exp}_x)_y z \in \mathrm{Exp}_x U_x$ for any $y, z\in V\cap T_xM$ (this will be denoted by V_x) for every $x\in M$, where $(d\ \mathrm{Exp}_x)_y$ is the derivative of Exp_x at y.

Let $\boldsymbol{\Gamma}(U)$ (resp. $\boldsymbol{\Gamma}(V)$) be the totality of $u\in \boldsymbol{\Gamma}(T_M)$ (cf., 0°) such that $u(x)\in U_x$ (resp. V_x) for every $x\in M$. $\boldsymbol{\Gamma}(U)$ and $\boldsymbol{\Gamma}(V)$ are open subsets of $\boldsymbol{\Gamma}(T_M)$ in C^0-topology.

It is not hard to verify that there is an open neighbourhood $\boldsymbol{U}'$ of 0 in C^1-topology such that $x\mapsto \mathrm{Exp}_x u(x)$ is a diffeomorphism for every $u\in \boldsymbol{U}'$. As a matter of fact, the structure of Frechet Lie group on $\mathcal{D}$ is given by this mapping [3], namely, define the mapping ξ of $\boldsymbol{U}'$ into $\mathcal{D}$ by

$$\xi(u)(x)=\mathrm{Exp}_x u(x)\,. \qquad \cdots (1)$$

Then, ξ is a homeomorphism of $\boldsymbol{U}'$ onto an open neighbourhood of the identity e in C^1-topology and can be regarded as a smooth chart of $\mathcal{D}$ at e [3]. Naturally we may assume that $\boldsymbol{U}'\subset \boldsymbol{\Gamma}(U)$. Since $\mathcal{D}$ is a topological group in C^1-topology, there is an open neighbourhood $\boldsymbol{V}'$ of 0 in C^1-topology such that $\boldsymbol{V}'\subset \boldsymbol{\Gamma}(V)$ and $\xi(u)\cdot\xi(v)\in\xi(\boldsymbol{U}')$ for every $u, v\in \boldsymbol{V}'$. Let U^{k_0+1} (resp. V^{k_0+1}) be an open, bounded neighbourhood of 0 in $\Gamma^{k_0+1}(T_M)$ such that $U^{k_0+1}\cap \boldsymbol{\Gamma}(T_M)\subset \boldsymbol{U}'$ (resp. $V^{k_0+1}\cap \boldsymbol{\Gamma}(T_M)\subset \boldsymbol{V}'$), where $k_0=\left[\frac{n}{2}\right]+1$. This is possible because of the Sobolev embedding theorem. This embedding theorem also shows that $U^{k_0+1}\cap \boldsymbol{\Gamma}(T_M)$ and $V^{k_0+1}\cap \boldsymbol{\Gamma}(T_M)$ are bounded in C^1-uniform norm. Put $\boldsymbol{U}=U^{k_0+1}\cap\boldsymbol{\Gamma}(T_M)$, $\boldsymbol{V}=V^{k_0+1}\cap\boldsymbol{\Gamma}(T_M)$, $U^s=U^{k_0+1}\cap\Gamma^s(T_M)$ and $V^s=V^{k_0+1}\cap\Gamma^s(T_M)$ for $s\geqq k_0+1$. Define a mapping η of $\boldsymbol{V}'\times\boldsymbol{V}'$ into $\boldsymbol{U}'$ by $\eta(u, v)=\xi^{-1}(\xi(U)\xi(v))$ and $\eta(u, v)$ has the following expression:

$$\eta(u, v)(x)=\mathrm{Exp}_x{}^{-1}\,\mathrm{Exp}\,u(\mathrm{Exp}_x v(X))\,, \qquad \cdots (2)$$

where we use the notation Exp in two different ways: Exp_x implies the mapping of T_xM and Exp does that of T_M into M. In the sense of Frechet Lie groups [3], η is a smooth mapping. Let $\boldsymbol{\Gamma}_g(T_M)$ be the space of the smooth sections of pull back $g^{-1}T_M$ of T_M by $g\in\mathcal{D}$. As a matter of course $\boldsymbol{\Gamma}_g(T_M)$ is the tangent space of $\mathcal{D}$ at g. Put $\gamma(T_M)=\cup\{\boldsymbol{\Gamma}_g(T_M)\,;\,g\in\mathcal{D}\}$ and $\gamma(T_M)$ has a structure of smooth Frechet manifold, because this is the tangent bundle of $\mathcal{D}$. Let R_g be the right translation of $g\in\mathcal{D}$, $dR_g:\boldsymbol{\Gamma}(T_M)\mapsto\boldsymbol{\Gamma}_g(T_M)$ the derivative of R_g at e and $dR:\boldsymbol{\Gamma}(T_M)\times\mathcal{D}\mapsto\gamma(T_M)$ the mapping defined by $dR(v, g)=dR_g v$. Put $\zeta(u, v)=d\xi^{-1}(dR_{\xi(v)}u)$. Then, $\zeta:\boldsymbol{\Gamma}(T_M)\times\boldsymbol{U}'\mapsto\boldsymbol{\Gamma}(T_M)$ can be regarded as the local expression of dR and it is easy to see that ζ has the following explicit expression:

$$\zeta(u, v)(x)=d\,\mathrm{Exp}_x{}^{-1}u(\mathrm{Exp}_x v(x)). \qquad \cdots (3)$$

So this mapping itself can be defined on $\boldsymbol{\Gamma}(T_M)\times\boldsymbol{\Gamma}(U)$.

Let $\boldsymbol{N}$ be the totality of $f\in\boldsymbol{\Gamma}(U, T_M)$ such that $f(v(x))\equiv 0$ for every $v\in\boldsymbol{U}$. Let $\Psi:\boldsymbol{\Gamma}(T_M)\mapsto\boldsymbol{\Gamma}(U, T_M)/\boldsymbol{N}$ be the mapping defined by

$$\Psi(u)(y)=d\,\mathrm{Exp}_{\pi y}^{-1}u(\mathrm{Exp}_{\pi y}y), \qquad y\in U,$$

where π is the projection of T_M. Strictly speaking, one has to project this to $\boldsymbol{\Gamma}(U, T_M)/\boldsymbol{N}$, because the right hand side in the defining equality is an element of $\boldsymbol{\Gamma}(U, T_M)$. However, the mapping Ψ is injective as a mapping of $\boldsymbol{\Gamma}(T_M)$ into $\boldsymbol{\Gamma}(U, T_M)$ and also as a mapping of $\boldsymbol{\Gamma}(T_M)$ into $\boldsymbol{\Gamma}(U, T_M)/\boldsymbol{N}$. So we may use the same notation Ψ.

Recollect that $\boldsymbol{U}$ is bounded in the C^1-uniform topology.

LEMMA 1. *Ψ can be extended to a bounded linear map of $\Gamma^{s+l}(T_M)$ into $\Gamma^{s,l}(U, T_M, U^{k_0+1})$ for $s\geqq k_0+1$, $l\geqq 0$, namely, there is a constant C_α, $\alpha=(s, l, \boldsymbol{U})$ such that*

$$\|\Psi(u)\|_\alpha\leqq C_\alpha\|u\|_{s+l}.$$

PROOF. Let $f\in\boldsymbol{\Gamma}(U, T_M)$ be an element defined by

$$f(y)=d\,\mathrm{Exp}_{\pi y}^{-1}u(\mathrm{Exp}_{\pi y}y), \qquad y\in U,\ u\in\boldsymbol{\Gamma}(T_M).$$

Since $\|\Psi(u)\|_\alpha=\inf\{|f+f'|_\alpha;\ f'\in\boldsymbol{N}\}$, we have only to show that $|f|_\alpha\leqq C_\alpha\|u\|_{s+l}$.

Put $x=\pi y$. $x^1\cdots x^n$ be the normal coordinate at x, $\xi^1,\cdots,\xi^n$ the linear coordinate of T_xM. By using parallel translations, $(x^1,\cdots,x^n,\xi^1,\cdots,\xi^n)$ is regarded as a local trivialization at x. $(0,\cdots,0,\xi_0^1,\cdots,\xi_0^n)$ denotes the coordinate of the point y. Let $z^1,\cdots,z^n$ be the normal coordinate at $\mathrm{Exp}_x y$. Then $z^i=z^i(x^1,\cdots,x^n,\xi^1,\cdots,\xi^n)$ and $z^i(0,\cdots,0,\xi_0^1,\cdots,\xi_0^n)=0$. For every fixed $x^1,\cdots,x^n$, ξ^i can be solved with respect to $z^1,\cdots,z^n$. This is because all points we consider are in U, hence Exp_x is a diffeomorphism for any $x\in M$. Denote $\hat{x}=(x^1,\cdots,x^n)$, $\hat{\xi}=(\xi^1,\cdots,\xi^n)$ and $\hat{z}=(z^1,\cdots,z^n)$.

Put $v(\hat{x},\hat{\xi})=d\,\mathrm{Exp}_{\hat{x}}{}^{-1}u(\mathrm{Exp}_{\hat{x}}\hat{\xi})$, and naturally we have $v(\hat{x},\hat{\xi})=\sum v^i(\hat{x},\hat{\xi})\dfrac{\partial}{\partial\xi^i}$. Therefore,

$$\acute{\nabla}^p\partial^q v(\hat{x},\hat{\xi})=\acute{\nabla}_{k_1}\cdots\acute{\nabla}_{k_p}\partial_{j_1}\cdots\partial_{j_q}v^i dx^{k_1}\otimes\cdots\otimes dx^{k_p}\otimes d\xi^{j_1}\otimes\cdots\otimes d\xi^{j_q}\otimes\frac{\partial}{\partial\xi^i},$$

where $\acute{\nabla}_{k i}=\acute{\nabla}_{\frac{\partial}{\partial x^{k i}}}$ and $\partial_{j i}=\frac{\partial}{\partial \xi^{j i}}$.

Use the following relations

$$\partial_{j i}=\sum \frac{\partial z^l}{\partial x^{j i}} \frac{\partial}{\partial z^l}, \quad \acute{\nabla}_{k i}=\sum \frac{\partial z^l}{\partial x^{j i}} \nabla_l \qquad \left(\nabla_l=\nabla_{\frac{\partial}{\partial z^l}}\right)$$

and

$$v^i(\hat{x}, \hat{\xi})=\sum \frac{\partial \xi^i}{\partial z^j} u^j(\hat{z}),$$

where $u^j(\hat{z})$ is u expressed by $\hat{z}$, namely $u(\hat{z})=\sum u^i(\hat{z}) \frac{\partial}{\partial z^i}$. Then, since $\partial^r z^j$ etc. are all bounded for $r \leqq p+q$. We have that there is constants C_k, $k=0$, 1, 2, $\cdots$, $p+q$, such that

$$|\acute{\nabla}^p \partial^q v(\hat{x}, \hat{\xi})|^2 \leqq \sum_{k=0}^{p+q} C_k |(\nabla^k u)(\hat{z})|^2 .$$

Therefore

$$\int_M |(\acute{\nabla}^p \partial^q f)(v(x))|^2 \mu \leqq \sum_{k=0}^{p+q} C_k \int_M |(\nabla^k u)(\mathrm{Exp}_x v(x))|^2 \mu .$$

Change the variable x into $y=\mathrm{Exp}_x v(x)$ in the right hand side. Since $\boldsymbol{U}$ is bounded in C^1-uniform topology, the Jacobian is uniformly bounded for $v \in \boldsymbol{U}$. Therefore the right hand side of above inequality is not larger than

$$\sum_{k=0}^{p+q} C_k' \int_M |(\nabla^k u)(x)|^2 \mu .$$

Thus, we have $|f|_\alpha^2 \leqq C_\alpha' \|u\|_{s+l}^2$.

Recollect the definition of Φ (see Theorem B in 0°), since $\zeta(u, v)=\Phi(\Psi(u), v)$, we have the following by assuming Theorems A, B:

COROLLARY 1. *$\zeta: \boldsymbol{\Gamma}(T_M) \times \boldsymbol{U} \rightarrow \boldsymbol{\Gamma}(T_M)$ can be extended to the C^l-mapping of $\Gamma^{s+l}(T_M) \times U^s$ into $\Gamma^s(T_M)$ for $s \geqq n+5$. Moreover, the following inequality holds.*

$$\|\zeta(u, v)\|_s^2 \leqq \|u\|_{k_0+1}^2 P'(\|v\|_{2k_0}^2)\|v\|_s^2+\|u\|_s^2 R_s'(\|v\|_{k_0+1}^2)$$
$$+\|u\|_{s-1}^2 S_s'(\|v\|_{s-1}^2), \qquad k_0=\left[\frac{n}{2}\right]+1,$$

where P', R_s', S_s' are polynomials with positive coefficients and P' does not depend on s.

