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Orbits of one-parameter groups I

(Plays in a Lie algebra)

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§1. Introduction.

By an analytic group and by an analytic subgroup of a Lie group, we mean a connected Lie group and a connected Lie subgroup of a Lie group, respectively. Unless specified otherwise, an analytic subgroup and its corresponding Lie subalgebra will be denoted by the same capital script and capital Roman letter, respectively. For example, if $\mathcal{G}$ denotes an analytic group and $\mathcal{L}$ denotes an analytic subgroup of $\mathcal{G}$, then G will denote the Lie algebra of $\mathcal{G}$, and L will denote the subalgebra of G corresponding to $\mathcal{L}$. We make the convention that the Lie algebra of a Lie group is the tangent space of that group at the identity.

The fields of real numbers and complex numbers will be denoted by $\boldsymbol{R}$ and $\boldsymbol{C}$, respectively.

Throughout this paper, we shall adopt the terminology and utilize the theorems in

N. Jacobson, *Lie algebras*, Tracts in Math. 10, Interscience, 1962,

and

C. Chevalley, *Theory of Lie groups* I, Princeton, 1946.

In particular, §2 and §3 below are connected with the former, and §4 and §5 with the latter.

Let $\mathcal{G}$ be an analytic group, and let $\mathcal{L}$ be an analytic subgroup of $\mathcal{G}$. Let X and Y be elements of G. The purpose of this paper is to give a necessary and sufficient condition, in terms of Lie algebras, for the validity of the equality $\exp \boldsymbol{R}X \cdot \mathcal{L} = \exp \boldsymbol{R}Y \cdot \mathcal{L}$. In particular, if $\mathcal{L}$ is a closed subgroup, this equality implies that the orbits of one-parameter groups $\exp \boldsymbol{R}X$ and $\exp \boldsymbol{R}Y$, passing through the point $\mathcal{L}$, coincide in the factor space $\mathcal{G}/\mathcal{L}$.

In order to explain our results, we first adopt the notation $(ad\ A)B = [A, B]$ for elements A and B of a Lie algebra, and introduce the following

DEFINITION. Let G be a Lie algebra, and let L be a subalgebra of G.

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Let X be an element of G. The set

$$P(X, L)=\{A\in L\,;\ (ad\ X)^n A\in L \text{ for } n=1, 2, \cdots\}$$

is called the *play* of X in L.

Obviously, $P(X, L)$ is a subalgebra of L, and X normalizes $P(X, L)$.

Let x and y be independent non-commutative indeterminates over the field of rational numbers. By the Campbell-Hausdorff formula, see Jacobson loc. cit., $\ln(\exp(-x)\cdot\exp y)$ can be written as a formal Lie power series. That is, we have

$$\exp(-x)\cdot\exp y=\exp\{\varphi_1(x, y)+\varphi_2(x, y)+\cdots\},$$

where $\varphi_n(x, y)$ is a homogeneous Lie polynomial of degree n over the field of rational numbers.

The significance of the notion of "*play*" is now made apparent by the following theorem:

THEOREM 1. *Let Φ be a field of characteristic 0, and let G be a Lie algebra over Φ. Let L be a subalgebra of G, and let X and Y be elements of G. Let φ_n be the homogeneous Lie polynomial of x and y, of degree n, such that*

$$\exp(-x)\cdot\exp y=\exp\{\varphi_1(x, y)+\varphi_2(x, y)+\cdots\}.$$

Then all the $\varphi_n(X, Y)$ belong to L if and only if $Y-X$ belongs to $P(X, L)$.

After establishing this, it is not difficult to translate it into the following form:

PROPOSITION 6. *Let $\mathfrak{G}$ be an analytic group, and let $\mathfrak{L}$ be an analytic subgroup of $\mathfrak{G}$. Let X and Y be elements of G. Then the one-parameter subgroup $\exp tY$, $t\in \boldsymbol{R}$, can be written in a form*

$$\exp tY=\exp tX\cdot l(t) \qquad t\in\boldsymbol{R}$$

where $l(t)$ is an analytic curve in the analytic group $\mathfrak{L}$, if and only if $Y-X$ belongs to the play $P(X, L)$.

