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A note on the vanishing of certain 'L^2-cohomologies'

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Introduction

Let G be a connected noncompact semisimple Lie group admitting a finite dimensional faithful representation. Let K be a maximal compact subgroup of G. Throughout, we assume that G/K is a hermitian symmetric space. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{k}$ the subalgebra of $\mathfrak{g}$ corresponding to K. Then, as is well known, rank of $\mathfrak{k}$ = rank of $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{k}$. For an ordering of the roots Σ of $(\mathfrak{h}^C, \mathfrak{g}^C)$ compatible with the complex structure on G/K, let P be the set of positive roots and P_k the set of positive compact roots. Let ρ be half the sum of the roots in P. Let $\mathcal{F}$ be the set of all integral linear forms on $\mathfrak{h}^C$. Let

$$\mathcal{F}' = \{\lambda \in \mathcal{F} \mid \langle \lambda+\rho, \alpha \rangle \neq 0, \text{ for } \alpha \in P\}$$

and

$$\mathcal{F}'_0 = \{\lambda \in \mathcal{F}' \mid \langle \lambda+\rho, \alpha \rangle > 0, \text{ for } \alpha \in P_k\}.$$

For $\lambda \in \mathcal{F}'_0$, let τ_λ be the irreducible unitary representation of K with highest weight λ on a vector space V_λ. Let τ_λ^* be the contragredient representation of K on the dual V_λ^* to V_λ and let $E_{V_\lambda^*}$ be the holomorphic vector bundle on G/K associated to τ_λ^*. Let $H_2^{0,q}(E_{V_\lambda^*})$ be the Hilbert space of square integrable harmonic forms of type $(0, q)$ with coefficients in $E_{V_\lambda^*}$ and let π_λ^q be the unitary representation of G on $H_2^{0,q}(E_{V_\lambda^*})$. If $\lambda+\rho$ is "sufficiently regular" it was proved in [5, Theorem 2, § 7] that $H_2^{0,q}(E_{V_\lambda^*})=0$, if $q \neq q_\lambda$, where q_λ is the number of non-compact positive roots α such that $\langle \lambda+\rho, \alpha \rangle > 0$ and that $[\pi_\lambda^{q_\lambda}] = \omega(\lambda+\rho)^*$ where $[\pi_\lambda^{q_\lambda}]$ denotes the equivalence class of the representation $\pi_\lambda^{q_\lambda}$ and $\omega(\lambda+\rho)^*$ is the discrete class contragredient to the discrete class $\omega(\lambda+\rho)$ which corresponds to λ (the correspondence being in the sense of Lemma 2.4 in [5]).

Define for $\lambda \in \mathcal{F}'_0$,

$$P^{(\lambda)} = \{\alpha \in \Sigma \mid \langle \lambda+\rho, \alpha \rangle > 0\}.$$

$P^{(\lambda)}$ is the set of positive roots with respect to a linear order in Σ. The main theorem (Theorem 1, § 1) of this note is that if every noncompact root in $P^{(\lambda)}$ is totally positive (in the sense of definition, p. 752 in [2.b]) in the above linear order then $H_2^{0,q}(E_{V_\lambda^*})=0$ if $q\neq q_\lambda$ and that $[\pi_\lambda^{q_\lambda}]=\omega(\lambda+\rho)^*$.

In § 2, from the results of [6] about "the spaces of square integrable Dirac Spinors" we deduce the vanishing Theorems for "the spaces of square integrable harmonic forms with coefficients in $E_{V_\lambda^*}$" under some condition on the parameter λ which is less restrictive than the one in [5, Theorem 2, § 7].

§ 1.

Let G be a noncompact semisimple Lie group. We assume that G has a finite dimensional faithful representation and that the complexification G^C of G is simply connected. Let K be a maximal compact subgroup of G. We assume that G/K is hermitian symmetric. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{k}$ the Lie subalgebra of $\mathfrak{g}$ corresponding to K. Then as is well known rank of $\mathfrak{k}=$ rank of $\mathfrak{g}$. We now fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{h}\subseteqq\mathfrak{k}$. Let $\mathfrak{g}^C$ be the complexification of $\mathfrak{g}$. Let B be the Killing form of $\mathfrak{g}^C$. We define

$$\mathfrak{p}=\{Y\in\mathfrak{g}\mid B(X, Y)=0 \text{ for every } X\in\mathfrak{k}\}\,.$$

Then we have

$$\mathfrak{g}=\mathfrak{k}+\mathfrak{p},\quad \mathfrak{k}\cap\mathfrak{p}=0,\quad [\mathfrak{p},\mathfrak{p}]\subseteqq\mathfrak{k}\quad\text{and}\quad [\mathfrak{k},\mathfrak{p}]\subseteqq\mathfrak{p}\,.$$

For any subset $\mathfrak{m}$ of $\mathfrak{g}^C$ we denote by $\mathfrak{m}^C$ the complex subspace of $\mathfrak{g}^C$ generated by $\mathfrak{m}$. We identify $\mathfrak{p}^C$ in the usual way with the complexification of the real tangent space at $\{K\}\in G/K$. We denote by $\mathfrak{p}_+$ and $\mathfrak{p}_-$ the subspaces of $\mathfrak{p}^C$ consisting of antiholomorphic and holomorphic tangent vectors respectively of $\mathfrak{p}^C$. Then one knows that

$$\mathfrak{p}^C=\mathfrak{p}_++\mathfrak{p}_-,\quad \mathfrak{p}_+\cap\mathfrak{p}_-=0\quad\text{and}\quad [\mathfrak{k}^C,\mathfrak{p}_\pm]\subseteqq\mathfrak{p}_\pm\,.$$

Now put $\mathfrak{g}_u=\mathfrak{k}+\sqrt{-1}\,\mathfrak{p}$. Then $\mathfrak{g}_u$ is a compact real form of $\mathfrak{g}^C$. If θ denotes the conjugation of $\mathfrak{g}^C$ with respect to $\mathfrak{g}_u$, we write $X^*=-\theta X$, for $X\in\mathfrak{g}^C$. We define an inner product $(\,,)$ in $\mathfrak{g}^C$ by $(X, Y)=B(X, Y^*)$, for $X, Y\in\mathfrak{g}^C$. Let Σ be the set of nonzero roots of $\mathfrak{g}^C$ with respect to $\mathfrak{h}^C$. For each root $\alpha\in\Sigma$, we choose an eigenvector X_α belonging to the root α such that $(X_\alpha, X_\alpha)=1$. A root $\alpha\in\Sigma$ is called compact if $X_\alpha\in\mathfrak{k}^C$ and noncompact if $X_\alpha\in\mathfrak{p}^C$. We choose, as we can, a linear order in Σ, such that if P is the set of positive roots and P_n the set of noncompact positive roots with respect to that linear order, then

$$\mathfrak{p}_+=\sum_{\alpha\in P_n}\mathfrak{g}^\alpha\,.$$

We denote by P_k the set of all compact positive roots. Then we have $P=P_k\cup P_n$. For any linear form λ on $\mathfrak{h}^C$, we shall denote by H_λ the element of $\mathfrak{h}^C$, such that $B(H_\lambda, H)=\lambda(H)$ for all $H\in\mathfrak{h}^C$. For any pair (λ, μ) of linear forms on $\mathfrak{h}^C$, we put $\langle\lambda, \mu\rangle=\lambda(H_\mu)$. Let $\mathcal{F}$ be the set of all integral linear forms on $\mathfrak{h}^C$, i.e. the set of linear forms λ on $\mathfrak{h}^C$ such that $2\frac{\langle\lambda, \alpha\rangle}{\langle\alpha, \alpha\rangle}$ is an integer for every root α. We put

$$\mathcal{F}'=\{\lambda\in\mathcal{F}\,;\,\langle\lambda+\rho, \alpha\rangle\neq 0,\ \text{for}\ \alpha\in\Sigma\}$$

and

$$\mathcal{F}'_0=\{\lambda\in\mathcal{F}'\,;\,\langle\lambda+\rho, \alpha\rangle> 0,\ \text{for}\ \alpha\in P_k\}$$

where $\rho=\frac{1}{2}\sum_{\alpha\in P}\alpha$. Then one can verify that any λ in $\mathcal{F}'$ is in $\mathcal{F}'_0$ if and only if λ is dominant with respect to P_k (i.e. $\langle\lambda, \alpha\rangle\geqq 0$ for every $\alpha\in P_k$).

