J. Math. Soc. Japan
Vol. 23, No. 3, 1971

On the characters and equivalence of continuous series representations

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(Received Aug. 11, 1970)

§ 1. Introduction.

In this paper we shall compute some explicit formulas for the characters and infinitesimal characters of general continuous series representations. We then apply these results to deduce some facts concerning equivalence and disjointness among representations from various series.

In more detail, let G be a connected semisimple Lie group and P a cuspidal parabolic subgroup of G. Then P has a Langland's decomposition $P=MAN$, where N is the "unipotent radical" of P, A is a maximal "split torus" and M is a reductive Lie group, not connected in general. Let λ be a unitary representation of P such that $\lambda(man)=\nu(a)\sigma(m)$, $m\in M$, $a\in A$, $n\in N$, where ν is a character of A and σ is a square-integrable irreducible representation of M. Such representations exist whenever P is cuspidal. The non-degenerate continuous series representations of G (corresponding tc P) are obtained by inducing these "cuspidal" representations λ from P to G.

Let $\pi=\mathrm{Ind}_P^G\lambda$. It is known that for $f\in C_0^\infty(G)$, $\pi(f)=\int_G f(g)\pi(g)dg$ is a trace class operator. Moreover, there exists a locally integrable function θ_π on G such that $\mathrm{Tr}\,\pi(f)=\int_G f(g)\theta_\pi(g)dg$, $f\in C_0^\infty(G)$. We are going to compute θ_π explicitly on an open subset G'_P of G. Specifically, let H be any Cartan subgroup of G such that $H\subseteqq P$, $H\cap AN=A$ and $H\cap M$ is a compact Cartan subgroup of M. Then $G'_P=\{g\in G: g$ is regular (see § 2 for the definition) and $g_1^{-1}gg_1\in H$ for some $g_1\in G\}$. The main steps in the computation are as follows: (i) extend Harish-Chandra's results on the discrete series of connected semisimple Lie groups to connected reductive Lie groups (§ 3); (ii) employ Mackey's theory in order to compute the discrete series of the disconnected group M (§ 4); (iii) develop an analog of the Weyl-Harish-Chandra integration formula for the group M (§ 7); (iv) define an appropriate class function on G (§ 8); and (v) combine various integral formulas to get the desired character formula (see 9.1). Although we do not evaluate θ_π on all of G, we shall

*) Supported in part by the NSF through GP-13871.

compute its support (Lemma 10.5). In addition, we compute the infinitesimal characters for the discrete series of M (at the end of § 4) and the continuous series of G (see Theorem 11.1).

Utilizing our character formula, we can deduce the following results (§ 10) on equivalence;

1) Let P_1, P_2 be two cuspidal parabolics ($P_2=G$ is allowed if G has a discrete series). Let π_1, π_2 be any representations in the series corresponding to P_1, P_2, respectively. Suppose P_1 and P_2 are not associate (see § 10 for the definition). Then π_1 and π_2 are not unitarily equivalent.

2) Let P be a proper cuspidal parabolic, $\pi_j=\mathrm{Ind}_P^G \lambda_j$, $j=1, 2$. Then $\pi_1 \cong \pi_2$ if and only if λ_1 and λ_2 are conjugate under the Weyl group $W_H=[\mathrm{Norm}(H) \cap G]/\mathrm{Cent}(H)$.

By an examination of the infinitesimal characters, we get the following additional result (§ 11):

3) Let P_1, P_2 be non-associate cuspidal parabolics ($P_2=G$ possible). Suppose $\dim A_1 \neq \dim A_2$.[1] Let π_1, π_2 be any representations in the corresponding series. Then π_1 and π_2 are disjoint, i. e. no irreducible constituent of π_1 is equivalent to any irreducible constituent of π_2.

We remark finally that our work generalizes [1a, the case of a minimal parabolic] and [3]. Both of these papers assume $M=M^0 \cdot \mathrm{Cent}(M)$, where M^0 denotes the connected component of the identity in M. We do *not* make that restrictive assumption here.

NOTATION. Let G be a Lie group. Set $G^0=$ the connected component of the identity in G, $Z_G=\mathrm{Cent}(G)$, and $\hat{G}=$ the space of unitary equivalence classes of irreducible unitary representations of G. Quite often, we blur the distinction between a given irreducible unitary representation π of G and its class $[\pi] \in \hat{G}$. Denote $\hat{G}_s=\{[\pi] \in \hat{G} : \pi \text{ is square-integrable}\}$.

Let $\mathfrak{g}=LA(G)$ be the Lie algebra of G. Suppose $S \subseteq G$, $\mathfrak{s} \subseteq \mathfrak{g}$ are subsets and $x \in G$. We denote $S^x=\{x^{-1}yx : y \in S\}$, $\mathfrak{s}^x=\{Ad_G\, x^{-1}(Y) : Y \in \mathfrak{s}\}$, $S^G=\bigcup_{x \in G} S^x$, $\mathfrak{s}^G=\bigcup_{x \in G} \mathfrak{s}^x$. We always use $N(\cdot)$, $Z(\cdot)$ to denote normalizers and centralizers, respectively. Finally, if V is a vector space, $\mathfrak{g}_V^*$ will denote the linear maps from $\mathfrak{g}$ to V.

§ 2. Semisimple groups and parabolic subgroups.

Let G be a connected semisimple Lie group with finite center. Let $\mathfrak{g}$ be its Lie algebra and let $\mathfrak{g}_c$ be the complexification. Suppose $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ is a Cartan decomposition. Let θ denote the corresponding Cartan involution of $\mathfrak{g}$(or G),

1) This restriction has been removed. See footnote 4).

and set $K=$ the maximal compact subgroup of G having $\mathfrak{k}$ as Lie algebra.

Next let P be a *parabolic* subgroup of G. This means: P is a closed subgroup of G such that (i) if $\mathfrak{P}=LA(P)$, then $P=N(\mathfrak{P})$ and (ii) $\mathfrak{P}_c$ contains a maximal solvable subalgebra of $\mathfrak{g}_c$. Let $N=$ the maximal normal subgroup of P such that $Ad(n)$ is unipotent for every $n\in N$. Set $\Xi=P\cap\theta P$, $A=$ the maximal connected split (i.e. $Ad(a)$ diagonizable over $\boldsymbol{R}$) abelian subgroup $\subseteqq Z_{\Xi}$. Then $\Xi=Z(A)\cap G$. Let $X(\Xi)=\{\chi:\Xi\to\boldsymbol{R}^*,\ \chi$ a continuous homomorphism$\}$. Set $M=\bigcap_{\chi\in X(\Xi)}\ker|\chi|$. Then M is reductive (i.e. $\mathfrak{m}=LA(M)$ is reductive), but not connected in general. Moreover $\Xi=MA$ is a direct product, and the map $(m, a, n)\to man$ is an analytic diffeomorphism of $M\times A\times N$ onto P.

Suppose P is *cuspidal*. By this we shall mean: there exists $\mathfrak{h}$, a θ-stable Cartan subalgebra of $\mathfrak{g}$, such that $\mathfrak{h}\cap\mathfrak{p}=\mathfrak{a}=LA(A)$. Let $H=Z(\mathfrak{h})\cap G$, a Cartan subgroup of G. We call any such H *compatible* with P. Set $B=H\cap K$. Then $H=BA$ is a direct product [1e, p. 481]. (Note: H and B are not necessarily connected or abelian [1d, p. 556].)

DEFINITION. A Cartan subgroup of M is the centralizer of a Cartan subalgebra of $\mathfrak{m}$.

LEMMA 2.1. *B is a compact Cartan subgroup of M.*

PROOF. B is clearly a compact group. Set $\mathfrak{b}=\mathfrak{h}\cap\mathfrak{k}$, so that $\mathfrak{h}=\mathfrak{b}\oplus\mathfrak{a}$. Let $b\in B$. Then $b\in\Xi$; but the map $b\to|\chi(b)|$, $\chi\in X(\Xi)$, is a continuous homomorphism of B into $\boldsymbol{R}_+^*$. Hence $B\subseteqq M$, and so $\mathfrak{b}\subseteqq\mathfrak{m}$. Moreover, it is clear that $\mathfrak{b}$ is a Cartan subalgebra of $\mathfrak{m}$. Next let $\beta\in Z(\mathfrak{b})\cap M$. Then $\beta\in H$, $\beta=ba$, $b\in B$, $a\in A$. If $a\neq 1$, choose $\chi\in X(\Xi)$ so that $|\chi(a)|\neq 1$. Then since $\beta\in M$, $b\in B\subseteqq M$, we have $1=|\chi(\beta)|=|\chi(b)|\,|\chi(a)|\neq 1$. Therefore $\beta=b\in B$; that is $B=Z(\mathfrak{b})\cap M$.

REMARK. It is clear that if B_1 is any θ-stable compact Cartan subgroup of M, then $H_1=B_1A$ is a Cartan subgroup G which is compatible with P.

Now let $\mathfrak{z}=LA(\Xi)=Z(\mathfrak{a})\cap\mathfrak{g}=\mathfrak{m}\oplus\mathfrak{a}$ (in fact $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{z}$ with respect to the Killing form). $\mathfrak{z}$ is of course reductive. By [1e, p. 481], $\Xi=\Xi_K\Xi_{\mathfrak{p}}$ where $\Xi_K=\Xi\cap K$ and $\Xi_{\mathfrak{p}}=\exp(\mathfrak{z}\cap\mathfrak{p})$. But $\Xi_K=M\cap K$, since for $\chi\in X(\Xi)$, $\xi\to|\chi(\xi)|$ is a continuous homomorphism of Ξ_K into $\boldsymbol{R}_+^*$. Also $\exp(\mathfrak{z}\cap\mathfrak{p})=\exp[(\mathfrak{m}+\mathfrak{a})\cap\mathfrak{p}]=\exp[(\mathfrak{m}\cap\mathfrak{p})+\mathfrak{a}]=\exp(\mathfrak{m}\cap\mathfrak{p})A$. It follows easily that $M=M_KM_{\mathfrak{p}}$, where $M_K=M\cap K$, $M_{\mathfrak{p}}=\exp(\mathfrak{m}\cap\mathfrak{p})$. It is well-known that $G=PK=MANK$. Since $P\cap K=M_K$, it is readily proven that $(m, a, n, k)\to mank$ is an analytic diffeomorphism of $M_{\mathfrak{p}}\times A\times N\times K$ onto G [1c, Lemma 11]. Finally let $\rho_P\in\mathfrak{a}_R^*$ be defined by $\rho_P(Y)=\frac{1}{2}\operatorname{trace}(ad\,Y)_{\mathfrak{n}}$, $Y\in\mathfrak{a}$, $\mathfrak{n}=LA(N)$.

Let ψ denote the involution of $\mathfrak{g}_c$ corresponding to $\mathfrak{g}$. Fix $i\in\mathfrak{g}_c$ such that $\psi(i)=-i$. We regard $\mathfrak{g}\subseteqq\mathfrak{g}_c=\mathfrak{g}+i\mathfrak{g}$ and so naturally $ad\,\mathfrak{g}\subseteqq ad\,\mathfrak{g}_c$. Let $\operatorname{Int}(\mathfrak{g}_c)$

be the *adjoint group* of $\mathfrak{g}_c$. We may identify $\mathrm{Int}(\mathfrak{g})$ = the adjoint group of $\mathfrak{g}$ with the (closed) analytic subgroup of $\mathrm{Int}(\mathfrak{g}_c)$ whose Lie algebra is $ad\,\mathfrak{g}$ [**2**, p. 155].

We denote the adjoint representation of G by $Ad_G: G \to \mathrm{Int}(\mathfrak{g})$. Define

$$\Gamma = Ad_G^{-1}(Ad_G(K) \cap \exp i\mathfrak{a}) .$$

THEOREM 2.2. (i) *Γ is a finite subgroup of B that commutes with M^0.*

(ii) *Γ is normal in M.*

(iii) $H = H^0\Gamma$.

PROOF. (i) Since $\ker Ad_G = Z_G$ is a finite group, it is easily seen that Γ is a finite subgroup of B. But $[\mathfrak{a}, \mathfrak{m}] = 0$ and so Γ and M^0 commute.

(ii) Since $M \subseteqq Z(\mathfrak{a})$, we must have $Ad_G(M)$ commutes with $Ad_G(K) \cap \exp i\mathfrak{a}$. But Γ is the complete inverse image (under Ad_G) of $Ad_G(K) \cap \exp i\mathfrak{a}$. It follows that M normalizes Γ.

(iii) The proof that follows strengthens the argument of [**1e**, Lemma 50] (see [**6**, p. 93] in this connection). Since $A \subseteqq H^0$, it is enough to prove $B = B^0\Gamma$. First assume $G \cong \mathrm{Int}(\mathfrak{g})$, that is $Z_G = \{e\}$. Let H_c be the Cartan subgroup of $\mathrm{Int}(\mathfrak{g}_c)$ corresponding to $\mathfrak{h}_c$ = the complexification of $\mathfrak{h}$. Let $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$ and U = the analytic subgroup of $G_c = \mathrm{Int}(\mathfrak{g}_c)$ having $\mathfrak{u}$ as Lie algebra. U is a maximal compact subgroup of G_c. But $U \cap H_c = U \cap Z(\mathfrak{h}_c)$ and $\mathfrak{h}_c$ is invariant under the Cartan involution of $\mathfrak{g}_c$ determined by $\mathfrak{u}$. By [**1e**, Lemma 27], $U \cap H_c$ is a connected compact Lie subgroup of G_c having Lie algebra $= \mathfrak{u} \cap \mathfrak{h}_c$. Therefore $\exp(\mathfrak{u} \cap \mathfrak{h}_c) = U \cap H_c$. But $\mathfrak{u} \cap \mathfrak{h}_c = (\mathfrak{h} \cap \mathfrak{k}) + i(\mathfrak{h} \cap \mathfrak{p}) = \mathfrak{b} + i\mathfrak{a}$. Let $b \in B \subseteqq U \cap H_c$. Then $b = b_1, b_2$, where $b_1 \in \exp \mathfrak{b} = B^0$ and $b_2 = b_1^{-1}b \in \exp i\mathfrak{a} \cap K = \Gamma$.