On the other hand, by complexifying the analytic groups $\mathfrak{G}$, $\mathfrak{L}$, $\exp tX$ and $\exp tY$, and using the fact that the only conformal automorphisms of the complex plane $\boldsymbol{C}$ are linear functions, we can prove the following proposition:

PROPOSITION 7. *Let $\mathfrak{G}$ be an analytic group, and let $\mathfrak{L}$ be an analytic subgroup of $\mathfrak{G}$. If $\exp \boldsymbol{R}X\cdot\mathfrak{L}=\exp\boldsymbol{R}Y\cdot\mathfrak{L}$ for $X, Y\in G$, then there exists a real number $\gamma\neq 0$ such that $Y-\gamma X\in P(X, L)$.*

Combining the above two propositions, we can realize our purpose as follows:

THEOREM 2. *Let $\mathfrak{G}$ be an analytic group, and let $\mathfrak{L}$ be an analytic subgroup of $\mathfrak{G}$. Let X and Y be elements of G. Then $\exp\boldsymbol{R}X\cdot\mathfrak{L}=\exp\boldsymbol{R}Y\cdot\mathfrak{L}$ if and only if there exists a non-zero real number γ such that $Y-\gamma X\in P(X, L)$. Moreover, in this case, we have $\exp(-t\gamma X)\cdot\exp tY\in\mathfrak{L}$ for $t\in\boldsymbol{R}$.*

If Q is an analytic group, and $\mathcal{P}$ a normal analytic subgroup of codimension one, then for any one-parameter subgroup $\mathcal{X}$ of Q, with the direction at the identity not in $\mathcal{P}$, we have $Q=\mathcal{X}\mathcal{P}$. Roughly speaking, Theorem 2 claims that this rather trivial situation must occur in case we have $\exp \boldsymbol{R}X\cdot\mathcal{L}=\exp \boldsymbol{R}Y\cdot\mathcal{L}$ in an analytic group $\mathcal{G}$. (Put $P=P(X, L)$ and $Q=\boldsymbol{R}X+P=\boldsymbol{R}Y+P$.)

REMARK. In general, $\exp \boldsymbol{R}Y\cdot\mathcal{L}$ can be a non-trivial proper part of $\exp \boldsymbol{R}X\cdot\mathcal{L}$. For example, a two-dimensional analytic group acts transitively on the straight line $\boldsymbol{R}$, by $y=\alpha x+\beta$ ($\alpha, \beta\in\boldsymbol{R}$, $\alpha>0$), and the only orbit of the translation group $y=x+t$ is $\boldsymbol{R}$ itself; on the other hand, the open interval $(-1, \infty)$ is the orbit of the one-parameter group $y=e^t x+e^t-1$, containing 0.

The paper is organized into five sections. In § 2, we introduce the notion of the length of a Lie polynomial without a linear term, which plays an important role in the proof of Theorem 1, in § 3. In § 4 we develop some machinery on analytic subgroups, which we apply in § 5 to obtain Proposition 6, Proposition 7 and Theorem 2. Finally, at the end of this paper, an example of a play in an infinite-dimensional Lie algebra will be given.

§ 2. Length of a Lie polynomial without a linear term.

Let Φ be a field of characteristic 0, and let $\mathfrak{F}$ be a free associative algebra generated by a pair $\{x, y\}$ over Φ. For a non-negative integer n, we denote by $\mathfrak{F}_n$ the n times tensor product of the two-dimensional vector space $\Phi x+\Phi y$. Then $\mathfrak{F}$ can be identified with the direct sum of the vector spaces $\mathfrak{F}_0, \mathfrak{F}_1, \mathfrak{F}_2, \cdots$. For f and g in $\mathfrak{F}$, we define the commutator $[f, g]$ by $[f, g]=fg-gf$, and obtain the Lie algebra $\mathfrak{F}_{\mathfrak{L}}$. Let $\mathfrak{F}\mathfrak{L}$ be the subalgebra of $\mathfrak{F}_{\mathfrak{L}}$ generated by x and y. By a Lie polynomial (of x and y) we mean an element of $\mathfrak{F}\mathfrak{L}$. $\mathfrak{F}\mathfrak{L}$ is a free Lie algebra generated by the $\{x, y\}$, and $\mathfrak{F}$ is a universal enveloping algebra of $\mathfrak{F}\mathfrak{L}$. $\mathfrak{F}\mathfrak{L}$ is the direct sum of subspaces $\mathfrak{F}\mathfrak{L}_n=\mathfrak{F}_n\cap\mathfrak{F}\mathfrak{L}$ ($n=1, 2, \cdots$), composed of homogeneous Lie polynomials of degree n.

When there is no danger of confusion, the expression $[a_1, [a_2, [\cdots, [a_{n-1}, a_n]] \cdots]]$ will be written simply as $a_1a_2\cdots a_{n-1}a_n$. A Lie polynomial is a linear combination of monomials of the form $a_1a_2\cdots a_{n-1}a_n$, where each a_i is either x or y. By a normalized monomial of degree $n\geqq 2$, we mean a monomial of the form $a_1a_2\cdots a_{n-1}a_n$, with $a_{n-1}=x$ and $a_n=y$. Then, a Lie polynomial without a linear term, i.e. an element in $\mathfrak{F}\mathfrak{L}_2+\mathfrak{F}\mathfrak{L}_3+\cdots$, can be written as a linear combination of normalized monomials. Note that formally distinct normalized monomials can be identical, e.g. $xyxy=yxxy$; however, we have the following proposition:

PROPOSITION 1. *Let $f_1, f_2, \cdots, f_k$, and $g_1, g_2, \cdots, g_l$ be normalized monomials. If*

$$\alpha_1 f_1+\alpha_2 f_2+\cdots+\alpha_k f_k=\beta_1 g_1+\beta_2 g_2+\cdots+\beta_l g_l$$

for $\alpha_1, \alpha_2, \cdots, \alpha_k$ *and* $\beta_1, \beta_2, \cdots, \beta_l$ *in* Φ, *then* $\alpha_1+\alpha_2+\cdots+\alpha_k=\beta_1+\beta_2+\cdots+\beta_l$.

