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Deformations of compact complex surfaces II

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§ 0. Introduction.

This is a continuation of the paper [4], referred to as Part I in this paper. We employ the notation of Part I, in which we call connected compact complex manifolds of complex dimension 2 " surfaces ". The purpose of the present paper is to prove

THEOREM III. *Plurigenera of surfaces are invariant under arbitrary holomorphic deformations.*

The proof is based heavily on the classification theory of all surfaces established by Italian algebraic geometers and K. Kodaira [7].

§ 1. Proof for surfaces of which all plurigenera vanish.

First, we consider an algebraic surface which is birationally equivalent to the product of an algebraic curve and a projective line. We call this ruled surface, extending the previous definition of ruled surfaces in [7, IV].

PROPOSITION 1. *Any deformation of a ruled surface is also ruled.*

PROOF. For a rational surface, this is Theorem I in Part I, and for an irrational ruled surface this follows immediately from the classification of surfaces in [7, IV]. q. e. d.

By Proposition I and the vanishing of all plurigenera of a ruled surface, *Theorem III is proved when a ruled surface appears as a deformation of the surface.*

Second, we shall consider the other surfaces of which all plurigenera vanish.

PROPOSITION 2. *All plurigenera of surfaces with $b_1=1$ and $P_{12}=0$ vanish and the class consisting of such surfaces is closed under deformations.*

PROOF. The former part of Proposition 2 has been proved in Theorem 35 in [7, II]. To prove the latter part, we recall the following Lemma A, which was used in Part I.

LEMMA A. *Let X and Y be complex manifolds, and let f be a proper and simple holomorphic map from X to Y, such that every fibre X_y is a surface. The following two assertions can be established.*

I. *If there exists an irreducible exceptional curve of the first kind* E_0 *in a fibre* X_0 *for a point* $o \in Y$, *then* E_0 *can be extended locally, i.e., there exist a neighborhood* U *and a complex submanifold* E *of* $f^{-1}(U)$, *whose restriction to* X_0 *is the curve* E_0. *We note that for any point* $y \in U$, $E_y = E \cap X_y$ *is an irreducible exceptional curve of the first kind on the surface* X_y.

II. *If there exists a complex submanifold* E *of* X *such that its restriction to* $X_y : E_y = E \cap X_y$ *is an irreducible exceptional curve of the first kind on* X_y *at any* $y \in Y$, *then we can construct a complex manifold* $\hat{X}$, *which is simple over* Y, *and a holomorphic map* $\mu : X \to \hat{X}$ *over* Y, *such that* $\mu|_{X_y} : X_y \to \hat{X}_y$ *shrinks* E_y *to a simple point in* $\hat{X}_y$ *for every point* $y \in Y$, *and such that* $\mu|_{X-E} : X - E \to \hat{X} - \mu(E)$ *is biholomorphic.*

PROOF. For the proofs of I and II, we refer to Theorem 5 in [6] and Appendix I, respectively. q. e. d.

Now, we continue the proof of Proposition 2. Let $f: X \to Y$ be a fibre space of connected complex manifolds describing a complex analytic family of deformations of S with $b_1 = 1$, $P_{12} = 0$, namely, f is proper, simple and surjective, and $X_0 = S$ for a point $o \in Y$. We denote by Σ the set of points $y \in Y$ such that $b_1(X_y) = 1$ and $P_{12}(X_y) = 0$. By the principle of upper semi-continuity, the set Σ is open. Under the assumption that $Y - \Sigma$ is not empty, we derive a contradiction. Since $P_{12}(X_y) > 0$ for $y \in Y - \Sigma$, the minimal model of X_y, which we denote by X_y^*, is elliptic, referring to Lemma 27 in [7, IV]. According to Lemma A we can choose a neighborhood U of y and construct a fibre space $X_U^* \to U$ describing a complex analytic family consisting of the surfaces $X_{y'}^*$, $y' \in U$. By Theorem 29 in [7, II], it follows that $P_{12}(X_{y'}^*) > 0$. Therefore, $Y - \Sigma$ is open. This contradicts the connectedness of Y.

COROLLARY. *Theorem III is proved for the surfaces with* $b_1 \equiv 1 \bmod 2$ *and* $P_m = 0$ *for all* $m \geqq 1$.

PROOF. This corollary is a direct conclusion of Lemma 27 in [7, II] and Proposition 2 above. q. e. d.

§ 2. Proof for surfaces of type (*i*) and for the algebraic surfaces of general type.

Third, we shall consider minimal surfaces with some plurigenera > 0.

PROPOSITION 3. *Minimal surfaces with some plurigenera* > 0 *can be classified into the following four classes.*

i) *the class of surfaces with* $p_g = 0$ *and* $12K \approx 0$[1];

ii) *the class of surfaces with* $K \approx 0$;

1) "K" denotes a canonical divisor of the surface under consideration and "$\approx$" indicates linear equivalence of divisors.

iii) *the class of elliptic surfaces of general type, defined to be elliptic surfaces with some plurigenera* $\geqq 2$;

iv) *the class of algebraic surfaces of general type, defined to be surfaces with* $P_2 > 0$ *and* $c_1^2 > 0$.

PROOF. This classification is easily derived from Table II in [7, IV].

PROPOSITION 4. *Each class in Proposition* 3 *is closed under deformations.*

PROOF. For the class ii), we have a straight proof of Proposition 4 depending on Lemma C below, which follows from the following Lemma B in the theory of complex spaces.