Now let $\lambda\in\mathcal{F}'_0$ and let τ_λ be the irreducible unitary representation of K with highest weight λ on a space V_λ. Let τ^*_λ denote the unitary representation of K contragredient to τ_λ on the dual space V^*_λ to V_λ. Let $E_{V^*_\lambda}$ denote the holomorphic vector bundle on G/K, associated with the representation τ^*_λ of K (see [5, § 1]). The inner product $(\,,)$ in $\mathfrak{g}^C$ restricts to a K invariant inner product on $\mathfrak{p}$ and gives rise to a hermitian metric on G/K. Also, the K invariant inner product in V^*_λ gives rise to a canonical hermitian metric on $E_{V^*_\lambda}$. Let $C^{0,q}(E_{V^*_\lambda})$ (resp. $L^{0,q}_2(E_{V^*_\lambda})$) denote the space of all C^∞ (resp. square integrable) differential forms of type $(0, q)$ with coefficients in $E_{V^*_\lambda}$. G acts on the bundle $E_{V^*_\lambda}$ and gives rise to an action of G on the space of forms on G/K with values in $E_{V^*_\lambda}$. The action of G on $C^{0,q}(E_{V^*_\lambda})$ gives rise to an action ν of $\mathfrak{g}^C$ on $C^{0,q}(E_{V^*_\lambda})$ which extends to an action, also denoted by ν, of the universal enveloping algebra $U(\mathfrak{g}^C)$ of $\mathfrak{g}^C$ on $C^{0,q}(E_{V^*_\lambda})$. Let $H^{0,q}_2(E_{V^*_\lambda})$ be the space of square integrable harmonic forms of type $(0, q)$ on G/K with coefficients in $E_{V^*_\lambda}$ (see § 4, in [5]). One knows that $H^{0,q}_2(E_{V_\lambda})\subseteqq C^{0,q}(E_{V^*_\lambda})$. Moreover, $H^{0,q}_2(E_{V^*_\lambda})$ is a closed subspace of $L^{0,q}_2(E_{V^*_\lambda})$ and hence a Hilbert space. Also $H^{0,q}_2(E_{V^*_\lambda})$ is invariant under the action of G on $L^{0,q}_2(E_{V^*_\lambda})$. Thus, we get a unitary representation, denoted π^q_λ, of G on $H^{0,q}_2(E_{V^*_\lambda})$.

Now, let $\mathcal{E}_d$ be the set of discrete class representations of G. (see § 1 in [5]). Denote by Ad^q_+ (resp. Ad^q_-) the representation of K on $\wedge^q\mathfrak{p}_+$ (resp. $\wedge^q\mathfrak{p}_-$) induced by the adjoint action of K on $\mathfrak{p}_+$ (resp. $\mathfrak{p}_-$) and put $\tau^q_\lambda=Ad^q_+\otimes\tau_\lambda$. We now have the following (Proposition 4.1 in [5])

PROPOSITION 1.1. *Put*

$$\mathcal{E}_d(\lambda)=\{\omega\in\mathcal{E}_d\,;\,\chi_\omega(\Omega)=\langle\lambda+2\rho, \lambda\rangle\}$$

where χ_ω denotes the infinitesimal character of ω and $\Omega\in U(\mathfrak{g}^C)$ the Casimir of G. Then, we have

$$[\pi^q_\lambda]=\bigoplus_{\omega\in\mathcal{E}_d(\lambda)}(\omega|K:[\tau^q_\lambda])\omega^*\quad(\textit{finite sum})$$

where $[\pi^q_\lambda]$ denotes the equivalence class of the representation π^q_λ of G (similarly for $[\tau^q_\lambda]$) and $(\omega|K:[\tau^q_\lambda])$ denotes the intertwining number of $\omega|K$ and $[\tau^q_\lambda]$ (see

§1 *in* [5]).

For any set Q of linear forms on $\mathfrak{h}^C$, we define

$$\langle Q \rangle = \sum_{\alpha \in Q} \alpha$$

($\langle Q \rangle = 0$ if $Q = \phi$, the empty set) and

$$[Q] = \text{the number of elements in the set } Q.$$

Fix any $\lambda \in \mathcal{F}_0'$. Let

$$Q_\lambda = \{\alpha \in P_n \mid \langle \lambda + \rho, \alpha \rangle > 0\}$$

and define

$$q_\lambda = [Q_\lambda].$$

Let $m = [P_n] = \frac{1}{2} \dim G/K$ and fix any q such that $0 \leqq q \leqq m$. Define

$$\Gamma_q = \{\langle Q \rangle \mid Q \subseteqq P_n, [Q] = q\}.$$

Define

$$P^{(\lambda)} = \{\alpha \in \Sigma \mid \langle \lambda + \rho, \alpha \rangle > 0\}.$$

Then $P^{(\lambda)}$ is the set of positive roots with respect to a linear order in Σ. Following [2.b], if Σ_+ is the set of positive roots with respect to some linear order in Σ, then we say that a noncompact root $\alpha \in \Sigma_+$ is *totally positive* (with respect to that ordering) if $\alpha + \beta$ is a noncompact root in Σ_+ for any compact root β such that $\alpha + \beta$ is a root. We now have the following

THEOREM 1. *Let $\lambda \in \mathcal{F}_0'$ and assume that with respect to the linear order in Σ for which $P^{(\lambda)}$ is the set of positive roots, every noncompact root in $P^{(\lambda)}$ is totally positive. Then*

$$H_2^{0,q}(E_{V_\lambda^*}) = 0, \qquad \text{if } q \neq q_\lambda$$

and

$$[\pi_\lambda^{q_\lambda}] = \omega(\lambda + \rho)^*$$

where $\omega(\lambda + \rho)^*$ is the discrete class contragredient to the discrete class $\omega(\lambda + \rho)$ which corresponds to λ, in the sense of Lemma 2.4 in [5] *and $[\pi_\lambda^{q_\lambda}]$ the equivalence class of the representation $\pi_\lambda^{q_\lambda}$ of G. Moreover,*

$$(\omega(\lambda + \rho) : [\tau_{\lambda + \langle Q_\lambda \rangle}]) = 1$$

where $\tau_{\lambda + \langle Q_\lambda \rangle}$ is the representation of K with $\lambda + \langle Q_\lambda \rangle$ as highest weight.