PROOF. We set

$$X=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \qquad Y=\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}, \qquad A=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}.$$

Then $\{X, Y\}$ forms a basis of a two-dimensional non-abelian Lie algebra and we have

$$[X, Y]=[X, A]=[Y, A]=A.$$

Hence, if we substitute $\{X, Y\}$ in place of $\{x, y\}$ in any normalized monomial, then it reduces to A. So under our assumptions we have

$$(\alpha_1+\alpha_2+\cdots+\alpha_k)A=(\beta_1+\beta_2+\cdots+\beta_l)A. \qquad \text{Q. E. D.}$$

By the Proposition 1, we can define the *length* of $\alpha_1 f_1+\alpha_2 f_2+\cdots+\alpha_k f_k$, as follows:

$$|\alpha_1 f_1+\alpha_2 f_2+\cdots+\alpha_k f_k|=\alpha_1+\alpha_2+\cdots+\alpha_k.$$

Next we extend the free algebra $\mathfrak{F}$ to the algebra $\overline{\mathfrak{F}}$ of formal power series in the $\{x, y\}$, and accordingly the free Lie algebra $\mathfrak{F}\mathfrak{L}$ to the Lie algebra $\overline{\mathfrak{F}\mathfrak{L}}$ of formal Lie power series in $\overline{\mathfrak{F}}$; and obtain the Campbell-Hausdorff formula

$$\exp x \cdot \exp y=\exp \psi(x, y),$$

where

$$\psi(x, y)=\psi_1(x, y)+\psi_2(x, y)+\cdots \in \overline{\mathfrak{F}\mathfrak{L}},$$

with

$$\psi_1(x, y)=x+y \in \mathfrak{F}\mathfrak{L}_1, \qquad \psi_2(x, y)=\frac{1}{2}[x, y] \in \mathfrak{F}\mathfrak{L}_2, \cdots.$$

We set $\varphi(x, y)=\psi(-x, y)$ and $\varphi_n(x, y)=\psi_n(-x, y)$ for $n=1, 2, \cdots$.

In order to compute the length of φ_n $(n \geqq 2)$, we make use of the two-dimensional Lie algebra in the proof of Proposition 1 again. Retaining the notations, the equality

$$\exp(-tX)\cdot \exp tY=\begin{pmatrix}e^{-t} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}e^{t} & e^{t}-1\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 1-e^{-t}\\ 0 & 1\end{pmatrix}$$

$$=\exp\Big(\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}t^n\Big)A,$$

leads to the following proposition:

PROPOSITION 2. $|\varphi_n|=\dfrac{(-1)^{n+1}}{n!}$ *for* $n \geqq 2$.

REMARK. For the length of ψ_n, we have the following formula:

$$t \tanh \frac{t}{2}=|\psi_2|t^2+|\psi_3|t^3+\cdots.$$

§ 3. Proof of Theorem 1.

Let G be a Lie algebra over the field Φ of characteristic 0, and let L be a subalgebra of G. Let X and Y be elements of G.

PROPOSITION 3. *If* $(ad\ X)^m(Y-X)\in L$ *for* $m=0, 1, \cdots, n-1$, $(n\geqq 1)$, *then for any normalized monomial* $f(x, y)$ *of degree* $n+1$, *we have*

$$f(X, Y)-(ad\ X)^n Y\in L.$$

PROOF (by induction on n). When $n=1$, $xy=(ad\ x)y$ is the only normalized monomial of degree 2, and it is obvious.

Let us suppose that the proposition is true for $k<n$. This means all the homogeneous Lie polynomials of degree k, $2\leqq k\leqq n$, of the $\{X, Y\}$, belong to L. Let $f(x, y)=a_1a_2\cdots a_{n-1}xy$ be a normalized monomial of degree $n+1$. If $a_1=a_2=\cdots=a_{n-1}=x$, there is nothing to prove. Otherwise, we can find the smallest integer i such that $a_i=y$. When $i=1$, we set $f_1(x, y)=xa_2\cdots a_{n-1}xy$ and get $f(x, y)-f_1(x, y)=[y-x, a_2\cdots a_{n-1}xy]$. Hence by the induction hypothesis, we have that $f(X, Y)-f_1(X, Y)\in L$. When $i>1$, we set $a_{i+1}a_{i+2}\cdots a_{n-1}xy=g(x, y)=g$. Then by Jacobi identity, we have that $xyg=yxg+[xy, g]$, from which it follows that