NOTE. If v is restricted in a bounded set in U^{2k_0}, then we have

$$\|\zeta(u, v)\|_s^2 \leqq C\|v\|_s^2+D_s\|u\|_s^2+\|u\|_{s-1}^2 S_s'(\|v\|_{s-1}^2) .$$

Now, we define the following mapping $\tilde{\xi}$ of $\boldsymbol{V}' \times \boldsymbol{V}'$ into $\mathcal{D}$:

$$\tilde{\xi}(u, v)(x)=\mathrm{Exp}\,(d\,\mathrm{Exp}_x)_{u(x)} v(x), \qquad \cdots (4)$$

where $(d\,\mathrm{Exp}_x)_{u(x)}$ is the derivative of Exp_x at $u(x)$. Obviously, we have

$$\tilde{\xi}(u, \zeta(v, u)) = \xi(v)\xi(u) \qquad \cdots (5)$$

and the local expression $\xi^{-1}\tilde{\xi}$ is given by

$$\xi^{-1}\tilde{\xi}(u, v)(x) = \mathrm{Exp}_x{}^{-1}\,\mathrm{Exp}\,(d\,\mathrm{Exp}_x)_{u(x)}v(x)\,. \qquad \cdots (6)$$

Restrict the domain $V'\times V'$ to $V\times V$ and we have the following:

LEMMA 2. *Suppose* $s\geqq n+5$. *The mapping* $\xi^{-1}\tilde{\xi}: V\times V\mapsto \Gamma(T_M)$ *can be extended to the smooth mapping of* $V^s\times V^s$ *into* $\Gamma^s(T_M)$. *Moreover the following inequality holds:*

$$\|\xi^{-1}\tilde{\xi}(u, v)\|_s \leqq P(\|u\|_{2k_0}^2+\|v\|_{2k_0}^2)(\|u\|_s^2+\|v\|_s^2)+S_s(\|u\|_{s-1}^2+\|v\|_{s-1}^2)\,,$$

where P, S_s *are polynomials with positive coefficients and* P *does not depen on* s.

PROOF. Recall that $V\subset V'\subset \Gamma(V)$. Define the mapping f by

$$f(y, z) = \mathrm{Exp}_x{}^{-1}\,\mathrm{Exp}\,(d\,\mathrm{Exp}_x)_y z\,, \qquad x=\pi y=\pi z\,.$$

f is a smooth mapping of $V\oplus V$ (Whitney sum) into T_M. Moreover f is an element of $\Gamma(V\oplus V, T_M)$. Since $V\oplus V$ is relatively compact in $T_M\oplus T_M$ (Whitney sum), we can use Theorem B. Let N be the totality of $f'\in \Gamma(V\oplus V, T_M)$ such that $f'(u(x), v(x))\equiv 0$ for any $u, v\in V$. Then, f is also an element of $\Gamma(V\oplus V, T_M)/N$, namely, $f\in \Gamma^{s,l}(V\oplus V, T_M, V^{k_0+1}\times V^{k_0+1})$ for any s, l.

Therefore Theorem B shows that $\xi^{-1}\tilde{\xi}$ can be extended to the smooth mapping of $V^s\times V^s$ into $T^s(T_M)$ for every $s\geqq n+5$, and Theorem A yields the desired inequality.

Since $\tilde{\xi}(u, \zeta(v, u)) = \xi(v)\cdot\xi(u)$ (cf. (5)), we have

COROLLARY 2. *If we restrict the domain of* η *to* $V\times V$, *then the mapping* $\eta: V\times V\mapsto \Gamma(T_M)$ *can be extended to the* C^l*-mapping of* $V^{s+l}\times V^s$ *into* $\Gamma^s(T_M)$ *for any* $s\geqq n+5$. *Furthermore, we have the following inequality,*

$$\|\eta(v, u)\|_s \leqq P(\|u\|_{2k_0}^2, \|v\|_{2k_0}^2)\|u\|_s^s+Q_s(\|u\|_{2k_0}^2, \|v\|_{2k_0}^2)\|v\|_s^2$$
$$+R_s(\|u\|_{s-1}^2, \|v\|_{s-1}^2)\,,$$

where P, Q_s, R_s *are polynomials of two variables with positive coefficients such that* P *does not depend on* s *and* Q_s, R_s *depend on* s.

NOTE. If u, v are restricted in a bounded subset in V^{2k_0}, then $\|\eta(u, v)\|_s^2 \leqq C\|u\|_s^2+D_s\|v\|_s^2+R_s(\|u\|_{s-1}^2, \|v\|_{s-1}^2)$.

Combining Corollary 1 and 2, we have the following:

COROLLARY 3. *There is an open neighbourhood* U' *of* 0 *of* $\Gamma(T_M)$ *in* C^1*-topology on which* ξ (cf. (1)) *can be defined and maps* U' *homeomorphically onto an open neighbourhood* $\tilde{U}$ *of the identity* e *in* $\mathcal{D}$ *in* C^1*-topology.*

(A) *There exists a bounded open neighbourhood* V^{k_0+1} *of* 0 *in* $\Gamma^{k_0+1}(T_M)$

such that $\xi(V^{k_0+1}\cap\boldsymbol{\Gamma}(T_M))^2\subset\tilde{U}_e$.

(B) *Put* $\boldsymbol{V}=V^{k_0+1}\cap\boldsymbol{\Gamma}(T_M)$, $V^s=V^{k_0+1}\cap\Gamma^s(T_M)$. *By defining the mapping* $\eta:\boldsymbol{V}\times\boldsymbol{V}\to\boldsymbol{\Gamma}(T_M)$ *by* $\eta(u,v)=\xi^{-1}(\xi(U)\xi(v))$, η *satisfies* (i)~(iii) *below: Suppose* $s\geqq n+5$,

(i) η *can be extended to the* C^l*-mapping of* $V^{s+l}\times V^s$ *into* $\Gamma^s(T_M)$.

(ii) *Put* $\eta_u(v)=\eta(v,u)$ *and* η_u *is a smooth map of* V^s *into* $\Gamma^s(T_M)$. (*This is because* $\tilde{\xi}(u,\zeta(v,u))=\xi(v)\xi(u)$.)

(iii) *Put* $\zeta(v,u)=(d\eta_u)_0(v)$ (*the derivative of* η_u *at* 0) *and* ζ *can be extended to the* C^l*-mapping of* $\Gamma^{s+l}(T_M)\times V^s$ *into* $\Gamma^s(T_M)$.

Moreover, if we restrict the domain of η *and* ζ *onto a bounded open subset* (*instead of* V^{k_0+1}) *in* $\Gamma^{2k_0}(T_M)$, *we have the following inequalities: Suppose* $s\geqq n+5$.

$$\|\eta(u,v)\|_s^2\leqq C\|v\|_s^2+D_s\|u\|_s^2+R_s(\|u\|_{s-1}^2,\|v\|_{s-1}^2)$$

$$\|\zeta(u,v)\|_s^2\leqq C'\|v\|_s^2+D_s'\|u\|_s^2+\|u\|_{s-1}^2S_s'(\|v\|_{s-1}^2),$$

where C, C' *are constants which do not depend on* s, D_s, D_s' *are constants depending on* s *and* R_s, S_s' *are polynomials with positive coefficients.*

By using these properties, we have the following:

LEMMA 3. *Suppose* $s\geqq n+5$. *Let* $\mathfrak{N}^s$ *be a basis of neighbourhoods of* 0 *in* $\Gamma^s(T_M)$ *such that every element* $W\in\mathfrak{N}^s$ *is contained in* V^s. *Then,* $\{\xi(W\cap\boldsymbol{\Gamma}(T_M));\ W\in\mathfrak{N}^s\}$ *defines a new topology on* $\mathcal{D}_0$ *which is weaker than the* C^∞*-topology and* $\mathcal{D}_0$ *becomes a topological group by this new topology, where* $\mathcal{D}_0$ *is the connected component of* $\mathcal{D}$ *containing* e *in* C^∞*-topology.*

PROOF. We have only to show the following (cf. [9] pp. 98-99 for example):

(a) $\cap\{\xi(W\cap\boldsymbol{\Gamma}(T_M));\ W\in\mathfrak{N}^s\}=\{e\}$. (trivial)

(b) If W_1, $W_2\in\mathfrak{N}^s$, then there is $W_3\in\mathfrak{N}^s$ such that $\xi(W_3\cap\boldsymbol{\Gamma}(T_M))\subset\xi(W_1\cap\boldsymbol{\Gamma}(T_M))\cap\xi(W_2\cap\boldsymbol{\Gamma}(T_M))$. (trivial)

(c) For any $W_1\in\mathfrak{N}^s$, there is W_2 such that $\xi(W_2\cap\boldsymbol{\Gamma}(T_M))^2\subset\xi(W_1\cap\boldsymbol{\Gamma}(T_M))$. (continuity of η)

(d) For any $W_1\in\mathfrak{N}^s$, there is W_2 such that $\xi(W_2\cap\boldsymbol{\Gamma}(T_M))^{-1}\subset\xi(W_1\cap\boldsymbol{\Gamma}(T_M))$. (see below)

(e) For any $W_1\in\mathfrak{N}^s$ and for any element $g\in\xi(W_1\cap\boldsymbol{\Gamma}(T_M))$, there is $W_2\in\mathfrak{N}^s$ such that $\xi(W_2\cap\boldsymbol{\Gamma}(T_M))\cdot g\subset\xi(W_1\cap\boldsymbol{\Gamma}(T_M))$. (continuity of η)

(f) For any $W_1\in\mathfrak{N}^s$ and for any $g\in\mathcal{D}_0$, there is $W_2\in\mathfrak{N}^s$ such that $g\cdot\xi(W_2\cap\boldsymbol{\Gamma}(T_M))\cdot g^{-1}\subset\xi(W_1\cap\boldsymbol{\Gamma}(T_M))$.

Since $\xi(W_1\cap\boldsymbol{\Gamma}(T_M))$ is an open neighbourhood of e in C^∞-topology and hence $\bigcup_{k=0}^{\infty}\xi(W_1\cap\boldsymbol{\Gamma}(T_M))^k=\mathcal{D}_0$ for any $W_1\in\mathfrak{N}^s$, the statement (f) can be changed into the following (f′):

(f′) For any $W_1 \in \mathfrak{N}^s$ and for any $g \in \xi(W_1 \cap \Gamma(T_M))$, there is $W_2 \in \mathfrak{N}^s$ such that $g \cdot \xi(W_2 \cap \Gamma(T_M)) \cdot g^{-1} \subset \xi(W_1 \cap \Gamma(T_M))$.

So this (f′) can be easily verified by using the continuity of η after we prove (d).

Therefore, the only thing we have to prove is (d). However, to prove this property, we have to do a similar computation as the proof of Theorem A (cf. 3°). So this will be done in the last section.

Now, let $\mathcal{D}_0^s$ ($s \geqq n+5$) be the completion of $\mathcal{D}_0$ by the right uniform topology defined by the above new topology. It is easy to see that $\mathcal{D}_0^s \supset \xi(U^s)$. ($\xi(U^s)$ makes sense because $s \geqq n+5$.) In the following, it will be proved that $\xi(U^s)$ is an open subset of $\mathcal{D}_0^s$.

Since $\xi^{-1}\tilde{\xi}$ is smooth, there exists an element $W \in \mathfrak{N}^s$ such that if u, v, v' are restricted in W, then

$$\|\xi^{-1}\tilde{\xi}(u, v') - \xi^{-1}\tilde{\xi}(u, v)\|_s \leqq C_s \|v' - v\|_s .$$

Since $\eta(v, u) = \xi^{-1}\tilde{\xi}(u, \zeta(v, u))$ and ζ is linear with respect to v, we have

$$\|\eta(v', u) - \eta(v, u)\|_s \leqq C_s \|\zeta(v' - v, u)\|_s$$

for u, v, v' restricted in a bounded open set $W' \in \mathfrak{N}^s$. By using Corollary 1, we have $\|\zeta(v'-v, u)\|_s \leqq C_s' \|v'-v\|_s$. Therefore,

$$\|\xi^{-1}(\xi(v')\xi(u)) - \xi^{-1}(\xi(v)\xi(u))\|_s \leqq K_s \|v' - v\|_s .$$

Assume $\xi(U^s)$ is not open. Then, there is a sequence $g_n \in \mathcal{D}_0^s$ converging to an element $\xi(u) \in \xi(U^s)$ such that $g_n \notin \xi(U^s)$. Let $V \in \mathfrak{N}^s$ such that $V \subset W'$ and $\xi(\bar{V})\xi(u) \subset \xi(U^s)$, where $\bar{V}$ is the closure of V. Since $\{g_n\}$ is a Cauchy sequence, there exists an integer k such that $g_k g_n^{-1} \in \xi(V)$ for any $n \geqq k$. Let $h_n = g_k g_n^{-1}$. This is also a Cauchy sequence. Therefore, for any $\varepsilon > 0$, there is an integer N such that $\|\xi^{-1}(h_n h_m^{-1})\|_s < \varepsilon$ for any $n, m \geqq N$. Thus, we may assume $h_n h_m^{-1} \in \xi(V)$ for $n, m \geqq N$. Let $u_n = \xi^{-1}(h_n)$. Then,

$$\begin{aligned} \|u_n - u_m\|_s &\leqq K_s \|\xi^{-1}(\xi(u_n)\xi(u_m)^{-1}) - \xi^{-1}(\xi(u_m)\xi(u_m)^{-1})\|_s \\ &\leqq K_s \varepsilon . \end{aligned}$$

This implies $\{u_n\}$ is a Cauchy sequence in V. Thus, $g_k g_n^{-1}$ converges in $\xi(\bar{V})$, hence $g_k \in \xi(\bar{V})\xi(u)$, contradicting the assumption. Hence, $\xi(U^s)$ is open.