PROOF. First we make a few observations which are consequences of our assumption about $P^{(\lambda)}$. Let W_G be the subgroup of the Weyl group of $(\mathfrak{h}^C, \mathfrak{g}^C)$ generated by reflections with respect to compact roots. Let

$$\bar{P}^{(\lambda)} = -\kappa P^{(\lambda)},$$

where κ is the unique element of W_G, such that $\kappa P_k = -P_k$. Clearly, $\bar{P}^{(\lambda)}$ is the set of positive roots with respect to a linear order in Σ. Note that

$$P^{(\lambda)} = P_k \cup Q_\lambda \cup -Q'_\lambda,$$

where Q'_λ is the complement of Q_λ in P_n. Since by assumption in the positive root system $P^{(\lambda)}$, every noncompact root is totally positive, we have $sP_n^{(\lambda)} = P_n^{(\lambda)}$, for every $s \in W_G$, where $P_n^{(\lambda)}$ is the set of noncompact roots in $P^{(\lambda)}$. Also, $sP_n = P_n$ for every $s \in W_G$. Since, clearly, we have $Q_\lambda = P_n \cap P_n^{(\lambda)}$ we see that for every $s \in W_G$,

(1.1) $$sQ_\lambda = sP_n \cap sP_n^{(\lambda)} = P_n \cap P_n^{(\lambda)} = Q_\lambda .$$

Thus,

$$\bar{P}^{(\lambda)} = P_k U - Q_\lambda U Q'_\lambda .$$

Now, assume that $H_2^{0,q}(E_{V_\lambda^*}) \neq 0$. We know that the representation π_λ^q of G on $H_2^{0,q}(E_{V_\lambda^*})$ decomposes into a finite direct sum of discrete class representations (Proposition 1.1). Let ω be a discrete class such that ω^* occurs in this decomposition. Let π be a member of the equivalence class ω and H the representation space of π. Denote also by π the derived representation of the enveloping algebra $U(\mathfrak{g}^C)$ of $\mathfrak{g}^C$ on the space of analytic vectors for π in H. Then, we assert that with respect to the linear order in Σ for which $\bar{P}^{(\lambda)}$ is the set of positive roots there exists a positive extreme weight vector (for definition see pp. 750-751 in [2.b]) with weight μ of the form $\lambda+\gamma$, for some $\gamma \in \Gamma_q$. For proving this we proceed as follows:

Let χ_ω be the infinitesimal character of ω. By Proposition 1.1 we know that

(1) $$\chi_\omega(\Omega) = \langle \lambda + 2\rho, \lambda \rangle$$

and

(2) $$(\omega | K : [\tau_\lambda^q]) \neq 0 .$$

The second condition implies that there exists an irreducible representation δ of K which is a subrepresentation of both $\pi | K$ and τ_λ^q. Let H_δ be the subspace of H spanned by elements which transform under $\pi | K$ according to δ. Fix a unit weight vector $\phi_\mu \in H_\delta$ belonging to the highest weight μ of δ. Then ϕ_μ is infinitely differentiable under π and we have by (1)

$$\pi(\Omega)\phi_\mu = \chi_\omega(\Omega)\phi_\mu = \langle \lambda + 2\rho, \lambda \rangle \phi_\mu .$$

From the fact that μ is the highest weight of an irreducible subrepresentation of τ_λ^q, one can show that

(1.2) $$\mu = \lambda + \gamma$$

for some $\gamma \in \Gamma_q$. In fact if σ is any finite dimensional representation of $\mathfrak{k}^C$, one can easily show that an irreducible subrepresentation of $\tau_\lambda \otimes \sigma$ has $\lambda + \nu$

as highest weight, where ν is a suitable weight of σ.

Now, choose a Weyl basis $\{E_\alpha\}_{\alpha\in\Sigma}$ of $\mathfrak{g}^C$ (mod $\mathfrak{h}^C$) with respect to a compact real form $\mathfrak{g}_u=\mathfrak{k}+\sqrt{-1}\,\mathfrak{p}$. Let Ω_k be the Casimir of K. Then we have

$$\Omega-\Omega_K=\sum_{\alpha\in P_n}(E_\alpha E_{-\alpha}+E_{-\alpha}E_\alpha)$$
$$=\sum_{\alpha\in\bar{P}_n^{(\lambda)}}(E_\alpha E_{-\alpha}+E_{-\alpha}E_\alpha)$$

where $\bar{P}_n^{(\lambda)}$ is the set of noncompact roots in $\bar{P}^{(\lambda)}$. After some computations (see proof of Theorem 2, §7 in [**5**]) we see that

$$-2\sum_{\alpha\in\bar{P}_n^{(\lambda)}}\|\pi(E_\alpha)\phi_\mu\|^2=\langle\lambda+2\rho,\lambda\rangle-\langle\mu-2\rho_n^{(\lambda)}+2\rho_k,\mu\rangle$$

where $\rho_n^{(\lambda)}=\frac{1}{2}\sum_{\alpha\in P_n^{(\lambda)}}\alpha=-\frac{1}{2}\sum_{\alpha\in\bar{P}_n^{(\lambda)}}\alpha$. Note that we have from (1.1), $\rho_n^{(\lambda)}=\langle Q_\lambda\rangle-\rho_n$. Substituting for μ from (1.2) we have

$$-2\sum_{\alpha\in\bar{P}_n^{(\lambda)}}\|\pi(E_\alpha)\phi_\mu\|^2$$
$$=\langle\lambda+2\rho,\lambda\rangle-\langle\lambda+\gamma-2\rho_n^{(\lambda)}+2\rho_k,\lambda+\gamma\rangle$$
$$=\langle\lambda+\rho,\lambda+\rho\rangle-\langle\rho,\rho\rangle-\langle\lambda+\gamma-\rho_n^{(\lambda)}+\rho_k,\lambda+\gamma-\rho_n^{(\lambda)}+\rho_k\rangle$$
$$+\langle\rho_k-\rho_n^{(\lambda)},\rho_k-\rho_n^{(\lambda)}\rangle$$
$$=\langle\lambda+\rho,\lambda+\rho\rangle-\langle\rho,\rho\rangle-\langle\lambda+\rho+\gamma-\rho_n-\rho_n^{(\lambda)},\lambda+\rho+\gamma-\rho_n-\rho_n^{(\lambda)}\rangle$$
$$+\langle\rho_k-\rho_n^{(\lambda)},\rho_k-\rho_n^{(\lambda)}\rangle.$$

Note that $\rho_k-\rho_n^{(\lambda)}=\frac{1}{2}\langle\bar{P}^{(\lambda)}\rangle=\sigma\cdot\rho$ for some σ belonging to the Weyl group of $(\mathfrak{h}^C,\mathfrak{g}^C)$ since $\bar{P}^{(\lambda)}$ is the set of positive roots with respect to some linear ordering in Σ. Hence, $\langle\rho_k-\rho_n^{(\lambda)},\rho_k-\rho_n^{(\lambda)}\rangle=\langle\rho,\rho\rangle$. Thus,

$$-2\sum_{\alpha\in\bar{P}_n^{(\lambda)}}\|\pi(E_\alpha)\phi_\mu\|^2=-2\langle\lambda+\rho,\gamma-\rho_n-\rho_n^{(\lambda)}\rangle-\langle\gamma-\rho_n-\rho_n^{(\lambda)},\gamma-\rho_n-\rho_n^{(\lambda)}\rangle$$
$$=-2\langle\lambda+\rho,\gamma-\langle Q_\lambda\rangle\rangle-\langle\gamma-\langle Q_\lambda\rangle,\gamma-\langle Q_\lambda\rangle\rangle.$$