$$f(x, y)=x^{i-2}xyg=x^{i-2}yxg+x^{i-2}[xy, g].$$

On the other hand, the last term is written as

$$\sum_{j=0}^{i-2}\binom{i-2}{j}[x^jxy, x^{i-2-j}g],$$

and by the induction hypothesis, reduces to an element of L, if we substitute the $\{X, Y\}$ in the $\{x, y\}$. Therefore putting $f_2(x, y)=x^{i-2}yxg$ we have $f(X, Y)-f_2(X, Y)\in L$. Repeating the process, the y which was in the i-th position can be moved to the first position; and it can be replaced by x using the case when $i=1$; that is, setting $f_3(x, y)=x^ig$, we have $f(X, Y)-f_3(X, Y)\in L$. When there exists a $j>i$ with $a_j=y$, we can proceed in a similar way. Q. E. D.

PROOF OF THEOREM 1. Suppose that $\varphi_n(X, Y)\in L$ for all n. Let us prove that $(ad\ X)^n(Y-X)\in L$ for $n=0, 1, 2, \cdots$, by induction on n. Since $\varphi_1(X, Y)=Y-X$, it is true for $n=0$. For $n\geqq 1$, we can write $\varphi_{n+1}(x, y)$ as a linear combination of normalized monomials, $f_1, f_2, \cdots, f_k$, as follows:

$$\varphi_{n+1}(x, y)=\alpha_1 f_1(x, y)+\alpha_2 f_2(x, y)+\cdots+\alpha_k f_k(x, y)$$

where $\alpha_i\in\Phi$, $1\leqq i\leqq k$. By the induction hypothesis $(ad\ X)^m(Y-X)\in L$ for $m=0, 1, \cdots, n-1$. Hence by Proposition 3, $f_i(X, Y)-(ad\ X)^n(Y-X)\in L$, for $i=1, 2, \cdots, k$, which implies that

$$\varphi_{n+1}(X, Y)-|\varphi_{n+1}|(ad\ X)^n(Y-X)\in L.$$

Since $|\varphi_{n+1}| \neq 0$ by Proposition 2, we have $(ad\ X)^n(Y-X) \in L$.

Conversely, if $(ad\ X)^n(Y-X) \in L$ for $n=0, 1, 2, \cdots$, then all Lie polynomials without a linear term, together with $Y-X$, belong to L, by Proposition 3.

§ 4. Propositions on analytic subgroups.

Let $\mathcal{G}$ be an analytic group of dimension r, and let e denote the identity of $\mathcal{G}$. Let $\mathcal{A}$ and $\mathcal{B}$ be analytic groups, and let us suppose there exist continuous (analytic) one-one homomorphisms α and β from $\mathcal{A}$ and $\mathcal{B}$ into $\mathcal{G}$, respectively. The images $\alpha(\mathcal{A})$ and $\beta(\mathcal{B})$ are analytic subgroups of $\mathcal{G}$. We shall study the properties of the double coset $\alpha(\mathcal{A})g\beta(\mathcal{B})$ for $g \in \mathcal{G}$.

We fix an element g of $\mathcal{G}$, and set $\mathcal{C}=\mathcal{C}(g)=\alpha(\mathcal{A}) \cap g\beta(\mathcal{B})g^{-1}$. Then $\mathcal{C}$ is a subgroup of $\mathcal{G}$. We consider a mapping $\rho=\rho(g)$ from the direct product group $\mathcal{A}\times\mathcal{B}$ into $\mathcal{G}$, defined by

$$\mathcal{A}\times\mathcal{B} \ni (a, b) \mapsto \alpha(a)^{-1}g\beta(b) \in \mathcal{G}\,.$$

We set $\mathcal{D}=\mathcal{D}(g)=\{(\alpha^{-1}(c), \beta^{-1}(g^{-1}cg))\,;\ c \in \mathcal{C}\}$. Then ρ is analytic and $\rho(a, b)=\rho(a', b')$ if and only if $(a', b') \in \mathcal{D}(a, b)$. Hence $\mathcal{D}$ is a closed subgroup of $\mathcal{A}\times\mathcal{B}$ and ρ induces a one-one analytic mapping $\tilde{\rho}=\tilde{\rho}(g)$ from $(\mathcal{A}\times\mathcal{B})/\mathcal{D}(g)$ into $\mathcal{G}$. We denote by $d(g)$ the dimension of the manifold $(\mathcal{A}\times\mathcal{B})/\mathcal{D}(g)$.