It is not hard to verify the following properties:

($\mathcal{D}_0$, 1) $\mathcal{D}_0^s$ is a smooth Hilbert manifold modelled on $\Gamma^s(T_M)$.

($\mathcal{D}_0$, 2) $\mathcal{D}_0^{s+1} \subset \mathcal{D}_0^s$ and the inclusion is a C^∞-map.

($\mathcal{D}_0$, 3) $\mathcal{D}_0 = \cap \mathcal{D}_0^s$ (use Sobolev embedding theorem).

($\mathcal{D}_0$, 4) The group multiplication $(g, h) \to g \cdot h$ is a C^l-mapping of $\mathcal{D}_0^{s+l} \times \mathcal{D}_0^s$ into $\mathcal{D}_0^s$.

($\mathcal{D}_0$, 5) For every $g \in \mathcal{D}_0^s$, the right translations $R_g : \mathcal{D}_0^s \to \mathcal{D}_0^s$ is smooth.

($\mathcal{D}_0$, 6) The group inversion $g \mapsto g^{-1}$ is a C^l-mapping of $\mathcal{D}_0^{s+l}$ into $\mathcal{D}_0^s$.

($\mathcal{D}_0$, 7) The mapping $dR: \Gamma^{s+l}(T_M) \times \mathcal{D}_0^s \mapsto T\mathcal{D}_0^s$ (the tangent bundle of $\mathcal{D}_0^s$) defined by $dR(v, g) = dR_g v$ is a C^l-map.

These are the properties which author called I. L. H.-Lie groups in [7] (see also [2]). However, comparing with Hilbert Lie groups, in Frechet Lie groups or even in I. L. H.-Lie groups, Lie algebras do not express the character of these groups so nicely (cf. the example mentioned in the introduction of [7]). We have to define somewhat different things to treat infinitesimal transformations. As in case of finite dimensional Lie groups it *must be invariant connections.*

Let V^{k_0+1} be the same open bounded neighbourhood of 0 in $\Gamma^{k_0+1}(T_M)$ which is defined in the first part of this section. Put $\boldsymbol{V} = V^{k_0+1} \cap \boldsymbol{\Gamma}(T_M)$, $V^s = V^{k_0+1} \cap \Gamma^s(T_M)$. Let $\tilde{\boldsymbol{V}} = \cup \{dR_g \boldsymbol{V};\ g \in \mathcal{D}_0\}$. $\tilde{\boldsymbol{V}}$ is an open subset of $\gamma_0(T_M)$ and of course a right invariant subset. Let $\tilde{V}^s = \cup \{dR_g V^s;\ g \in \mathcal{D}_0^s\}$ for $s \geqq n+5$. $\tilde{V}^s$ is also an open, right invariant subset of the tangent bundle $T\mathcal{D}_0^s$. Let p be the projection of the tangent bundle $\gamma_0(T_M)$ onto $\mathcal{D}_0$. (Of course $\gamma_0(T_M) = \cup \{dR_g \boldsymbol{\Gamma}(T_M);\ g \in \mathcal{D}_0\} = \cup \{\boldsymbol{\Gamma}_g(T_M);\ g \in \mathcal{D}_0\}$ is the tangent bundle of $\mathcal{D}_0$.) Define a mapping $\Xi: \tilde{\boldsymbol{V}} \to \mathcal{D}_0$ by

$$\Xi(v)(x) = \mathrm{Exp}_{(pv)(x)} v(x)\,. \qquad \cdots (7)$$

Obviously, we have

$$\Xi(dR_g v) = R_g \Xi(v)\,. \qquad \cdots (8)$$

Since the local trivialization of $\gamma_0(T_M)$ is given by $d\xi: \boldsymbol{U}' \times \boldsymbol{\Gamma}(T_M) \mapsto \gamma_0(T_M)$, the local expression $\xi^{-1}\Xi d\xi$ of Ξ has the following explicit expression:

$$\xi^{-1}\Xi d\xi(u, v)(x) = \mathrm{Exp}_x^{-1} \mathrm{Exp}\,(d\,\mathrm{Exp}_x)_{u(x)} v(x)\,, \qquad \cdots (9)$$

namely, $\xi^{-1}\Xi d\xi = \xi^{-1}\tilde{\xi}$. By Lemma 2, we have the following:

PROPOSITION 1. *Ξ can be extended to the smooth mapping of $\tilde{V}^s$ into $\mathcal{D}_0^s$. Moreover, Ξ is right invariant, that is, $\Xi dR_g = R_g \Xi$. Let $V^s g = dR_g V^s$ for $g \in \mathcal{D}_0^s$ and Ξ_g the restriction of Ξ to $V^s g$. Then Ξ_g is a smooth diffeomorphism of $V^s g$ onto an open neighbourhood of g in $\mathcal{D}_0^s$, where $s \geqq n+5$. Of course Ξ_g is a smooth diffeomorphism of $\boldsymbol{V}g (= dR_g \boldsymbol{V})$ onto an open neighbourhood of $g \in \mathcal{D}_0$ in the sense of Frechet Lie groups.*

By this proposition, the pair $(\Xi_g, \boldsymbol{V}g)$ can be regarded as a coordinate (or smooth chart) around $g \in \mathcal{D}_0$ for every g. So consider this coordinate as a normal coordinate at $g \in \mathcal{D}_0$ and we get the concept of right invariant connection on $\mathcal{D}_0$, namely, the covariant derivative at g is the natural derivative with respect to this coordinate $(\Xi_g, \boldsymbol{V}g)$. Let $\tilde{\nabla}$ denote the obtained right invariant connection of $\mathcal{D}_0$. Since Ξ can be extended to the smooth mapping of $\tilde{V}^s$ into $\mathcal{D}_0^s$, this connection $\tilde{\nabla}$ can be extended to the smooth connection of $\mathcal{D}_0^s$ for $s \geqq n+5$, namely, $\tilde{\nabla}$ is an I. L. H.-connection [8].

For any $v \in \boldsymbol{\Gamma}(T_M)$ let $d\xi v$ denote the vector field around $e \in \mathcal{D}_0$ defined by $(d\xi)_w v$ for $w \in \boldsymbol{U}'$. $(d\xi)_w$ is the derivative of ξ at w. Put $\Gamma_w(u, v) = d\xi^{-1}\tilde{\nabla}_{(d\xi)_w u} d\xi v$. This is regarded as a local expression of $\tilde{\nabla}$, corresponding to Γ_{jk}^i of that of connections in finite dimensional case.

Let $(d^2\mathrm{Exp}_x)_{w(x)}$ be the 2nd derivative of Exp_x at $w(x)$.

LEMMA 4. *$\Gamma_w(u, v)$ has the following explicit expression*

$$\Gamma_w(u, v)(x) = d\,\mathrm{Exp}_x^{-1}(d^2\,\mathrm{Exp}_x)_{w(x)}((d\,\mathrm{Exp}_x)_{w(x)}u(x), v(x))\,. \qquad \cdots (10)$$

PROOF. Since $\tilde{\nabla}$ is right invariant,

$$d\xi^{-1}\tilde{\nabla}_{(d\xi)_w u} d\xi v = d\xi^{-1} dR_{\xi(w)} \tilde{\nabla}_{dR_{\xi(w)}^{-1}(d\xi)_w u}(dR_{\xi(w)}^{-1} d\xi v)\,.$$

Put $\tilde{v} = dR_{\xi(w)}^{-1} d\xi v$, $u' = dR_{\xi(w)}^{-1}(d\xi)_w u$. Then, letting $y = \xi(w)^{-1}(x)$,

$$\begin{cases} \tilde{v}(w')(x) = ((d\xi)_{w'}v)(y) = (d\,\mathrm{Exp}_y)_{w'(y)}v(y)\,, \\ u'(x) = (d\,\mathrm{Exp}_y)_{w(y)}u(y)\,. \end{cases}$$

Therefore, by definition of $\tilde{\nabla}$

$$\begin{aligned}(\tilde{\nabla}_{u'}\,\tilde{v})(x) &= \lim_{\lambda \to 0} \frac{1}{\lambda}\,\{(d\,\mathrm{Exp}_y)_{w(y)+\lambda u'(y)}v(y) - (d\,\mathrm{Exp}_y)_{w(y)}v(y)\} \\ &= (d^2\,\mathrm{Exp}_y)_{w(y)}(u'(y), v(y))\,.\end{aligned}$$

Thus

$$(d\xi^{-1}\tilde{\nabla}_{(d\xi)_w u} d\xi v)(x) = d\,\mathrm{Exp}_x^{-1}(d^2\,\mathrm{Exp}_x)_{w(x)}((d\,\mathrm{Exp}_x)_{w(x)}u(x), v(x))\,.$$

Let π be the projection of T_M. Define an element $\gamma \in \boldsymbol{\Gamma}(U, T_M^* \otimes T_M^* \otimes T_M)$ by $\gamma(p)(q, r) = d\,\mathrm{Exp}_{\pi p}^{-1}(d^2\,\mathrm{Exp}_{\pi p})_p((d\,\mathrm{Exp}_{\pi p})_p q, r)$, where $p \in U_{\pi p}$, $q, r \in T_{\pi p}M$. Then, obviously $\Gamma_w(u, v)(x) = \gamma(w(x))(u(x), v(x))$. The derivative $(d\Gamma)_w$ of Γ at w is given by

$$(d\Gamma)_w(w')(u, v) = (d\gamma)_{w(x)}(w'(x))(u(x), v(x))\,,$$

where $(d\gamma)_{w(x)}$ is the derivative of γ at $w(x)$. Therefore by Theorem A we have the following:

COROLLARY 4. *Let W be a bounded neighbourhood of 0 in $\Gamma^{2k_0}(T_M)$ such that $W \cap \boldsymbol{\Gamma}(T_M) \subset \boldsymbol{\Gamma}(U)$. Put $W^s = W \cap \Gamma^s(T_M)$ for $s \geqq n+5$. Let $w \in W^s$, u, v, $w' \in \Gamma^s(T_M)$. Then the following inequalities hold:*

$$(\Gamma, 1) \qquad \begin{aligned}\|\Gamma_w(u, v)\|_s \leqq\; & C\{\|u\|_s\|v\|_{k_0} + \|u\|_{k_0}\|v\|_s\} + C^2\|w\|_s\|u\|_0\|v\|_0 \\ & + P_s(\|w\|_{s-1})\|u\|_{s-1}\|v\|_{s-1}\,,\end{aligned}$$

$$(\Gamma, 2) \qquad \begin{aligned}\|(d\Gamma)_w(w')(u, v)\|_s \leqq\; & C\{\|u\|_s\|v\|_{k_0}\|w'\|_{k_0} + \|u\|_{k_0}\|v\|_s\|w'\|_{k_0} + \|u\|_{k_0}\|v\|_{k_0}\|w'\|_s\} \\ & + D\|w\|_s\|u\|_{k_0}\|v\|_{k_0}\|w'\|_{k_0} \\ & + Q_s(\|w\|_{s-1})\|u\|_{s-1}\|v\|_{s-1}\|w'\|_{s-1}\,,\end{aligned}$$

where C, D are constants which do not depend on s and P_s, Q_s are polynomials with positive coefficients.

REMARK. (Γ1), (Γ2) imply that the connection $\tilde{\nabla}$ satisfies the conditions which are mentioned in [8] (hence $\tilde{\nabla}$ is regular in the sense of [8]). However, the regularity of this connection is trivial in this case, because the curve $c(t)$ in $\mathcal{D}_0$ is geodesic if and only if $c(t)(x)$ is geodesic in M for any $x \in M$.