Since $\lambda+\rho$ is a regular integral linear form which is dominant with respect to $P^{(\lambda)}$, we see that $\lambda+\rho-\rho^{(\lambda)}$ is dominant with respect to $P^{(\lambda)}$. Since $\gamma\in\Gamma_q$, $\gamma=\langle Q\rangle$ for some subset Q of P_n. Thus

$$\gamma-\langle Q_\lambda\rangle=\langle Q\cap Q'_\lambda\rangle-\langle Q_\lambda\cap Q'\rangle$$

where Q'_λ and Q' are respectively the complements of the sets Q_λ and Q in P_n. Now,

$$-2\sum_{\alpha\in\bar{P}_n^{(\lambda)}}\|\pi(E_\alpha)\phi_\mu\|^2=-2\langle\lambda+\rho-\rho^{(\lambda)},\gamma-\langle Q_\lambda\rangle\rangle$$
$$-\langle\gamma-\langle Q_\lambda\rangle+2\rho^{(\lambda)},\gamma-\langle Q_\lambda\rangle\rangle.$$

Consider the two terms on the right hand side. Since $\gamma-\langle Q_\lambda\rangle=\langle Q\cap Q'_\lambda\rangle-\langle Q_\lambda\cap Q'\rangle$ and since $P^{(\lambda)}=P_k\cup Q_\lambda\cup(-Q'_\lambda)$ it follows using the fact that $\lambda+\rho-\rho^{(\lambda)}$ is dominant with respect to $P^{(\lambda)}$ that $-2\langle\lambda+\rho-\rho^{(\lambda)}, \gamma-\langle Q_\lambda\rangle\rangle$ is nonnegative. Also

$$\begin{aligned}-\langle\gamma-\langle Q_\lambda\rangle+2\rho^{(\lambda)}, \gamma-\langle Q_\lambda\rangle\rangle\\ &=-\langle\gamma-\langle Q_\lambda\rangle+\rho^{(\lambda)}, \gamma-\langle Q_\lambda\rangle+\rho^{(\lambda)}\rangle+\langle\rho^{(\lambda)}, \rho^{(\lambda)}\rangle\\ &=-\langle\rho^{(\lambda)}-\langle Q_1\rangle, \rho^{(\lambda)}-\langle Q_1\rangle\rangle+\langle\rho^{(\lambda)}, \rho^{(\lambda)}\rangle,\end{aligned}$$

where $Q_1=(Q_\lambda\cap Q')\cup(-Q'_\lambda\cap-Q)$. Note that $Q_1\subset P^{(\lambda)}$. By [4, Lemma 5.9], $\rho^{(\lambda)}-\langle Q_1\rangle$ is a weight of the irreducible representation of $\mathfrak{g}^C$ with $\rho^{(\lambda)}$ as highest weight with respect to $P^{(\lambda)}$. Hence $\langle\rho^{(\lambda)}-\langle Q_1\rangle, \rho^{(\lambda)}-\langle Q_1\rangle\rangle\leqq\langle\rho^{(\lambda)}, \rho^{(\lambda)}\rangle$ so that $-\langle\gamma-\langle Q_\lambda\rangle+2\rho^{(\lambda)}, \gamma-\langle Q_\lambda\rangle\rangle$ is nonnegative. Thus it follows that $-2\sum_{\alpha\in\bar{P}_n^{(\lambda)}}\|\pi(E_\alpha)\psi_\mu\|^2$ is nonnegative. But, clearly $-2\sum_{\alpha\in\bar{P}_n^{(\lambda)}}\|\pi(E_\alpha)\psi_\mu\|^2\leqq 0$. Hence, we conclude that $\sum_{\alpha\in\bar{P}_n^{(\lambda)}}\|\pi(E_\alpha)\psi_\mu\|^2=0$. Thus, $\pi(E_\alpha)\psi_\mu=0$ for every $\alpha\in\bar{P}_n^{(\lambda)}$. But by the choice of ψ_μ, $\pi(E_\alpha)\psi_\mu=0$, $\forall\alpha\in P_k$. Since $\bar{P}^{(\lambda)}=P_k\cup\bar{P}_n^{(\lambda)}$ our claim in the beginning is proved.

Now, suppose $H_2^{0,q}(E_{V_\lambda^*})\neq 0$ and $H_2^{0,q'}(E_{V_\lambda^*})\neq 0$ where q and q' are distinct and $0\leqq q, q'\leqq n$. If ω (resp. ω') is a discrete class such that ω^* (resp. ω'^*) occurs in the decomposition of $[\pi_\lambda^q]$ (resp. $[\pi_\lambda^{q'}]$) then we assert that $\omega\neq\omega'$. This can be proved as follows. The representations of $U(\mathfrak{g}^C)$ on the spaces of analytic vectors for the representations ω and ω' possess positive extremal weight vectors (for definition see pp. 750-751 in [2.b]) of weights μ and μ' respectively with respect to $\bar{P}^{(\lambda)}$ where μ and μ' are of the form $\mu=\lambda+\gamma$ and $\mu'=\lambda+\gamma'$ with $\gamma\in\Gamma_q$ and $\gamma'\in\Gamma_{q'}$. Let $J\in\mathfrak{h}$ be the unique element such that $ad\,J|\mathfrak{p}$ gives the complex structure on the real tangent space at $\{K\}\in G/K$, when $\mathfrak{p}$ is identified with that tangent space in the usual way. (One knows that such a J exists. See [3, Theorem 4.5].) Thus

$$\mathfrak{p}_+=\{Y\in\mathfrak{p}^C\mid ad\,J(Y)=-iY\}$$

and

$$\mathfrak{p}_-=\{Y\in\mathfrak{p}^C\mid ad\,J(Y)=iY\}.$$

Hence, we have $\alpha(J)=-i$ for every $\alpha\in P_n$. Thus

$$\begin{aligned}\mu(J)&=\lambda(J)+\gamma(J)\\ &=\lambda(J)-qi\end{aligned}$$

since $\gamma\in\Gamma_q$. Similarly, $\mu'(J)=\lambda(J)-q'i$. Since $q\neq q'$, we then conclude that $\mu\neq\mu'$. Then, using [Lemma 2, 2.b], we conclude that the representations of $U(\mathfrak{g}^C)$ on the spaces of analytic vectors for ω and ω' are not equivalent.

Consequently $\omega\neq\omega'$ and hence also $\omega^*\neq\omega'^*$.

Since for each q, $[\pi_\lambda^q]$ decomposes into a finite direct sum of irreducible unitary representations of G (Proposition 1.1), we now conclude using the "alternating sum formula," (i.e. [5, Theorem 2, § 6]) and [2.a, Theorem 6 and its corollary] that there exists q_0 such that $H_2^{0,q}(E_{V_\lambda^*})=0$ for $q\neq q_0$ and that $[\pi_\lambda^{q_0}]=\omega(\lambda+\rho)^*$. Now, it follows using Proposition 1.1, that if $0\leqq q\leqq n$ and ω is a discrete class such that $\chi_\omega(\Omega)=\langle\lambda+2\rho,\lambda\rangle$ where χ_ω is the infinitesimal character of ω and $\Omega\in U(\mathfrak{g}^C)$ the Casimir, then

$$(\omega\,|\,K:[\tau_\lambda^q])=0 \qquad \text{if } \omega\neq\omega(\lambda+\rho)$$

and

$$(\omega(\lambda+\rho)\,|\,K:[\tau_\lambda^q])=\begin{cases}0 & \text{if } q\neq q_0\\ 1 & \text{if } q=q_0.\end{cases}$$

We now prove that $q_0=q_\lambda$. From the assumption about $P^{(\lambda)}$ and from the fact that $\bar{P}^{(\lambda)}=-\kappa P^{(\lambda)}$, it is easy to see that with respect to the linear order in Σ in which $\bar{P}^{(\lambda)}$ is the set of positive roots, every noncompact root in $\bar{P}^{(\lambda)}$ is totally positive.