Let us denote by $Ad(g)$ the automorphism of the Lie algebra G induced by the inner automorphism: $\mathcal{G} \ni h \mapsto ghg^{-1} \in \mathcal{G}$. We also denote by $\alpha(A)$ and $\beta(B)$ the Lie algebras of $\alpha(\mathcal{A})$ and $\beta(\mathcal{B})$, respectively, and we set $C=C(g)=\alpha(A) \cap (Ad(g))\beta(B)$. Then $C(g)$ is a subalgebra of G.

Let $\mathcal{D}^0$ be the connected component containing the identity of $\mathcal{D}$. We denote by $\mathcal{C}^0=\mathcal{C}^0(g)$ the image by α of the projection of $\mathcal{D}^0$ into $\mathcal{A}$. Then $\mathcal{C}^0$ is an analytic subgroup of $\mathcal{G}$. Since $\mathcal{C}^0 \subset \alpha(\mathcal{A}) \cap g\beta(\mathcal{B})g^{-1}=\mathcal{C}(g)$, the Lie algebra of $\mathcal{C}^0$ is contained in $C(g)$. On the other hand, for any X in $C(g)$, we have $\exp \boldsymbol{R}X \subset \mathcal{C}(g)$, and $(\alpha^{-1}(\exp tX), \beta^{-1}(g^{-1}(\exp tX)g)) \in \mathcal{D}^0$ for all $t \in \boldsymbol{R}$. Hence $\exp tX \subset \mathcal{C}^0$. Hence $C(g)$ is the Lie algebra of the analytic subgroup $\mathcal{C}^0(g)$. Since we have a continuous one-one homomorphism from $\mathcal{D}^0$ onto $\mathcal{C}^0$, we have that

$$\dim \mathcal{D}=\dim \mathcal{D}^0=\dim \mathcal{C}^0=\dim C\,.$$

This implies, in particular, that

$$d(g)=\dim \mathcal{A}+\dim \mathcal{B}-\dim \mathcal{D}=\dim(\alpha(A)+(Ad(g))\beta(B))\,.$$

Let us take the maximum d_0 of all $d(g)$, $g \in \mathcal{G}$, and we set $\mathcal{S}=\{g \in \mathcal{G}\,;\ d(g)<d_0\}$. We are, now, going to prove that $\mathcal{S}$ is an analytic set in $\mathcal{G}$. For that purpose, we take a basis $\{X_1, \cdots, X_k\}$ of $\alpha(A)$, and extend it into a basis $\{X_1, \cdots, X_k, X_{k+1}, \cdots, X_r\}$ of G. Let $Y_1, Y_2, \cdots, Y_l$ be a basis of $\beta(B)$, and

we set

$$Ad(g)Y_i = \sum_{j=1}^{r} \xi_{ji}(g)X_j \qquad \text{for} \quad i=1, 2, \cdots, l,$$

and consider the $(k+l)\times r$ matrix $M(g)$, whose j-th row is $(0, \cdots, 0, \overset{j\text{-th}}{1}, 0, \cdots, 0)$ for $1 \leqq j \leqq k$, and $(k+i)$-th row is $(\xi_{1i}(g), \xi_{2i}(g), \cdots, \xi_{ri}(g))$, for $1 \leqq i \leqq l$. Then $g \in \mathcal{S}$ if and only if all the $d_0 \times d_0$ minor determinants of $M(g)$ vanish.

Let us call g regular if $g \notin \mathcal{S}$. Then the set $\mathcal{G}-\mathcal{S}$ of all regular points is an open, everywhere dense, submanifold of $\mathcal{G}$.

For X in G, we denote by $*X$ (or X^*) the left (or right) invariant vector field with the value X at e. We denote the value of a vector field U at g by $U(g)$, and for a vector subspace M of G we use the notation $*M = \{*X;\ X \in M\}$ and $*M(g) = \{*X(g);\ X \in M\}$. M^*, $M^*(g)$ will be defined in a similar way. We define the analytic mappings Φ_g, Ψ_g and J by $\Phi_g h = gh$, $\Psi_g h = hg$, and $Jg = g^{-1}$, respectively. These are all analytic automorphisms of the analytic manifold $\mathcal{G}$ onto itself.

Since $(Ad(g)X)^*(g) = d\Psi_g d\Psi_{g^{-1}} d\Phi_g X^*(e)$ and $X^*(e) = X$, we have $(Ad(g)X)^*(g) = *X(g)$, which implies that

$$\begin{aligned}\dim(\alpha(A)^*(g) + *\beta(B)(g)) &= \dim(\alpha(A)^*(g) + (Ad(g)\beta(B))^*(g)) \\ &= \dim(\alpha(A) + (Ad(g))\beta(B)) = d(g).\end{aligned}$$

Since $\Phi_g \circ \Psi_h = \Psi_h \circ \Phi_h$ for every pair (g, h) in $\mathcal{G}$, we have $[\alpha(A)^*, *\beta(B)] = 0$, and since $\alpha(A)^*$ and $*\beta(B)$ are Lie algebras,

$$\Delta(g) = \alpha(A)^*(g) + *\beta(B)(g)$$

defines a d_0-dimensional analytic involutive distribution Δ in $\mathcal{G}-\mathcal{S}$.