LEMMA B. *Let* $f: X \to Y$ *be a fibre space of complex spaces such that* f *is proper, and let* $\mathcal{F}$ *be a* f*-flat, coherent* $\mathcal{O}_X$*-Module. If* $\dim H^0(X_y, \mathcal{F}_y)$ *is independent of* $y \in Y$, *then* $f_*(\mathcal{F})$ *is a locally free* $\mathcal{O}_Y$*-Module whose rank is equal to* $\dim H^0(X_y, \mathcal{F}_y)$ *and the canonical homomorphism* $f_*(\mathcal{F})_y \otimes_{\mathcal{O}_{Y,y}} k(y) \to H^0(X_y, \mathcal{F}_y)$ *is isomorphic, where* $k(y)$ *is the field of complex numbers regarded as* $\mathcal{O}_{Y,y}$*-Module by the map* $f \mapsto f(y)$ *for the germ of* f: $\{f\} \in \mathcal{O}_{Y,y}$.

PROOF. We refer to Satz 5 in [2]. q. e. d.

LEMMA C. *Under the same condition as in Lemma B, we assume further that* Y *is connected,* X_y *is reduced and irreducible for every point* $y \in Y$, $\mathcal{F}$ *is invertible and that* $\mathcal{F}_0 \simeq \mathcal{O}_{X_0}$ *for a fixed point* $o \in Y$. *Then,* $\mathcal{F}_y \simeq \mathcal{O}_{X_y}$ *for every point* $y \in Y$.

PROOF. By Y', we denote the set of points y such that $\mathcal{F}_y \simeq \mathcal{O}_{X_y}$ as $\mathcal{O}_{X_y}$-Module. Then Y' is closed by the upper semi-continuity of $\dim H^0(X_y, \mathcal{F}_y^{-1})$ as a function of $y \in Y$. We shall show that Y' is open, which proves Lemma C. For an arbitrarily fixed point $1 \in Y'$, we can choose a non-vanishing section $s \in H^0(X_1, \mathcal{F}_1)$, then s extends to a section $\tilde{s}$ of $\mathcal{F}$ on the inverse image $f^{-1}(U)$ of a neighborhood U by Lemma B. Since $\tilde{s}_z \in H^0(X_z, \mathcal{F}_z)$ is a non-vanishing section for every point z in a small neighborhood V of y in U, it follows that $\mathcal{F}_z \simeq \mathcal{O}_{X_z}$ for every $z \in V$. Therefore Y' is open. q. e. d.

To prove Proposition 3 for the class i), we define *a surface of type* (i) *to be a surface whose i-canonical divisor: iK is linearly equivalent to* 0 *but jK is not so for* $0 < j < i$. We shall show that *any deformation of a surface of type* (i) *is also of type* (i). Let $f: X \to Y$ be a fibre space of complex manifolds describing a complex analytic family of deformations of a surface S of type (i), namely, $X_0 = S$ for a point $o \in Y$. In this proof, we denote *the underlying topological manifold of a complex manifold* M *by the symbol* (M). Then, the topological fibre space $f: (X) \to (Y)$ is a fibre bundle. Hence, there exists a connected open covering $\{U_\lambda\}_{\lambda \in \Lambda}$ of Y, such that $(f^{-1}(U_\lambda))$ is homeomorphic to the product $(S) \times (U_\lambda)$ for every $\lambda \in \Lambda$. On the other hand, $X_0 = S$ has an unramified covering manifold $\tilde{S}$, such that $\tilde{S}$ is of type (1) and the degree of the covering holomorphic map $\mu: \tilde{S} \to S$ is i, referring to Theorem 33 in [7, II]. Now we get a complex manifold $\tilde{X}_\lambda$ and an unramified holomorphic map μ_λ:

$\tilde{X}_\lambda \to f^{-1}(U_\lambda)$ such that $(\tilde{X}_\lambda)$ is homeomorphic to $(\tilde{S})\times(U_\lambda)$ and the map μ_λ is equal to the product of μ and id_{U_λ} for each $\lambda \in \Lambda$. Then, the holomorphic map $f\circ\mu_\lambda\colon \tilde{X}_\lambda \to U_\lambda$ is proper, simple and surjective. Moreover, $(f\circ\mu_\lambda)^{-1}(o)$ is biholomorphic to the surface $\tilde{S}$ for any suffix λ such that $o\in U_\lambda$. Therefore, $(f\circ\mu_\lambda)^{-1}(y)$ is of type (1) by the previous argument for any point $y\in U_\lambda$ and any suffix λ such that $o\in U_\lambda$. Since Y is connected and $\{U_\lambda\}_{\lambda\in\Lambda}$ is a covering of Y, $(f\circ\mu_\lambda)^{-1}(y)$ is of type (1) for every $y\in U_\lambda$ and $\lambda\in\Lambda$. Thus, X_y is of type (i). q. e. d.

To prove Proposition 3 for the classes iii) and iv), we define $n(S)$ to be *the number of quadric transforms required to obtain the non-rational surface S from one of its relative minimal model* S^*. By $f\colon X\to Y$, we denote a fibre space describing a family of surfaces. Then, by Lemma A, $n(X_y)$ is lower semi-continuous as a function of $y\in Y$. If X_0 is in the class iii) for a point $o\in Y$, the set of points $y\in Y$, such as $n(X_y)>0$ is open. This set coincides with the set of $y\in Y$ such that the minimal model of X_y is of general type. Hence, this set is closed by the upper semi-continuity of $P_m(X_y)$ as a function of $y\in Y$. If X_0 is in the class iv) for a point $o\in Y$, then X_y^* is also in the class iv) for every $y\in Y$, because if X_y^* is in the class iii), $c_1^2(X_0)=c_1^2(X_y)\leqq c_1^2(X_y^*)=0$. Now, from the vanishing of $H^1(X_y^*, (\Omega^2)^{\otimes 2})$ (which has been proved in Theorem 5 in [7]), it follows immediately that $n(X_y)=\dim H^1(X_y, (\Omega^2)^{\otimes 2})$. Hence, $n(X_y)$ is an upper semi-continuous function of $y\in Y$. Consequently, $n(X_y)=n(X_0)=0$. This completes the proof of Proposition 3.