Now let

$$\bar{\rho}^{(\lambda)}=\frac{1}{2}\sum_{\alpha\in\bar{P}^{(\lambda)}}\alpha$$

and let

$$\lambda'=\lambda+\rho-\bar{\rho}^{(\lambda)}.$$

Note that $\bar{\rho}^{(\lambda)}=\rho_k-\frac{1}{2}\langle Q_\lambda\rangle+\frac{1}{2}\langle Q'_\lambda\rangle$, so that one has

$$\lambda'=\lambda+\langle Q_\lambda\rangle.$$

Observe that λ' is dominant with respect to P_k; as a matter of fact λ is dominant with respect to P_k since $\lambda\in\mathcal{F}'_0$ and also $\langle Q_\lambda\rangle$ is dominant with respect to P_k (see proof of [5, Corollary 2, § 7]). Let $\tau_{\lambda'}$ be the irreducible representation of K with λ' as highest weight. Let $\bar{\mathfrak{p}}^{(\lambda)}=\sum_{\alpha\in\bar{P}_n^{(\lambda)}}\mathfrak{g}^\alpha$. Since with respect to the linear order in Σ for which $\bar{P}^{(\lambda)}$ is the set of positive roots every noncompact root in $\bar{P}^{(\lambda)}$ is totally positive it follows that $\bar{\mathfrak{p}}^{(\lambda)}$ is stable under the adjoint action of K. Denote by $\bar{\tau}$ the representation of K on $\wedge\bar{\mathfrak{p}}^{(\lambda)}$ induced by the adjoint action of K on $\bar{\mathfrak{p}}^{(\lambda)}$. We assert that

$$\tau_{\lambda'}\otimes\bar{\tau}\cong\sum_{q=0}^{n}\tau_\lambda\otimes Ad_+^q$$

where Ad_+^q is the representation of K on $\wedge^q\mathfrak{p}_+$ induced by the adjoint action of K on $\mathfrak{p}_+$. This can be proved as follows: For any representation δ of K we denote by Trace δ the character of the representation δ. Then we have by Weyl's character formula

$$\text{Trace}\ \tau_\lambda|_H = \frac{\sum_{s\in W_G} \varepsilon(s) e^{s(\lambda+\rho_k)}}{\sum_{s\in W_G} \varepsilon(s) e^{s\rho_k}}$$

and similarly

$$\text{Trace}\ \tau_{\lambda'}|_H = \frac{\sum_s \varepsilon(s) e^{s(\lambda'+\rho_k)}}{\sum_s \varepsilon(s) e^{s\rho_k}}.$$

Also

$$\text{Trace}\ Ad_+^q|_H = \sum_{Q\subseteq P_n,\ [Q]=q} e^{\langle Q\rangle}$$

and

$$\text{Trace}\ \bar\tau|_H = \sum_{Q\subseteq \bar P_n^{(\lambda)}} e^{\langle Q\rangle}.$$

Now,

$$\begin{aligned}
\text{Trace}\ (\tau_{\lambda'}\otimes\bar\tau)|_H &= (\text{Trace}\ \tau_{\lambda'}|_H)\cdot(\text{Trace}\ \bar\tau|_H) \\
&= \frac{\sum_{s\in W_G} \varepsilon(s) e^{s(\lambda+\langle Q_\lambda\rangle+\rho_k)}}{\sum_s \varepsilon(s) e^{s\rho_k}} \cdot \sum_{Q\subseteq \bar P_n^{(\lambda)}} e^{\langle Q\rangle} \\
&= e^{\langle Q_\lambda\rangle} \cdot \frac{\sum_s \varepsilon(s) e^{s(\lambda+\rho_k)}}{\sum_s \varepsilon(s) e^{s\rho_k}} \cdot \sum_{Q\subset \bar P_n^{(\lambda)}} e^{\langle Q\rangle} \\
&\qquad (\text{since } s\langle Q_\lambda\rangle = \langle Q_\lambda\rangle \text{ by } (1.1)) \\
&= \frac{\sum_s \varepsilon(s) e^{s(\lambda+\rho_k)}}{\sum_s \varepsilon(s) e^{s\rho_k}} \cdot \sum_{Q\subset \bar P_n^{(\lambda)}} e^{\langle Q\rangle+\langle Q_\lambda\rangle} \\
&= \frac{\sum_s \varepsilon(s) e^{s(\lambda+\rho_k)}}{\sum_s \varepsilon(s) e^{s\rho_k}} \cdot \sum_{Q\subset P_n} e^{\langle Q\rangle}.
\end{aligned}$$

Thus the characters of the representations $\tau_{\lambda'}\otimes\bar\tau$ and $\bigoplus_q (\tau_\lambda\otimes Ad_+^q)$ are equal and hence these two representations are equivalent.

One can introduce a new G invariant complex structure on G/K such that when the complexification of the real tangent space at $\{K\}\in G/K$ is as usual identified with $\mathfrak{p}^C$, the space of antiholomorphic tangent vectors in $\mathfrak{p}^C$ is precisely $\sum_{\alpha\in\bar P_n^{(\lambda)}} \mathfrak{g}^\alpha$. (We show this when $\mathfrak{g}$ is simple, the general case being easily deducible from this. Thus let P be a positive root system in Σ compatible with a G invariant complex structure on G/K. Let P_k and P_n be the set of compact and noncompact roots in P. Then one knows that $P_k\cup(-P_n)$ is also a positive root system in Σ which is compatible with a G invariant complex structure on G/K. Now let $\tilde P$ be any positive root system in Σ such that every noncompact root in $\tilde{P}$ is totally positive and such that $\tilde{P}_k = P_k$ where $\tilde{P}_k$ is the set of compact roots in $\tilde{P}$. One knows that there exists exactly one noncompact simple root α in $\tilde{P}$. (See [2.b, Corollary 2, § 5].) Then it is clear that $\tilde{P} = P$ or $\tilde{P} = P_k \cup (-P_n)$ according as $\alpha \in P_n$ or $\alpha \in -P_n$.) Let $\bar{\tau}^q$ be the representation of K on $\wedge^q \bar{\mathfrak{p}}^{(\lambda)}$ which is induced by the adjoint action of K on $\bar{\mathfrak{p}}^{(\lambda)}$. The representation $\tau^*_{\lambda'}$ of K on $V^*_{\lambda'}$ induces a vector bundle $E_{V^*_{\lambda'}}$ on G/K and as in [5, § 1] $E_{V^*_{\lambda'}}$ can be made into a holomorphic vector bundle (the complex structure of G/K being the new complex structure). We have $\lambda' + \bar{\rho}^{(\lambda)} = \lambda + \rho$. Since $\langle \lambda' + \bar{\rho}^{(\lambda)}, \alpha \rangle > 0$ for every compact root in $\bar{P}^{(\lambda)}$ and $\langle \lambda' + \bar{\rho}^{(\lambda)}, \alpha \rangle < 0$ for every noncompact root in $\bar{P}^{(\lambda)}$ one knows that (see [2.c] or [1]) the space H of square integrable (with respect to a hermitian metric on $E_{V^*_{\lambda'}}$ induced by a K invariant metric on $V^*_{\lambda'}$) holomorphic sections of $E_{V^*_{\lambda'}}$ is nonzero. H is nothing but the space of square integrable harmonic forms of type (0, 0) (See [5, § 4] for definition) on G/K with coefficients in the vector bundle $E_{V^*_{\lambda'}}$. If π denotes the action of $U(\mathfrak{g}^C)$ on the space $C(E_{V^*_{\lambda'}})$ of C^∞ sections of the bundle $E_{V^*_{\lambda'}}$ which is derived from the action of G on $C(E_{V^*_{\lambda'}})$, then by [5, Lemma 1.1] we have