Let Y be in A, and let $*Y$ be the left invariant vector field of $\mathcal{A}\times\mathcal{B}$ with the value Y at the identity. We extend the homomorphism α into the homomorphism $\tilde{\alpha}$ from $\mathcal{A}\times\mathcal{B}$ into $\mathcal{G}$ by $\tilde{\alpha}(a, b) = \alpha(a)$. We denote the tangential map corresponding to α from A into G by $d\alpha_1$, and set $d\alpha_1(Y) = X$. Then we have $d\tilde{\alpha}*Y(a, b) = *X(\alpha(a))$. For elements in $\mathcal{A}\times\mathcal{B}$, we have $\rho(a, b) = (\Psi_{g\beta(b)} \circ J \circ \tilde{\alpha})(a, b)$. On the other hand, we have $dJ(*X(g)) = -X^*(g^{-1})$. Hence we have

$$d\rho*Y(a, b) = -X^*(\alpha(a)^{-1}g\beta(b)).$$

Similarly for Y in B and the image $d\beta_1(Y) = X \in \beta(B)$ we have

$$d\rho*Y(a, b) = *X(\alpha(a)^{-1}g\beta(b)).$$

We denote by $(\mathcal{A}\times\mathcal{B})(a, b)$ the tangent space of $\mathcal{A}\times\mathcal{B}$ at (a, b). Then we have

$$d\rho \cdot (\mathcal{A}\times\mathcal{B})(a, b) = \Delta(\alpha(a)^{-1}g\beta(b)).$$

Hence, changing the notations we have

PROPOSITION 4. *Let $\mathcal{G}$ be an analytic group of dimension r, and let A and B be subalgebras of G. Let $\mathcal{S}$ be the set of all elements g in $\mathcal{G}$ such that* $\dim(A+Ad(g)B)$ *is not maximal. Then $\mathcal{S}$ is an analytic set of dimension $\leqq r-1$, and $A^*+{}^*B=\Delta$ defines an analytic involutive distribution in the open submanifold $\mathcal{G}-\mathcal{S}$. For g in $\mathcal{G}-\mathcal{S}$, we can define a closed subgroup $\mathcal{D}(g)$ in $\mathcal{A}\times\mathcal{B}$ such that*

$$\tilde{\rho}(g):\quad (\mathcal{A}\times\mathcal{B})/\mathcal{D}(g)\rightarrow\mathcal{A}g\mathcal{B}$$

defines an integral manifold of Δ through g.

REMARK 1. It is easy to see that $\mathcal{A}g\mathcal{B}$ is an analytic submanifold of $\mathcal{G}$, even if g is in $\mathcal{S}$.

REMARK 2. When e is not in $\mathcal{S}$, we say that the pair (A, B) is in a *generic position*. When $A\cap B=\{0\}$, the pair is always in a generic position.

REMARK 3. If, in particular, $A=\{0\}$, then the set $\mathcal{S}$ is empty.

PROPOSITION 5. *Let f be an analytic function from an open interval in $\boldsymbol{R}$ into $\mathcal{G}-\mathcal{S}$. If there exists a subinterval, for which the image of f is contained in $\mathcal{A}g_0\mathcal{B}$, $g_0\in\mathcal{G}-\mathcal{S}$, then $f(t)$ is completely contained in $\mathcal{A}g_0\mathcal{B}$, and $\tilde{\rho}(g_0)^{-1}f(t)$ is analytic from the interval into $(\mathcal{A}\times\mathcal{B})/\mathcal{D}(g_0)$.*

PROOF. Suppose it is not true, and we assume that $f(t)\in\mathcal{A}g_0\mathcal{B}$ if $-\varepsilon<t<\varepsilon$, and $f(t)\notin\mathcal{A}g_0\mathcal{B}$ for some $t>0$, without loss of generality. Let t_1 be the greatest lower bound of the set $\{t\in\boldsymbol{R};\ f(t)\notin\mathcal{A}g_0\mathcal{B}$ and $t>0\}$. Then t_1 is positive. We can find a neighborhood $\mathcal{U}$ of $f(t_1)=g_1$, and an analytic coordinate system $\{\xi^1, \xi^2, \cdots, \xi^r\}$ defined in $\mathcal{U}$, such that $\xi^{d_0+1}=\gamma^1, \cdots, \xi^r=\gamma^{r-d_0}$ defines a slice for $(\gamma^1, \cdots, \gamma^{r-d_0})$ sufficiently close to the origin in $\boldsymbol{R}^{r-d_0}$. We note that $\mathcal{A}g_0\mathcal{B}\cap\mathcal{U}$ is composed of at most countably many slices. We can take a positive number δ, small enough, such that $f(t)\in\mathcal{U}$, $t_1-\delta<t<t_1+\delta$. Then the curve $f(t)$, $t_1-\delta<t<t_1$ is in one of the slices, and we have $\xi^{d_0+j}(f(t))=\gamma^j$ for $j=1, 2, \cdots, r-d_0$. Hence the equation must be true for all t, i.e. $f(t)$ is in $\mathcal{A}g_0\mathcal{B}$ for all t with $t_1-\delta<t<t_1+\delta$. This contradicts the choice of t_1.