Now drop the assumption $Z_G = \{e\}$. Consider the adjoint representation $Ad_G: G \to \mathrm{Int}(\mathfrak{g})$. Clearly $Ad_G(Z(\mathfrak{h}) \cap G) = Z(\mathfrak{h}) \cap \mathrm{Int}(\mathfrak{g})$. Also $Ad_G(Z(\mathfrak{h}) \cap G)^0 \subseteqq [Z(\mathfrak{h}) \cap \mathrm{Int}(\mathfrak{g})]^0$. But these are both connected Lie subgroups of $\mathrm{Int}(\mathfrak{g})$ having Lie algebra $\mathfrak{h}$. Therefore they are equal. Part (iii) is thus a consequence of the following general

LEMMA 2.3. *Let $\tau: G_1 \to G_2$ be a continuous homomorphism onto. Suppose H_j, Γ_j are subgroups of G_j, $j = 1, 2$ with the properties: $G_2 = H_2\Gamma_2$, $\tau(H_1) = H_2$, and $\Gamma_1 = \tau^{-1}(\Gamma_2)$. Then $G_1 = H_1\Gamma_1$.*

PROOF. If $x \in G_1$, then $\tau(x) \in G_2 \Rightarrow \tau(x) = h_2\gamma_2$ for some $h_2 \in H_2$, $\gamma_2 \in \Gamma_2$. Then there are $h_1 \in H_1$, $\gamma_1 \in \Gamma_1$ such that $\tau(h_1) = h_2$, $\tau(\gamma_1) = \gamma_2$. So $\tau(x) = \tau(h_1\gamma_1) \Rightarrow (h_1\gamma_1)^{-1}x \in \ker \tau \subseteqq \Gamma_1$. Therefore $x \in H_1\Gamma_1$.

REMARK. When P is a *minimal parabolic*, then (a) M is compact and (b) $M = M^0\Gamma$ (see [**4**, Lemma 3.1]). Neither (a) nor (b) is true in general.

Let $n = \dim \mathfrak{g}$, $\mathfrak{h} \subseteqq \mathfrak{g}$ a θ-stable Cartan subalgebra, $l = \dim \mathfrak{h}$. For an indeterminate t, consider $\det(t + 1 - Ad_G(x)) = D_0(x) + \cdots + D_n(x)t^n$, $x \in G$. The first non-zero coefficient will be $D_l(x)$. Set $D(x) = D_l(x)$. The *regular* elements are $G' = \{x \in G : D(x) \neq 0\}$, a dense open submanifold. The following are obvious:

$D(xz)=D(x)$, $z\in Z_G$, and $D(gxg^{-1})=D(x)$, g, $x\in G$. For any subset $S\subseteq G$, we denote $S'=S\cap G'$. In particular, if H is a Cartan subgroup of G, let $H'=H\cap G'$. Consider the map

$$\varphi_H:\quad Z_H\backslash G\times H'\to G'$$

defined by $\varphi_H(g^*, h)=g^{-1}hg$. The image $G'_H=(H')^G$ is an open submanifold of G and the map

$$\varphi_H:\quad Z_H\backslash G\times H'\to G'_H$$

is *proper* (i. e. the inverse image of a compact set is compact). More precisely, let $W_H=N(H)/Z_H$. Then W_H acts effectively on $(Z_H\backslash G\times H')$ and G'_H is diffeomorphic to $(Z_H\backslash G\times H')/W_H$ (see [1e, p. 488]).

Next consider the roots of $(\mathfrak{g}, \mathfrak{h})$. By definition these are the linear forms $\alpha\in\mathfrak{h}_c^*$ such that $\mathfrak{g}_c^\alpha=\{X\in\mathfrak{g}: [Y, X]=\alpha(Y)X$ for all $Y\in\mathfrak{h}_c\}$ is non-empty. Choose an ordering on the roots. Set $W(\mathfrak{g}, \mathfrak{h})=$ the group of automorphisms of $\mathfrak{h}_c$ generated by the reflections corresponding to a simple root system. Then W_H may be identified with a subgroup of $W(\mathfrak{g}, \mathfrak{h})$.

Suppose λ is a linear form on $\mathfrak{h}$. Then there exists at most one homomorphism $\xi_\lambda: H\to \boldsymbol{C}$ such that

$$\xi_\lambda(\exp Y)=e^{\lambda(Y)}, \qquad Y\in\mathfrak{h}.$$

If α is a root of $(\mathfrak{g}, \mathfrak{h})$ then ξ_α always exists. Let Q denote a choice of positive roots and set $\rho=\rho_{\mathfrak{h}}=\frac{1}{2}\sum_{\alpha\in Q}\alpha$. We assume henceforth that G is *acceptable*, that is ξ_ρ exists. (This is independent of the choice of $\mathfrak{h}$—see [1e, p. 484].) Let $Q_-=\{\alpha\in Q: \alpha|_{\mathfrak{h}\cap\mathfrak{p}}\equiv 0\}$, $Q_+=Q-Q_-$. We can now define several C^∞-functions: For $h\in H$, let

$$\Delta(h)=\Delta_H(h)=\xi_\rho(h)\prod_{\alpha\in Q}(1-\xi_\alpha(h^{-1})),$$

$$\Delta_-(h)=\xi_\rho(h_-)\prod_{\alpha\in Q_-}(1-\xi_\alpha(h^{-1})),$$

$$\Delta_+(h)=\Delta(h)/\Delta_-(h)=\xi_\rho(h_+)\prod_{\alpha\in Q_+}(1-\xi_\alpha(h^{-1})),$$

where $h=h_-h_+$, $h_-\in H\cap K$, $h_+\in\exp(\mathfrak{h}\cap\mathfrak{p})$. It is well-known [1e, p. 504] that $D(h)=(-1)^p\Delta(h)^2$, $h\in H$, $p=\#(Q)$; and so $H'=\{h\in H: \Delta_H(h)\neq 0\}$. It also follows that (up to sign) Δ is independent of the choice of ordering on the roots.

Suppose $\pi\in\hat{G}$. Then for every $f\in C_0^\infty(G)$, $\pi(f)=\int_G f(g)\pi(g)\,dg$ is trace class and there exists a locally integrable function θ_π such that $\theta_\pi|_{G'}$ is real analytic and

$$\operatorname{Tr}\pi(f)=\int_G f(g)\theta_\pi(g)\,dg, \qquad f\in C_0^\infty(G). \tag{2.1}$$

θ_π is called the *character* of π. One of our goals is to make some fairly explicit computations on θ_π for certain "continuous series" representations π.

We also wish to consider *infinitesimal characters*. Let $\mathfrak{U}$ = the universal enveloping algebra of $\mathfrak{g}_c$, $\mathfrak{Z}$ = the center of $\mathfrak{U}$. $\mathfrak{Z}$ may be identified with the commutative algebra of left and right invariant differential operators on G. We recall how to construct characters of $\mathfrak{Z}$. Let $\mathfrak{h} \subseteq \mathfrak{g}$ be any Cartan subalgebra, $\mathfrak{h}_c$ its complexification. Denote by $I(\mathfrak{h}_c)$ the $W(\mathfrak{g}, \mathfrak{h})$-invariant polynomial functions on $\mathfrak{h}_c^*$. Then there is a natural isomorphism $\gamma_\mathfrak{h} : \mathfrak{Z} \to I(\mathfrak{h}_c)$ [1b, Lemma 19]. If $\lambda \in \mathfrak{h}_c^*$, define $\chi_\lambda^\mathfrak{h} : \mathfrak{Z} \to \boldsymbol{C}$ by $\chi_\lambda^\mathfrak{h}(z) = \gamma_\mathfrak{h}(z)(\lambda)$, $z \in \mathfrak{Z}$. We obtain all homomorphisms of $\mathfrak{Z}$ into $\boldsymbol{C}$ this way, and $\chi_\lambda^\mathfrak{h} = \chi_{\lambda'}^\mathfrak{h}$ if and only if $\lambda = s\lambda'$ for some $s \in W(\mathfrak{g}, \mathfrak{h})$. Now it is well-known (and easy to see) that for $\pi \in \hat{G}$, θ_π is an *eigendistribution* of $\mathfrak{Z}$. Thus there exists $\chi_\pi : \mathfrak{Z} \to \boldsymbol{C}$ such that

$$z\theta_\pi = \chi_\pi(z)\theta_\pi, \qquad z \in \mathfrak{Z}. \tag{2.2}$$

(2.2) may be understood in the sense of distribution theory or as a differential equation on the manifold G'. χ_π is called the infinitesimal character of π.

§ 3. Discrete series for connected reductive groups.

Let G be as before, a connected semisimple Lie group with finite center and acceptable. Suppose there is $\mathfrak{b} \subseteq \mathfrak{k}$, a Cartan subalgebra of $\mathfrak{g}$. Then $B = Z(\mathfrak{b})$ is a compact Cartan subgroup—moreover, it is abelian and connected. Let $\hat{B}$ = the character group. Every $\hat{b} \in \hat{B}$ determines $\lambda \in \mathfrak{b}_{iR}^*$ by $\hat{b}(\exp Y) = e^{\lambda(Y)}$, $Y \in \mathfrak{b}$. The collection $\mathcal{L} \subseteq \mathfrak{b}_{iR}^*$ thus obtained is a lattice; in fact,

$$\mathcal{L} = \{\lambda \in \mathfrak{b}_{iR}^* : \lambda(Y) \in 2\pi i \boldsymbol{Z} \text{ whenever } \exp Y = e,\ Y \in \mathfrak{b}\}.$$

Let ω be the polynomial function on $\mathfrak{b}_c$ defined by $\omega = \prod_{\alpha \in Q} \alpha$, Q a system of positive roots for $(\mathfrak{g}, \mathfrak{b})$. Identifying $\mathfrak{b}_c$ and $\mathfrak{b}_c^*$ via the Killing form, we single out the regular elements $\mathcal{L}' = \{\lambda \in \mathcal{L} : \omega(\lambda) \neq 0\}$. For $s \in W(\mathfrak{g}, \mathfrak{b})$, set $\omega^s = \varepsilon(s)\omega$. Note finally that W_B leaves $\mathcal{L}'$ invariant. The following theorem is due to Harish-Chandra [1h, Theorem 16].

THEOREM 3.1. *Let $\sigma \in \hat{G}_s$. Then there exists $\lambda \in \mathcal{L}'$ such that*

$$\theta_\sigma(b) = \frac{c}{\Delta_B(b)} \sum_{W_B} \varepsilon(s) e^{s\lambda(Y)}, \qquad \exp Y = b \in B', \tag{3.1}$$

$c = c(\sigma) = (-1)^q \operatorname{sgn} \omega(\lambda)$, $q = \frac{1}{2} \dim G/K$. *Moreover λ is uniquely determined up to an element of W_B and the infinitesimal character of σ is $\chi_\lambda^\mathfrak{b}$. Conversely if $\lambda \in \mathcal{L}'$, then there exists $\sigma(\lambda) \in \hat{G}_s$ such that $\theta_{\sigma(\lambda)}$ is given by* (3.1) *on B' and the infinitesimal character $\chi_{\sigma(\lambda)}$ is precisely $\chi_\lambda^\mathfrak{b}$.*

COROLLARY. *Let $\sigma \in \hat{G}_s$ and let $\lambda \in \mathcal{L}'$ be a corresponding linear form.*

Then

$$\sigma(z)=\hat{b}(z)\xi_\rho(z^{-1})I_{\dim\sigma}, \qquad z\in Z_G, \tag{3.2}$$

$\hat{b}(\exp Y)=e^{\lambda(Y)}$, $Y\in\mathfrak{b}$.

PROOF. Since σ is irreducible, there is some character $\chi\in\hat{Z}_G$ such that $\sigma(z)=\chi(z)I_{\dim\sigma}$, $z\in Z=Z_G$. It follows from a simple calculation that $\theta_\sigma(zg)=\chi(z)\theta_\sigma(g)$, $g\in G'$, $z\in Z$. But for $b\in B'$,

$$\theta_\sigma(bz)=\frac{c}{\Delta_B(bz)}\sum_{W_B}\varepsilon(s)(s\cdot\hat{b})(bz)$$
$$=\frac{c\hat{b}(z)}{\Delta_B(bz)}\sum\varepsilon(s)(s\cdot\hat{b})(b),$$

since W_B leaves Z pointwise fixed. But $\Delta_B(bz)=\xi_\rho(z)\Delta_B(b)$ [1f, p. 299]. This proves the corollary (once we choose $b\in B'$ such that $\theta_\sigma(b)\neq 0$).

REMARK. It follows from $D(b)=(-1)^p\Delta_B(b)^2$, $D(bz)=D(b)$, and $\Delta_B(bz)=\xi_\rho(z)\Delta_B(z)$ that $\xi_\rho(z)^2=1$. Hence $\xi_\rho(z)=\pm 1$, $z\in Z$.

We wish to extend Theorem 3.1 to the reductive case. So let G be a connected reductive Lie group, $\mathfrak{g}=LA(G)$. Let $\mathfrak{g}_1=[\mathfrak{g},\mathfrak{g}]$ be the semisimple part, and $\mathfrak{c}=\operatorname{Cent}\mathfrak{g}$. Then $\mathfrak{g}=\mathfrak{g}_1+\mathfrak{c}$. Let $\mathfrak{g}_1=\mathfrak{k}_1+\mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}_1$. Set $\mathfrak{k}=\mathfrak{k}_1+\mathfrak{c}$ so that $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$. Suppose $\mathfrak{h}_1\subseteq\mathfrak{g}_1$ is a Cartan subalgebra. Then $\mathfrak{h}=\mathfrak{h}_1+\mathfrak{c}$ is a Cartan subalgebra of $\mathfrak{g}$. Let G_1 and C be the analytic subgroups of G corresponding to $\mathfrak{g}_1$ and $\mathfrak{c}$. Then $G=G_1C$, and G_1 (resp. C) is closed in G since it is the commutator subgroup of G (resp. Z_G^0).

Once again, consider $\det(t+1-Ad_G(x))$, $x\in G$. The lowest non-vanishing coefficient of t^j will be the l-th, where $l=\dim\mathfrak{h}$. Set $D_G(x)=$ the coefficient of t^l. The regular elements are $G'=\{x\in G: D(x)\neq 0\}$, again an open dense submanifold such that $D(g^{-1}xg)=D(x)$. More importantly, $D(x\zeta)=D(x)$, $x\in G$, $\zeta\in Z_G$. In fact, it is easy to see that $D_G(g_1\zeta)=D_{G_1}(g_1)$, $g_1\in G_1$, $\zeta\in C$. So $g=g_1\zeta$ is regular in $G\Leftrightarrow g_1$ is regular in G_1.

Let $H=Z(\mathfrak{h})$. Then $H=H_1C$ where $H_1=H\cap G_1=Z(\mathfrak{h}_1)\cap G_1$. It is immediate that

$$G'_H=(G_1)'_{H_1}\cdot C, \tag{3.3}$$

where G'_H is as usual the set of elements in G conjugate to $H'=G'\cap H$. Equation (3.3) will be useful later.