$$H = \{\varphi \in C(E_{V^*_{\lambda'}}) \mid \varphi \text{ square integrable, } \pi(\Omega)\varphi = \langle \lambda' + 2\bar{\rho}^{(\lambda)}, \lambda' \rangle \varphi\}$$

where $\Omega \in U(\mathfrak{g}^C)$ is the Casimir of $\mathfrak{g}^C$. Let $[\beta]$ be the equivalence class of the unitary representation β of G on H. Then we know that $[\beta]$ is a finite sum of discrete classes of G (Proposition 1.1). Since $H \neq 0$ there exists a discrete class ω_0 of G such that ω_0^* occurs in the decomposition of $[\beta]$. Note that by Proposition 1.1 we have

i) $$\chi_{\omega_0}(\Omega) = \langle \lambda' + 2\bar{\rho}^{(\lambda)}, \lambda' \rangle$$

and

ii) $$(\omega_0 | K : [\tau_{\lambda'}]) \neq 0 .$$

Observe that since $\lambda' + \bar{\rho}^{(\lambda)} = \lambda + \rho$, we have

$$\begin{aligned}\langle \lambda' + 2\bar{\rho}^{(\lambda)}, \lambda' \rangle &= |\lambda' + \bar{\rho}^{(\lambda)}|^2 - |\bar{\rho}^{(\lambda)}|^2 = |\lambda + \rho|^2 - |\rho|^2 \\ &= \langle \lambda + 2\rho, \lambda \rangle .\end{aligned}$$

The representation $\tau_{\lambda'} = \tau_{\lambda + \langle Q_\lambda \rangle}$ is a subrepresentation of the representation $\tau_\lambda^{q_\lambda}$ of K on $V_\lambda \otimes \wedge^{q_\lambda} \mathfrak{p}_+$. (This is because, as we already saw, $\langle Q_\lambda \rangle$ is the highest weight of an irreducible component of the representation of K on $\wedge^{q_\lambda} \mathfrak{p}_+$.) Thus from i) and ii) we have

a) $$\chi_{\omega_0}(\Omega) = \langle \lambda + 2\rho, \lambda \rangle$$

and

b) $$(\omega_0 | K : [\tau_\lambda^{q_\lambda}]) \neq 0 .$$

Thus, using Proposition 1.1, it follows that ω_0^* occurs in the decomposition of $[\pi_\lambda^{q_\lambda}]$. Then, by what we have proved already it follows that

i) $$q_0 = q_\lambda$$

ii) $$[\pi_\lambda^{q_\lambda}] = \omega_0^* = \omega(\lambda+\rho)^*$$

and

iii) $$(\omega(\lambda+\rho)\,|\,K : [\tau_{\lambda+<Q_\lambda>}]) = 1\,.$$

(Q. E. D.)

REMARK. Now, choose an element π in the equivalence class $\omega(\lambda+\rho)$ and let H be the representation space of π. Let $\delta = \tau_{\lambda+<Q_\lambda>}$ and let H_δ be the subspace spanned by the set of vectors in H which transform under $\pi|K$ according to δ. Then the proof of Theorem 1 actually shows that if $\psi_{\lambda+<Q_\lambda>}$ is a nonzero element of H which belongs to the highest weight $\lambda+\langle Q_\lambda\rangle$ then $\pi(E_\alpha)\psi_{\lambda+<Q_\lambda>} = 0$ for every $\alpha \in \bar{P}^{(\lambda)}$.

REMARK. The set of discrete classes realized in Theorem 1 by L^2-cohomology method, i. e.,
$\{\omega \in \mathcal{E}_d \mid \omega = \omega(\lambda+\rho)^*$ for some $\lambda \in \mathcal{F}_0'$ such that any non-compact root in the positive root system $\{\alpha \in \Sigma \mid \langle \lambda+\rho, \alpha\rangle > 0\}$ is totally positive$\}$
is just the subset of $\mathcal{E}_d$ whose classes are constructed by Harish-Chandra in [2.c]. When G is simple, the set of $\lambda \in \mathcal{F}_0'$ satisfying the condition of Theorem 1 is just the set

$$\{\lambda \in \mathcal{F}_0' \mid q_\lambda = 0 \text{ or } n(=[P_n])\}\,.$$

§ 2.

We continue with the notation of the previous section. Let $so(\mathfrak{p})$ be the Lie subalgebra of $\mathrm{End}(\mathfrak{p})$ which corresponds to the rotation group $SO(\mathfrak{p}) \subseteqq \mathrm{Aut}(\mathfrak{p})$, under the positive definite bilinear form $B|\mathfrak{p}$, where B is the Killing form of $\mathfrak{g}^C$. We make the following observations the details of which can be found in [6].

Let $\sigma : so(\mathfrak{p}) \to \mathrm{End}(L)$ be the spin representation of $so(\mathfrak{p})$. Then σ is the direct sum of two subrepresentations $\sigma^\pm : so(\mathfrak{p}) \to \mathrm{End}(L^\pm)$ which are called the half spin representations of $so(\mathfrak{p})$. Let $\alpha : \mathfrak{k} \to so(\mathfrak{p})$ be the homomorphism induced by the adjoint action of $\mathfrak{k}$ on $\mathfrak{p}$. Let χ, χ^+ and χ^- be the representations of $\mathfrak{k}$ defined by $\chi = \sigma\circ\alpha$, $\chi^+ = \sigma^+\circ\alpha$ and $\chi^- = \sigma^-\circ\alpha$. The sets Γ, Γ^+ and Γ^- of weights of the representations χ, χ^+ and χ^- respectively are given by

$$\Gamma = \{\rho_n - \langle Q\rangle \mid Q \subseteqq P_n\}\,,$$

$$\Gamma^+ = \{\rho_n - \langle Q\rangle \mid Q \subseteqq P_n,\ [Q] \text{ is even}\}$$

and

$$\Gamma^- = \{\rho_n - \langle Q\rangle \mid Q \subseteqq P_n,\ [Q] \text{ is odd}\}\,.$$

Define a subset W^1 of the Weyl group $W(\mathfrak{h}^C, \mathfrak{g}^C)$ by setting

$$W^1=\{\sigma\in W(\mathfrak{h}^C, \mathfrak{g}^C) \mid \sigma(-P)\cap P\subseteqq P_n\}$$

where $P\subseteqq \Sigma$ is the fixed positive root system compatible with the complex structure on G/K and P_n is the set of noncompact roots in P. For $\sigma\in W^1$ we define

$$j(\sigma)=+, \qquad \text{if } [\sigma(-P)\cap P] \text{ is even}$$

and

$$j(\sigma)=-, \qquad \text{if } [\sigma(-P)\cap P] \text{ is odd}.$$

For every $\sigma\in W^1$, $\sigma\rho-\rho_k$ is dominant with respect to P_k and the representation of $\mathfrak{k}^C$ with $\sigma\rho-\rho_k$ as highest weight occurs in χ. We have further the following

LEMMA 2.1. *For every $\sigma\in W^1$, the representation $\tau_{\sigma\rho-\rho_k}$ of $\mathfrak{k}^C$ with $\sigma\rho-\rho_k$ as highest weight occurs with multiplicity one in χ and we have a decomposition*

$$L=\sum_{\sigma\in W^1} V_{\sigma\rho-\rho_k}. \tag{2.1}$$

Moreover in the same notation

$$L^+=\sum_{\sigma\in W^1,\ j(\sigma)=+} V_{\sigma\rho-\rho_k}$$

and

$$L^-=\sum_{\sigma\in W^1,\ j(\sigma)=-} V_{\sigma\rho-\rho_k}$$

(For proof see [**6**, Lemma 9.1 and Remark 9.2]).