Now the last statement is obvious. Q. E. D.

COROLLARY. *For $g\notin\mathcal{S}$, $\mathcal{A}g\mathcal{B}$ is a maximal integral manifold of Δ.*

PROOF. Suppose there exists a connected analytic manifold $\mathcal{M}$, containing $(\mathcal{A}\times\mathcal{B})/\mathcal{D}(g)$ as a proper open submanifold, and an analytic mapping $\bar{\rho}$ which is an extension of $\tilde{\rho}$, such that $(\mathcal{M}, \bar{\rho})$ defines an integral manifold of Δ. Let p be a point in the boundary of $(\mathcal{A}\times\mathcal{B})/\mathcal{D}(g)$ in $\mathcal{M}$, and let $\mathcal{U}$ be a neighborhood of p with an analytic coordinate system $(\xi^1, \xi^2, \cdots, \xi^{d_0})$, with $\xi^i(p)=0$ for $i=1, 2, \cdots, d_0$, and $-1<\xi^1<1, \cdots, -1<\xi^{d_0}<1$. Let q be a point in $\mathcal{U}\cap(\mathcal{A}\times\mathcal{B})/\mathcal{D}(g)$, with $\xi^i(g)=\gamma^i$ for $i=1, 2, \cdots, d_0$. For t, with $|t|<1$, we associate a point p_t in $\mathcal{U}$ such that $\xi^i(p_t)=\gamma^i t$ $(i=1, 2, \cdots, d_0)$. Then $\bar{\rho}(p_t)$ defines an analytic curve in $\mathcal{G}-\mathcal{S}$, which satisfies the conditions in Proposition 5.

§ 5. Proof of Theorem 2.

We shall prove Propositions 6 and 7, see § 1, which imply Theorem 2.

PROOF OF PROPOSITION 6. If $\exp(tY)=\exp(tX)\cdot l(t)$, $l(t)\in\mathcal{L}$ for t in $\boldsymbol{R}$, we can find a positive number ε such that

$$(*)\qquad \exp(-tX)\cdot\exp(tY)=\exp\{\varphi_1(X,Y)t+\varphi_2(X,Y)t^2+\cdots\}$$
$$\varphi_1(X,Y)t+\varphi_2(X,Y)t^2+\cdots\in L$$

for $|t|<\varepsilon$. Thus we have $\varphi_n(X,Y)\in L$ for $n=1,2,\cdots$, and it follows that $Y-X\in P(X,L)$ by Theorem 1.

Conversely, we assume that $Y-X\in P(X,L)$ and put $\exp(-tX)\cdot\exp(tY)=l(t)$. Then for t, sufficiently close to 0, $l(t)$ is given by the right side of $(*)$, and is contained in $\mathcal{L}$. Since l is analytic, $l(t)\in\mathcal{L}$ for all $t\in\boldsymbol{R}$, by Proposition 5. Q. E. D.

PROOF OF PROPOSITION 7. Let G^C be the complexification of G, i. e. $G^C=G\otimes_R\boldsymbol{C}$. G^C is a Lie algebra over $\boldsymbol{C}$, and contains G in a natural way. The complexification L^C of L can be canonically imbedded in G^C, with $G\cap L^C=L$.

Let $\mathcal{G}^C$ be one of the complex analytic groups with the Lie algebra G^C. $\mathcal{G}^C$ contains an analytic subgroup $\mathcal{L}^C$, whose Lie algebra is L^C.

Let us suppose that X is not in L. Then for $\mathcal{A}=\exp(\boldsymbol{R}X)$ and $\mathcal{B}=\mathcal{L}$, in the discussions of § 4, $\mathcal{D}(e)$ is a discrete subgroup of $\mathcal{A}\times\mathcal{B}$. We denote by π the homomorphism $\boldsymbol{R}\ni t\mapsto\exp(tX)\in\mathcal{A}$. Then $\sigma:\boldsymbol{R}\times\mathcal{B}\ni(t,b)\mapsto\mathcal{D}(e)(\pi(t),b)\in(\mathcal{A}\times\mathcal{B})/\mathcal{D}(e)$ is a covering mapping. Since $\exp(tY)$ is an analytic curve in $\mathcal{A}\mathcal{B}$, we can find a unique analytic curve $(f(t),g(t))$ in $\boldsymbol{R}\times\mathcal{B}$ such that

$$\tilde{\rho}(e)\pi(f(t),g(t))=\exp(tY)\qquad t\in\boldsymbol{R}\,,$$

and

$$f(0)=g(0)=\text{the identity}\,,$$

i. e. we have $\exp(tY)=\exp(f(t)X)\cdot g(t)$, $g(t)\in\mathcal{B}$.