PROPOSITION 5. *Plurigenera of surfaces in each class in Proposition* 3 *are as follows:*

TABLE III

class	plurigenera ($m=1, 2, \cdots$)
i)	$P_m=0$ when $m\not\equiv 0 \bmod i$, $P_m=1$ when $m\equiv 0 \bmod i$, where $i=2, 3, 4$ and 6.
ii)	$P_m=1$.
iii)	$P_\nu\geqq 2$ for an index $\nu\geqq 1$. (For an explicit formula of P_m, see Proposition 8.)
iv)	$P_{m+1}=\frac{1}{2}m(m+1)c_1^2+1-q+p_g$.

PROOF. The formula for the class iv) is found in Theorem 5 in [8]. The verification is easy for the other classes. q. e. d.

We note that the invariance of plurigenera for the class i) is proved in the proof of Proposition 3. Hence, for our purpose it is sufficient to prove Theorem III for the elliptic surfaces of general type.

§ 3. Proof for elliptic surfaces of general type.

In the following Propositions 6, 7, 8, 9 and 10, we denote by S an elliptic surface of general type.

PROPOSITION 6. *$P_m(S)$ becomes arbitrarily large, when "m" grows to infinity.*

PROOF. It is immediately proved that $P_{m\nu}(S)\to\infty$ $(m\to\infty)$ for any integer ν satisfying $P_\nu \geqq 2$. Combining this with the formula (40) in [7, I], Proposition 6 is proved.

PROPOSITION 7. *The irreducible algebraic pencil of elliptic curves on S is unique. This is determined by the canonical fibration of the elliptic surface S.*

PROOF. The m-canonical system $|mK|$ for sufficiently large m is composite with an irreducible algebraic pencil of elliptic curves $\{F_y\}_{y\in\Delta}$. Every member of this pencil $\{F_y\}$ is a fibre of the canonical fibring $\Psi: S\to\Delta$, where we denote by Δ and Ψ the base curve of the elliptic surface S and the canonical map of S, respectively. Now, let $\{E_\lambda\}_{\lambda\in B}$ be an arbitrarily given irreducible algebraic pencil of elliptic curves on S. If E_λ is not contained in $\{F_y\}$, then the intersection number of E_λ and F_y : $E_\lambda\cdot F_y>0$. Hence,

$$E_\lambda K=\frac{1}{m}E_\lambda\cdot mK=\frac{1}{m}\sum_y E_\lambda F_y>0\,. \tag{1}$$

Combining $E_\lambda^2\geqq 0$ with this, we have the following false relation:

$$0=2\pi(E_\lambda)^{2)}-2=E_\lambda^2+KE_\lambda>0\,. \tag{2}$$

Thus $E_\lambda\in\{F_y\}$. By the irreducibility of these pencils, we get $\{E_\lambda\}=\{F_y\}$. Note that Δ corresponds to B birationally and bijectively.

PROPOSITION 8. *The m-genus $P_m(S)$ $(m=2,3,4,\cdots)$ is given by the following formula:*

$$P_m(S)=m(2\pi-1-q+p_g)+\sum_{\lambda=1}^{s}\Big[m\Big(1-\frac{1}{m_\lambda}\Big)\Big]+1-\pi\,, \tag{3}$$

where we denote by π, q, p_g, and $m_1,\cdots,m_s$, respectively, the genus of the base curve Δ of S, the irregularity of S, the geometric genus of S and the multiplicities of the multiple fibres $\Psi^*(a_1),\cdots,\Psi^*(a_s)$ of the canonical map Ψ of S.

The bracket [ ] denotes the Gauss symbol.

PROOF. By the formula (40) in [7, I], an m-canonical divisor mK is given as follows:

2) $\pi(E_\lambda)$ denotes the arithmetic genus of E_λ.

$$mK = m\Psi^*(\mathfrak{k}-\mathfrak{f}) + \sum_{\lambda=1}^{s} m(m_\lambda - 1)\Psi^{-1}(a_\lambda) \tag{4}$$

where we denote by $\mathfrak{k}$ and $\mathfrak{f}$ a canonical divisor and a divisor of degree $1-q+p_g$, respectively. Let $\mathfrak{d}_m = \mathfrak{d}_m(S)$ indicate the divisor:

$$m(\mathfrak{k}-\mathfrak{f}) + \sum_{\lambda=1}^{s}\Big[m\Big(1-\frac{1}{m_\lambda}\Big)\Big]a_\lambda \quad \text{on } \Delta .$$

Then the equality $P_m(S) = \dim|\mathfrak{d}_m(S)|+1$ holds. Therefore, to prove the formula in Proposition 8, it suffices to show that $\deg(\mathfrak{k}-\mathfrak{d}_m)<0$, since this implies that the complete linear system $|\mathfrak{d}_m|$ is not special.

PROOF OF THE INEQUALITY: $\deg(\mathfrak{k}-\mathfrak{d}_m)<0$.

We obtain immediately the following formula (5) where t denotes the non-negative integer $1-q+p_g$.

$$\deg(\mathfrak{k}-\mathfrak{d}_m) = 2\pi - 2 - m(2\pi-2) - mt - \sum_{\lambda=1}^{s}\Big[m\Big(1-\frac{1}{m_\lambda}\Big)\Big]. \tag{5}$$

On the other hand, we infer from Proposition 6 that the following inequality (6) holds:

$$2\pi - 2 + \sum_{\lambda=1}^{s}\Big(1-\frac{1}{m_\lambda}\Big) + t > 0, \tag{6}$$

where t denotes $1-q+p_g$, too. We recall that in the classical theory of automorphic functions, it has been proved that the right side of the formula (5) is negative for every set of integers $\{\pi; m_1, \cdots, m_s; t\}$ satisfying the inequality (6) in calculating the dimensions of the vector spaces consisting of all Fuchsian forms of weight $m \geqq 2$ attached to a Fuchsian group of genus π which has elliptic points $a_1, \cdots, a_s$ with respective orders $m_1, \cdots, m_s$ and certain parabolic points $b_1, \cdots, b_t$. q. e. d.