Now let α be a root of $(\mathfrak{g}_1,\mathfrak{h}_1)$. Extend α to $\mathfrak{h}_c$ by setting it equal to zero on $\mathfrak{c}_c$. The resulting forms are the roots of $(\mathfrak{g},\mathfrak{h})$. Furthermore $W(\mathfrak{g},\mathfrak{h})$ is the group of automorphisms of $\mathfrak{h}_c$ obtained from $W(\mathfrak{g}_1,\mathfrak{h}_1)$ by letting each $s\in W(\mathfrak{g}_1,\mathfrak{h}_1)$ fix the elements of $\mathfrak{c}_c$.

Next assume $G_1\cap C$ is finite (as is the case for example if C is compact). Then ξ_α can be defined as usual for any root α [1e, p. 483]. *Assume* in addition that G is acceptable; that is ξ_ρ can be defined. In particular, since the roots vanish on $\mathfrak{c}$,

$$\xi_\rho|_{G_1 \cap C} \equiv 1 . \tag{3.4}$$

Defining Δ as in the semisimple case, it is easily verified that

$$\Delta_H(h_1\zeta) = \Delta_{H_1}(h_1) , \qquad h_1 \in H_1 , \qquad \zeta \in C . \tag{3.5}$$

Now suppose there exists a Cartan subalgebra $\mathfrak{b}_1 \subseteqq \mathfrak{k}_1$ of $\mathfrak{g}_1$. Set $\mathfrak{b} = \mathfrak{b}_1 + \mathfrak{c}$ and $B = Z(\mathfrak{b}) \cap G$. Suppose B is compact. Then C is compact and Z_{G_1} must be finite. Also $B = B_1 C$, where $B_1 = B \cap G_1 = Z(\mathfrak{b}_1) \cap G_1$ is a torus. Therefore B itself is a torus. Moreover $N(B) \cap G = [N(B_1) \cap G_1] \cdot C$ and $N(B_1) \cap G_1 \cap C = B_1 \cap C$. Therefore $W_B \cong W_{B_1}$.

Let $\hat{B}$ be the character group of B. Then $\hat{b} \in \hat{B}$ determines $\lambda \in \mathfrak{b}_{iR}^*$ by $\hat{b}(\exp Y) = e^{\lambda(Y)}$, $Y \in \mathfrak{b}$. The collection so obtained is the lattice

$$\mathcal{L} = \{\lambda \in \mathfrak{b}_{iR}^* : \lambda(Y) \in 2\pi i \boldsymbol{Z} \text{ whenever } \exp Y = e,\ Y \in \mathfrak{b}\} .$$

For $\hat{b} \in \hat{B}$, set $\hat{b}_1 = \hat{b}|_{B_1}$ and $\chi = \hat{b}|_C$. Let $\lambda_1 \in (\mathfrak{b}_1)_{iR}^*$, $\chi \in \mathfrak{c}_{iR}^*$ be defined by

$$\hat{b}_1(\exp Y) = e^{\lambda_1(Y)} , \quad Y \in \mathfrak{b}_1 ; \qquad \chi(\exp Y) = e^{\chi(Y)} , \quad Y \in \mathfrak{c} .$$

Note we use χ both for the element of $\hat{C}$ as well as its differential $\in \mathfrak{c}_{iR}^*$. In any event $\lambda = \lambda_1 + \chi$ and we also have

$$\mathcal{L} = \{\lambda_1 + \chi : \lambda_1(Y_1) - \chi(Y_2) \in 2\pi i \boldsymbol{Z} \text{ whenever } \exp Y_1 = \exp Y_2,\ Y_1 \in \mathfrak{b}_1,\ Y_2 \in \mathfrak{c}\} . \tag{3.6}$$

Defining ω (or ω_1) as the product of the positive roots of $(\mathfrak{g}, \mathfrak{b})$ (or $(\mathfrak{g}_1, \mathfrak{b}_1)$), we check easily that $\omega(\lambda) = \omega_1(\lambda_1)$, $\lambda = \lambda_1 + \chi$. Letting $\mathcal{L}' = \{\lambda \in \mathcal{L} : \omega(\lambda) \neq 0\}$, we see that $\lambda = \lambda_1 + \chi$ is regular if and only if λ_1 is regular.

We obtain the irreducible unitary representations of G as follows. Let $\sigma_1 \in \hat{G}_1$, $\chi \in \hat{C}$ be such that

$$\sigma_1(g) = \chi(g) I_{\dim \sigma_1} , \qquad g \in G_1 \cap C . \tag{3.7}$$

Define $\sigma = \sigma_1 \otimes \chi \in \hat{G}$. More precisely, the space of σ is the space of σ_1 and $\sigma(g_1\zeta) = \sigma_1(g_1)\chi(\zeta)$, $g_1 \in G_1$, $\zeta \in C$. σ is well-defined because of (3.7), and is easily seen to be in $\hat{G}$. Moreover changing the class of either σ_1 or χ changes the class of σ. Conversely, every representation of $\hat{G}$ is obtained in this way. Finally, since C is compact, σ is square-integrable if and only if σ_1 is square-integrable.

Suppose θ_{σ_1} is the character of σ_1. Then a simple computation shows that: for every $f \in C_0^\infty(G)$, $\operatorname{Tr} \sigma(f)$ exists and

$$\operatorname{Tr} \sigma(f) = \int_G f(g)\theta_\sigma(g)dg , \quad \text{where}$$

$$\theta_\sigma(g_1\zeta) = \theta_{\sigma_1}(g_1)\chi(\zeta) , \qquad g_1 \in G_1' , \qquad \zeta \in C . \tag{3.8}$$

Now let $\sigma\in\hat{G}_s$, $\sigma=\sigma_1\otimes\chi$. Let $\lambda_1\in\mathcal{L}_1'$ be a linear form in $(\mathfrak{b}_1)_{iR}^*$ determined by σ_1 (according to Theorem 3.1). Set $\lambda=\lambda_1+\chi\in\mathcal{L}'$. It follows from (3.5), (3.8) and Theorem 3.1 that

$$\theta_\sigma(b)=\theta_\sigma(b_1\zeta)=\frac{c}{\Delta_{B_1}(b_1)}\sum_{W_{B_1}}\varepsilon(s)e^{s\lambda_1(Y_1)}\cdot\chi(\zeta)\,,\qquad Y_1\in\mathfrak{b}_1\,,\ \exp Y_1=b_1\in B_1',\ \zeta\in C$$

$$=\frac{c}{\Delta_B(b)}\sum_{W_B}\varepsilon(s)e^{s\lambda(Y)}\,,\qquad Y\in\mathfrak{b}\,,\ \exp Y=b\in B'\,.$$

Clearly $\lambda\in\mathcal{L}'$ is uniquely determined up to an element of W_B. Also $c=c(\sigma)=c(\sigma_1)$ because $\omega(\lambda)=\omega_1(\lambda_1)$ and $\frac{1}{2}\dim\mathfrak{g}/\mathfrak{k}=\frac{1}{2}\dim\mathfrak{g}_1/\mathfrak{k}_1=\frac{1}{2}\dim\mathfrak{p}$.

Conversely, let $\lambda\in\mathcal{L}'$, $\lambda=\lambda_1+\chi$, $\lambda_1\in\mathcal{L}_1'$, $\chi\in\hat{C}$. By Theorem 3.1, there exists $\sigma_1\in(G_1)_s^\wedge$ such that

$$\theta_{\sigma_1}(b_1)=\frac{c}{\Delta_{B_1}(b_1)}\sum_{W_{B_1}}\varepsilon(s)e^{s\lambda_1(Y_1)}\,,\qquad Y_1\in\mathfrak{b}_1\,,\qquad \exp Y_1=b_1\in B_1'\,.$$

Claim: σ_1 and χ satisfy (3.7). In fact, by the Corollary to Theorem 3.1 and formulas (3.4) and (3.6)

$$\begin{aligned}\sigma_1(g)&=\hat{b}_1(g)\xi_\rho(g)^{-1}I_{\dim\sigma_1}\,,\qquad \hat{b}_1=e^{\lambda_1}\\&=\hat{b}_1(g)\,I_{\dim\sigma_1}\\&=\chi(g)I_{\dim\sigma_1}\,,\qquad\text{if}\quad g\in G_1\cap C\,.\end{aligned}$$

Here we used the fact that the ρ function is the same for $(\mathfrak{g},\mathfrak{b})$ and $(\mathfrak{g}_1,\mathfrak{b}_1)$. Therefore $\sigma=\sigma_1\otimes\chi$ is a member of $\hat{G}_s$ whose character satisfies

$$\theta_\sigma(b)=\frac{c}{\Delta_B(b)}\sum_{W_B}\varepsilon(s)e^{s\lambda(Y)}\,,\qquad Y\in\mathfrak{b}\,,\qquad \exp Y=b\in B'\,.$$

Finally we compute the infinitesimal character of $\sigma=\sigma_1\otimes\chi$. Let $\mathfrak{U}=$ the universal enveloping algebra of $\mathfrak{g}_c$, $\mathfrak{Z}$ its center. Set $\mathfrak{U}_1$, $\mathfrak{C}=$ the subalgebras of $\mathfrak{U}$ generated by $(1,(\mathfrak{g}_1)_c)$ and $(1,\mathfrak{c}_c)$ respectively. Then $\mathfrak{U}=\mathfrak{U}_1\mathfrak{Z}$ and $\mathfrak{Z}=\mathfrak{Z}_1\mathfrak{C}$, direct products, where $\mathfrak{Z}_1$ is the center of $\mathfrak{U}_1$. $\mathfrak{U}_1$ is isomorphic to the enveloping algebra of $(\mathfrak{g}_1)_c$ and $\mathfrak{C}$ is isomorphic to the enveloping (i.e. symmetric) algebra $S(\mathfrak{c}_c)$. Once again, if $\mathfrak{h}\subseteq\mathfrak{g}$ is a Cartan subalgebra, there is a natural isomorphism $\gamma_{\mathfrak{h}}:\mathfrak{Z}\to I(\mathfrak{h}_c)=$ the $W(\mathfrak{g},\mathfrak{h})$-invariant polynomials in $S(\mathfrak{h}_c^*)$. We obtain all characters of $\mathfrak{Z}$ by $\chi_\lambda^{\mathfrak{h}}(z)=\gamma_{\mathfrak{h}}(z)(\lambda)$, $z\in\mathfrak{Z}$ and

$$\chi_\lambda^{\mathfrak{h}}=\chi_{\lambda'}^{\mathfrak{h}}\iff s\lambda=\lambda'\,,\qquad\text{some}\quad s\in W(\mathfrak{g},\mathfrak{h})\tag{3.9}$$

(see [1b, Lemma 9] and [1e, § 12]). Let I_1, I_2 be the subalgebras of $I(\mathfrak{h}_c)$ generated by $(1,(\mathfrak{h}_1)_c)$ and $(1,\mathfrak{c}_c)$, respectively. These are isomorphic to $I((\mathfrak{h}_1)_c)$ and $S(\mathfrak{c}_c)$. Moreover $\gamma_{\mathfrak{h}}|_{\mathfrak{Z}_1}:\mathfrak{Z}_1\to I_1$, and $\gamma_{\mathfrak{h}}|_{\mathfrak{C}}:\mathfrak{C}\to I_2$. With these observations it is easy to check that for

$$\sigma=\sigma_1\otimes\chi\,,\qquad \lambda=\lambda_1+\chi\,,\qquad z\theta_\sigma=\chi_\lambda^{\mathfrak{b}}(z)\theta_\sigma\,,\qquad z\in\mathfrak{Z}\,.$$

So we have proven Harish-Chandra's theorem for reductive groups. More precisely, we have

THEOREM 3.2. *Let G be a connected reductive Lie group. Suppose G has a compact Cartan subgroup B and that G is acceptable. Let $\sigma\in\hat{G}_s$. Then there is $\lambda\in\mathcal{L}'$ such that*

$$\theta_\sigma(b)=\frac{c}{\Delta_B(b)}\sum_{W_B}\varepsilon(s)e^{s\lambda(Y)}\,,\qquad Y\in\mathfrak{b}\,,\qquad \exp Y=b\in B' \tag{3.10}$$

and λ is uniquely determined up to an element of W_B. Conversely, if $\lambda\in\mathcal{L}'$, there exists $\sigma(\lambda)\in\hat{G}_s$ such that $\theta_{\sigma(\lambda)}$ is a locally integrable function on G, analytic on G', and given by (3.10) *on B'; $c=c(\sigma)=(-1)^q\operatorname{sgn}\omega(\lambda)$, $q=\frac{1}{2}\dim\mathfrak{p}$. Moreover, the infinitesimal character of $\sigma=\sigma(\lambda)$ is precisely $\chi_\lambda^{\mathfrak{b}}$.*

REMARK. We shall have occasion in the sequel to use the following fact: all compact Cartan subgroups of G are conjugate under G. This is proven easily by using: (a) the corresponding fact which is known for compact G, and (b) the conjugacy of all maximal compact subgroups.

§4. Discrete series for M.

We begin with a word on Weyl groups. Let $\mathfrak{g}$ be a reductive Lie algebra with Cartan subalgebra $\mathfrak{h}$. $W(\mathfrak{g},\mathfrak{h})$ is the group of automorphisms of $\mathfrak{h}_c$ generated by the reflections corresponding to a simple root system. Let $G_c^1=\operatorname{Int}(\mathfrak{g}_c)$. Then every $s\in W(\mathfrak{g},\mathfrak{h})$ may be realized by an element of G_c^1, i.e. there is $y\in G_c^1$ such that $y|_{\mathfrak{h}_c}=s$. Conversely, if $y\in G_c$ leaves $\mathfrak{h}_c$ invariant, then $y|_{\mathfrak{h}_c}\in W(\mathfrak{g},\mathfrak{h})$. More generally, let G_c be any connected Lie group with $\mathfrak{g}_c$ as Lie algebra (G_c^1 has Lie algebra equal to the semisimple part of $\mathfrak{g}_c$, but that is immaterial). Then

$$W(\mathfrak{g},\mathfrak{h})\cong[N(\mathfrak{h}_c)\cap G_c]/[Z(\mathfrak{h}_c)\cap G_c]\,.$$

Now let G be a connected semisimple Lie group, with finite center and acceptable. Return to the notation of §2: $P=MAN$, $H=BA$ a Cartan, etc. Let $B_1=B\cap M^0=Z(\mathfrak{b})\cap M^0$, a compact connected Cartan subgroup of the connected reductive group M^0. But then B_1 and B^0 are both connected Lie subgroups of M^0 having $\mathfrak{b}$ as Lie algebra. Therefore $B^0=B\cap M^0$. By [**1e**, Lemma 30] M^0 is acceptable; and so Theorem 3.2 holds for M^0. We wish, however, to compute the discrete series for M. We accomplish this by applying Mackey's theory of normal subgroups.