2° Vector bundles over $\mathcal{D}_0$

Let E be a finite dimensional smooth riemannian vector bundle over M, $\boldsymbol{\Gamma}(E)$ the space of smooth sections of E and $\boldsymbol{\Gamma}_g(E)$ the space of smooth sections of the pull back $g^{-1}E$ of E by $g \in \mathcal{D}_0$. Put $\gamma(E) = \cup \{\boldsymbol{\Gamma}_g(E)\,;\, g \in \mathcal{D}_0\}$. This is a sort of vector bundle over $\mathcal{D}_0$, if we forget about the topology. Denote by p the projection of $\gamma(E)$ onto $\mathcal{D}_0$. Since g is a diffeomorphism, for every $v \in \boldsymbol{\Gamma}_g(E)$ there is a unique $v' \in \boldsymbol{\Gamma}(E)$ such that $v(x) = v'(g(x))$ and conversely $v'(g(x))$ is an element of $\boldsymbol{\Gamma}_g(E)$ for any $v' \in \boldsymbol{\Gamma}(E)$. Moreover, $v(g(x))$ is an element of $\boldsymbol{\Gamma}_{h\cdot g}(E)$ for every $v \in \boldsymbol{\Gamma}_h(E)$. Therefore $\gamma(E)$ is a vector bundle on which $\mathcal{D}_0$ acts (from right) as bundle automorphism. Denote by R_g^* the action of g, that is, $(R_g^* v)(x) = v(g(x))$. (Strictly speaking, we should use the terminology *anti-action*, because $R_{gh}^* = R_h^* \cdot R_g^*$.) Recollect the definition of $\xi : \boldsymbol{U}' \mapsto \mathcal{D}_0$. Using this notation, $\tau'(\xi(u)(x))v(x)$ denotes the parallel translation of $v(x) \in E$ along the curve $\xi(tu)(x)$, $t \in [0, 1]$, in M. Of course $\tau'(\xi(u)(x))v(x)$ is an element of the fibre of E at $\xi(u)(x)$. So if we regard x as a variable, $\tau'(\xi(u)(x))v(x)$ is an element of $\boldsymbol{\Gamma}_{\xi(u)}(E)$ for every $v \in \boldsymbol{\Gamma}(E)$ and $u \in \boldsymbol{U}'$. Put

$$\tau(u, v)(x) = \tau'(\xi(u)(x))v(x)\,. \qquad \cdots (11)$$

Then $\tau : \boldsymbol{U}' \times \boldsymbol{\Gamma}(E) \mapsto \gamma(E)$ is a bijective mapping of $\boldsymbol{U}' \times \boldsymbol{\Gamma}(E)$ onto $p^{-1}(\xi \boldsymbol{U}')$ and if we put $\tau_{\xi(u)} v = \tau(u, v)$, $\tau_{\xi(u)}$ is a linear bijection of $\boldsymbol{\Gamma}(E)$ onto $\boldsymbol{\Gamma}_{\xi(u)}(E)$. So τ *can be regarded as a local trivialization of* $\gamma(E)$.

Let $R^* : \boldsymbol{\Gamma}(E) \times \mathcal{D}_0 \mapsto \gamma(E)$ be the right translation defined by $R^*(v, g) = R_g^* v$. Then, the local expression of R^* is the following:

Put $R'(v, u) = \tau_{\xi(u)}{}^{-1} R^*(v, \xi(u))$.

$$R'(v, u)(x) = \tau'(\xi(u)(x))^{-1} v(\xi(u)(x))\,. \qquad \cdots (12)$$

Let $\boldsymbol{U}$ be the same open set as in Lemma 1.

LEMMA 5. *$R' : \boldsymbol{\Gamma}(E) \times \boldsymbol{U} \mapsto \boldsymbol{\Gamma}(E)$ can be extended to the C^l-mapping of $\boldsymbol{\Gamma}^{s+l}(E) \times U^s \mapsto \boldsymbol{\Gamma}^s(E)$ for $s \geqq n+5$. Moreover, the following inequality holds:*

$$\|R'(v, u)\|_s^2 \leqq \|v\|_{k_0+1}^2 P'(\|u\|_{2k_0}^2)\|u\|_s^2 + \|v\|_s^2 R_s'(\|u\|_{k_0+1}^2)$$

$$+\|v\|_{s-1}^2 S_s'(\|u\|_{s-1}^2), \qquad k_0 = \left[\frac{n}{2}\right]+1,$$

where P', R_s', S_s' are polynomials with positive coefficients and P' does not depend on s.

PROOF. Let $\tau'(\mathrm{Exp}_x y)$ denote the parallel translation along the curve $\mathrm{Exp}_x ty$, $t \in [0, 1]$. Let $\boldsymbol{N}$ be the totality of $f \in \boldsymbol{\Gamma}(U, E)$ such that $f(u(x)) \equiv 0$ for every $u \in \boldsymbol{U}$. Let $\Psi : \boldsymbol{\Gamma}(E) \to \boldsymbol{\Gamma}(U, E)/\boldsymbol{N}$ be the mapping defined by

$$\Psi(v)(y) = \tau'(\mathrm{Exp}_{\pi y} y)^{-1} v(\mathrm{Exp}_{\pi y} y), \qquad y \in U,$$

where π is the projection of T_M. We have only to show that Ψ can be extended to a bounded linear map of $\Gamma^{s+l}(T_M)$ into $\Gamma^{s,l}(U, E, U^{k_0+1})$ (cf. Lemma 1 and Corollary 1). Let $\alpha = (s, l, \boldsymbol{U})$ we have to prove $\|\Psi(v)\|_\alpha \leqq C_\alpha \|v\|_{s+l}$.

As in Lemma 1, let $\hat{x} = (x^1, \cdots, x^n)$ be the normal coordinate at $x = \pi y$, $\hat{\xi} = (\xi^1, \cdots, \xi^n)$ the linear coordinate of $T_x M$ and $\hat{z} = (z^1, \cdots, z^n)$ the normal coordinate at $\mathrm{Exp}_x y$. Put $w(\hat{x}, \hat{\xi}) = \tau'(\mathrm{Exp}_{\hat{x}} \hat{\xi}) v(\mathrm{Exp}_{\hat{x}} \hat{\xi})$. Letting $e^1, \cdots, e^m$ be a basis of $E_{\hat{x}}$ (the fibre at $\hat{x}$) which is obtained by the parallel translation by a basis of E_x, we have

$$w(\hat{x}, \hat{\xi}) = \sum w^i(\hat{x}, \hat{\xi}) e^i .$$

Thus by the same computation as in Lemma 1 we have the desired result.

Now, the local expression of right translation $R^*_{\xi(u)}$ is given by the following: Suppose $u, v \in \boldsymbol{V}$, $w \in \boldsymbol{\Gamma}(E)$. Putting $R''_{\xi(u)}(w, v) = \tau^{-1}_{\xi(v)\xi(u)} R^*_{\xi(u)} \tau_{\xi(v)} w$, we have

$$\left\{\begin{aligned} &R''_{\xi(u)}(w, v)(x) = \tau'(\xi(\eta(v, u)(x)))^{-1} \tau'(\xi(v)(y)) w(y) \\ &\qquad y = \xi(u)(x) . \end{aligned}\right. \qquad \cdots (13)$$

This is a rather complicated expression. So we define a mapping $\tau_\Delta(v, u)$ of $\boldsymbol{\Gamma}(E)$ onto itself by

$$\left\{\begin{aligned} &(\tau_\Delta(v, u) w)(x) = \tau'(\xi(\eta(v, u))(x))^{-1} \tau'(\xi(v)(y)) \tau'(\xi(u)(x)) w(x) \\ &\qquad y = \xi(u)(x) . \end{aligned}\right. \qquad \cdots (14)$$

This is easy to understand (cf. Fig. 1).

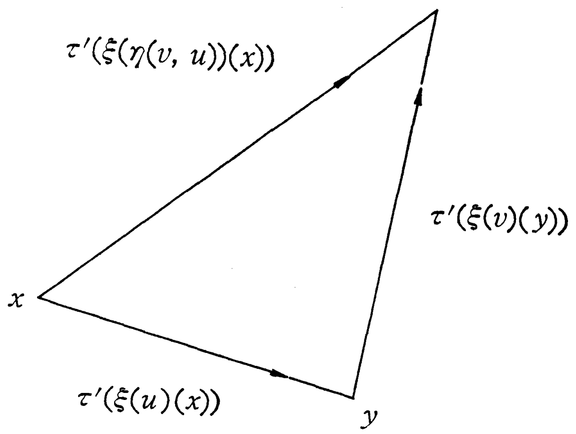

Fig. 1

Using this notation, we have $R''_{\xi(u)}(w, v) = \tau_\Delta(v, u)R'(w, u)$.

For any $y, z \in V_x$ let $\tau'(\mathrm{Exp}_x y)$, $\tau'(\mathrm{Exp}(d\,\mathrm{Exp}_x)_y z)$ and $\tau'(\mathrm{Exp}\, f(y, z))$ be the parallel translations along $\mathrm{Exp}_x ty$, $t \in [0, 1]$, $\mathrm{Exp}(d\,\mathrm{Exp}_x)_y tz$, $t \in [0, 1]$ and $\mathrm{Exp}\, tf(y, z)$, $t \in [0, 1]$ respectively, where $f(y, z) = \mathrm{Exp}_x^{-1}\mathrm{Exp}(d\,\mathrm{Exp}_x)_y z$. Define a smooth mapping T of $V \oplus V$ into $E^* \otimes E$ by

$$T(y, z) = \tau'(\mathrm{Exp}\, f(y, z))^{-1}\tau'(\mathrm{Exp}(d\,\mathrm{Exp}_x)_y z)\tau'(\mathrm{Exp}_x y).$$

Then, $T \in \boldsymbol{\Gamma}(V \oplus V, E^* \otimes E)$. Denote by $\tilde{\tau}_\Delta$ the mapping defined by $\tilde{\tau}_\Delta(u, v)(x) = T(u(x), v(x))$. Then we have

$$\tau_\Delta(v, u) = \tilde{\tau}_\Delta(u, \zeta(v, u)) \qquad \cdots (15)$$

(Compare this equality with (5)).

LEMMA 6. *Suppose $s \geqq n+5$. The mapping $\tilde{\tau}_\Delta: V \times V \to \boldsymbol{\Gamma}(E^* \otimes E)$ can be extended to the smooth mapping of $V^s \times V^s$ into $\Gamma^s(E^* \otimes E)$.*

Moreover, the following inequality holds:

$$\begin{aligned}\|\tilde{\tau}_\Delta(u, v)w\|_s^2 \leqq\ & P(\|u\|_{2k_0}^2 + \|v\|_{2k_0}^2)(\|u\|_s^2 + \|v\|_s^2)\|w\|_{k_0}^2 \\ & + Q(\|u\|_{2k_0}^2 + \|v\|_{2k_0}^2)\|w\|_s^2 \\ & + R_s(\|u\|_{s-1}^2 + \|v\|_{s-1}^2)\|w\|_{s-1}^2 .\end{aligned}$$

The proof of this lemma is completely parallel to that of Lemma 2. The inequality is an immediate conclusion of Theorem A.

Let $\tau_\Delta \circ R': V \times \boldsymbol{\Gamma}(E) \times V \to \boldsymbol{\Gamma}(E)$ be the mapping defined by $\tau_\Delta \circ R'(v, w, u) = \tau_\Delta(v, u)R'(w, u) (= R''_{\xi(u)}(w, v))$. Then combining Lemmas 5 and 6, we have the following:

PROPOSITION 2. *$\tau_\Delta \circ R'$ can be extended to C^l-map of $V^{s+l} \times \Gamma^{s+l}(E) \times V^s$ into $\Gamma^s(E)$. If $u \in V^s$ is fixed, then this is a smooth map of $V^s \times \Gamma^s(E)$ into $\Gamma^s(E)$, where s is provided $\geqq n+5$.*

PROOF. $\tau_\Delta \circ R'(v, w, u) = \tilde{\tau}_\Delta(u, \zeta(v, u))R'(w, u)$. Apply Lemma 1, Corollary 1, Lemma 5 and Lemma 6. Then, we get the first statement. If u is fixed, then $w \to R'(w, u)$ and $v \to \zeta(v, u)$ are linear, hence smooth. So we get the second statement.

Since $\mathcal{D}_0^{n+5}$ is a topological group, there is an open neighbourhood W^{n+5} of 0 in $\Gamma^{n+5}(E)$ such that $\xi(W^{n+5})(\xi(W^{n+5}))^{-1} \subset \xi(V^{n+5})$.

Put $W^s = W^{n+5} \cap \Gamma^s(E)$ for $s \geqq n+5$. We define an equivalence relation on the disjoint union $\cup \{\xi(W^s)g \times \Gamma^s(E)\,;\, g \in \mathcal{D}_0^s\}$ by the following: $(\xi(u)g, w) \sim (\xi(u')g', w')$ (called equivalent), if and only if $\xi(u)g = \xi(u')g'$ and $w = R''_{g'g^{-1}}(w', u')$. Since $g'g^{-1}$ is contained in $\xi(V^s)$, this is well defined. $R''_{g'g^{-1}}(w', u')$ is smooth and linear with respect to w'.

Let $\gamma^s(E)$ be the equivalence classes $\cup \{\xi(W^s)g \times \Gamma^s(E)\,;\, g \in \mathcal{D}_0^s\}/\sim$. Obviously $\gamma^s(E)$ is a smooth fibre bundle over $\mathcal{D}_0^s$ with the fibre $\Gamma^s(E)$. It is not hard to verify the following properties:

(γ1) $\gamma^s(E)$ is a smooth fibre bundle over $\mathcal{D}_0^s$ with the fibre $\Gamma^s(E)$.

(γ2) $\gamma^{s+1}(E) \subset \gamma^s(E)$ and the inclusion is a smooth bundle map.