COROLLARY. *There exists an integer α such that the complete linear system $|\mathfrak{d}_m(S)|$ on the curve Δ is very ample for every elliptic surface of general type S and for every integer $m \geqq \alpha$. Moreover we can choose $\alpha = 86$.*

PROOF. It is sufficient for very ampleness of $|\mathfrak{d}_m(S)|$ to choose α such that $\deg(\mathfrak{d}_m(S)) \geqq 2\pi+1$ for every $m \geqq \alpha$. Hence, by the following elementary assertion we can prove Corollary.

For any integers $\pi \geqq 0$, $s \geqq 0$, $m_1, \cdots, m_s \geqq 2$, $t \geqq 0$ among which the inequality (6) *holds, there exists an integer α such that for every $m \geqq \alpha$ the following inequality:*

$$m(2\pi-2) + \sum_{\lambda=1}^{s}\Big[m\Big(1-\frac{1}{m_\lambda}\Big)\Big] + mt \geqq 2\pi+1$$

holds. Moreover, α may be chosen to be 3, 6, 14, 8 *and* 86, *in the cases in which* i) $t+\pi \geqq 2$, ii) $\pi = 1$ *and* $s \geqq 1$, iii) $\pi = 0$ *and* $s+t \geqq 4$, iv) $\pi = 0$, $t = 1$ *and* $s = 2$, *and* v) $\pi = 0$ *and* $s = 3$, *respectively.*

PROOF. We can prove this in the former four cases easily. In the last case, we recall the inequality:

$$-2+\sum_{\lambda=1}^{3}\left(1-\frac{1}{m_\lambda}\right)\geqq\frac{1}{42}$$

which may be found in the usual proof of the estimates of the order of the automorphism group of the curve with the genus $\geqq 2$. Letting $\alpha_m(\lambda)$ be $m\left(1-\frac{1}{m_\lambda}\right)$, we have

$$[\alpha_m(\lambda)]-\alpha_m(\lambda)\geqq-\left(1-\frac{1}{m_\lambda}\right)\qquad\text{for}\quad\lambda=1,2,3\,.$$

Then, we have

$$\sum_{\lambda=1}^{3}[\alpha_m(\lambda)]=\sum_{\lambda=1}^{3}\alpha_m(\lambda)+\sum_{\lambda=1}^{3}\{[\alpha_m(\lambda)]-\alpha_m(\lambda)\}$$
$$\geqq(m-1)\sum_{\lambda=1}^{3}\left(1-\frac{1}{m_\lambda}\right)=(m-1)\left(2+\frac{1}{\varepsilon}\right),$$

where we denote the reciprocal of $\sum_{\lambda=1}^{3}\left(1-\frac{1}{m_\lambda}\right)-2$ by ε. The inequality $(m-1)\left(2+\frac{1}{\varepsilon}\right)>2\cdot m$ implies $m>2\varepsilon+1$. On the other hand, 2ε is smaller than or equal to 84. Thus, for every $m\geqq 86$ we have the inequality:

$$\sum_{\lambda=1}^{3}\left[m\left(1-\frac{1}{m_\lambda}\right)\right]\geqq 2m\,. \qquad\text{q. e. d.}$$

Note that a surface with $\pi=0$, $m_1=2$, $m_2=3$, $m_3=7$, $t=0$ has the following plurigenera: $P_4=P_{43}=0$, $P_6=P_{85}=1$, $P_{84}=P_{86}=2$. Such a surface can be obtained from $E\times\boldsymbol{P}^1$ by logarithmic transformations, where E is an elliptic curve.

PROPOSITION 9. *$\pi(S)$ is invariant under deformations.*

PROPOSITION 10. *The set $\{m_1(S),\cdots,m_s(S)\}$ is invariant under deformations.*

PROOF OF PROPOSITIONS 9 AND 10. We shall describe a complex analytic family of elliptic surfaces of general type as in Lemma B. Thus, let X and Y be connected complex manifolds, and let f be a proper, simple and surjective holomorphic map from X onto Y, such that every fibre X_y is an elliptic surface of general type. We denote by $\mathcal{F}$ the invertible sheaf $(\Omega^2_{X/Y})^{\otimes\alpha}$, where $\Omega^2_{X/Y}$ is the sheaf of germs of holomorphic 2-forms over $f: X\to Y$ (cf. p. 14-08 in [3, VII]). Then, we have $P_\alpha(X_y)=\dim H^0(X_y,\mathcal{F}_y)$. Since $P_\alpha(X_y)$ is an upper semi-continuous function of $y\in Y$, there exists a non-empty open subset Y' of Y, on which the function $P_\alpha(X_y)$ is constant. Hence, for $y\in Y'$, it follows from Lemma B that $f_*(\mathcal{F})\otimes_{\mathcal{O}_{Y,y}}k(y)\simeq H^0(X_y,\mathcal{F}_y)$. Let Q be the cokernel of the natural homomorphism $\sigma_{\mathcal{F}}: f^*f_*\mathcal{F}\to\mathcal{F}$, and let Σ be its support: $\Sigma=\operatorname{supp} Q$. Then $\operatorname{codim}(\Sigma)\geqq 1$. Since $\sigma_{\mathcal{F}}$ is surjective on $X-\Sigma$, the canonical holomorphic map $j: X-\Sigma \rightarrow \boldsymbol{P}(f_*(\mathcal{F}))$ is defined by Proposition 2.1 in [3, V]. On the other hand, the holomorphic map j_y induced from j at each point $y \in Y'$ is the restriction to the open set $X_y-\Sigma_y$ of the holomorphic map h_y associated with the complete linear system $|\alpha K_y|$, where K_y denotes the canonical divisor of X_y. Furthermore, at every point $p \in Y-Y'$, j_p is also the restriction to $X_p-\Sigma_p$ of the holomorphic map h_p, which is associated with the sub-linear system of $|\alpha K_p|$. This sub-linear system consists of the effective divisors defined by the sections of $f_*(\mathcal{F}) \otimes k(p)$ on X_p. We note that X_p does not coincide with Σ_p because we have, by the principle of upper semi-continuity, the inequality: $2 \leqq P_\alpha(X_y) = \dim f_*(\mathcal{F}) \otimes k(y) \leqq \dim f_*(\mathcal{F}) \otimes k(p)$ for every $y \in Y'$ and $p \in Y-Y'$.