Let $\sigma\in(M^0)_s^\wedge$ and $\xi\in\Xi$. Define $\sigma^\xi\in(M^0)_s^\wedge$ by

$$\sigma^\xi(m)=\sigma(\xi^{-1}m\xi)\,,\qquad m\in M^0\,.$$

THEOREM 4.1. (1) *σ^{ξ} is unitarily equivalent to σ if and only if $\xi\in\Xi^0\Gamma$.*
(2) *σ^{ξ} has the same infinitesimal character as σ for* every $\xi\in\Xi$.

PROOF. First let $\xi\in\Xi^0\Gamma$. Then $\xi=m^0a\gamma$, $m^0\in M^0$, $a\in A$, $\gamma\in\Gamma$ and $\sigma^{\xi}=\sigma^{m^0a\gamma}=\sigma^{m^0}\cong\sigma$. Here, we are using the properties of Γ derived in Theorem 2.2.

Next, we wish to compute the character and infinitesimal character of σ^{ξ}, $\sigma\in(M^0)^{\wedge}_s$, $\xi\in\Xi$. Before proceeding, we establish the following notation: $G'=$ the regular elements of G, $(M^0)''=$ the regular elements of M^0. Now let $\lambda\in\mathcal{L}'$ (= the regular forms on $\mathfrak{b}$) be such that

$$\theta_\sigma(b)=\frac{c}{\Delta_{B^0}(b)}\sum_{W_{B^0}}\varepsilon(s)e^{s\lambda(Y)}\,,\qquad Y\in\mathfrak{b}\,,\qquad \exp Y=b\in(B^0)''$$

and $\chi^{\mathfrak{b}}_\lambda$ is the infinitesimal character. As usual, λ is unique up to an element of W_{B^0}. It is trivial to check that the character θ_{σ^ξ} of σ^ξ is given by

$$\theta_{\sigma^\xi}(m)=\theta_\sigma(\xi^{-1}m\xi)\,,\qquad m\in M^0\,.$$

Now the compact Cartan subgroups of M^0 are all conjugate under M^0. Hence (modifying ξ by an element of M^0 if necessary), we may assume that ξ leaves B^0 (and $\mathfrak{b}$) invariant. Then

$$\theta_{\sigma^\xi}(b)=\frac{c}{\Delta_{B^0}(\xi^{-1}b\xi)}\sum_{W_{B^0}}\varepsilon(s)e^{s\lambda^{\xi}(Y)}\,,\qquad Y\in\mathfrak{b}\,,\quad \exp Y=b\in(B^0)'' \tag{4.1}$$

where $\lambda^\xi(Y)=\lambda(Ad\,\xi^{-1}(Y))$, $Y\in\mathfrak{b}$.

Now consider the adjoint representation of G, $Ad_G: G\to\mathrm{Int}(\mathfrak{g})\subseteq\mathrm{Int}(\mathfrak{g}_c)$. Then $Ad_G(\Xi)\subseteq Z(\mathfrak{a}_c)\cap G_c$, $G_c=\mathrm{Int}(\mathfrak{g}_c)$. But $\mathfrak{a}_c$ is invariant under the involution of $\mathfrak{g}_c$ determined by the compact real form $\mathfrak{u}=\mathfrak{k}+i\mathfrak{p}$. Hence $Z(\mathfrak{a}_c)\cap G_c$ is the connected Lie subgroup of G_c having $Z(\mathfrak{a}_c)\cap\mathfrak{g}_c$ as Lie algebra [1e, Lemma 27]. One checks easily that $Z(\mathfrak{a}_c)\cap\mathfrak{g}_c=(Z(\mathfrak{a})\cap\mathfrak{g})_c=\mathfrak{z}_c$. That is, the Lie algebra of $Z(\mathfrak{a}_c)$ is $\mathfrak{z}_c=\mathfrak{m}_c+\mathfrak{a}_c$. Let M^1_c be the analytic subgroup of G_c having $\mathfrak{m}_c$ as Lie algebra and set $A_c=\exp\mathfrak{a}_c$. Then $Z(\mathfrak{a}_c)=M^1_cA_c$. Consider $Ad_G(\xi)=m_1a_1$, $m_1\in M^1_c$, $a_1\in A_c$. Since $[\mathfrak{b},\mathfrak{a}]=0$, it is clear that the action of ξ on $\mathfrak{b}$ under Ad_G coincides with that of m_1. But the discussion at the beginning of this section shows that m_1 determines an element $s_\xi\in W(\mathfrak{m},\mathfrak{b})$. Therefore in (4.1) we have: $\Delta_{B^0}(\xi^{-1}b\xi)=\varepsilon(s_\xi)\Delta_{B^0}(b)$, $c(\sigma^\xi)=(-1)^q\operatorname{sgn}\omega(s_\xi\lambda)=\varepsilon(s_\xi)c(\sigma)$, and $\lambda^\xi=s_\xi\lambda\in\mathcal{L}'$. Thus the regular form on $\mathfrak{b}$ determined (up to W_{B^0}) by σ^ξ is precisely $\lambda^\xi=s_\xi\lambda\in\mathcal{L}'$. It follows immediately from (3.9) that σ and σ^ξ have the same infinitesimal character. This proves (2).

It remains to prove (1), i.e. that $\sigma^\xi\cong\sigma$ if and only if $\xi\in\Xi^0\Gamma$. We have already observed that $\xi\in\Xi^0\Gamma\Rightarrow\sigma^\xi\cong\sigma$. Conversely, suppose $\sigma^\xi\cong\sigma$. Then $\theta_{\sigma^\xi}=\theta_\sigma$, and by (4.1), there must exist $s_1\in W_{B^0}$ such that $\lambda^\xi=s_1\lambda$. But $W_{B^0}=(N(B^0)\cap M^0)/B^0$; and so (modifying ξ by an element of M^0 if necessary) we may assume that $\lambda^\xi=\lambda$. But $\lambda^\xi=s_\xi\lambda$, $s_\xi\in W(\mathfrak{m},\mathfrak{b})$. The regularity of λ

therefore implies that $s_\xi = 1$. That is, in $Ad_G(\xi) = m_1 a_1$, we must have $m_1 \in Z(\mathfrak{b}_c)$. But then $\xi \in Z(\mathfrak{h}) \cap \Xi = H \cap \Xi = H^0 \Gamma \subseteq \Xi^0 \Gamma$. q. e. d.

We are ready to compute the discrete series of M. First consider the subgroup $M^0\Gamma$. Since M^0 and Γ commute, the irreducible unitary representations of $M^0\Gamma$ are obtained as follows: take $\sigma \in (M^0)^\wedge$, $\omega \in \hat{\Gamma}$ such that $\sigma|_{M^0 \cap \Gamma}$, $\omega|_{M^0 \cap \Gamma}$ act via the same scalar (in different dimensions, of course). Then $\sigma \otimes \omega \in (M^0\Gamma)^\wedge$ and we get all representations of $M^0\Gamma$ this way. If $\sigma \not\cong \sigma_1$ or $\omega \not\cong \omega_1$, then $\sigma \otimes \omega \not\cong \sigma_1 \otimes \omega_1$. It is trivial to verify that σ is square-integrable if and only if $\sigma \otimes \omega$ is square-integrable. The character is easily computed:

$$\theta_{\sigma\otimes\omega}(m^0\gamma) = \theta_\sigma(m^0)\theta_\omega(\gamma), \qquad m^0 \in M^0, \qquad \gamma \in \Gamma,$$

where θ_σ, θ_ω are the characters of σ, ω respectively. Finally, since Γ is finite, we have

$$z\theta_{\sigma\otimes\omega} = \chi_\sigma(z)\theta_{\sigma\otimes\omega}, \qquad z \in \mathfrak{Z}(\mathfrak{M}),$$

where $\mathfrak{Z}(\mathfrak{M})$ = the center of the universal enveloping algebra $\mathfrak{M}$ of $\mathfrak{m}_c$, and χ_σ is the infinitesimal character of σ. That is, the infinitesimal character of $\sigma \otimes \omega$ is the same as that for σ.

Now $M^0\Gamma$ is a normal subgroup of M. Let $\sigma \in (M^0\Gamma)^\wedge_s$, $m \in M$. Consider the representation $\sigma^m \in (M^0\Gamma)^\wedge_s$ defined by $\sigma^m(m_1) = \sigma(m^{-1}m_1 m)$, $m_1 \in M^0\Gamma$. It follows from Theorem 4.1 that $\sigma^m \cong \sigma$ if and only if $m \in M^0\Gamma$. Applying Mackey's theory [**5**, Theorem 8.1], we conclude: $\mathrm{Ind}_{M^0\Gamma}^M \sigma \in \hat{M}$ and $\mathrm{Ind}_{M^0\Gamma}^M \sigma \cong \mathrm{Ind}_{M^0\Gamma}^M \sigma'$ if and only if there exists $m \in M$ such that $\sigma' \cong \sigma^m$. Moreover, $\mathrm{Ind}_{M^0\Gamma}^M \sigma$ is square-integrable; and since $[M : M^0\Gamma] < \infty$, we certainly obtain all of $\hat{M}_s$ in this way.[2)]

Next we compute the character and infinitesimal character of a representation in $\hat{M}_s$. To do that we need the following

THEOREM 4.2. *Let G be a unimodular Lie group, $H \subseteq G$ an open normal subgroup of finite index. Let π be a unitary representation of H in a separable Hilbert space. Suppose that for every $f \in C_0^\infty(H)$, $\pi(f) = \int_H f(h)\pi(h)dh$ is trace class and that there is a locally integrable function θ_π on H such that* $\mathrm{Tr}\,\pi(f) = \int_H f(h)\theta_\pi(h)dh$, *$f \in C_0^\infty(H)$. Let $T^\pi = \mathrm{Ind}_H^G \pi$. Then $T^\pi(f)$ is trace class for every $f \in C_0^\infty(G)$ and* $\mathrm{Tr}\,T^\pi(f) = \int_G f(x)\Theta_\pi(x)dx$ *where*

$$\Theta_\pi(x) = \begin{cases} \sum_{H\backslash G} \theta_\pi(gxg^{-1}) & x \in H \\ 0 & x \notin H. \end{cases}$$

2) This is a consequence of the fact that for finite (indeed compact) extensions, the normal subgroup is always "regularly embedded."

This result is perhaps known, but we have not been able to locate it in the literature. The basic idea of the following proof is due to A. Kleppner.

Suppose π acts in the separable space $\mathcal{V}$. Let $\mathcal{H}(\pi)$ be the space of functions $f: G \to \mathcal{V}$ such that $f(hx)=\pi(h)f(x)$. $\mathcal{H}(\pi)$ is a Hilbert space under the inner product $(f, f')=\sum_{H\backslash G}\langle f(g), f'(g)\rangle$, where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathcal{V}$. The induced representation T^{π} acts on $\mathcal{H}(\pi)$ via $T^{\pi}(g)f(x)=f(xg)$. Let $g \in G$, $v \in \mathcal{V}$ and define

$$f_{g,v}(x)=\begin{cases} \pi(h)v & \text{if} \quad x=hg,\ h \in H \\ 0 & \text{otherwise}. \end{cases}$$

It is clear that $f_{g,v} \in \mathcal{H}(\pi)$.

LEMMA 4.3.

$$(f_{g,v}, f_{g',v'})=\begin{cases} \langle \pi(g'g^{-1})v, v'\rangle & \text{if} \quad g'g^{-1} \in H \\ 0 & \textit{otherwise}. \end{cases}$$

PROOF.

$$(f_{g,v}, f_{g',v'})=\sum_{x\in H\backslash G}\langle f_{g,v}(x), f_{g',v'}(x)\rangle=0\,,$$

if g and g' are not in the same right coset. If on the other hand $g=hg'$, then

$$\begin{aligned}(f_{g,v}, f_{g',v'}) &= \langle f_{hg',v}(g'), f_{g',v'}(g')\rangle \\ &= \langle \pi(h^{-1})v, v'\rangle \\ &= \langle \pi(g'g^{-1})v, v'\rangle\,.\end{aligned}$$

LEMMA 4.4. *Let $g_1, \cdots, g_r$ denote a choice of representatives for $H\backslash G$ and $\{v_j\}$ an orthonormal basis for $\mathcal{V}$. Then $\{f_{g_i,v_j}\}$ forms an orthonormal basis of $\mathcal{H}(\pi)$.*

PROOF. It follows from Lemma 4.3 that the system is orthonormal. In fact, it is also complete. For suppose $(f, f_{g_i,v_j})=0$, $1 \leqq i \leqq r$, $j \geqq 1$. Then

$$\begin{aligned}0=(f, f_{g_i,v_j}) &= \sum_{x\in H\backslash G}\langle f(x), f_{g_i,v_j}(x)\rangle \\ &= \langle f(g_i), f_{g_i,v_j}(g_i)\rangle=\langle f(g_i), v_j\rangle\,, \qquad 1\leqq i \leqq r\,,\quad j \geqq 1\,.\end{aligned}$$

Since $\{v_j\}$ is complete in $\mathcal{V}$, $f(g_i)=0$, $1 \leqq i \leqq r$. But f is completely determined by its values on g_i, $1 \leqq i \leqq r$; and so $f \equiv 0$.

The next result is almost obvious and we omit the proof.