(γ3) $\gamma(E) = \cap \gamma^s(E)$ (use the Sobolev embedding theorem).

(γ4) For every $g \in \mathcal{D}_0^s$, the right translation R_g^* is smooth. (Proposition 2).

(γ5) $R^* : \boldsymbol{\Gamma}(E) \times \mathcal{D}_0 \to \gamma(E)$ can be extended to a C^l-mapping of $\Gamma^{s+l}(E) \times \mathcal{D}_0^s$ into $\gamma^s(E)$. (Lemma 5).

For any s, t such that $n+5 \leqq s \leqq t$, $\gamma^{t,s}(E)$ implies the pull back of $\gamma^s(E)$ by the inclusion $\mathcal{D}_0^t \subset \mathcal{D}_0^s$. Obviously $\gamma^{t,s}(E)$ is a smooth vector bundle over $\mathcal{D}_0^t$ with the fibre $\Gamma^s(E)$. This is the bundle defined by $\cup \{\xi(W^t)g \times \Gamma^s(E)\,;\, g \in \mathcal{D}_0^t\}/\sim$, where $(\xi(u)g, w) \sim (\xi(u')g', w')$ if and only if $\xi(u)g = \xi(u')g'$ and $w = R''_{g'g^{-1}}(w', u')$. Since $g'g^{-1} \in \xi(V^t) \subset \xi(V^s)$ this is well defined. In Lemma 5, take U^t instead of U^s. Then, $U^t \subset U^s$ implies that $R' : \boldsymbol{\Gamma}(E) \times \boldsymbol{U} \to \boldsymbol{\Gamma}(E)$ can be extended to the C^l-map of $\Gamma^{s+l}(E) \times U^t$ into $\Gamma^s(E)$. Similarly in Proposition 2 we can take V^{t+l}, V^t instead of V^{s+l}, V^s because $V^{t+l} \subset V^{s+l}$, $V^t \subset V^s$. Therefore, we have the following additional properties:

(γ6) For every $g \in \mathcal{D}_0^t$, the right translation R_g^* is a smooth map of $\gamma^{t,s}(E)$ onto itself.

(γ7) $R^* : \boldsymbol{\Gamma}(E) \times \mathcal{D}_0 \to \gamma(E)$ can be extended to a C^l-mapping of $\Gamma^{s+l}(E) \times \mathcal{D}_0^t$ into $\gamma^{t,s}(E)$.

Here we would like to illustrate why we need to lift the concept of vector bundles over M onto that over $\mathcal{D}_0$. Mainly, it is because many subalgebra of the Lie algebra of $\mathcal{D}_0$ is given by the kernell of differential operators. The Lie algebra of $\mathcal{D}_0$ is of course $\boldsymbol{\Gamma}(T_M)$. Let $\mathfrak{G}$ be a subalgebra of $\boldsymbol{\Gamma}(T_M)$. Then, the distribution of $\{dR_g\mathfrak{G}\,;\, g \in \mathcal{D}_0\}$ obtained by the right translation is smooth in the sense of Frechet Lie groups. However, in the category of Frechet manifolds the Frobenius' theorem does not hold in general case. So, take the closure $\mathfrak{G}^s$ of $\mathfrak{G}$ in $\Gamma^s(T_M)$ and consider the distribution $\{dR_g\mathfrak{G}^s\,;\, g \in \mathcal{D}_0^s\}$. This is a distribution on Hilbert manifold $\mathcal{D}_0^s$, on which the Frobenius' theorem holds. However, at this time the differentiability might lose.

If we take $\{dR_g\mathfrak{S}^{s+1};\ g\in\mathcal{D}_0^s\}$, then we get the differentiability but lose the closedness of the distribution.

Now, consider a linear mapping $A:\boldsymbol{\Gamma}(T_M)\to\boldsymbol{\Gamma}(E)$ which can be extended to the bounded linear operator of $\Gamma^s(T_M)$ into $\Gamma^{s-d}(E)$. Define a right invariant bundle morphism $\tilde{A}$ of $\gamma^s(T_M)$ into $\gamma^{s,s-d}(E)$ by $A_g=R_g^*AdR_g^{-1}$. This is only continuous in general case. However, suppose $\tilde{A}$ happens to be smooth and $A\Gamma^s(T_M)$ is closed in $\Gamma^{s-d}(E)$. Then it is not hard to verify that $\operatorname{Ker}\tilde{A}$ is a smooth distribution on $\mathcal{D}_0^s$. In fact, we have the following smooth extension theorem: *If $A:\boldsymbol{\Gamma}(T_M)\to\boldsymbol{\Gamma}(E)$ is a differential operator of order d with smooth coefficients, then $\tilde{A}$ is a smooth bundle morphism.* To prove this theorem we need more complicated arguments about jet bundles and higher order connections. So this will be proved in the next paper.

3° Proof of Theorem A, B

Notations are as in the introduction (cf. the statements of Theorems A, B). We have to begin with the precise definition of partial covariant derivatives $\acute{\nabla}f$, ∂f.

Let $c(t)$ be a smooth curve in M such that $c(o)=x$ and $\left.\frac{d}{dt}\right|_{t=0}c(t)=X\in T_xM$ (the tangent space of M at x). For every $p\in W_x$, p_t denotes the curve in W obtained by the parallel displacement of p along the curve $c(t)$. Define $(\acute{\nabla}_Xf)(p)$ by

$$(\acute{\nabla}_Xf)(p)=\left.\frac{\nabla}{dt}\right|_{t=0}f(p_t) \qquad \cdots (16)$$

where $\frac{\nabla}{dt}$ implies the covariant derivative in F. It is easy to see that $(\acute{\nabla}_Xf)(p)$ does not depend on the choice of curves $c(t)$ as far as $c(o)=x$, $\left.\frac{d}{dt}\right|_{t=0}c(t)=X$.

Since f preserves the fibres, that is, $f(W_x)\subset F_x$, we define $(\partial_qf)(p)$ for every $q\in E$ such that $\pi p=\pi q$ (π is the projection of E) by the following:

$$(\partial_qf)(p)=\lim_{\delta\to0}\frac{1}{\delta}\,\{f(p+\delta q)-f(p)\}\,. \qquad \cdots (17)$$

Since $(\acute{\nabla}_Xf)(p)$ is linear with respect to X, $\acute{\nabla}f$ can be regarded as a mapping of $\overline{W}$ into $T_M^*\otimes E$, where T_M^* is the cotangent bundle of M. Similarly, ∂f can be regarded as a mapping of $\overline{W}$ into $E^*\otimes F$, where E^* is the dual vector bundle of E. As a matter of course, $T_M^*\otimes F$ and $E^*\otimes F$ have connections extended naturally by the rule of derivations from the connections of T_M, E and F. So iterating these operations $\acute{\nabla}$, ∂, we get $\acute{\nabla}^kf$ (resp. ∂^kf) is a mapping of $\overline{W}$ into $T_M^{*k}\otimes F$ (resp. $E^{*k}\otimes F$), where $T_M^{*k}=T_M^*\otimes\cdots\otimes T_M^*$ (k-tensor product of T_M^*) and $E^{*k}=E^*\otimes\cdots\otimes E^*$.

LEMMA 7. $\acute{\nabla}\partial=\partial\acute{\nabla}$.

PROOF. Let $\tau_E(c(t))$, $\tau_F(c(t))$, $\tau_{E^*\otimes F}(c(t))$ be parallel displacements in E, F, $E^*\otimes F$ respectively along a curve $c(t)$. Then, we have

$$(\tau_{E^*\otimes F}(c(t)A))(e)=\tau_F(c(t))A(\tau_E(c(t))^{-1}e)\,, \qquad \cdots (18)$$

for every $A\in E_x^*\otimes F_x$ and $e\in F_{c(t)}$. This is because of the way of definition of the connection ∇ of $E^*\otimes F$. It is defined by the rule of derivation, and in fact, since the concept of connections can be defined naturally from that of parallel displacements, we may take the equality (18) as the defining equation of the connection of $E^*\otimes F$.

Let $X=\frac{d}{dt}\Big|_{t=0}c(t)$ and $p_t=\tau_E(c(t))p$, that is, the parallel displacement of $p\in W_x$ along the curve $c(t)$. Put $q_t=\tau_E(c(t))q$ for any $q\in E_x$. Then, we have

$$\begin{aligned}(\acute{\nabla}_X\partial_q f)(p)&=\lim_{t\to 0}\frac{1}{t}\{\tau_F(c(t))^{-1}(\partial_{q_t}f)(p_t)-(\partial_q f)(p)\}\\ &=\lim_{\substack{t\to 0\\ s\to 0}}\frac{1}{st}\{\tau_F(c(t))^{-1}[f(p_t+sq_t)-f(p_t)]-[f(p+sq)-f(p)]\}\\ &=\lim_{s\to 0}\frac{1}{s}\{(\acute{\nabla}_x f)(p+sq)-(\acute{\nabla}_x f)(p)\}=(\partial_q\acute{\nabla}_X f)(p) \qquad \text{q. e. d.}\end{aligned}$$

Obviously, $\acute{\nabla}^k\partial^m f$ is a smooth mapping of $\overline{W}$ into $T_M^{*k}\otimes E^{*q}\otimes F$. On the other hand, if v is a smooth section of E, then the covariant derivative v can be regarded as a smooth section of $T_M^*\otimes E$ and hence $\nabla^k v$ is a section of $T^{*k}\otimes E$. $(\nabla^k v)^m$ denotes the m-tensor product $\nabla^k v\otimes\cdots\otimes\nabla^k v$. Obviously $(\nabla^k v)^m$ is a section of $T_M^{*km}\otimes E^m$. Since there is the natural pairing $\cdot: E^*\times E\to R$ (real numbers), we can define the natural pairing $\cdot: E^{*m}\otimes F\times E^m\to F$ and hence

$$\cdot: T_M^{*k}\otimes E^{*m}\otimes F\times T_M^{*k'}\otimes E^m\to T_M^{*k+k'}\otimes F\,,$$

where $E^m=E\otimes\cdots\otimes E$ (m-tensor product of E). Denote by $A\cdot B$ the natural pairing of $A\in T_M^{*k}\otimes E^{*m}\otimes F$ and $B\in T_M^{*k'}\otimes E^m$. Using these notations, we have

LEMMA 8. *If v is a smooth section of E such that $v(x)\in W_x$ for every $x\in M$, then*

$$\nabla^k f(v(x))=\sum C_{p,q,a_1,\cdots,a_k}(\nabla^p\partial^q f)(v(x))\cdot(\nabla v)^{a_1}\otimes(\nabla^2 v)^{a_2}\otimes\cdots\otimes(\nabla^k v)^{a_k}\,,$$

where the summation runs through all non negative integers such that $q=a_1+\cdots+a_k$, $k=p+a_1+2a_2+\cdots+ka_k$ and $C_{p,q,a_1,\cdots,a_k}$ is a universal constant such that $C_{0,1,0,\cdots,0,1}=1$. (Remark that if $a_k\neq 0$, then $a_k=1$ and $p=a_1=a_2=\cdots=a_{k-1}=0$.)

PROOF. Obviously, $\nabla f(v(x))=(\acute{\nabla}f)(v(x))+(\partial f)(v(x))\cdot(\nabla v)(x)$. By induction, we have $q=a_1+\cdots+a_k$, $k=p+a_1+2a_2+\cdots+ka_k$ and $C_{0,1,0,\cdots,0,1}=1$. (As for other constants, computations are a little complicated. However these are completely parallel to that of $\dfrac{d^k}{dx^k} f(x, u(x))$ for a two variable function f and a single variable function u.)

COROLLARY 5. *Let s be an integer such that $s\geqq n+l$ $(n=\dim M)$. For every k $(k\leqq s)$ and the constants $C_{p,q,a_1,\cdots,a_k}$ in Lemma 8 we have $\Sigma^{(l)}a_t\leqq 1$, where $\Sigma^{(l)}$ denotes the summation for $t\geqq s-(n+l-1)/2$. Of course, if $k<s-(n+l-1)/2$, we put $\Sigma^{(l)}a_t=0$.*

PROOF. Assume $\Sigma^{(l)}a_t\geqq 2$. Then $q=\sum_{t=1}^{k}a_t\geqq 2$. Thus,

$$s-2\geqq k-2\geqq k-(p+q)=\sum_{t=2}^{k}(t-1)a_t\geqq 2(s-1-(n+l-1)/2)$$

$$=2s-(n+l+1).$$

This implies $n+l-1\geqq s\geqq n+l$. q. e. d.

COROLLARY 6. *Notations being as above,*

(i) *if $t<s-(n+l-1)/2$ and $l\geqq 3$, then $t+n/2+1\leqq s-1$ and*

(ii) *if $\Sigma^{(l)}a_t=1$ and $l\geqq 5$, then $p+q+n/2+1\leqq s-1$.*

PROOF.