We shall show that j will extend to a holomorphic map h from X to $\boldsymbol{P}(f_*(\mathcal{F}))$. First, we define h as a set-theoretical map by $h(x) = h_{f(x)}(x)$ for each $x \in X$. Next, by the following Lemma D, we see that h is holomorphic.

LEMMA D. *Let h be a function of complex variables z_1, z_2 and w, satisfying the following two conditions:*

1. *For an arbitrarily fixed w, $h(z_1, z_2, w)$ is holomorphic with respect to the complex variables z_1, z_2.*

2. *h is holomorphic on the complement of A, where A is an analytic set of codimension 1, such that A contains no analytic set defined by $w =$ const. Then, h is holomorphic.*

PROOF. In a neighborhood of a general point p of A, A is defined by $\zeta_1 = 0$ after choosing a suitable system of local coordinates $\zeta_1, \zeta_2, \omega = w - w(p)$ with the center p according to the condition 2. Hence, h is given by means of the Laurent series in ζ_1: $h = \sum_{i=1}^{\infty} A_i(\zeta_2, \omega)/\zeta_1^i + \sum_{j=0}^{\infty} B_j(\zeta_2, \omega)\zeta_1^j$, where A_i, B_j (for $i > 0$, $j \geqq 0$) are all holomorphic functions. By the condition 1, all A_i (for $i > 0$) have to vanish. Namely, h is holomorphic at any general point of A. Then, it follows from a theorem of Hartogs that h is holomorphic everywhere. q. e. d.

Thus, we have a fibre space $h: X \rightarrow Z$ over Y, where we denote by Z the reduced sub-complex space of $\boldsymbol{P}(f_*(\mathcal{F}))$ with $h(X)$ as the underlying topological space. For every $y \in Y'$, $h_y: X_y \rightarrow Z_y$ is the canonical fibring of the elliptic surface X_y, but for a point $p \in Y-Y'$, Z_p may have a singular point. If we assume that $\pi(X_y)$ is independent of $y \in Y$, Z_p is non-singular. We shall prove this as follows. For $y \in Y'$ and for $p \in Y-Y'$, the equalities $\pi(X_p) = \pi(X_y) = \pi(Z_y) = \pi(Z_p)$ follow from the assumption above and the independence of the arithmetic genus $\pi(Z_y)$ of $y \in Y$. On the other hand, the effective genus of Z_p is equal to $\pi(X_p)$ by Proposition 7. Therefore, X_p is non-singular.

We shall, now, show that, *for every multiple fibre $h_p^*(q) = mL_p$ of multiplicity m (whose reduced connected component is L_p), $(m-1)L_p$ extends locally* *to an effective divisor* $\mathcal{L}$ *on* $f^{-1}(U)$, *where* U *is a small neighborhood of* p, *and that* $\mathcal{L}|_{X_y}$ *decomposes into a sum of multiple fibres for* $y\in U$; $\mathcal{L}|_{X_y}=\sum_{i=1}^{t}(m^{(i)}-1)L_y^{(i)}$, *where* $m^{(i)}\geqq 2$, $L_y^{(i)}$ *is the reduced component of the multiple fibre* $h_y^*(q_y^{(i)})$ *of* $h_y: X_y\to Z_y$ *over* $q_y^{(i)}\in Z_y$, *and that the multiplicity of* $h_y^*(q_y^{(i)})=m_y^{*(i)}$ *is greater than or equal to* $m^{(i)}$ *for each* $1\leqq i\leqq t$. We shall show furthermore that $m^{(1)}$ *is equal to* m *under the assumption* $t=1$.

If the above is proved, then *Propositions* 9 *and* 10 *can be derived as follows:*

By Proposition 8 we have

$$P_m(X_p)=m(2\pi-1-q+p_g)+\sum_{\lambda=1}^{s}\Big[m\Big(1-\frac{1}{m_\lambda}\Big)\Big]+1-\pi\,,$$

where $\pi=\pi(X_p)$, $q=q(X_p)=q(X_y)$, and $p_g=p_g(X_p)=p_g(X_y)$. Moreover, by Proposition 8 and by the preceding assertion, we have

$$P_m(X_y)=m(2\pi'-1-q+p_g)+\sum_{\lambda=1}^{s}\sum_{i=1}^{t_\lambda}\Big[m\Big(1-\frac{1}{m_\lambda^{*(i)}}\Big)\Big]+1-\pi'\,,$$

where $\pi=\pi(X_p)$. We note that $\pi=\pi(X_p)=$ the effective genus of $Z_p\leqq\pi(Z_p)=\pi(Z_y)=\pi(X_y)=\pi'$.