LEMMA 4.5.

$$T^{\pi}(x)f_{g,v}=f_{gx^{-1},v}\,, \qquad x, g \in G\,, \quad v \in \mathcal{V}\,.$$

Now let $\varphi \in C_0^{\infty}(G)$, $T^{\pi}(\varphi)=\int_G \varphi(x)T^{\pi}(x)dx$. Then

$$
\begin{aligned}
(T^{\pi}(\varphi)f_{g,v}, f_{g,v}) &= \sum_{y\in H\backslash G}\left\langle \int_G \varphi(x)T^{\pi}(x)f_{g,v}(y)dx,\ f_{g,v}(y)\right\rangle \\
&= \sum_{y\in H\backslash G}\left\langle \int_G \varphi(x)f_{gx^{-1},v}(y)dx,\ f_{g,v}(y)\right\rangle \\
&= \int_G \sum_{H\backslash G}\langle f_{gx^{-1},v}(y),\ f_{g,v}(y)\rangle \varphi(x)dx \\
&= \int_G (f_{gx^{-1},v}, f_{g,v})\varphi(x)dx\,.
\end{aligned}
$$

But

$$
\begin{aligned}
(f_{gx^{-1},v}, f_{g,v}) &= \begin{cases} \langle \pi(gxg^{-1})v, v\rangle & \text{if } gxg^{-1}\in H \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \langle \pi(h)v, v\rangle & \text{if } x = g^{-1}hg \\ 0 & \text{otherwise}. \end{cases}
\end{aligned}
$$

Now if dg denotes a choice of Haar measure on G, then $dg|_H$ is a Haar measure on H. Therefore

$$
\begin{aligned}
(T^{\pi}(\varphi)f_{g,v}, f_{g,v}) &= \int_H \langle \pi(h)v, v\rangle \varphi(g^{-1}hg)dh \\
&= \langle \pi(\varphi^g)v, v\rangle\,,
\end{aligned}
$$

where $\varphi^g \in C_0^\infty(H)$ is defined by $\varphi^g(h) = \varphi(g^{-1}hg)$. Hence, for $g\in G$ fixed

$$
\begin{aligned}
\sum_{j\geqq 1}(T^{\pi}(\varphi)f_{g,v_j}, f_{g,v_j}) &= \sum_{j\geqq 1}\langle \pi(\varphi^g)v_j, v_j\rangle \\
&= \int_H \varphi(g^{-1}hg)\theta_\pi(h)dh\,.
\end{aligned}
$$

Finally

$$
\begin{aligned}
\sum_{\substack{1\leqq i\leqq r \\ j\geqq 1}}(T^{\pi}(\varphi)f_{g_i,v_j}, f_{g_i,v_j}) &= \sum_{y\in H\backslash G}\int_H \varphi(y^{-1}hy)\theta_\pi(h)dh \\
&= \sum_{H\backslash G}\int_H \varphi(h)\theta_\pi(yhy^{-1})dh \\
&= \int_G \varphi(x)\Theta_\pi(x)dx\,.
\end{aligned}
$$

We now apply Theorem 4.2 to the case $M^0\Gamma \subsetneqq M$.

THEOREM 4.6. *Let* $\sigma\in \hat{M}_s$, $\sigma = \mathrm{Ind}_{M^0\Gamma}^M \sigma_1$, $\sigma_1 \in (M^0\Gamma)_s^\wedge$. *Then for every* $f\in C_0^\infty(M)$, $\mathrm{Tr}\,\sigma(f)$ *exists and* $\mathrm{Tr}\,\sigma(f) = \int_M f(m)\theta_\sigma(m)dm$, *where*

$$
\theta_\sigma(m_1) = \begin{cases} \sum_{M^0\Gamma\backslash M}\theta_{\sigma_1}(mm_1m^{-1}) & m_1\in M^0\Gamma \\ 0 & \textit{otherwise}. \end{cases} \tag{4.2}
$$

It is easy to write down the value of θ_σ on $B = B^0\Gamma$. Indeed, if $\sigma_1 = \sigma^0\otimes\omega$,

$\sigma^0 \in (M^0)_s\hat{}$, $\omega \in \hat{\Gamma}$, then

$$\theta_\sigma(b) = \theta_\sigma(b^0\gamma) = \sum_{M^0\Gamma\backslash M} \theta_{\sigma^0}(mb^0m^{-1})\theta_\omega(m\gamma m^{-1}) .$$

Moreover, we have the

COROLLARY. *The characters of $\hat{M}_s$ corresponding to inequivalent representations are linearly independent on B.*

PROOF. We have the corresponding fact for characters of inequivalent representations of $(M^0)_s\hat{}$ on B^0. This is easily extended to $(M^0\Gamma)_s\hat{}$, i.e. the characters of inequivalent representations of $(M^0\Gamma)_s\hat{}$ are linearly independent on $B^0\Gamma = B$.

Now suppose $\sigma_1, \cdots, \sigma_r$ are non-equivalent representations in $\hat{M}_s$, $\sigma_j = \mathrm{Ind}_{M^0\Gamma}^M \tau_j$,

$$\theta_{\sigma_j}(b) = \sum_{M^0\Gamma\backslash M} \theta_{\tau_j}(mbm^{-1}) , \qquad b \in B .$$

Assume $\sum_{j=1}^r c_j\theta_{\sigma_j} = 0$, a.e. on B. Then

$$c_1 \sum_{M^0\Gamma\backslash M} \theta_{\tau_1}(mbm^{-1}) + \cdots + c_r \sum_{M^0\Gamma\backslash M} \theta_{\tau_r}(mbm^{-1}) = 0 .$$

By the corresponding result for $M^0\Gamma$, either $c_j = 0$, $1 \leqq j \leqq r$ or

$$\tau_i^m \cong \tau_j^{m'} \quad \text{for some} \quad i,\ j,\ m,\ m' .$$

(Recall $\tau^m(m_1) = \tau(m^{-1}m_1m)$, $m_1 \in M^0\Gamma$.) If $i = j$ and m, m' are distinct as elements of $M^0\Gamma\backslash M$, then $\tau_i^m \ncong \tau_i^{m'}$ by Theorem 4.1. On the other hand, if $i \neq j$ and $\tau_i^m \cong \tau_j^{m'}$, then $\sigma_i = \mathrm{Ind}_{M^0\Gamma}^M \tau_i \cong \mathrm{Ind}_{M^0\Gamma}^M \tau_i^m \cong \mathrm{Ind}_{M^0\Gamma}^M \tau_j^{m'} \cong \mathrm{Ind}_{M^0\Gamma}^M \tau_j = \sigma_j$. Hence the constants c_j must all be zero and the corollary is proven.

Finally, what is the infinitesimal character of $\sigma = \mathrm{Ind}_{M^0\Gamma}^M \tau$? Let $\tau = \sigma^0 \otimes \omega$, $\sigma^0 \in (M^0)_s\hat{}$, $\omega \in \Gamma$. The infinitesimal character χ_τ is equal to χ_{σ^0}. But it follows immediately from Theorem 4.1 part (2) and equation (4.2) that

$$z\theta_\sigma = \chi_\tau(z)\theta_\sigma , \qquad z \in \mathfrak{Z}(\mathfrak{M}) .$$

Thus the infinitesimal character of σ is the same as that for τ and σ^0.

§ 5. The continuous series.

Let $P = MAN$ be a cuspidal parabolic as in § 2. Take $\sigma \in \hat{M}_s$ (see § 4), and a linear form $\nu \in \mathfrak{a}_{iR}^*$. Set $\pi = \pi(\sigma, \nu) = \mathrm{Ind}_P^G \sigma \otimes \nu$, where $(\sigma \otimes \nu)(man) = e^{\nu(\log a)}\sigma(m)$. The representations π so obtained are called the *non-degenerate continuous series* corresponding to P. (In case P is a *minimal* parabolic, it is more common to say *principal series*.) It is known a priori (at least when π is irreducible) that these representations have characters which are locally integrable functions, analytic on G' [1e]. Our goal is to compute these functions explicitly on H' (and so on G'_H) where H is a Cartan subgroup, compatible with P (see § 2).

Let us write down π more explicitly now and obtain a first approximation for $\mathrm{Tr}\,\pi(f)$. If $\mathcal{H}_\sigma=$ the space of σ (a separable Hilbert space), then we may take

$$\mathcal{H}(\pi)=\left\{\begin{array}{ll} & f \text{ Borel-measurable} \\ f\colon G\to\mathcal{H}_\sigma, & f(man\,x)=e^{(\nu+\rho)(\log a)}\sigma(m)f(x)\,, \quad man\in P,\ x\in G \\ & \|f\|^2=\int_K\|f(k)\|^2dk<\infty\,. \end{array}\right.$$

Here we identify functions equal a. e. and $\rho=\rho_P$ comes in because

$$\text{left Haar measure on } P \text{ is } dm\,da\,dn\,,$$

$$\text{right Haar measure on } P \text{ is } e^{2\rho(\log a)}dm\,da\,dn\,.$$

The representation $\pi=\mathrm{Ind}_P^G\,\sigma\otimes\nu$ acts on $\mathcal{H}(\pi)$ by right translation.

Suppose $\varphi\in C_0^\infty(G)$, $\pi(\varphi)=\int_G\varphi(g)\pi(g)\,dg$. The functions in $\mathcal{H}(\pi)$ are uniquely determined by their values on K. Thus if $f\in\mathcal{H}(\pi)$

$$\begin{aligned}\pi(\varphi)f(k)&=\int_G\varphi(g)\pi(g)f(k)dg\\ &=\int_G\varphi(k^{-1}g)f(g)dg\\ &=\int_{P\times K}\varphi(k^{-1}man\,\kappa)f(man\,\kappa)dm\,da\,dn\,d\kappa\\ &=\int_K\Lambda_\varphi(k,\kappa)f(\kappa)d\kappa\end{aligned}\tag{5.1}$$

where

$$\Lambda_\varphi(k,\kappa)=\int_P\varphi(k^{-1}man\,\kappa)e^{(\nu+\rho)(\log a)}\sigma(m)\,dm\,da\,dn\,.$$

In (5.1) we used the fact that $G=PK$, $P\cap K=M_K$ is compact, and a well-known integral decomposition.[3] Now it can be shown that under our assumptions, $\pi(\varphi)$ is trace class and

$$\mathrm{Tr}\,\pi(\varphi)=\int_K\mathrm{Tr}\,\Lambda_\varphi(k,k)dk\,.$$

This is worked out for example in [3, § 4]. But then

$$\mathrm{Tr}\,\pi(\varphi)=\int_K\mathrm{Tr}\left(\int_M\phi_k(m)\sigma(m)dm\right)dk$$

where

3) N. Bourbaki, Livre VI, Intégration, Ch. 7, p. 66.

$$\psi_k(m)=\int_{AN}\varphi(k^{-1}man\,k)e^{(\nu+\rho)(\log a)}da\,dn\in C_0^\infty(M)\,.$$

Therefore

$$\begin{aligned}\operatorname{Tr}\pi(\varphi)&=\int_K\int_M\psi_k(m)\theta_\sigma(m)dm\,dk\\&=\int_{K\times P}\varphi(k^{-1}man\,k)e^{(\nu+\rho)(\log a)}\theta_\sigma(m)dm\,da\,dn\,dk\\&=\int_\Xi\theta_{\sigma,\nu}(\xi)h_\varphi(\xi)d\xi\end{aligned}$$

where $\theta_{\sigma,\nu}$ denotes the character of $\sigma\otimes\nu$ on $\Xi=MA$ and

$$\begin{aligned}&h_\varphi(\xi)=d(\xi)\int_{K\times N}\varphi(k^{-1}\xi nk)\in C_0^\infty(\Xi)\\&d(\xi)=|\det Ad\,\xi_{\mathfrak{n}}|^{1/2}=e^{\rho(\log a)}\,,\qquad \xi=ma\,.\end{aligned}\tag{5.2}$$

These formulas have been obtained already by Harish-Chandra [**1i**, p. 19].

§ 6. The support of h_φ.

In this section we make a calculation regarding the support of the function defined by (5.2).

THEOREM 6.1. *Suppose* $\varphi\in C_0(G'_H)$, i.e. φ *is a continuous function on* G *with compact support contained in* G'_H. *Then* $\operatorname{Supp}h_\varphi\subseteqq(H')^\Xi$.

PROOF. By [**1g**, p. 93], if ξ is regular

$$\int_{K\times N}\varphi(k^{-1}\xi nk)dk\,dn=\delta(\xi)\int_{K\times N}\varphi(k^{-1}n^{-1}\xi nk)dk\,dn\,,\qquad\varphi\in C_0(G)$$

$$\delta(\xi)=|\det(Ad(\xi^{-1})-1)_{\mathfrak{n}}|\,.$$

Now let $\varphi\in C_0(G'_H)$. It will suffice to prove that $\operatorname{Supp}h'_\varphi\subseteqq(H')^\Xi$, where

$$h'_\varphi(\xi)=\int_{K\times N}\varphi(k^{-1}n^{-1}\xi nk)dk\,dn\,.\tag{6.1}$$

Let $\xi\in\Xi$ and suppose $h'_\varphi(\xi)\neq0$. Then from (6.1) there is $g\in G$ such that $g\xi g^{-1}\in H'$. In particular $\xi\in G'$. We shall show that there exists $\xi_1\in\Xi$ such that $\xi_1\xi\xi_1^{-1}\in H'$. Set $Z_\xi(\mathfrak{g})=\{X\in\mathfrak{g}: Ad_G\xi(X)=X\}$. By [**1e**, p. 460], $Z_\xi(\mathfrak{g})$ is a Cartan subalgebra of $\mathfrak{g}$ (since ξ is regular). But $\xi\in\Xi\Rightarrow\mathfrak{a}\subseteqq Z_\xi(\mathfrak{g})$. Since $Z_\xi(\mathfrak{g})$ is abelian, $[Z_\xi(\mathfrak{g}),\mathfrak{a}]=0$. But $\mathfrak{z}=LA(\Xi)$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$. Therefore $Z_\xi(\mathfrak{g})\subseteqq\mathfrak{z}$ and it follows that $Z_\xi(\mathfrak{g})$ is also a Cartan subalgebra of $\mathfrak{z}$. Set $\mathfrak{h}_1=Z_\xi(\mathfrak{g})$ and $H_1=Z(\mathfrak{h}_1)$, a Cartan subgroup of G. Since $\mathfrak{a}\subseteqq\mathfrak{h}_1$, we have that H_1 is also a Cartan subgroup of Ξ.

Now $\xi\in H_1\cap(H')^G$; therefore H and H_1 are conjugate under G [**1e**, p. 505]. Let $x\in G$ be such that $H_1=H^x$. Then $H_1=(BA)^x=B^xA^x$. Consider $H_1^0=(H^x)^0$.

Claim: $(H^x)^0=(H^0)^x$. Indeed, $(H^0)^x$ is a connected Lie subgroup of $H^x \Rightarrow (H^0)^x \subseteqq (H^x)^0$. But they have the same Lie algebra, namely $\mathfrak{h}^x$, and so they are equal. Hence $H_1^0=(B^0A)^x=(B^0)^x A^x$. Since B^0 is a maximal compact subgroup of H^0, $(B^0)^x$ must be a maximal compact subgroup of H_1^0.