(i) $t+\dfrac{n}{2}+1<s-\dfrac{n+l-1}{2}+\dfrac{n}{2}+1=s-\dfrac{l}{2}+\dfrac{3}{2}\leqq s.$

(ii) $p+q=k-\sum_{t=2}^{k}(t-1)a_t\leqq k-\left(s-\dfrac{n+l-1}{2}-1\right)\leqq\dfrac{n+l-1}{2}+1.$

Hence

$$p+q+\frac{n}{2}+1\leqq\frac{n+l-1}{2}+1+\frac{n}{2}+1\leqq s-\frac{l}{2}+\frac{3}{2}\leqq s-1.$$

Now, let $\Gamma(E)$ (resp. $\Gamma(F)$) be the space of smooth sections of E (resp. F) and $\Gamma(W)$ the subset of $\Gamma(E)$ consisting of the elements $u\in\Gamma(E)$ such that $u(x)\in W_x$ for every $x\in M$. Define $\langle\,,\rangle_k$, $\Gamma^k(E)$, $\Gamma^k(F)$ and $\Gamma^k(W)$ as in the introduction. By the Sobolev embedding theorem we have the following:

LEMMA 9. *For $k=\left[\dfrac{n}{2}\right]+1+r$, there is a constant e_k such that*

$$e_k\|v\|_k\geqq\max_x|v(x)|+\max_x|(\nabla v)(x)|+\cdots+\max_x|(\nabla^r v)(x)|.$$

REMARK. In the following part of this section, the constant e_{k_0} for $k_0=\left[\dfrac{n}{2}\right]+1$ will be appearing very often. Of course $|\ |$ implies the length of the tensors.

Suppose $s\geqq n+5$ and $k\leqq s$ and let $t_0=s-1-(n+4)/2$ (cf. Corollary 6) and $k_0=\left[\dfrac{n}{2}\right]+1$. Since $\sum_{t\geqq t_0+1}a_t\leqq 1$ in the equality of Lemma 8, denote by r the number such that $r \geqq t_0+1$ and $a_r=1$ if such a number exists, that is, in the case of $\sum_{t \geqq t_0+1} a_x = 1$. Put $m_t = \max_x |(\nabla^t v)(x)|$ for $t \leqq t_0$ and $m_{p,q} = \max_x |(\acute{\nabla}^p \partial^q f)(v(x))|$ in case of $\sum_{t \geqq t_0+1} a_t = 1$. Then, by Corollary 6 together with Lemma 9, we have

$$m_t \leqq e_{k_0} \|\nabla^t v\|_{k_0} \leqq e_{k_0} \|v\|_{t+k_0} \leqq e_{k_0} \|v\|_{s-1} \qquad \cdots (19)$$

$$m_{p,q} \leqq e_{k_0} \|(\acute{\nabla}^p \partial^q f)(v)\|_{k_0} \qquad \cdots (20)$$

and $p+q+k_0 \leqq s-1$.

LEMMA 10. *Notations being as above, we have*

$$|\nabla^k f(v(x))| \leqq m_{0,1} |\nabla^k v| + \Sigma_{(1)} C_{p,q,q,0,\cdots,0} |(\acute{\nabla}^p \partial^q f)(v(x))| m_1^q$$
$$+ \Sigma_{(2)} C_{p,q,a_1,\cdots,a_{t_0},0,\cdots,0} |(\acute{\nabla}^p \partial^q f)(v(x))| m_1^{a_1} m_2^{a_2} \cdots m_{t_0}^{a_{t_0}}$$
$$+ \Sigma_{(3)} C_{p,q,a_1,\cdots,a_{t_0},0,\cdots,1,0,\cdots,0} m_{p,q} m_1^{a} m_2^{a_2} \cdots m_{t_0}^{a_{t_0}} |\nabla^r v|,$$

where the summation $\Sigma_{(i)}$, $i=1, 2, 3$, runs through all non-negative integers as follows:

$\Sigma_{(1)}$: $p+q=k$ *(Remark that if $p+q=k$, then $a_2=\cdots=a_k=0$)*

$\Sigma_{(2)}$: $p+q<k$, $q=a_1+\cdots+a_{t_0}$ *and* $k=p+a_1+\cdots+t_0 a_{t_0}$.

(Remark that this is the case of $\sum_{t \geqq t_0+1} a_t = 0$.)

$\Sigma_{(3)}$: $a_k=0$, $\sum_{t \geqq t_0+1} a_t = 1$, $q=a_1+\cdots+a_{k-1}$

$k=p+a_1+\cdots+(k-1)a_{k-1}$.

(Remark that this is the case of $\sum_{t \geqq t_0+1} a_t = 1$ *and* $a_k=0$. *If* $a_k \neq 0$, *then* $a_k=1$ *and* $p=a_1=\cdots=a_{k-1}=0$. *This case is expressed by the first term.)*

Recall the definition in 0°. Let $\boldsymbol{\Gamma}(W, F)$ be the linear space of the smooth mappings f of $\overline{W}$ into F such that $f(W_x) \subset F_x$ for every $x \in M$. Let V^{k_1} be a bounded open set of $\Gamma^{k_1}(W)$ for $k_0 \leqq k_1 \leqq n+5$. $\boldsymbol{N}$ denotes the totality of f such that $f(v(x)) \equiv 0$ for any $v \in V^{k_1} \cap \boldsymbol{\Gamma}(E)$.

LEMMA 11. *If $f \in \boldsymbol{N}$, then $(\acute{\nabla}^p \partial^q f)(v(x)) \equiv 0$ for any $v \in V^{k_1} \cap \boldsymbol{\Gamma}(E)$.*

PROOF. By definition, we have

$$(\partial_{w(x)} f)(v(x)) = \lim_{\delta \to 0} \frac{1}{\delta} \{f(v(x)+\delta w(x)) - f(v(x))\}.$$

So we have $(\partial f)(v(x)) \equiv 0$ if $f \in \boldsymbol{N}$. On the other hand, $\nabla f(v(x)) \equiv 0$ implies $(\nabla f)(v(x)) \equiv 0$, because $\nabla f(v(x)) = \acute{\nabla} f + \partial f \cdot \nabla v$. Thus we get the desired result by induction.

Put $\boldsymbol{V} = V^{k_1} \cap \boldsymbol{\Gamma}(E)$, $V^s = V^{k_1} \cap \Gamma^s(E)$ for $s \geqq n+5$. $\Gamma^{k,l}(W, F, V^{k_1})$ denotes the completion of $\boldsymbol{\Gamma}(W, F)/\boldsymbol{N}$ by $\| \ \|_\alpha$, $\alpha=(k, l, \boldsymbol{V})$.

PROPOSITION 3. *Suppose* $s \geqq n+5$. *If* $\bar{f} \in \Gamma^{s,0}(W, F, V^{k_1})$ *and* $v \in V^s$, *then* $\bar{f}(v(x))$ *is contained in* $\Gamma^s(F)$ *and the following inequality holds:*

$$\|\bar{f}(v(x))\|_s^2 \leqq 2m_{0,1}{}^2\|v\|_s^2 + C_s(m_1+1)^{2s}\|\bar{f}\|_\alpha^2 + \|\bar{f}\|_{\alpha'}^2 P_s(\|v\|_{s-1}^2),$$
$$\alpha = (s, 0, V), \qquad \alpha' = (s-1, 0, V),$$

where C_s *is a constant depending on* s *and* P_s *is a polynomial with positive coefficients.*

To prove this, we need one more lemma (Lemma 12) and by this lemma, it becomes enough to prove the inequality. The essential point of the above inequality is that $m_{0,1}$ does not depend on s and the maximal terms $\| \ \|_s$ in norm come in like linear mappings.

LEMMA 12. *There is a constant* $\bar{C}_{k_0}$ *depending only on* k_0 *such that*

$$m_{p,q} \leqq \bar{C}_{k_0}(1+m_1+\cdots+m_{k_0})^{k_0}\|\bar{f}\|_\beta, \qquad \beta = (k_0+p+q, 0, V),$$

where $m_{p,q} = \max_x |(\acute{\nabla}^p \partial^q f)(v(x))|$ *for an* $f \in \bar{f}$. (Remark that $m_{p,q}$ depends only on $\bar{f}$ by Lemma 11.)

PROOF. By Lemma 11, we see that $\|\bar{f}\|_\beta = \|f\|_\beta$ for any $f \in \bar{f}$. So by the inequality (20), we have only to consider $\|(\acute{\nabla}^p\partial^q f)(v(x))\|_{k_0}$. In Lemma 8, replace f by $\acute{\nabla}^p\partial^q f$ and we have

$$|\nabla^t(\acute{\nabla}^p\partial^q f)(v(x))| \leqq \sum C_{i,j,b_1,\cdots,b_t}|\acute{\nabla}^{i+p}\partial^{j+q}f|\cdot|\nabla v|^{b_1}\cdots|\nabla^t v|^{b_t}.$$

Therefore for any $t \leqq k_0$, we have that the right hand side is not larger than

$$\sum C_{i,j,b_1,\cdots,b_t}|(\acute{\nabla}^{i+p}\partial^{j+q}f)(v(x))|(1+m_1+\cdots+m_{k_0})^{k_0}.$$

Thus,

$$m_{p,q} \leqq e_{k_0}\|(\acute{\nabla}^p\partial^q f)(v(x))\|_{k_0} \leqq \bar{C}_{k_0}\|\bar{f}\|_\beta(1+m_1+\cdots+m_{k_0})^{k_0}.$$

PROOF OF PROPOSITION 3. Since $s > 2k_0$ and $m_j \leqq e_{k_0}\|v\|_{j+k_0}$ (cf. (19)), if $v \in \Gamma^s(E)$, $\bar{f} \in \Gamma^{s,0}(W, F, V^{k_1})$, then $m_1 < \infty$. By Lemma 12 we get

$$\begin{aligned}|\nabla^k f(v(x))| \leqq\ & m_{0,1}|\nabla^k v| + C'_k(1+m_1)^k \sum_{p+q=k}|(\acute{\nabla}^p\partial^q f)(v(x))| \\ & + D_k\Sigma_{(2)}|(\acute{\nabla}^p\partial^q f)(v(x))|\, e_{k_0}^q\|v\|_{s-1}^q \\ & + E_k\Sigma_{(3)}\|f\|_\beta(e_{k_0}\|v\|_{s-1})^{a_1+\cdots+a_{t_0}}|\nabla^r v|,\end{aligned}$$

where C'_k, D_k and E_k are constants depending on k.

Since $p+q+k_0 \leqq s-1$ in the summation $\Sigma_{(3)}$ (Corollary 6), this term can be replaced by

$$E_k\|f\|_{\alpha'}\Sigma_{(3)}(e_{k_0}\|v\|_{s-1})^{q-1}|\nabla^r v|.$$

On the other hand,

$$|\nabla^k f(v(x))|^2 \leqq 2m_{0,1}^2|\nabla^k v|^2 + 2(C'_k(1+m_1)^k\Sigma + D_k\Sigma_{(2)} + E_k\Sigma_{(3)})^2.$$

The second term II can be estimated very roughly by the following:

Let C''_k, D'_k and E'_k are constants depending on k.

$$II \leqq C''_k(1+m_1)^{2k} \sum_{p+q=k} |(\acute{\nabla}^p \partial^q f)(v(x))|^2$$
$$+D'_k \sum_{p+q<k} |(\acute{\nabla}^p \partial^q f)(v(x))|^2 \sum_{q \leqq k} e_{k0}^{2q} \|v\|_{s-1}^{2q}$$
$$+E'_k \|f\|_{\alpha'}^2 \sum_{q<k} (e_{k_0}\|v\|_{s-1})^{2(q-1)} \sum_{r<k} |\nabla^r v|^2 .$$

Thus we have

$$\|f(v(x))\|_s^2 \leqq 2m_{0,1}^2 \|v\|_s^2 + C_s(1+m_1)^{2s} \|f\|_\alpha^2 + D''_s \|f\|_\alpha^2 (\sum_{q \leqq s} e_{k_0}\|v\|_{s-1})^{2q}$$
$$+E''_s \|f\|_{\alpha'}^2 \sum_{q<s} (e_{k_0}\|v\|_{s-1})^{2(q-1)} \|v\|_{s-1}^2 .$$

This completes the proof of Proposition 3.

The following lemma is easy to prove:

LEMMA 13. *Suppose* $s \geqq n+3$ ($n = \dim M$). *Let* e_{k_0} *be the same constant as in* (19) *and* (20).

(i) *If* $u \in \boldsymbol{\Gamma}(E)$ *and* $v \in \boldsymbol{\Gamma}(F)$, *then*

$$\|u \otimes v\|_s^2 \leqq e_{k0}^2 \{\|u\|_s \|v\|_{k0}^2 + \|u\|_{k0}^2 \|v\|_s^2\} + C_s \|u\|_{s-1}^2 \|v\|_{s-1}^2 .$$

(ii) *If* $f \in \boldsymbol{\Gamma}(E^* \otimes F)$ *and* $u \in \boldsymbol{\Gamma}(E)$, *then*

$$\|f(u)\|_s^2 \leqq e_{k0}^2 \{\|f\|_s^2 \|u\|_{k0}^2 + \|f\|_{k0}^2 \|u\|_s^2\} + C_s \|f\|_{s-1}^2 \|u\|_{s-1}^2 .$$

As well as Proposition 3, the essential part of this lemma is that e_{k_0} does not depend on s.