If there exist some indices λ, such that $t_\lambda\geqq 2$ or $t_\lambda=1$ and $m_\lambda^{*(1)}>m_\lambda$, then

$$\sum_{i=1}^{t_\lambda}\Big[m\Big(1-\frac{1}{m_\lambda^{*(i)}}\Big)\Big]>\Big[m\Big(1-\frac{1}{m_\lambda}\Big)\Big]$$

for sufficiently large m, while $P_m(X_y)\leqq P_m(X_p)$ by the principle of upper-semicontinuity. It follows that $t_\lambda=1$, $m_\lambda^{*(1)}=m_\lambda$ for any λ and $\pi'=\pi$. Thus we prove *the invariance of plurigenera of elliptic surfaces of general type under deformations.*

We shall proceed to prove the assertion mentioned above. We take Y to be a complex manifold of complex dimension 1 and p to be a point on the boundary of Y'. First, we shall consider the case in which $c_2(X_p)=c_2(X_y)\neq 0$. Since $\pi(X_p)=\pi(X_y)$ for every y, by the formula (12.8) in [**5**, II], the holomorphic map $g: Z\to Y$ is simple. We can choose a point $q\in Z_p$ and a section $U=\{q_y\}_{y\in Y}$ of g such that q_p is q and each fibre in X_y over q_y is regular for every $y\in Y$. We have the induced divisor $V=h^*(U)$. In the usual way, we can associate the invertible sheaf $\mathcal{O}(V)$ with the divisor V on the complex manifold. Now we let $\mathcal{F}$ be the invertible sheaf $\Omega^2_{X/Y}\otimes\mathcal{O}(V)^{\otimes 2}$ on X, then we have

$$\begin{aligned}\dim H^0(X_y,\mathcal{F}_y)&=\dim H^0(Z_y,\mathcal{O}(\mathfrak{k}_y-\mathfrak{f}_y+2q_y))\\&=\pi+2-q+p_g=p_g+2\,.\end{aligned}$$

This is independent of $y\in Y$. Hence we can take a section s of $\mathcal{F}_p$ on X_p such that the divisor of $s=(m_p-1)L_p+F'$, where F' does not contain L_p any more by a theorem of Bertini. Then, by Lemma B, we can extend s to a section $\tilde{s}$ of $\mathcal{F}$ on the inverse image $f^{-1}(U)$ of a neighborhood. Therefore, we can choose a divisor $\mathcal{L}$ on $f^{-1}(U)$ which induces on X_p the divisor $(m-1)L_p$. For every point y in a neighborhood of p, we can write the divisor on X_y induced from $\mathcal{L}$ in the form $\sum_{i=1}^{t}(m^{(i)}-1)L_i$, where $m^{(i)}\geqq 2$ and L_i is the reduced connected component of the multiple fibre on X_y. Next we shall show that $m^{(1)}=m$ under the assumption $t=1$. Since the multiple fibre has the form ${}_mI_b$ by Theorem (6.2) in [**5**, II], there exists a system of local coordinates (σ, z, w) of X with the center x lying on $h_p^{-1}(q)$-{finite points}, such that σ is a local uniformization variable on Y with the center p and $z=0$ is a local equation of L_p. We denote by (σ, τ) a system of local coordinates on Z with the center q. Then h is written in the form:

$$\tau = z^m+\sigma B(\sigma, z, w)\,, \tag{8}$$

where B is a holomorphic function of (σ, z, w). Moreover, let $\tilde{A}(\sigma, z, w)=0$ be a local equation of s with the center x. Then

$$\tilde{A}(\sigma, z, w)=z^{m-1}+\sigma A(\sigma, z, w)\,, \tag{9}$$

where A is a holomorphic function of (σ, z, w) and therefore, $\tilde{A}(0, z, 0)=z^{m-1}$. By the Weierstrass preparation theorem $\tilde{A}$ may be written by use of a unit ε and holomorphic functions $A_1, \cdots, A_m$ of (σ, w) in the form of a pseudo-polynomial:

$$\tilde{A}=\varepsilon(z^{m-1}+\sigma A_1 z^{m-2}+\cdots+\sigma A_{m-1})\,. \tag{10}$$

Obviously we may assume $\varepsilon=1$. Letting R be the Puiseux series ring of σ, we consider $\tilde{A}(\sigma, z, w)$ to be a holomorphic function of (z, w) over R. Such a consideration is the same as studying locally $\mathcal{L}|_{X_y}$ for any point y in a small neighborhood of p. Therefore, the number of distinct roots of $A(\sigma, z, 0)$ as an equation of z over R is equal to the number of branches of $\mathcal{L}|_{X_y}$. By the assumption $t=1$, $\tilde{A}(\sigma, z, 0)=0$ has one root $\alpha(\sigma)\in R$. Hence, if we rewrite $\tilde{A}$ by replacing z with $Z=z-\alpha(\sigma)$, $\tilde{A}(\sigma, z, w)$ becomes a pseudo-polynomial of Z with coefficients $\bar{A}_1, \cdots, \bar{A}_{m-1}$ such that $\bar{A}_i$ is a holomorphic function of w over R, and $\bar{A}_i(0)=0$ for every i, $1\leqq i\leqq m-1$. By the local irreducibility of $L_y^{(1)}$, $\tilde{A}$ has one and only one irreducible factor as a pseudo-polynomial of Z:

$$\tilde{A}=(Z^e+A_1^*Z^{e-1}+\cdots+A_e^*)^l\,. \tag{10}$$

Therefore, we have $el=m-1$. Note that, to prove $m^{(1)}=m$, it suffices to show $e=1$. *We shall prove $el'=m$ for an integer l'. Combined with $el=m-1$, this proves $e=1$.* We let τ_0 be $\alpha(\sigma)^m+\sigma B(\sigma, \alpha(\sigma), 0)\in R$, then we can write $z^m+\sigma B-\tau_0$, up to a unit factor ε', in the form of a pseudo-polynomial of Z whose coefficients $\bar{B}_1, \cdots, \bar{B}_m$ are holomorphic functions of w over R such that

$\bar{B}_1(0)=\cdots=\bar{B}_m(0)=0$:

(11) $$z^m+\sigma B-\tau_0=\varepsilon'(Z^m+\bar{B}_1Z^{m-1}+\cdots+\bar{B}_m)\,.$$

According to the fact that $\mathcal{L}|_{X_y}$ is a fibre of $h_y: X_y\to Z_y$, $Z^m+\bar{B}_1Z^{m-1}+\cdots+\bar{B}_m$ has one and only one irreducible factor which is the same as in (10):

(12) $$Z^m+\bar{B}_1Z^{m-1}+\cdots+\bar{B}_m=(Z^e+A_1^*Z^{e-1}+\cdots+A_e^*)^{l'}\,.$$

Therefore, we have $el'=m$.