Next let $B_1=H_1^0 \cap M^0$. Since $\mathfrak{a} \subseteqq \mathfrak{h}_1$, it follows readily that H_1^0 is a direct product $H_1^0=B_1A$. Let B_2 be a maximal compact subgroup of the connected abelian Lie group B_1. Then there is a vector group V such that $B_1=B_2V$. Moreover, $H_1^0=B_2VA$, and B_2 is also a maximal compact subgroup of H_1^0. Hence $\dim B_2=\dim (B^0)^x$. But $\dim A=\dim A^x$; thus $\dim V=0$, i. e. $B_2=B_1$.

Now $(B^0)^x$ and B_1 are both maximal compact subgroups of the connected abelian Lie group H_1^0. Hence they are conjugate in H_1^0, and so in fact equal $B_1=(B^0)^x$. Therefore B^0 and B_1 are two compact Cartan subgroups of the connected reductive Lie group M^0. By the remark at the end of § 3, there exists $m \in M^0$ such that

$$B_1=(B^0)^m. \tag{6.2}$$

Finally, let $\mathfrak{b}_1=\mathfrak{h}_1 \cap \mathfrak{m}$. Then $\mathfrak{b}_1$ is the Lie algebra of B_1 and $\mathfrak{h}_1=\mathfrak{b}_1+\mathfrak{a}$, a direct sum. It follows from (6.2) that $\mathfrak{b}_1=\mathfrak{b}^m$. Then $\mathfrak{h}_1=\mathfrak{b}_1+\mathfrak{a}=\mathfrak{b}^m+\mathfrak{a}=\mathfrak{h}^m$. Taking centralizers in G, we get

$$H_1=Z(\mathfrak{h}_1)=Z(\mathfrak{h}^m)=Z(\mathfrak{h})^m=H^m.$$

In particular, $m\xi m^{-1} \in H \cap G'=H'$. So $\xi \in (H')^{\Xi}$.

Now let $\xi \in \operatorname{Supp} h'_\varphi$. Choose ξ_j such that $h'_\varphi(\xi_j) \neq 0$, $\xi_j \in (H')^{\Xi}$, $\xi_j \to \xi$. Then there exists $g_j \in G$ such that $\varphi(g_j \xi g_j^{-1}) \neq 0$. Moreover, it is clear from (6.1) that we may restrict g_j to a fixed compact set in G (depending only on φ). So we may assume $g_j \to g$. But then $g\xi g^{-1} \in \operatorname{Supp} \varphi \subseteqq G'_H$. It follows that ξ is regular and that $\xi \in G'_H$. Reasoning as in the preceding case, we conclude once again that $\xi \in (H')^{\Xi}$. The proof is now complete.

§ 7. An analog of Weyl's integration formula.

Let G be a connected semisimple Lie group, with finite center and acceptable. Let $H \subseteqq G$ be a Cartan subgroup. Choose Haar measures dg, dh on G, H respectively. Since Z_H is open in H, $dh|_{Z_H}$ is a Haar measure on Z_H. The following result is due to Harish-Chandra [1h, Appendix].

THEOREM 7.1. *Normalize the invariant measure dg^* on $G^*=Z_H \backslash G$ such that*

$$\int_G f(g)dg=\int_{G^*} dg^* \int_{Z_H} f(hg)dh, \qquad f \in C_0(G).$$

Then for any $f \in C_0(G'_H)$, we have

$$\int_G f(g)dg=w^{-1}\int_H |\Delta_H(h)|^2 dh \int_{G^*} f(h^{g^*})dg^*,$$

where $w = w_H = \#(W_H)$.

In case G is compact and H is a maximal torus, this is precisely Weyl's classical integration formula.

Next suppose $G = G_1C$ is a connected reductive Lie group as in § 3. Let H be a Cartan subgroup $H = H_1C$, $H_1 = H \cap G_1$. It follows from (3.3), (3.5) and the facts $W_H \cong W_{H_1}$, $Z_H \backslash G \cong Z_{H_1} \backslash G_1$ that Theorem 7.1 also holds for such reductive groups.

Let $P = MAN$ be a cuspidal parabolic subgroup of G as usual. For $\xi \in \Xi$, let $D_\Xi(\xi)$ be the first non-zero coefficient in $\det(t+1-Ad_\Xi(\xi))$. (Here $Ad_\Xi : \Xi \to \mathrm{Aut}(\mathfrak{z})$ and $Ad_\Xi(\xi) \in \mathrm{Int}(\mathfrak{z})$ whenever $\xi \in \Xi^0\Gamma$.) Define the regular elements $\Xi'' = \{\xi \in \Xi : D_\Xi(\xi) \neq 0\}$. Since $\mathfrak{g} = \mathfrak{z} + \mathfrak{n} + \theta\mathfrak{n}$, each of which is left invariant by Ξ, it is easy to see that $\Xi' \subseteq \Xi''$. Note that $D_\Xi(\xi a) = D_\Xi(\xi)$, $\xi \in \Xi$, $a \in A$, so that $H'' = B''A$.

Next we remark that Q_- can be identified with the positive roots of $(\mathfrak{z}, \mathfrak{h})$. Since $\gamma \in \Gamma \Rightarrow Ad_G\gamma \in \exp i\mathfrak{a}$, it follows readily that $\xi_\alpha(\gamma) = 1$, $\alpha \in Q_-$. Also $\gamma \to |\xi_\rho(\gamma)|$ is a homomorphism of Γ into $\boldsymbol{R}_+^*$, and so $|\xi_\rho(\gamma)| = 1$, $\gamma \in \Gamma$. Therefore

$$|\Delta_-(h\gamma)| = |\Delta_-(h)|\,, \qquad h \in H\,, \qquad \gamma \in \Gamma\,. \tag{7.1}$$

In particular, $H'' = (H^0)''\Gamma$. Set $\Xi''_H = (H'')^\Xi$ and ${}^0\Xi''_H = [(H^0)'']^{\Xi^0}$.

Now choose a Haar measure $d\xi$ on Ξ (and so also on Ξ^0). We have already chosen dh on H (and so also on H^0). We have the following

THEOREM 7.2. (1) $\Xi''_H = {}^0\Xi''_H\Gamma$.

(2) *Normalize the invariant measure* $d\xi^*$ *on* $\Xi^* = H^0 \backslash \Xi^0 \cong B^0 \backslash M^0$ *so that*

$$\int_{\Xi^0} f(\xi)d\xi = \int_{\Xi^*} d\xi^* \int_{H^0} f(h\xi)dh\,, \qquad f \in C_0(\Xi^0)\,.$$

Then for any $f \in C_0(\Xi''_H)$, *we have*

$$\int_\Xi f(\xi)d\xi = w_0^{-1}\int_H |\Delta_-(h)|^2 dh \int_{\Xi^*} f(h^{\xi^*})d\xi^*\,,$$

where $w_0^{-1} = \#(W_{H^0})$, $W_{H^0} = [N(H^0) \cap \Xi^0]/H^0$.

PROOF. (1) Consider H^Ξ. Since M normalizes Γ, we have $H^\Xi = (BA)^{MA} = B^M A = (B^0)^M \Gamma A$. But $(B^0)^M = (B^0)^{M^0}$. In fact, let $m^{-1}bm \in (B^0)^M$. The map $b_1 \to m^{-1}b_1 m$, $b_1 \in B^0$, is a continuous homomorphism; thus $(B^0)^m$ is another compact Cartan subgroup of M^0. Hence there exists $m_1 \in M^0$ such that $m^{-1}B^0 m = m_1^{-1}B^0 m_1$, that is $m^{-1}bm \in (B^0)^{M^0}$. Therefore $(H'')^\Xi = [(B^0)'']^M A\Gamma = [(B^0)'']^{M^0} A\Gamma = {}^0\Xi''_H\Gamma$.

(2) If f has support in Ξ''_H, then using (1)

$$\int_\Xi f(\xi)d\xi = \sum_{\gamma \in \Gamma} \int_{{}^0\Xi''_H} f(\xi^0\gamma)\, d\xi^0$$

$$=\sum_{\Gamma} w_0^{-1}\int_{H^0}|\Delta_-(h)|^2 dh\int_{\Xi^*}f(h^{\xi^*}\gamma)d\xi^*$$

$$=w_0^{-1}\int_{H}|\Delta_-(h)|^2 dh\int_{\Xi^*}f(h^{\xi^*})d\xi^* .$$

Here, we have applied Theorem 7.1 to the pair (Ξ^0, H^0) and used formula (7.1).

§ 8. A class function on G.

Let G, $P=MAN$ be as before, $\pi=\pi(\sigma, \nu)$ in the continuous series corresponding to P. Set

$$\theta_{\sigma,\nu}(h)=e^{\nu(\log a)}\theta_\sigma(b), \qquad h=ba\in H', \qquad b\in B, \qquad a\in A,$$

as in § 5. Let

$$\Psi_1(h)=\frac{1}{|\Delta_+(h)|}\theta_{\sigma,\nu}(h), \qquad h\in H',$$

$$\Psi(h)=\sum_{s\in W_H}\Psi_1(h^s), \qquad h\in H'.$$

LEMMA 8.1. *Ψ is a G-class function on H'; that is, if for $h_1, h_2\in H'$ there exists $g\in G$ such that $g^{-1}h_1g=h_2$, then $\Psi(h_1)=\Psi(h_2)$.*

PROOF. Recall that $W_H=N(H)/Z_H\cong[N(H)\cap K]/[Z_H\cap K]$ (see [**1e**, p. 488]). Therefore W_H leaves $\mathfrak{a}$ invariant. In particular, W_H stabilizes Q_+. Hence $|\Delta_+(h^s)|=|\Delta_+(h)|$, $s\in W_H$.

Next recall the w-to-1 mapping $\varphi_H: G^*\times H'\to G'_H$, $\varphi_H(g^*, h)=g^{-1}hg$. But the equation $g^{-1}h_1g=h_2$ says precisely that $\varphi_H(g^*, h_1)=\varphi_H(e^*, h_2)$. Hence there is $s\in W_H$ such that $h_1=h_2^s$. From these facts it is clear that $\Psi(h_1)=\Psi(h_2)$ and

$$\Psi(h)=\frac{1}{|\Delta_+(h)|}\sum_{W_H}\theta_{\sigma,\nu}(h^s), \qquad h\in H'.$$

§ 9. Computation of the character.

Continuing with the same situation as in § 8, define

$$\theta_\pi(x)=\begin{cases}\theta_\pi(yxy^{-1}), & y\in G, \quad x\in G'_H\\ 0 \quad , & x\notin G'_H\\ \dfrac{c}{w_0}\dfrac{1}{|\Delta_+(h)|}\sum_{W_H}\theta_{\sigma,\nu}(h^s), & x=h\in H',\end{cases} \tag{9.1}$$

where c is a fixed positive constant (whose precise value will be commented on later). θ_π is a class function on G (by Lemma 8.1).

THEOREM 9.1. *Let $\pi=\pi(\sigma,\nu)$ be a continuous series representation. Let $f\in C_0^\infty(G'_H)$. Then*

$$\operatorname{Tr}\pi(f)=\int_G f(x)\theta_\pi(x)dx\,.$$

PROOF. By the results of § 5

$$\operatorname{Tr}\pi(f)=\int_\Xi \theta_{\sigma,\nu}(\xi)h_f(\xi)d\xi$$

with h_f given by (5.2). By Theorem 6.1, the support of h_f is contained in $(H')^\Xi\subseteqq(H'')^\Xi=\Xi''_H$. Applying Theorem 7.2, we obtain

$$\begin{aligned}\operatorname{Tr}\pi(f)&=w_0^{-1}\int_H|\Delta_-(h)|^2\theta_{\sigma,\nu}(h)dh\int_{\Xi^*}h_f(h^{\xi^*})d\xi^*\\&=w_0^{-1}\int_H|\Delta_-(h)|^2\theta_\sigma(b)e^{(\nu+\rho)(\log a)}dh\int_{\Xi^*}d\xi^*\int_{K\times N}f(k^{-1}(ba)^{\xi^*}nk)dk\,dn\,.\end{aligned}\tag{9.2}$$

Now

$$\begin{aligned}\int_G f(x)\theta_\pi(x)dx&=\frac{c}{w_0w}\int_H|\Delta(h)|^2\theta_\pi(h)dh\int_{G^*}f(h^{x^*})dx^*\\&=\frac{c}{w_0w}\sum_{W_H}\int_H|\Delta(h)|^2\Psi_1(h^s)dh\int_{G^*}f(h^{x^*})dx^*\\&=\frac{c}{w_0w}\sum_{W_H}\int_H|\Delta(h^{s^{-1}})|^2\Psi_1(h)dh\int_{G^*}f((h^{s^{-1}})^{x^*})dx^*\\&=\frac{c}{w_0}\int_H|\Delta(h)|^2\Psi_1(h)dh\int_{G^*}f(h^{x^*})dx^*\end{aligned}$$

because $|\Delta|$ is invariant under W_H and dx^* is G-invariant. Therefore

$$\int_G f(x)\theta_\pi(x)dx=\frac{c}{w_0}\int_H|\Delta_-(h)|^2\,|\Delta_+(h)|\,\theta_{\sigma,\nu}(h)dh\int_{G^*}f(h^{x^*})dx^*\,.$$

But [1g, p. 94, Corollary 2] says that there exists a positive constant c such that for $h\in H'$

$$c\varepsilon_R(h)\Delta_+(h)\int_{G^*}f(h^{x^*})dx^*=e^{\rho(\log a)}\int_{\Xi^*}d\xi^*\int_{K\times N}f(k^{-1}h^{\xi^*}nk)\tag{9.3}$$

(see [1e, §§ 19 & 22] for the definition of ε_R). Choosing $f\geqq 0$ in (9.3), we see that $\varepsilon_R(h)\Delta_+(h)\geqq 0$, that is $\varepsilon_R(h)\Delta_+(h)=|\Delta_+(h)|$. (This could also be deduced from [1d, Lemma 9].) Thus

$$\begin{aligned}&\int_G f(x)\theta_\pi(x)\,dx\\&=w_0^{-1}\int_H|\Delta_-(h)|^2\theta_\sigma(b)e^{(\nu+\rho)(\log a)}dh\int_{\Xi^*}d\xi^*\int_{K\times N}f(k^{-1}h^{\xi^*}nk)dk\,dn\,.\end{aligned}\tag{9.4}$$

Comparing (9.2) and (9.4), we obtain the theorem.