Now, let $f \in \boldsymbol{\Gamma}(E^{*l} \otimes F)$. Then f is an l-linear map of $E \times \cdots \times E$ into F and also f can be regarded as a linear map of $E^l (= E \otimes \cdots \otimes E)$ into F.

LEMMA 14. *Notations being as above, suppose* $s \geqq n+3$ *and let* $\bar{e}_{k_0} = \max\{1, e_{k_0}\}$. *Then*

$$\|f(v_1, \cdots, v_l)\|_s^2 \leqq \bar{C}_{k0}^{2(l+1)} \|f\|_s^2 \|v_1\|_{k0}^2 \cdots \|v_l\|_{k0}^2$$
$$+C_{k0}^{2(l+1)} \|f\|_{k0}^2 \sum_{j=1}^{l} \|v_1\|_{k0}^2 \cdots \|v_{j-1}\|_{k0}^2 \|v_j\|_s^2 \|v_{j+1}\|_{k0}^2 \cdots \|v_l\|_{k0}^2$$
$$+C_s \|f\|_{s-1}^2 \|v_1\|_{s-1}^2 \cdots \|v_l\|_{s-1}^2 ,$$

where C_s *is a constant depending on* s.

PROOF. $f(v_1, \cdots, v_l) = f(v_1 \otimes \cdots \otimes v_l)$. Thus by (ii) in Lemma 13, we have

$$\|f(v_1, \cdots, v_l)\|_s^2 \leqq e_{k0}^2 \{\|f\|_s^2 \|v_1 \otimes \cdots \otimes v_l\|_{k0}^2 + \|f\|_{k0}^2 \|v_1 \otimes \cdots \otimes v_l\|_s^2\}$$
$$+C_s \|f\|_{s-1}^2 \|v_1 \otimes \cdots \otimes v_l\|_{s-1}^2 .$$

Using (i) of Lemma 13 successively, we get the desired result.

Now, Theorem A is an immediate conclusion of Proposition 3 together with Lemma 12 and 14.

Using this Theorem A, we have

LEMMA 15. *Let* $f \in \boldsymbol{\Gamma}(W, F)$. *Then there exist positive valued continuous functions* γ_1, γ_2 *defined on the real half line* $[0, \infty)$ *and a constant* C *all of them depending on* f *such that*

$$\| f(u(x)+v(x))-f(u(x))\|_s^2 \leqq C\int_0^1 \|u+\theta u\|_s^2 d\theta \|v\|_{k0}^2 + \int_0^1 \gamma_1(\|u+\theta v\|_{k0}^2) d\theta \|v\|_s^2 + \int_0^1 \gamma_2(\|u+\theta v\|_{s-1}^2\| d\theta \|v\|_{s-1}^2 .$$

PROOF. Since $f(u(x)+v(x))-f(u(x)) = \int_0^1 (\partial f)(u(x)+\theta v(x))\cdot v(x) d\theta$, we have

$$\| f(u(x)+v(x))-f(u(x))\|_s^2 = \sum_{k=0} \int_M \left| \int_0^1 \nabla^k [(\partial f)(u(x)+\theta v(x))\cdot v(x)] d\theta \right|^2 \mu \leqq \sum_{k=0}^{s} \int_M \int_0^1 |\nabla^k [(\partial f)(u(x)+\theta v(x))\cdot v(x)]|^2 d\theta \mu$$

using Schwartz inequality. Thus,

$$\| f(u(x)+v(x))-f(u(x))\|_s^2 \leqq \int_0^1 \|(\partial f)(u(x)+\theta v(x))\cdot v(x)\|_s^2 d\theta .$$

So Theorem A gives the desired result.

Now we define a mapping $\boldsymbol{\Phi}: \boldsymbol{\Gamma}(W, F)/\boldsymbol{N} \times \boldsymbol{V} \mapsto \boldsymbol{\Gamma}(F)$ by $\boldsymbol{\Phi}(\bar{f}, u)(x) = f(u(x))$, where $f \in \bar{f}$. Then, Proposition 3 shows that if $s \geqq n+5$, then $\boldsymbol{\Phi}$ can be extended as a mapping $\boldsymbol{\Phi}: \Gamma^{s,0}(W, F, V^{k_1}) \times \Gamma^s(W) \mapsto \Gamma^s(F)$.

LEMMA 16. $\boldsymbol{\Phi}: \Gamma^{s,0}(W, F, V^{k_1}) \times V^s \mapsto \Gamma^s(F)$ *is continuous for every* $s \geqq n+5$.

PROOF. Let $\bar{f} \in \Gamma^{s,0}(W, F, V^{k_1})$, $u \in V^s$. Suppose ε be an arbitrary positive number. There is a positive number δ_1 such that if $\|\bar{f}'-\bar{f}\|_\alpha < \delta_1$, $\alpha = (s, 0, \boldsymbol{V})$, then $\|(\bar{f}'-\bar{f})(u'(x))\|_s < \varepsilon/3$ for every u' such that $\|u-u'\|_s \leqq 1$. This is an immediate conclusion of Proposition 3. Now, we can choose an element $\bar{f}' \in \boldsymbol{\Gamma}(W, F)/\boldsymbol{N}$ such that $\|\bar{f}'-\bar{f}\|_\alpha < \delta_1$. So Lemma 15 tells us that there is δ_2 such that if $\|u'-u\|_s < \delta_2$, then $\|f'(u(x))-f'(u'(x))\|_s < \varepsilon/3$. We may assume that $\delta_2 \leqq 1$. Therefore

$$\|\bar{f}(u(x))-\bar{f}(u'(x))\|_s \leqq \|(\bar{f}-\bar{f}')(u(x))\|_s + \|\bar{f}'(u(x))-\bar{f}'(u'(x))\|_s + \|(\bar{f}'-\bar{f})(u'(x))\|_s < \varepsilon ,$$

whenever $\|u-u'\|_s < \delta_2$. Since $\boldsymbol{\Phi}$ is linear with respect to $\bar{f}$, this inequality together with Proposition 3 again shows the continuity of $\boldsymbol{\Phi}$.

PROOF OF THEOREM B. First of all, if $\boldsymbol{\Phi}$ is differentiable at (f, u), then the k-th derivative of $\boldsymbol{\Phi}$ must be given by the following:

$$\begin{aligned}(d^k\Phi)_{(f,u)}&(f_1, v_1)\cdots(f_k, v_k)(x)\\ &=\sum_{j=1}^{k}(\partial^{k-1}f_j)(u(x))\cdot v_1\otimes\cdots\otimes v_{j-1}\otimes v_{j+1}\otimes\cdots\otimes v_k\\ &\quad+(\partial^k f)(u(x))\cdot v_1\otimes\cdots\otimes v_k\,,\end{aligned}$$

where v_j implies of course $v_j(x)$. Now, let $k\leqq l$. Then $\partial^k f\in\Gamma^{s,l-k}(W, E^{*k}\otimes F, V^{k_1})\subset\Gamma^{s,0}(W, E^{*k}\otimes F, V^{k_1})$. Hence Theorem A and the continuity of Lemma 16 show that $d^k\Phi$ is continuous, if it exists. Thus we have only to show the following (see [5], p. 7):

Let $\delta=\sqrt{\|f'\|_\beta^2+\|v\|_s^2}$, $\beta=(s, l, V)$.

$$\lim_{\delta\to 0}\frac{1}{\delta^l}\left\|(f+f')(u(x)+v(x))-\left\{\sum_{k=0}^{l-1}\frac{1}{k!}(\partial^k f')(u)\cdot v^k+\sum_{k=0}^{l}\frac{1}{k!}(\partial^k f)(u)\cdot v^k\right\}\right\|_s=0\,.$$

The second term in $\|\ \|_s$ is the Taylor series of Φ up to l.

To do this it is enough to prove the following two equalities:

$$\lim_{\delta\to 0}\frac{1}{\|v\|_s^l}\left\|f(u(x)+v(x))-\sum_{k=0}^{l}\frac{1}{k!}(\partial^k f)(u)\cdot v^k\right\|_s=0\,. \tag{1}$$

$$\lim_{\delta\to 0}\frac{1}{\delta^l}\left\|f'(u(x)+v(x))-\sum_{k=0}^{l-1}\frac{1}{k!}(\partial^k f')(u)\cdot v^k\right\|_s=0\,. \tag{2}$$

By Taylor's theorem ([1] p. 186), we have

$$\begin{aligned}f'(u(x)+v(x))&=\sum_{k=0}^{l-1}\frac{1}{k!}(\partial^k f')(u(x))\cdot v(x)^k\\ &\quad+\int_0^1\frac{(1-\theta)^{l-1}}{(l-1)!}(\partial^l f')(u(x)+\theta v(x))\cdot v(x)^l d\theta\,.\end{aligned}$$

So, as for (2), we have only to show

$$\lim_{\delta\to 0}\frac{1}{\delta^{2l}}\left\|\int_0^1\frac{(1-\theta)^{l-1}}{(l-1)!}(\partial^l f')(u(x)+\theta v(x))\cdot v(x)^l d\theta\right\|_s^2=0\,.$$

Using Schwartz inequality as in Lemma 15, we have that the left hand side of the above equality is not larger than

$$C\lim_{\delta\to 0}\frac{1}{\delta^{2l}}\int_0^1\|(\partial^l f')(u(x)+\theta v(x))\cdot v(x)^l\|_s^2 d\theta\,.$$

Therefore by Theorem A we see that the above quantity is 0.

As for (1), since $\displaystyle\int_0^1\frac{(1-\theta)^{l-1}}{(l-1)!}d\theta=\frac{1}{l!}$, we have also to prove

$$\lim_{\|v\|_s\to 0}\frac{1}{\|v\|_s^l}\left\|\int_0^1\frac{(1-\theta)^{l-1}}{(l-1)!}[(\partial^l f)(u(x)+\theta v(x))-(\partial^l f)(u(x))]\cdot v(x)^l d\theta\right\|_s=0\,.$$

It is enough to prove

$$\lim_{\|v\|_s \to 0} \|(\partial^l f)(u+\theta v)-(\partial^l f)(u)\|_s = 0 .$$

Since $\partial^l f \in \Gamma^{s,0}(W, F, V^{k_1})$, the continuity in Lemma 16 shows the desired results.

4° Estimate for group inversion

Recall the definition of ξ and $\boldsymbol{U}$ in 1°. We may assume that $\boldsymbol{U}$ is a star shaped neighborhood of 0 in $\boldsymbol{\Gamma}(T_M)$. Let $\varphi = \xi(u)$, $u \in \boldsymbol{U}$. φ is a diffeomorphism and obviously, there is $v \in \boldsymbol{\Gamma}(T_M)$ such that $\varphi^{-1}(x) = \mathrm{Exp}_x\, v(x)$. Put $\psi = \varphi^{-1}$.

For an arbitrarily fixed $x \in M$, let $y = \varphi(x)$. Clearly, $\rho(x, y) = |u(x)| = |v(y)|$, where ρ is the distance function.

Let $x(t)$ be the geodesic defined by $\mathrm{Exp}\, t\dfrac{u(x)}{|u(x)|}$, $t \in [0, |u(x)|]$. Parameters for geodesics are always arc length. Consider a family of geodesics $x_\alpha(t)$ such that $x_0(t) \equiv x(t)$. Then, $w(t) = \dfrac{\partial}{\partial\alpha}\Big|_0 x_\alpha(t)$ satisfies

$$\frac{\nabla^2}{dt^2} w(t) + A(t)w(t) = 0 , \qquad \cdots (21)$$

where $A(t)X = -R(\dot{x}(t), X)\dot{x}(t)$ and R is the curvature tensor. $A(t)$ is self-adjoint. If we use frames along $x(t)$ which are obtained by parallel displacement of a frame at $x(0)$ along the curve $x(t)$, and express the vector $w(t)$, as an n-tuple ($n = \dim M$) of functions (this will be denoted by $w(t)$ again), then the above equation is changed into

$$\frac{d^2}{dt^2} w(t) + A(t)w(t) = 0 , \qquad \cdots (22)$$

where $A(t)$ is a symmetric matrix at each $t \in [0, |u(x)|]$.