To complete the proof of the previous assertion, we shall prove that the multiplicity $m^{*(i)}$ of $h_y^*(q_y^{(i)})$ in X_y is not smaller than $m^{(i)}$. We choose a non-Weierstrass point z on Z_0 and a section $U=\{z_y\}_{y\in Y}$ such that z_p coincides with z and such that each fibre over z_y of $h_y: X_y\to Z_y$ is regular for $y\in Y$. Then we have the induced divisor $V=h^*(U)$. Now, denoting by $\mathcal{G}$ the invertible sheaf $\mathcal{O}(\mathcal{L})\otimes\mathcal{O}(V)^{\otimes\pi}$, we have

(13) $$\dim H^0(X_p, \mathcal{G}_p)=\dim H^0(Z_p, \mathcal{O}(\pi z))=1\,.$$

Hence, for any point y in a small neighborhood of p, we have

(14) $$\dim H^0\Big(X_y, \mathcal{O}\Big(\sum_{i=1}^{t}(m^{(i)}-1)L_i+h_y^{-1}(z_y)\Big)\Big)=\dim H^0(X_y, \mathcal{G}_y)=1\,.$$

On the other hand, $m^{*(i)}$ is smaller than $m^{(i)}$ for some indices i if and only if the following inequality holds:

(15) $$\dim H^0\Big(X_y, \mathcal{O}\Big(\sum_{i=1}^{t\lambda}(m^{(i)}-1)L_i+\pi h_y^{-1}(z_y)\Big)\Big)$$
$$=\dim H^0(Z_y, \mathcal{O}(q_y^{(i)}+\cdots+\pi z_y))\geqq 2\,,$$

because z_y is also a non-Weierstrass point of Z_y.

In the case where $c_2(X_p)=c_2(X_y)=0$, the proof is easy. Since the multiple fibre has the form ${}_mI_0$ by Table II in [5, III], we make use of Proposition 11 below employing the notation in the previous case.

PROPOSITION 11. *If a multiple fibre is mL_p, where L_p is a non-singular elliptic curve, L_p is extended locally and uniquely.*

PROOF. We denote by N the normal bundle of L_p in X_p. Then we have $(m-1)N\not\approx 0$ and $mN\approx 0$ on L_p. Hence, we have $H^0(L_p, \mathcal{O}(N))=H^1(L_p, \mathcal{O}(N))=0$. By Theorem 5 in [6], L_p can be uniquely extended to a divisor $\mathcal{L}$ of X, by making Y smaller if necessary. Moreover, we let $\mathcal{N}$ be the normal bundle of $\mathcal{L}$ on X, then we have $\mathcal{N}_p=N$ and $\dim H^0(\mathcal{L}_y, \mathcal{O}((m-1)\mathcal{N}_y))=\dim H^0(L_p, \mathcal{O}((m-1)N))=0$ for any point y in a small neighborhood of 0.

This completes the proof of Theorem III for the class iii).

§ 4. Completion of the proof of Theorem III.

Finally, we shall consider non-minimal surfaces. Let $f: X \to Y$ be a fibre space of complex manifolds describing a complex analytic family of deformations of the surface S. Thus $X_0 = S$ for a point $o \in Y$. We denote by the symbol S^* the minimal model of the surface S. In the case in which X_0^* is in one of the classes i), ii) and iii) in Proposition 3, then $n(X_y)$ is independent of y by the same reasoning as in the proof of Proposition 5. By Lemma A, there exists an open covering $\{U_i\}$ of Y, such that we can construct for each index i a fibre space $\tilde{X}_i \to U_i$ describing a complex analytic family consisting of $X_y^*, y \in U_i$. Thus, we have proved Theorem III in this case by the uniqueness of the minimal models of non-ruled surfaces. In the case in which X_0^* is in the class iv), we infer from the closedness of other classes under deformations that X_y^* is also in the class iv). For any $y \in Y$, there exists a neighborhood U_1 of y and a fibre space $\tilde{X}_1 \to U_1$ of a complex analytic family of surfaces such that $\tilde{X}_{1,y}$ is X_y^* and $\tilde{X}_{1,z}$ is dominated by X_z for $z \in U_1$, according to Lemma A. $\tilde{X}_{1,z}$ is also minimal by Proposition 5. Therefore, $n(X_y)$ is independent of $y \in Y$. Consequently, $P_m(X_y) = n(X_y) + \frac{m(m-1)}{2} c_1^2(X_y) + 1 - q(X_y) + p_g(X_y)$ is independent of $y \in Y$. This completes the proof of Theorem III.

APPENDIX 1. A proof of Lemma A, II.

For any point $o \in Y$, letting U_0 be a relatively compact, holomorphically convex and open neighborhood of E_0 in U_0, we have $H^1(U_0, \Theta) = 0$, where we denote by Θ the sheaf of germs of holomorphic vector fields. Hence, by a theorem of Andreotti and Vesentini in [**1**], there exist an open neighborhood N of o and an open set U in $f^{-1}(N)$ such that $U \cap X_0 = U_0$ and such that U is biholomorphic to $U_0 \times N$ over N. We may assume that $U \cap X_y$ is the open neighborhood of E_y in X_y for $y \in N$. On the other hand, we can construct a complex manifold $\tilde{U}_0$ such that $U_0 = Q_p(\tilde{U}_0)$ and $E_0 = Q_p(p)$, where Q_p denotes the quadric transformation with the center p on $\tilde{U}_0$. Therefore, replacing U by $\tilde{U}_0 \times N$, we obtain from $f^{-1}(N)$ the complex manifold $\tilde{X}_N$ over N.

This proof is due to T. Suwa.

APPENDIX 2. A supplement to the proof of Theorem II in Part I.

We might add the following Proposition 12 to the proof of Theorem II in Part I.