REMARK. The absolute constant c in (9.1) comes from the constant in formula (9.3). Suppose that H is abelian, $H=Z_H$. This is always the case for example if G has a faithful matrix representation. Then we can compute c as follows:

$$\begin{aligned}\int_{G^*} f(h^{x^*})\,dx^* &= \int_{H\backslash G} f(x^{-1}hx)dx \\ &= \int_{\Xi\backslash G} d\bar{g}\int_{H\backslash\Xi} f(x^{-1}\xi^{-1}h\xi x)d\bar{\xi} \\ &= \int_{K\times N} dk\,dn\int_{H\backslash\Xi} f(k^{-1}n^{-1}\xi^{-1}h\xi nk)d\bar{\xi} \\ &= \int_{K\times N} dk\,dn\sum_{\Xi^0\Gamma\backslash\Xi}\int_{H\backslash\Xi^0\Gamma} f(k^{-1}n^{-1}\xi_1^{-1}\xi^{-1}h\xi\xi_1 nk)d\xi^*\,. \qquad (9.5)\end{aligned}$$

But $\Xi^0\Gamma\backslash\Xi \cong M^0\Gamma\backslash M \cong (M^0\Gamma\cap K)\backslash M_K$. Therefore

$$\begin{aligned}(9.5) &= \#(\Xi^0\Gamma\backslash\Xi)\int_{K\times N} dk\,dn\int_{\Xi^*} f(k^{-1}n^{-1}\xi^{-1}h\xi nk)d\xi^* \\ &= \#(\Xi^0\Gamma\backslash\Xi)\frac{e^{\rho(\log a)}}{|\Delta_+(h)|}\int_{K\times N} dk\,dn\int_{\Xi^*} f(k^{-1}h^{\xi^*}nk)\,d\xi^*\,.\end{aligned}$$

Here we used [1g, p. 94, Corollary 1]. So the constant $c=1/\#(\Xi^0\Gamma\backslash\Xi)$. Unfortunately, in the general case $H\neq Z_H$, we have not been able to pin down c precisely.

§ 10. Equivalence.

In this section we would like to determine when equivalence can occur among the continuous series representations we have been considering.

First of all, suppose P_1, P_2 are conjugate cuspidal parabolics. Let $\eta=\sigma\otimes\nu$ be a cuspidal representation of P_1 as usual. Choose $x\in G$ such that $P_1^x=P_2$. Consider the representation η^x of P_2 defined by $\eta^x(m_2a_2n_2)=\eta(xm_2a_2n_2x^{-1})$. Then, it is well-known and easy to see that $\pi(\eta)$ and $\pi(\eta^x)$ are unitarily equivalent. Next suppose P_1, P_2 are only *associate*: that is, $P_j=M_jA_jN_j$, $j=1, 2$, and there is $x\in G$ (equivalently $x\in K$) such that $A_1^x=A_2$. Conjugating by x, we may assume (see the proof of Lemma 10.3) that $P_1=MAN_1$, $P_2=MAN_2$. Let $\theta_\pi^{P_j}$ be the characters of the representations $\pi^{P_j}=\pi^{P_j}(\sigma, \nu)$ $=\mathrm{Ind}_{P_j}^G\,\sigma\otimes\nu$. (In § 9, we denoted $\theta_\pi^P=$ character of π-characteristic function of G'_H, H compatible with P. Henceforth θ_π denotes the full character as in (2.1).)

THEOREM 10.1. (Harish-Chandra [1i, p. 20].) $\theta_\pi^{P_1}=\theta_\pi^{P_2}$. *Consequently* π^{P_1} *is unitarily equivalent to* π^{P_2}.

We obtain now some necessary conditions for equivalence. First suppose P is a cuspidal parabolic and $\sigma_j \otimes \nu_j$, $j=1, 2$ are cuspidal representations of P. Then we have

THEOREM 10.2. *$\pi_1 = \mathrm{Ind}_P^G \sigma_1 \otimes \nu_1$ is unitarily equivalent to $\pi_2 = \mathrm{Ind}_P^G \sigma_2 \otimes \nu_2$ if and only if there is $s \in W = W_H$ such that $s\sigma_1 \cong \sigma_2$ and $s\nu_1 = \nu_2$.*

PROOF. Let $s \in W$ exist. Then clearly π_1 is unitarily equivalent to the representation of G induced from $P^s = MAN^s$ by $\sigma_2 \otimes \nu_2$. The result then follows from Theorem 10.1. This implication is also a consequence of [1i, Lemma 9].

Conversely, assume $\pi_1 \cong \pi_2$. Then in particular $\theta_{\pi_1} = \theta_{\pi_2}$ on H'. By Theorem 9.1 and the explicit formula for the character, we see that

$$\sum_W e^{s\nu_1(\log a)} \theta_{s\sigma_1}(b) = \sum_W e^{s\nu_2(\log a)} \theta_{s\sigma_2}(b), \qquad ba \in H'. \tag{10.1}$$

Now multiply both sides of (10.1) by $\Delta_-(b)$. But for any $\sigma \in \hat{M}_s$, the function $\Delta_- \theta_\sigma$ extends to a bounded continuous function on B (by our results on $\theta_{\sigma|B'}$). Putting $b=1$ and using the fact that exponentials of distinct linear forms are linearly independent, we see that there is $s_0 \in W$ such that $\nu_2 = s_0 \nu_1$.

Let $W_0 = \{t \in W: t\nu_2 = \nu_2\}$, a subgroup of W. Then from (10.1)

$$\sum_{t \in W_0} \theta_{ts_0\sigma_1} = \sum_{t \in W_0} \theta_{t\sigma_2} \qquad \text{a.e. on } B.$$

Now apply the Corollary to Theorem 4.6. There must exist $t_1 \in W_0$ such that $t_1 s_0 \sigma_1 \cong \sigma_2$. Let $s = t_1 s_0$. Then $s\sigma_1 \cong \sigma_2$ and $s\nu_1 = t_1 \nu_2 = \nu_2$. This completes the proof.

We do not need the full strength of the next result, but we include it for its own sake.

LEMMA 10.3. *Let $P_1 = M_1 A_1 N_1$, $P_2 = M_2 A_2 N_2$ be two cuspidal parabolics. Suppose H_1, H_2 are corresponding compatible Cartan subgroups. Then P_1 is associate to P_2 if and only if H_1 is conjugate to H_2.*

PROOF. First suppose P_1 and P_2 are associate. Then there is $x \in G$ such that $A_1 = A_2^x$. Then $Z(A_1) = Z(A_2^x) = Z(A_2)^x$. But $M_1 = \cap \ker|\chi| : \chi \in X(Z(A_1))$. Therefore

$$\begin{aligned} M_1^{x^{-1}} &= \cap (\ker |\chi|)^{x^{-1}} : \ \chi \in X(Z(A_2)^x) \\ &= \cap (\ker |\chi|) \quad : \ \chi \in X(Z(A_2)) = M_2 . \end{aligned}$$

Hence $M_1 = M_2^x$.

Consider B_2^x and its Lie algebra $\mathfrak{b}_2^x$. Then $\mathfrak{b}_1$ and $\mathfrak{b}_2^x$ are compact Cartan subalgebras of $\mathfrak{m}_1$. So there exists $k \in M_1^0 \cap K$ such that $\mathfrak{b}_2^x = \mathfrak{b}_1^k$. Moreover, $B_2^x = Z(\mathfrak{b}_2)^x = Z(\mathfrak{b}_2^x) = Z(\mathfrak{b}_1^k) = B_1^k$. Finally, $H_2^{xk^{-1}} = (B_2 A_2)^{xk^{-1}} = B_2^{xk^{-1}} A_2^{xk^{-1}} = B_1 A_1^{k^{-1}} = B_1 A_1 = H_1$.

Conversely, suppose H_1, H_2 are conjugate. The Lie algebras $\mathfrak{h}_1, \mathfrak{h}_2$ are also conjugate. But they are θ-invariant Cartan subalgebras. Hence $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are also conjugate under K. Let $k \in K$ be such that $\mathfrak{h}_1 = \mathfrak{h}_2^k$. Then $H_1 = H_2^k$ and

$$A_2^k = (H_2 \cap \exp \mathfrak{p})^k = H_2^k \cap \exp \mathfrak{p} = H_1 \cap \exp \mathfrak{p} = A_1 .$$

That is A_1 and A_2 are conjugate, and so P_1, P_2 are associate.

We now begin work on the following

THEOREM 10.4. *Let P_1, P_2 be non-associate parabolics; π_1, π_2 representations in the continuous series corresponding to P_1, P_2 respectively. Then π_1 and π_2 are not unitarily equivalent.*

We shall prove this by showing that $\theta_{\pi_1}^{P_1}$ and $\theta_{\pi_2}^{P_2}$ have different supports. We begin by making a

DEFINITION. Let $P = \Xi N$ be a cuspidal parabolic, and let $H \subseteqq G$ be *any* Cartan subgroup. We say P *surrounds* H and write $H < P$ if there is $x \in G$ such that $H^x \subseteqq \Xi$—or equivalently, if $\mathfrak{h} = LA(H)$, there is $x \in G$ such that $\mathfrak{h}^x$ is a Cartan subalgebra of $\mathfrak{z} = LA(\Xi)$.

Suppose H is not surrounded by P. Claim: $G'_H \cap \Xi = \phi$. Indeed, suppose there is $\xi \in G'_H \cap \Xi$. Then, as in § 6, $\mathfrak{h}_1 = \{X \in \mathfrak{g} : Ad_G(\xi)X = X\}$ is a Cartan subalgebra of $\mathfrak{z}$. Setting $H_1 = Z(\mathfrak{h}_1)$, we obtain a Cartan subgroup of Ξ and $\xi \in G'_H \cap H_1$. Therefore H and H_1 are conjugate, contradicting the fact that H is not surrounded by P.

Fix a cuspidal parabolic P. Let $H_1, \cdots, H_r$ be a complete list of non-conjugate Cartan subgroups of G. Suppose $H_1, \cdots, H_s < P$ and $H_{s+1}, \cdots, H_r \nless P$. Then we may assume $H_1, \cdots, H_s \subseteqq \Xi$ and $G'_{H_j} \cap \Xi = \phi$, $j = s+1, \cdots, r$. Let $\sigma \otimes \nu$ be a cuspidal representation of P and $\pi = \pi(\sigma, \nu)$ as usual.

LEMMA 10.5. *θ_π is identically zero on $\bigcup_{j=s+1}^{r} G'_{H_j}$.*

PROOF. Let $f \in C_0^\infty(G)$. Then from our work in § 5 and § 6, we know

$$\int_G f(g)\theta_\pi(g)dg = \operatorname{Tr} \pi(f) = \int_\Xi \theta_{\sigma,\nu}(\xi) h_f(\xi) d\xi$$

where

$$h_f(\xi) = d(\xi)\,\delta(\xi) \int_{K \times N} f(k^{-1}n^{-1}\xi n k) dk\, dn .$$

But for any $\xi \in \Xi$, we have $\xi \notin G'_{H_j}$, $j > s$. Therefore, if $\operatorname{Supp} f \subseteqq G'_{H_j}$, $j > s$, we must have $h_f(\xi) = 0$ and the lemma is proven.

LEMMA 10.6. *Suppose $H < P$ and H is θ-invariant. Then we may choose $x \in K$ such that $H^x \subseteqq \Xi$.*

PROOF. By assumption there is $x \in G$ such that $H^x \subseteqq \Xi$. Let $H_1 = H^x$, a Cartan subgroup of Ξ and G. If H_1 is also θ-invariant, then we know H and H_1 are K-conjugate. Claim: there is $\xi \in \Xi$ such that H_1^ξ is θ-invariant. To show this, let $\mathfrak{c}$ be the center of $\mathfrak{m}$. Since $\mathfrak{m}$ has a compact Cartan subalgebra, we must have $\mathfrak{c} \subseteqq \mathfrak{m} \cap \mathfrak{k}$. Therefore $\mathfrak{m} = (\mathfrak{m} \cap \mathfrak{k}) + (\mathfrak{m} \cap \mathfrak{p})$ is a Cartan decomposition of the reductive Lie algebra $\mathfrak{m}$. Now let $\mathfrak{h}_1 = LA(H_1)$, a Cartan subalgebra of $\mathfrak{z}$. Hence $\mathfrak{h}_1 = \mathfrak{h}_2 \oplus \mathfrak{a}$, where $\mathfrak{h}_2 = \mathfrak{h}_1 \cap \mathfrak{m}$ is a Cartan subalgebra of $\mathfrak{m}$. It follows that there exists $\xi \in M^0$ such that $\mathfrak{h}_2^\xi$ is invariant under the appropriate Cartan involution, namely $\theta|_{\mathfrak{m}}$ [1b, p. 100]. Hence $\mathfrak{h}_1^\xi = \mathfrak{h}_2^\xi + \mathfrak{a}$ and $H_1^\xi = Z(\mathfrak{h}_1^\xi)$ are both θ-invariant. But then $H^{x\xi} = H_1^\xi = H_2$ is θ-invariant and contained in Ξ. Therefore H and H_2 are conjugate under K.

The following result will enable us to complete the proof of the theorem.

LEMMA 10.7. *Suppose $P_1 = \Xi_1 N_1$, $P_2 = \Xi_2 N_2$ are cuspidal parabolics; and H_1, H_2 are compatible Cartan subgroups. Suppose $H_1 < P_2$ and $H_2 < P_1$. Then P_1 and P_2 are associate.*

PROOF. By Lemma 10.6, there exist $x, y \in K$ such that

$$H_1^x \subseteqq \Xi_2 , \qquad H_2^y \subseteqq \Xi_1 .$$

Or equivalently $\mathfrak{h}_1^x \subseteqq \mathfrak{z}_2$, $\mathfrak{h}_2^y \subseteqq \mathfrak{z}_1$. That is

$$\mathfrak{h}_1^x \subseteqq \mathfrak{z}_2 = Z(\mathfrak{h}_2 \cap \mathfrak{p}) \cap \mathfrak{g} , \qquad \mathfrak{h}_2^y \subseteqq \mathfrak{z}_1 = Z(\mathfrak{h}_1 \cap \mathfrak{p}) \cap \mathfrak{g} .$$

Therefore

$$[\mathfrak{h}_1^x, \mathfrak{h}_2 \cap \mathfrak{p}] = 0 , \qquad [\mathfrak{h}_2^y, \mathfrak{h}_1 \cap \mathfrak{p}] = 0 .$$

But $\mathfrak{h}_1^x, \mathfrak{h}_2^y$ are Cartan subalgebras of $\mathfrak{g}$; in particular they are maximal abelian subalgebras $\Rightarrow$

$$\mathfrak{h}_2 \cap \mathfrak{p} \subseteqq \mathfrak{h}_1^x , \qquad \mathfrak{h}_1 \cap \mathfrak{p} \subseteqq \mathfrak{h}_2^y .$$

Since $x \in K$, $\mathfrak{h}_2 \cap \mathfrak{p} \subseteqq \mathfrak{h}_1^x \cap \mathfrak{p} = (\mathfrak{h}_1 \cap \mathfrak{p})^x$. Similarly $\mathfrak{h}_1 \cap \mathfrak{p} \subseteqq (\mathfrak{h}_2 \cap \mathfrak{p})^y$. Therefore

$$\dim (\mathfrak{h}_2 \cap \mathfrak{p}) \leqq \dim (\mathfrak{h}_1 \cap \mathfrak{p})^x = \dim (\mathfrak{h}_1 \cap \mathfrak{p}) \leqq \dim (\mathfrak{h}_2 \cap \mathfrak{p})^y = \dim (\mathfrak{h}_2 \cap \mathfrak{p}) .$$

Thus we get equality throughout. In particular

$$\dim (\mathfrak{h}_2 \cap \mathfrak{p}) = \dim (\mathfrak{h}_1 \cap \mathfrak{p})^x .$$

But $\mathfrak{h}_2 \cap \mathfrak{p} \subseteqq (\mathfrak{h}_1 \cap \mathfrak{p})^x$. By dimensionality, $\mathfrak{h}_2 \cap \mathfrak{p} = (\mathfrak{h}_1 \cap \mathfrak{p})^x$. Taking exponentials, we obtain $A_2 = A_1^x$, that is P_1 and P_2 are associate.