Let D be the set of all linear forms λ on $\mathfrak{h}^C$ such that $\dfrac{2\langle\lambda, \alpha\rangle}{\langle\alpha, \alpha\rangle}$ is a nonnegative integer for every $\alpha\in P_+$. Then one can easily see that the map $D\times W^1\to \mathcal{F}'_0$ given by

$$(\lambda, \sigma)\longmapsto \lambda^{(\sigma)}$$

where $\lambda^{(\sigma)}=\sigma(\lambda+\rho)-\rho$, is a bijection (see [**4**, Lemma 6.4]). Now, choose $\lambda\in D$ and $\sigma\in W^1$ and consider $\lambda^{(\sigma)}\in \mathcal{F}'_0$. $\lambda^{(\sigma)}+\rho_n$ is dominant with respect to P_k. Let $\tau_{\lambda^{(\sigma)}+\rho_n}$ be the irreducible representation of $\mathfrak{k}$ with $\lambda^{(\sigma)}+\rho_n$ as highest weight on a space $V_{\lambda^{(\sigma)}+\rho_n}$. The representation $\chi^{\pm}\otimes\tau_{\lambda^{(\sigma)}+\rho_n}$ can be integrated to a representation, also denoted by $\chi^{\pm}\otimes\tau_{\lambda^{(\sigma)}+\rho_n}$, of K. Let $E_{L^+\otimes V_{\lambda^{(\sigma)}+\rho_n}}$ and $E_{L^-\otimes V_{\lambda^{(\sigma)}+\rho_n}}$ be the homogeneous vector bundles on G/K induced by the representations $\chi^+\otimes\tau_{\lambda^{(\sigma)}+\rho_n}$ and $\chi^-\otimes\tau_{\lambda^{(\sigma)}+\rho_n}$ of K. We choose K invariant hermitian metrics in the spaces $L^+\otimes V_{\lambda^{(\sigma)}+\rho_n}$ and $L^-\otimes V_{\lambda^{(\sigma)}+\rho_n}$ and induce metrics on the fibres of $E_{L^{\pm}\otimes V_{\lambda^{(\sigma)}+\rho_n}}$. Let $H_2^{\pm}(E_{V_{\lambda^{(\sigma)}+\rho_n}})$ denote the Hilbert spaces of square integrable sections φ, which are infinitely differentiable and such that

$$\pi(\Omega)\cdot\varphi=\langle\lambda+2\rho, \lambda\rangle\varphi$$

where $\pi(\Omega)$ denotes the action of the Casimir $\Omega \in U(\mathfrak{g}^C)$ on the space of C^∞ sections of $E_{L^\pm \otimes V_{\lambda^{(\sigma)}+\rho_n}}$. In view of [6, Proposition 4.2] these are the spaces of 'square integrable Dirac spinors' defined in [6, § 7]. Suppose

$$\langle \sigma\lambda, \alpha \rangle \neq 0, \qquad \text{for} \quad \alpha \in P_n .$$

Let $j = +$ or $-$. Then in [6] it was proved that

$$H_2^j(E_{V_{\lambda^{(\sigma)}+\rho_n}}) = 0, \qquad \text{if} \quad j \neq j(\sigma).$$

The proof of [6, Theorem 2, § 9] actually yields a sharper form which we will now state. The decomposition (2.1) gives rise to a decomposition

$$E_{L\otimes V_{\lambda^{(\sigma)}+\rho_n}} = \bigoplus_{\xi \in W^1} E_{V_{\xi\rho-\rho_k} \otimes V_{\lambda^{(\sigma)}+\rho_n}}$$

of the vector bundle $E_{L\otimes V_{\lambda^{(\sigma)}+\rho_n}}$. Let $H_2^\xi(E_{V_{\lambda^{(\sigma)}+\rho_n}})$ be the Hilbert space of square integrable infinitely differentiable sections φ of $E_{V_{\xi\rho-\rho_k}\otimes V_{\lambda^{(\sigma)}+\rho_n}}$ which satisfy

$$\pi(\Omega)\varphi = \langle \lambda + 2\rho, \lambda \rangle \varphi$$

where $\pi(\Omega)$ denotes the action of Ω on the space of C^∞ sections of the bundle $E_{V_{\xi\rho-\rho_k}\otimes V_{\lambda^{(\sigma)}+\rho_n}}$. Then we have the following

THEOREM 2. *Let $\lambda \in D$ and $\sigma \in W^1$, so that $\lambda^{(\sigma)} \in \mathcal{F}_0'$. Then, for $\xi \in W^1$,*

$$H_2^\xi(E_{V_{\lambda^{(\sigma)}+\rho_n}}) = 0, \qquad \textit{if} \quad \xi \neq \sigma,$$

if $\langle \sigma\lambda, \alpha \rangle \neq 0$, for $\alpha \in P_n$.

From this we now deduce the following

THEOREM 3. *Let $\lambda \in D$ and $\sigma \in W^1$ and let $\mu = \sigma(\lambda+\rho)-\rho$ so that $\mu \in \mathcal{F}_0'$. Assume that*

$$\langle \sigma\lambda, \alpha \rangle \neq 0 \qquad \textit{for any} \quad \alpha \in P_n . \tag{2.2}$$

Then

$$H_2^{0,q}(E_{V_\mu^*}) = 0 \qquad \textit{if} \quad q \neq q_\mu$$

when q_μ is the number of $\alpha \in P_n$ such that $\langle \mu+\rho, \alpha \rangle > 0$.