Now, there is a constant K such that $\|A(t)\| \leqq K$ for any geodesic and for any t. This is because M is compact. For simplicity, we use the notation $w'(t)$, $w''(t)$ for differentiation. Define a function $F(t)$ by

$$F(t) = \int_0^t \int_0^s |w'(\tau)|\, d\tau\, ds .$$

Then,

$$|w'(t)| - |w'(0)| \leqq \int_0^t |w''(s)|\, ds \leqq KF(t) + K|w(0)|t .$$

Therefore,

$$F''(t) \leqq KF(t) + K|w(0)|t + |w'(0)| .$$

Putting $F(t) = g(t)e^{\sqrt{K}t}$,

$$g''(t) = 2\sqrt{K}g'(t) \leqq K|w(0)|t + |w'(0)| .$$

By assuming $\max_x |u(x)| \leqq 1$, that is, restricting u in such a neighborhood of 0 in $\Gamma(T_M)$, we have that there is a constant C such that

$$F'(t) \leqq C\{|w(0)|t^2 + |w'(0)|t\}.$$

Since $t \leqq |u(x)|$ and

$$|w(t)-w(0)| \leqq \int_0^t |w'(s)|ds = F'(t),$$

$$|w(t)-w(0)| \leqq C\{|w(0)||u(x)|^2 + |w'(0)||u(x)|\}.$$

Now, notice that $d\varphi_x X$ is given by the value $w(|u(x)|)$ of the solution $w(t)$ of (21) such that $w(0)=X$ and $w'(0)=\nabla_X u$, where $X \in T_x M$. So expressing $d\varphi_x$ by the same frame obtained by the parallel displacement mentioned above and denoting by the same notation, we have

$$|d\varphi_x X - X| \leqq C\{|X||u(x)|^2 + |\nabla_X u||u(x)|\}.$$

Hence

$$\|d\varphi_x - I\| \leqq C\{|u(x)|^2 + |(\nabla u)(x)||u(x)|\}, \qquad \cdots (23)$$

where I is the identity matrix.

Assume furthermore that

$$\max_x \{|u(x)| + |(\nabla u)(x)|\} < \frac{1}{2C},$$

that is, restrict u in such a neighborhood of 0 in $\Gamma(T_M)$. Then, it is easy to see that

$$\|d\varphi_x^{-1} - I\| \leqq 2\|d\varphi_x - I\|. \qquad \cdots (24)$$

Obviously, $d\varphi_x^{-1} = d\psi_y$, where $d\psi_y$ is again expressed by the same frame along $x(t)$. ψ is given by $\psi(z) = \operatorname{Exp} v(z)$. Let $\psi_\theta(z) = \operatorname{Exp} \theta v(z)$. Then ψ_θ is also a diffeomorphism for every $\theta \in [0, 1]$, because if not, we can find some θ at which $\xi(\theta u)$ is not a diffeomorphism.

Consider the geodesic $x(t) = \operatorname{Exp} t \dfrac{v(y)}{|v(y)|}$, $t \in [0, |v(y)|]$. In fact this is the same geodesic as above, if you change the parameter t to $|u(x)|-t$. By this change of parameter the same equality as (21) or (22) holds.

Since M is compact, there is a constant K' depending only on M such that $\|A(t)-A(0)\| \leqq K't$.

Let $\bar{w}(t)$ be the solution of

$$\bar{w}''(t) + A(0)\bar{w}(t) = 0 \qquad \cdots (26)$$

with the same initial condition $\bar{w}(0) = w(0)$, $\bar{w}'(0) = w'(0)$. Since $A(0)$ is symmetric, we may assume that $A(0)$ is a diagonal matrix. Therefore, by a direct computation, we see that there are C_1 and δ_1 such that

$$|\bar{w}(t)| \leqq C_1\{|w'(0)|t + |w(0)|\}$$

for any $t \leqq \delta_1$, where these constants depend only on M. By using the inequality in Theorem 5, [5] p. 41, we can compare solutions of (22) and (26), that is,

$$|w(t)-\bar{w}(t)| \leqq C_2\{K'|w'(0)|t^3+K'|w(0)|t^2\} ,$$

where C_2 is a constant depending only on M. Therefore

$$|w(t)-w(0)| \geqq |\bar{w}(t)-\bar{w}(0)|-C_2K'\{|w'(0)|t^3+|w(0)|t^2\}$$

for any $t<\delta_1$.

By a direct computation, we have that there are C_3, C_4 and δ_2 such that

$$|\bar{w}(t)-\bar{w}(0)| \geqq C_3|\bar{w}'(0)|t-C_4|\bar{w}(0)|t^2$$

for any $t \leqq \delta_2$. Thus,

$$|w(t)-w(0)| \geqq (C_3-C_2K't^2)t|w'(0)|-(C_2K'+C_4)t^2|w(0)| .$$

Take δ so that it may satisfy $\delta \leqq \min\{\delta_1, \delta_2, 1\}$ and $C_3-C_2K'\delta^2 \geqq C_3/2$. Then, for any $t \leqq \delta$,

$$|w(t)-w(0)| \geqq \frac{C_3}{2}t|w'(0)|-C't^2|w(0)| .$$

So restrict u in a neighborhood $\tilde{\boldsymbol{U}}$ such that

$$\tilde{\boldsymbol{U}}=\{u \in \boldsymbol{U};\ |u(x)| \leqq \delta,\ |u(x)|+|(\nabla u)(x)|<\frac{1}{2C} \ \text{ for any } x \in M\} .$$

Since $|u(x)|=|v(y)|$, all arguments above do work well. Therefore

$$|d\phi_y X-X| \geqq \frac{C_3}{2}|v(y)||\nabla_X v|-C'|v(y)|^2|X| .$$

This implies

$$\|d\phi_y-I\| \geqq \frac{C_3}{2}|u(x)||(\nabla v)(y)|-C'|u(x)|^2 .$$

Combining with (23) and (24), we have that there are constants C_4, C_5 depending only on M such that

$$|(\nabla v)(y)| \leqq C_4|u(x)|+C_5|(\nabla u)(x)| \qquad \cdots (27)$$

for any $u \in \tilde{\boldsymbol{U}}$.

Notice that the restriction for u is only that in C^1-uniform topology. Therefore, we can find an open neighborhood $\hat{U}^{k_0+1}$ of 0 in $\Gamma^{k_0+1}(T_M)$, $k_0=\left[\frac{n}{2}\right]+1$, such that $\hat{U}^{k_0+1} \cap \boldsymbol{\Gamma}(T_M) \subset \boldsymbol{U}$. Hereafter, u is always restricted in this set $\hat{U}^{k_0+1} \cap \boldsymbol{\Gamma}(T_M)$.

Let $x^1, \cdots, x^n, y^1, \cdots, y^n$ be coordinate at the points x and y respectively. We may assume without loss of generality that the coordinate neighborhood of x contains y and that of y contains x. Since

$$\frac{\partial\varphi^i}{\partial x^k}(\psi(y))\frac{\partial\psi^k}{\partial y^j}(y)=\delta^i_j,$$

we have

$$\frac{\partial^2\psi^i}{\partial y^j\partial y^k}=-\frac{\partial^2\varphi^c}{\partial x^a\partial x^b}\frac{\partial\psi^a}{\partial x^j}\frac{\partial\psi^b}{\partial x^k}\frac{\partial\psi^i}{\partial x^c}.$$

After this point, computations are very regular. Namely, denoting by B, $\partial^k\psi$ the tensors $B^i_j=\frac{\partial\psi^i}{\partial y^j}(y)$ and $\frac{\partial^k\psi^i}{\partial y^{i_1}\partial y^{i_2}\cdots\partial y^{i_k}}(y)$ respectively, and using the notation $\cdot$ in Lemma 8, we have

$$\partial^k\psi=-\Sigma C'_{a_2,\cdots,a_k}(\partial^2\psi)^{a_2}\otimes\cdots\otimes(\partial^k\psi)^{a_k}\cdot B\otimes B\otimes\cdots\otimes B, \qquad \cdots (28)$$

where $k\geqq 2$ and the summation runs through all non-negative integers such that $a_2+2a_3+\cdots+(k-1)a_k=k-1$. $C'_{a_2,\cdots,a_k}$ are universal constants which satisfy

$$C'_{0,\cdots 0,1}\equiv 1.$$

These properties can be proved by induction. In the last part $B\cdots B$, we have no need to care about how the indices are arranged, because after all, we take norms of those.

Now, let s be an integer in the range $\geqq n+5$. This s is the number in $\Gamma^s(T_M)$ of which we want to prove the continuity of the inversion. For any $k\leqq s$, $(k\geqq 2)$, write the equality (28) as follows:

$$\partial^k\psi=-\partial^k\varphi\cdot B\otimes\cdots\otimes B+\Sigma I^{a_2,\cdots,a_{k-1}}\cdot B\otimes\cdots\otimes B.$$

Then, by the property $a_2+2a_3+\cdots+(k-1)a_k=k-1$ and by a similar method as in Corollary 6, we have the following:

In $I^{a_2,\cdots,a_j,0\cdots 0}$,

(i) if $a_j\geqq 2$, then all $\partial^i\varphi$ in this term satisfies

$$i+\left[\frac{n}{2}\right]+1\leqq s-1.$$

(ii) if $a_j=1$, then all $\partial^i\varphi$ other than $\partial^j\varphi$ satisfies

$$i+\left[\frac{n}{2}\right]+1\leqq s-1.$$

Let $f(x^1,\cdots,x^n,p^1,\cdots,p^n)$ be the coordinate expression of Exp by the coordinate at x, where $p^1,\cdots,p^n$ is a linear coordinate of T_xM. Then $\varphi(z)$ is expressed by $f(z,u(z))$. Therefore, we can use Lemma 8, and $\partial^k\varphi$ can be expressed by $(\nabla u)(x),\cdots,(\nabla^k u)(x)$. Insert the results into (28) and we get

$$(\partial^k\psi)(y)=-\partial f\cdot(\nabla^k u)(x)\cdot B\otimes\cdots\otimes B$$
$$+\Sigma J^{a_1,\cdots,a_{k-1}}\cdot B\otimes\cdots\otimes B.$$

Of course $J^{a_1,\cdots,a_q}$ is the term which contains

$$(\nabla u)^{a_1}\otimes\cdots\otimes(\nabla^q u)^{a_q}.$$

This $J^{a_1,\cdots,a_q}$ also satisfies the same property as in (i), (ii) replacing $I^{a_1,\cdots,a_q}$ into $J^{a_1,\cdots,a_q}$.

Let $f'(y^1,\cdots,y^n,q^1,\cdots,q^n)$ be the coordinate expression of Exp by the coordinate at y, where $q^1,\cdots,q^n$ is a linear coordinate of T_yM. By using Lemma 8, we have

$$(\partial^k\varphi)(y)=\partial f'\cdot(\nabla^k v)(y)+\Sigma K^{a_1,\cdots,a_{k-1}},$$

where $K^{a_1,\cdots,a_{k-1}}$ also satisfies the same property as in (i), (ii). Therefore,

$$\begin{aligned}\partial f'\cdot(\nabla^k v)(y)&+\Sigma K^{a_1,\cdots,a_{k-1}}\\&=\partial f\cdot(\nabla^k u)(x)\cdot B\otimes\cdots\otimes B+\Sigma J^{a_1,\cdots,a_{k-1}}\cdot B\otimes\cdots\otimes B.\end{aligned}$$

By (23) and (24), we have that there is a constant L which depends only on $\tilde{U}$ such that $\|B\|\leqq L$. Since $\partial f'$ and ∂f are bounded, because ∂f (or $\partial f'$) is the derivative along the fibres and so has the meaning independent of the choice of coordinates (cf. first part of 3°), we have

$$|(\nabla^k v)(y)|-\Sigma|K^{a_1,\cdots,a_{k-1}}|\leqq C|(\nabla^k u)(x)|+D\Sigma|J^{a_1,\cdots,a_{k-1}}|. \qquad \cdots (29)$$

Replace all $|(\nabla^i u)(x)|$ by $e_{k_0}\|u\|_{i+k_0}$, if $i+k_0\leqq s-1$, where $k_0=\left[\frac{n}{2}\right]+1$ and e_{k_0} is the same constant as in Lemma 9. Then we have

$$|(\nabla^k v)(y)|\leqq C|(\nabla^k u)(x)|+P_k+Q_k+\Sigma|K^{a_1,\cdots,a_{k-1}}|,$$

where

$P_k=$ linear function of $|(\nabla^1 u)(x)|,\cdots,|(\nabla^{k-1}u)(x)|$ whose coefficients are polynomials of $\|u\|_{k_0},\cdots,\|u\|_{s-1}$.

$Q_k=$ polynomial of $\|u\|_{s-1}$ without constant term.

Use this relation successively, and combine this with (27). We have

$$|(\nabla^k v)(y)|\leqq C|(\nabla^k u)(x)|+P'_k+Q'_k \qquad \text{for any } k\geqq 0,$$

where P'_k, Q'_k satisfies the same property as P_k, Q_k above. Therefore, we have

PROPOSITION 3. *Let $i:\mathcal{D}_0\to\mathcal{D}_0$ be the group inversion. Then putting $j=\xi^{-1}i\xi$, j satisfies*

$$\|ju\|_s^2\leqq C'\|u\|_s^2+P_s(\|u\|_{s-1})\|u\|_{s-1}$$

for any $u\in U^{k_0+1}\cap\Gamma(T_M)$, where C' does not depend on s.

This Proposition shows the property (d) in 1°, Lemma 3.

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