PROOF OF THEOREM 10.4. Suppose π_1 and π_2 were unitarily equivalent. Then $\theta_{\pi_1}^{P_1} = \theta_{\pi_2}^{P_2}$. Let H_1, H_2 be Cartan subgroups, compatible with P_1, P_2. But $H_j \subseteqq \operatorname{Supp} \theta_{\pi_j}^{P_j}$, $j = 1, 2$. Therefore $\theta_{\pi_1}^{P_1} \neq 0$ on G'_{H_2} and $\theta_{\pi_2}^{P_2} \neq 0$ on G'_{H_1}. By Lemma 10.5, it follows that $H_1 < P_2$ and $H_2 < P_1$. Hence, by Lemma 10.7, P_1 and P_2 are associate. This contradicts the hypothesis and completes the proof.

REMARK. Theorem 10.4 remains valid in case $P_2 = G$ is cuspidal. That is, suppose G has a compact Cartan subgroup so that G has a discrete series. By a representation π corresponding to $P_2 = G$, we mean any irreducible square-integrable unitary representation. So let P_1 be a proper cuspidal parabolic and suppose $P_2 = G$. Let π_1 and π_2 be in the continuous (respectively discrete) series corresponding to P_1 (respectively P_2). Then π_1 and π_2 are not unitarily equivalent. In fact, $\theta_{\pi_2}^{P_2}$ has support in the whole group (see Harish-

Chandra's computation of these characters [**1**f, part II]). But $\theta_{\pi_1}^{P_1}$ does not. Indeed, if H_2 is a compact Cartan subgroup, it is easily seen that $H_2 \not\prec P_1$. Therefore Lemma 10.5 insures that $\theta_{\pi_1}^{P_1}$ is zero on the open set G'_{H_2}.

§ 11. Disjointness.

Let P_1, P_2 be non-associate cuspidal parabolics ($P_2=G$ allowed). Suppose π_1, π_2 are representations in the series corresponding to P_1, P_2 respectively. Then we know (Theorem 10.4) that these representations are not unitarily equivalent. If they are irreducible, they give rise to distinct points of $\hat{G}$. On the other hand, suppose one or both are reducible. We would like to show that they have distinct constituents, i. e. π_1 and π_2 are *disjoint*. Since any subrepresentation of π_j has the same infinitesimal character as π_j, it is enough to show that π_1 and π_2 have distinct infinitesimal characters. Unfortunately, we can only prove that result for some parabolics.[4]

THEOREM 11.1. *Let P_1, P_2 be non-associate cuspidal parabolics; π_1, π_2 representations in the corresponding series. Assume* $\dim A_1 \neq \dim A_2$[4] ($\dim A_2=0$ *if* $P_2=G$). *Then π_1 and π_2 have distinct infinitesimal characters.*

PROOF. First, let us compute the infinitesimal characters. If $P_2=G$, we already have Harish-Chandra's computations (see § 3). Otherwise, let $P=MAN$ be a proper cuspidal parabolic, $\pi=\pi(\sigma,\nu)$ as usual. Let $\mathfrak{Z}$ and $\mathfrak{Z}(\mathfrak{M})$ be the centers of the enveloping algebras $\mathfrak{U}(\mathfrak{g}_c)$ and $\mathfrak{U}(\mathfrak{m}_c)$ respectively. Then there is a canonical homomorphism $\mu:\mathfrak{Z}\to\mathfrak{Z}(\mathfrak{M})$ of $\mathfrak{Z}$ onto $\mathfrak{Z}(\mathfrak{M})$ [**1**e, § 12].

Next, recall the equation

$$\theta_\pi(f)=\int_G f(g)\theta_\pi(g)dg=\int_\Xi \theta_{\sigma,\nu}(\xi)h_f(\xi)d\xi=\theta_{\sigma,\nu}(h_f)\,.$$

Using this formula together with a result stated [**1**e, Lemma 52] and proved [**1**g, § 10] by Harish-Chandra, we obtain

$$z\theta_\pi(f)=\mu(z)\theta_{\sigma,\nu}(h_f)\,,\qquad f\in C_0^\infty(G)\,,\qquad z\in\mathfrak{Z}\,.$$

It follows that $\chi_\pi(z)=\chi_{\sigma,\nu}(\mu(z))$. Now we use the results of § 4. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{z}$ (and $\mathfrak{g}$) compatible with P. Let λ be a linear form on $\mathfrak{b}$ determined by σ, and set $\eta=$ the linear form on $\mathfrak{h}$ such that

$$\eta=\begin{cases}\lambda & \text{on } \mathfrak{b}\\ \nu & \text{on } \mathfrak{a}\,.\end{cases}$$

4) By examining our proof, one sees that to complete the argument in case $\dim A_1=\dim A_2$, the following fact is required: If $\mathfrak{h}_1$, $\mathfrak{h}_2$ are $\mathfrak{g}_c$-conjugate Cartan subalgebras of $\mathfrak{g}$, then they are $\mathfrak{g}$-conjugate. A proof of this has been communicated to me by Joe Wolf.

Then $\chi_{\sigma,\nu}(z)=\chi_{\eta}^{\mathfrak{h}(\mathfrak{m})}(z)$, $z\in\mathfrak{Z}(\mathfrak{M})$.

Now let

$$\gamma:\ \mathfrak{Z}\longrightarrow I_{\mathfrak{g}}(\mathfrak{h}_c)$$
$$\gamma_{\mathfrak{m}}:\ \mathfrak{Z}(\mathfrak{M})\longrightarrow I_{\mathfrak{m}}(\mathfrak{h}_c)$$

be the canonical isomorphisms (see § 2). Also let $i: I_{\mathfrak{g}}(\mathfrak{h}_c)\to I_{\mathfrak{m}}(\mathfrak{h}_c)$ be the injection. Then $\mu=\gamma_{\mathfrak{m}}^{-1}\circ i\circ\gamma$ [1e, § 12]. If $z\in\mathfrak{Z}$, we compute

$$\begin{aligned}\chi_{\pi}(z)&=\chi_{\eta}^{\mathfrak{h}(\mathfrak{m})}(\mu(z))\\&=\gamma_{\mathfrak{m}}(\mu(z))(\eta)\\&=\gamma(z)(\eta)\\&=\chi_{\eta}^{\mathfrak{h}}(z)\,.\end{aligned}$$

Thus we have shown that $\chi_{\pi}(z)=\chi_{\eta}^{\mathfrak{h}}(z)$, $z\in\mathfrak{Z}$, where η is the linear form on $\mathfrak{h}$ such that $\eta=\lambda$ on $\mathfrak{b}$, $\eta=\nu$ on $\mathfrak{a}$.

NOTE. By the results in § 4, the form λ is regular. That is $s\lambda\neq\lambda$ for all $s\in W(\mathfrak{m},\mathfrak{b})$, $s\neq 1$.

It is enough now to prove

LEMMA 11.2. *Let $\mathfrak{h}_1,\mathfrak{h}_2$ be non-conjugate, θ-invariant Cartan subalgebras of $\mathfrak{g}$, $\mathfrak{h}_j=\mathfrak{b}_j+\mathfrak{a}_j$, where $\mathfrak{b}_j=\mathfrak{h}_j\cap\mathfrak{k}$, $\mathfrak{a}_j=\mathfrak{h}_j\cap\mathfrak{p}$. Let η_j be linear forms on $\mathfrak{h}_j$ such that*

$$\eta_j=\begin{cases}\lambda_j & on\ \ \mathfrak{b}_j\\ \nu_j & on\ \ \mathfrak{a}_j\,.\end{cases}$$

Suppose the λ_j are regular, i.e. $s\lambda_j\neq\lambda_j$ for all $s\in W(\mathfrak{m}_j,\mathfrak{b}_j)$, $s\neq 1$. (Here we may take $\mathfrak{m}_j=$ the orthogonal complement with respect to the Killing form of $\mathfrak{a}_j$ in $\mathfrak{z}_j=Z(\mathfrak{a}_j)$. Of course $\mathfrak{m}_2=\mathfrak{g}$ in case $\mathfrak{a}_2=\{0\}$.) Then $\chi_{\eta_1}^{\mathfrak{h}_1}\neq\chi_{\eta_2}^{\mathfrak{h}_2}$.

PROOF. Unfortunately, to give the proof, we have to make the additional assumption $\dim\mathfrak{a}_1\neq\dim\mathfrak{a}_2$.[4] Now by the symmetry of the hypotheses, we may suppose $\dim\mathfrak{a}_2<\dim\mathfrak{a}_1$. Let $\mathfrak{h}_j^c$ be the complexifications of $\mathfrak{h}_j$ and suppose Q_j denotes a choice of positive roots of $(\mathfrak{g}_c,\mathfrak{h}_j^c)$. Set $Q_j^+=\{\alpha\in Q_j:\ \alpha|_{\mathfrak{a}_j}\neq 0\}$, $\Sigma_j=$ the simple roots of Q_j. Then $\#(\Sigma_j\cap Q_j^+)=\dim_R\mathfrak{a}_j$. Choose $y\in\mathrm{Int}(\mathfrak{g}_c)$ such that $(\mathfrak{h}_1^c)^y=\mathfrak{h}_2^c$. Then $\chi_{\eta_1}^{\mathfrak{h}_1}=\chi_{\eta_1^y}^{\mathfrak{h}_2}$ where $\eta_1^y(X)=\eta_1(y^{-1}\cdot X)$, $X\in\mathfrak{h}_2^c$.

Suppose $\chi_{\eta_1}^{\mathfrak{h}_1}=\chi_{\eta_2}^{\mathfrak{h}_2}$. Then $\chi_{\eta_1^y}^{\mathfrak{h}_2}=\chi_{\eta_2}^{\mathfrak{h}_2}$; therefore there exists $s\in W(\mathfrak{g},\mathfrak{h}_2)$ such that $\eta_1^y=s\eta_2$. Choose $y_1\in\mathrm{Int}(\mathfrak{g}_c)\cap N(\mathfrak{h}_2^c)$ such that y_1 acts on $\mathfrak{h}_2^c$ by the same automorphism as s. Setting $z=yy_1^{-1}\in\mathrm{Int}(\mathfrak{g}_c)$, we see that $\eta_1^z=\eta_2$, $\eta_1^z(X)=\eta_1(z^{-1}\cdot X)$, $X\in\mathfrak{h}_2^c$.

Now fix a choice of positive roots Q_1 on $(\mathfrak{g},\mathfrak{h}_1)$. Then Q_1^z is a set of positive roots Q_2 for $(\mathfrak{g},\mathfrak{h}_2)$, $\alpha^z(Y)=\alpha(z^{-1}\cdot Y)$, $Y\in\mathfrak{h}_2^c$, $\alpha\in Q_1$. Let $\alpha_1,\cdots,\alpha_r\in Q_1^+\cap\Sigma_1$, $r=\dim\mathfrak{a}_1$. Then $\alpha_1^z,\cdots,\alpha_r^z\in\Sigma_2$. Suppose $\alpha_1^z,\cdots,\alpha_r^z\in Q_2^+$. Since these are linearly independent forms, it would follow that $\dim\mathfrak{a}_2\geqq\dim\mathfrak{a}_1$.

This is contrary to assumption. Hence we may assume that

$$\beta_1 = \alpha_1^z \in Q_2^- = \{\beta \in Q_2 : \beta|_{\mathfrak{a}_2} \equiv 0\} .$$

Next for $\xi \in (\mathfrak{h}_j^c)^*$, choose $Y_\xi \in \mathfrak{h}_j^c$ so that $\xi(Y) = (Y, Y_\xi)$, $Y \in \mathfrak{h}_j^c$, $(\cdot, \cdot) =$ the Killing form. When $\alpha \in Q_j^+$, then $Y_\alpha \in \mathfrak{a}_j$; and when $\alpha \in Q_j^-$, then $Y_\alpha \in i\mathfrak{b}_j$. Consider $Y_{\alpha_1} \in \mathfrak{a}_1$. Then

$$\eta_1(Y_{\alpha_1}) = \nu(Y_{\alpha_1}) \subseteqq i\boldsymbol{R} .$$

But $\eta_1^z = \eta_2$ and so

$$\eta_1(Y_{\alpha_1}) = \eta_2(z \cdot Y_{\alpha_1}) = \eta_2(Y_{\beta_1}) = \lambda(Y_{\beta_1}) \subseteqq \boldsymbol{R} ,$$

since $\beta_1 = \alpha_1^z \in Q_2^- =$ the positive roots of $(\mathfrak{z}_2, \mathfrak{h}_2) =$ the positive roots of $(\mathfrak{m}_2, \mathfrak{b}_2)$. Therefore $\eta_2(Y_{\beta_1}) = 0$; that is $s_{\beta_1}\eta_2 = \eta_2$, where $s_{\beta_1} \in W(\mathfrak{m}_2, \mathfrak{b}_2)$ is the reflection through the simple root β_1. This contradicts the regularity of η_2 and concludes the proof.

REMARK. In particular, the proof works for $\dim \mathfrak{a}_2 = 0$. So every irreducible constituent of any continuous series representation is not equivalent to a discrete series representation.

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