PROOF. Put $\mu' = -\kappa(\mu+\rho)-\rho$, where κ is the unique element of W_G which takes P_k into $-P_k$. One can easily verify that $\mu' \in \mathcal{F}_0'$. Let $\varphi \in D$ and $\sigma' \in W^1$ be the unique elements such that $\mu' = \sigma'(\varphi+\rho)-\rho$. Then by Theorem 2 above one has, for $\xi \in W^1$,

$$H_2^\xi(E_{V_{\mu'+\rho_n}}) = 0, \qquad \text{if} \quad \xi \neq \sigma'$$

provided

$$\langle \sigma'\varphi, \alpha \rangle \neq 0, \qquad \text{for} \quad \alpha \in P_n . \tag{2.3}$$

Since $\mu'+\rho = \sigma'(\varphi+\rho)$, we have

$$\sigma'\varphi=\mu'+\rho-\sigma'\rho\,.$$

Note that $\sigma'\rho=\frac{1}{2}\langle P'\rangle$, where

$$P'=\{\alpha\in P\cup -P\,|\,\langle\mu'+\rho,\,\alpha\rangle>0\}\,.$$

But since $\mu'+\rho=-\kappa(\mu+\rho)=-\kappa\sigma(\lambda+\rho)$, we have

$$\{\alpha\in P\cup -P\,|\,\langle\mu'+\rho,\,\alpha\rangle>0\}=-\kappa\sigma P\,.$$

Hence,

$$\sigma'\rho=\frac{1}{2}\langle P'\rangle=-\kappa\sigma\rho\,. \tag{2.4}$$

Thus

$$\sigma'\varphi=\mu'+\rho+\kappa\sigma\rho=-\kappa(\mu+\rho)+\kappa\sigma\rho$$
$$=-\kappa\sigma(\lambda+\rho)+\kappa\sigma\rho=-\kappa\sigma\lambda\,.$$

Thus the condition (2.3) is just that

$$\langle -\kappa\sigma\lambda,\,\alpha\rangle\neq 0 \qquad \text{for} \quad \alpha\in P_n$$

i. e.

$$\langle \sigma\lambda,\,\alpha\rangle\neq 0 \qquad \text{for} \quad \alpha\in P_n\,.$$

Thus, when (2.2) is satisfied, for $\xi\in W^1$

$$H_2^{\xi}(E_{V_{\mu'+\rho_n}})=0 \qquad \text{if} \quad \xi\neq\sigma'\,. \tag{2.5}$$

Consider the irreducible representation τ_{ρ_n} of $\mathfrak{k}$ which occurs in the decomposition (2.1) of χ. One knows that $\langle\rho_n,\,\alpha\rangle=0$ for every $\alpha\in P_k$. (This is clear since $s_\alpha\rho_n=\rho_n$ where $s_\alpha\in W_G$ denotes the reflection with respect to α.) It follows from this that τ_{ρ_n} is a one dimensional representation of $\mathfrak{k}$. One knows that the set of weights of χ is $\{\rho_n-\langle Q\rangle\,|\,Q\subseteqq P_n\}$ (See [**6**, Remark 3.1 and Remark 3.2]). On the other hand, the set of weights of the representation $Ad^{\underline{q}}$ of K on $\wedge^q\mathfrak{p}_-$ is $\{-\langle Q\rangle\,|\,Q\subseteqq P_n,\,[Q]=q\}$. Thus it follows that

$$\textstyle\sum_q\tau_{\rho_n}\otimes Ad^{\underline{q}}=\chi$$

on comparing the set of weights of the representations on the two sides. Because of Lemma 2.1, it follows that

$$\tau_{\rho_n}\otimes Ad^{\underline{q}}=\bigoplus_{\nu\in S_q}\tau_{\nu\rho-\rho_k} \tag{2.6}$$

where S_q $(q=1,2,\cdots,n)$ are subsets of W^1 such that

$$S_q\cap S_{q'}=\phi \qquad \text{if} \quad q\neq q' \ \text{and} \ \bigcup_q S_q=W^1\,.$$

Since τ_{ρ_n} is one dimensional, $\tau_{\mu+\rho_n}=\tau_\mu\otimes\tau_{\rho_n}$. One knows that if τ is an irreducible representation of $\mathfrak{k}$ with highest weight λ, then the representation τ^* which is dual to τ has highest weight $-\kappa\lambda$. Thus,

$$\tau_{\mu'+\rho_n}=\tau_{-\kappa(\mu+\rho_n)}=\tau^*_{\mu+\rho_n}=\tau^*_\mu\otimes\tau^*_{\rho_n}\,.$$

Thus, for $\nu\in W^1$,

$$\tau_{\nu\rho-\rho_k}\otimes\tau_{\mu'+\rho_n}=\tau_{\nu\rho-\rho_k}\otimes\tau^*_\mu\otimes\tau^*_{\rho_n}\,.$$

Hence from (2.6) it follows that

$$\bigoplus_{\nu\in S_q}\tau_{\nu\rho-\rho_k}\otimes\tau_{\mu'+\rho_n}=Ad^q_-\otimes\tau^*_\mu\,.$$

From the definitions of the spaces $H^\nu_2(E_{V_{\mu'+\rho_n}})$ (for $\nu\in W^1$) and $H^{0,q}_2(E_{V^*_\mu})$ and from [5, Lemma 1.1] it now follows that

$$\bigoplus_{\nu\in S_q}H^\nu_2(E_{V_{\mu'+\rho_n}})=H^{0,q}_2(E_{V^*_\mu})\,.$$

Now, in view of (2.5), in order to prove that $H^{0,q}_2(E_{V^*_\mu})=0$ if $q\neq q_\mu$, it is enough to prove that $\sigma'\notin S_q$ if $q\neq q_\mu$, or equivalently that $\sigma'\in S_{q_\mu}$.

For this we prove that $\tau_{\sigma'\rho-\rho_k}$ is a subrepresentation of $\tau_{\rho_n}\otimes Ad^{q_\mu}_-$. Since by (2.4) $\sigma'\rho=-\kappa\sigma\rho$, we have $\sigma'\rho-\rho_k=-\kappa(\sigma\rho-\rho_k)$. Note that $\sigma\rho=\frac{1}{2}\langle\sigma P\rangle$. But

$$\begin{aligned}\sigma P&=\{\alpha\in P\cup-P\mid\langle\sigma\rho,\alpha\rangle>0\}\\&=\{\alpha\in P\cup-P\mid\langle\sigma(\lambda+\rho),\alpha\rangle>0\}\\&=\{\alpha\in P\cup-P\mid\langle\mu+\rho,\alpha\rangle>0\}\\&=P_k\cup Q_\mu\cup(-Q'_\mu)\end{aligned}$$

where Q'_μ is the complement of Q_μ in P_n. Thus,

$$\begin{aligned}\sigma\rho=\frac{1}{2}\langle\sigma P\rangle&=\rho_k+\frac{1}{2}\langle Q_\mu\rangle-\frac{1}{2}\langle Q'_\mu\rangle\\&=\rho_k+\langle Q_\mu\rangle-\rho_n\,.\end{aligned}$$

Hence $\sigma\rho-\rho_k=\langle Q_\mu\rangle-\rho_n$. Thus,

$$\begin{aligned}\sigma'\rho-\rho_k&=-\kappa(\sigma\rho-\rho_k)=-\kappa(\langle Q_\mu\rangle-\rho_n)\\&=\rho_n-\kappa\langle Q_\mu\rangle\,.\end{aligned}$$

Thus

$$\begin{aligned}\tau_{\sigma'\rho-\rho_k}&=\tau_{\rho_n}\otimes\tau_{-\kappa\langle Q_\mu\rangle}\\&=\tau_{\rho_n}\otimes\tau^*_{\langle Q_\mu\rangle}\,.\end{aligned}$$

But $\tau_{\langle Q_\mu\rangle}$ is a subrepresentation of $Ad^{q_\mu}_+$ (see proof of [5, Corollary 2, §7]). Since obviously $Ad^{q_\mu}_-$ is the representation of K dual to the representation $Ad^{q_\mu}_+$ of K, we then see that $\tau_{-\kappa\langle Q_\mu\rangle}$ is a subrepresentation of $Ad^{q_\mu}_-$. Thus $\tau_{\sigma'\rho-\rho_k}$ is a subrepresentation of $\tau_{\rho_n}\otimes Ad^{q_\mu}_-$.

This concludes the proof of Theorem 3.

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