PROPOSITION 12. *Any surface with $c_1^2 = c_2 = 0$ is free from exceptional curves of the first kind.*

PROOF. When a surface S with $c_1^2 = c_2 = 0$ dominates a surface S^* which is free from exceptional curves, then S^* has the following invariants: $c_1^2(S^*) > 0$, $1 - q(S^*) + p_g(S^*) = 0$. This is impossible by a formula in [**9**], which shows that $1-q(S^*)+p_g(S^*)>0$ when $c_1^2(S^*)>0$. q. e. d.

APPENDIX 3. On the topological invariance of plurigenera of some special surfaces.

The following Proposition 13 is a partial solution of the problem, "*Are all plurigenera topological invariants?*"

PROPOSITION 13. *All plurigenera $P_m(S)$ of a surface S is determined by certain topological invariants, if S is homeomorphic to an irrational ruled surface or a hyperelliptic surface or a complex torus of dimension 2 or a Hopf surface or an elliptic surface of which fundamental group is finite but is neither an abelian group generated by at most two elements nor a dihedral group of order $4k$ for an integer $k\geqq 1$ (which we call an F-surface).*

PROOF. Using Theorem II and Remark B in Part I, we can immediately prove Proposition 13 for the first four cases. To prove Proposition 13 for the last case, we prove the following:

THEOREM IV. *All F-surfaces can be classified as shown in Table IV below, and any surface which is homeomorphic to a surface in Table IV is an F-surface, that is, one of the surfaces classified in Table IV.*

TABLE IV

class	$\pi_1/\mathrm{Cent}\,\pi_1$	$\{m_1, \cdots, m_s\}$	P_3	P_4	P_5
I_m	dihedral group of order $2m$, $m\geqq 2$	$\{2, 2, m\}$ or $\{2, 2, 2m\}$ according as $\mathrm{Cent}\,\pi_1$ is cyclic or not	$3p_g+1$ or $3p_g+2$		
II	tetrahedral group	$\{2, 3, 3\}$	$3p_g+3$	$4p_g+3$	$5p_g+4$
III	octahedral group	$\{2, 3, 4\}$	$3p_g+3$	$4p_g+4$	$5p_g+4$
IV	icosahedral group	$\{2, 3, 5\}$	$3p_g+3$	$4p_g+4$	$5p_g+5$

In this table, by Cent G *we denote the center of the group G, and we use the notation in Proposition* 8. *We note that the elliptic surface in Table IV is of general type, hence, all P_m can be calculated by use of p_g and π_1.*

PROOF. Let S be an elliptic surface with a base curve Δ and a canonical map $\Psi: S\rightarrow\Delta$, and let $m_1, \cdots, m_s$ be multiplicities of multiple fibres of $\Psi: S\rightarrow\Delta$ at $a_1, \cdots, a_s\in\Delta$, respectively. If $\pi=\pi(\Delta)$ is equal to zero and $\Psi: S\rightarrow\Delta$ has at most two multiple fibres, then the fundamental group $\pi_1(S)$ of S is an abelian group generated by at most two elements. Otherwise, we can construct a universal ramified covering manifold $\tilde{\Delta}$ of Δ which has ramification indices $m_1, \cdots, m_s$ at $a_1, \cdots, a_s$, respectively. Hence S has an unramified covering manifold $\tilde{S}=S\times_{\Delta}\tilde{\Delta}$ and a holomorphic surjective map $\tilde{\Psi}:\tilde{S}\to\tilde{\Delta}$ which has no multiple fibres. If $\pi_1(S)$ is finite, then the degree of the covering $\tilde{\Delta}\to\Delta$ is finite and $\pi_1(S)$ is finite cyclic. Therefore, if we assume that S is an F-surface, then we conclude that $\pi(S)$ is equal to zero and that $\pi_1(S)/\mathrm{Cent}\,\pi_1(S)$ is the dihedral group of order m or $2m$ $(m\geqq 2)$, the tetrahedral group, the octahedral group and the icosahedral group, according as the set of multiplicities $\{m_1,\cdots,m_s\}$ is $\{2,2,m\}$, $\{2,3,3\}$, $\{2,3,4\}$ and $\{2,3,5\}$, respectively.

To determine all the complex structures, which the underlying topological manifold of S admits, we consider the above elliptic surface $\tilde{S}=S\times_{\Delta}\tilde{\Delta}$ which is a finite unramified covering manifold of S. We can easily verify that $1+p_g(\tilde{S})$ is equal to $r(1+p_g(S))$, where r is the degree of the covering: $\tilde{\Delta}\to\Delta$. The canonical divisor $\tilde{K}$ of $\tilde{S}$ may be given as follows:

$$\tilde{K}=\tilde{\Psi}^*((p_g(\tilde{S})-1)\tilde{q}),$$

where $\tilde{\Psi}:\tilde{S}\to\tilde{\Delta}$ is the canonical projection of the elliptic fibre space of $\tilde{S}$ and $\tilde{q}$ is a point on $\tilde{\Delta}$. Therefore, $\tilde{K}=0$ in $H^2(\tilde{S},\boldsymbol{Z}/(2))$, because r is an even integer. This shows that *the Stiefel-Whitney class of* $\tilde{S}$ *is zero.* It should be noticed that *the Stiefel-Whitney class is a topological invariant.* Consequently, the underlying topological manifold of $\tilde{S}$ cannot admit a complex structure of a non-minimal surface. Hence, any surface which is homeomorphic to $\tilde{S}$ is elliptic. Furthermore if a surface S' is homeomorphic to the surface S above, then a finite unramified covering manifold of S', which is denoted by $\tilde{S}'$, is elliptic. Hence, $P_m(\tilde{S}')$ is asymptotically equal to αm, for a positive number α, when "m" grows to infinity. By the inequality: $P_m(\tilde{S}')\geqq P_m(S')$, S' cannot be birationally equivalent to an algebraic surface of general type. Consequently, S' is elliptic and S' has the group $\pi_1(S)$ as its fundamental group. q. e. d.

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