We denote the exponential function from G^C into $\mathcal{G}^C$ also by exp. Since $f(t)$ is an analytic function defined on $\boldsymbol{R}$, it can be extended to an entire function on $\boldsymbol{C}$. We shall denote the extension also by f. Then $\xi(z)=\exp(-f(z)X)\cdot\exp(zY)$ is an analytic function from $\boldsymbol{C}$ into $\mathcal{G}^C$. Let $X_1, X_2, \cdots, X_k$ be a basis of L. Then for a real number t sufficiently close to 0, we have $\xi(t)=g(t)=\exp\{l_1(t)X_1+l_2(t)X_2+\cdots+l_k(t)X_k\}$, where $l_1(t), l_2(t),\cdots,l_k(t)$ are power series of t. Hence for a complex number z sufficiently close to 0, we have

$$\xi(z)=\exp\{l_1(z)X_1+l_2(z)X_2+\cdots+l_k(z)X_k\}\,,$$

i. e. $\xi(z)\in\mathcal{L}^C$. Hence by Proposition 5, $\xi(z)$ is completely contained in $\mathcal{L}^C$.

So we have

$$\exp(zY) = \exp(f(z)X) \cdot \xi(z), \qquad z \in \boldsymbol{C}.$$

Similarly we have

$$\exp(zX) = \exp(h(z)Y) \cdot \eta(z), \qquad \eta(z) \in \mathcal{L}^C, \quad z \in \boldsymbol{C}.$$

Hence we have

$$\exp(zX) = \exp(f(h(z))X) \cdot \xi(h(z))\eta(z).$$

Since $X \notin L^C$, at least for z sufficiently close to 0, we have $z = f(h(z))$, and the functional relation must be valid for all z. Hence $f(z)$ must be a linear function. On the other hand, since $f(0)=0$, we have $f(z) = \gamma z$. Q. E. D.

EXAMPLE. Let Γ_n denote the set composed of all germs of analytic vector fields of $\boldsymbol{R}^n$, $n \geqq 1$, at the origin $o = (0, 0, \cdots, 0)$. Then Γ_n is a Lie algebra of infinite dimension over $\boldsymbol{R}$. The set of elements of Γ_n which vanish at o forms a subalgebra Λ_n.

For an element X of Γ_n which can be expressed as

$$X = X_1 \frac{\partial}{\partial x_1} + X_2 \frac{\partial}{\partial x_2} + \cdots + X_n \frac{\partial}{\partial x_n},$$

we associate the following analytic ordinary differential equation

$$\text{(1)} \qquad \frac{dx_i}{dt} = X_i\,(x_1, x_2, \cdots, x_n) \qquad i = 1, 2, \cdots, n,$$

and consider the (germ of) solution of (1) with the initial condition

$$\text{(2)} \qquad x_1(0) = x_2(0) = \cdots = x_n(0) = 0.$$

When X is in Λ_n, the solution is trivial. If X does not vanish at o, by choosing a suitable analytic coordinate system around o, we may assume that $X = \frac{\partial}{\partial x_1}$. Let A be an element of Λ_n: $A = A_1 \frac{\partial}{\partial x_1} + A_2 \frac{\partial}{\partial x_2} + \cdots + A_n \frac{\partial}{\partial x_n}$. If the solution $(t, 0, \cdots, 0)$ of X is also a solution of $X + A$, then $A_1(x_1, 0, \cdots, 0) = A_2(x_1, 0, \cdots, 0) = A_n(x_1, 0, \cdots, 0) = 0$, and vice versa.

On the other hand, since we have

$$(ad\ X)^k A = \frac{\partial^k A_1}{\partial x_1^k} \frac{\partial}{\partial x_1} + \frac{\partial^k A_2}{\partial x_1^k} \frac{\partial}{\partial x_2} + \cdots + \frac{\partial^k A_n}{\partial x_1^k} \frac{\partial}{\partial x_n}, \qquad k = 1, 2, \cdots,$$

A is contained in the play $P(X, \Lambda_n)$ if and only if all $\frac{\partial^k A_i}{\partial {x_1}^k}(o) = 0$, i.e. $A_i(x_1, 0, \cdots, 0) = 0$ for $i = 1, 2, \cdots, n$. Thus we have the following conclusion: *Two germs of analytic vector fields X and Y, at the origin of $\boldsymbol{R}^n$, give the same solution, with (2), if and only if $Y - X \in P(X, \Lambda_n)$.*

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