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The resolution of an irregularity of boundary points in the boundary problem for Markov processes

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§ 1. Introduction.

The purpose of this paper is to solve completely one of the open problems in M. Motoo's paper [17].

The object that we shall consider is the boundary problem of Markov processes, which can be formulated in the following way. Let $\boldsymbol{M}^{\min}$ be a Markov process on a space $\bar{D}$ whose path functions stop as soon as they arrive at the boundary V of D (Such a process is called a *minimal process* in this paper). Then, the problem is to determine the class of all Markov processes whose stopped path functions at the boundary V coincide with path functions of the given minimal process $\boldsymbol{M}^{\min}$.

Let S be a locally compact Hausdorff topological space with the axiom of second countability and D be an open subset of S having closure S and non-empty compact boundary $V=S-D$. Suppose that we are given a Markov process $\boldsymbol{M}^{\min}=(W, P_x^{\min}; x\in S)$ on S satisfying the following conditions ($\boldsymbol{M}^{\min}$.1), ($\boldsymbol{M}^{\min}$.2) and ($\boldsymbol{M}^{\min}$.3).

($\boldsymbol{M}^{\min}$.1) $\boldsymbol{M}^{\min}$ is a Hunt process on S.

($\boldsymbol{M}^{\min}$.2) $P_\xi^{\min}(x_t=\xi,\ 0\leqq t<\infty)=1$ for any $\xi\in V$.

($\boldsymbol{M}^{\min}$.3) There exists a measure m_0 on D such that for any $E\in\boldsymbol{B}(D)$, $m_0(E)=0$ is equivalent to $G_\alpha^0(x, E)=0$ for any $\alpha>0$ and $x\in D$, where G_α^0 is the kernel defined by

$$G_\alpha^0 f(x)=E_x^{\min}\Big(\int_0^{\sigma_V} e^{-\alpha t}f(x_t(w))dt\Big) \qquad (\alpha>0,\ x\in S,\ f\in B(S))$$

and moreover $\sigma_V(w)$ is the time when the path w first arrives at V; that is,

$$\sigma_V(w)=\inf\{t>0;\ x_t(w)\in V\}.$$

Then, our purpose is to characterize the Markov process $\boldsymbol{M}=(W, P_x; x\in S)$ on S whose stopped path functions at V coincide with path functions of $\boldsymbol{M}^{\min}$, that is, satisfying the following conditions ($\boldsymbol{M}$.1), ($\boldsymbol{M}$.2) and ($\boldsymbol{M}$.3).

($\boldsymbol{M}$.1) $\boldsymbol{M}$ is a Hunt process on S.

($\boldsymbol{M}$.2) Let G_α be the Green kernel of $\boldsymbol{M}$. There exists a measure m on

S such that for any $E\in \boldsymbol{B}(S)$, $m(E)=0$ is equivalent to $G_\alpha(x, E)=0$ for any $\alpha>0$ and $x\in S$.

($\boldsymbol{M}$.3) The process $\boldsymbol{M}$ stopped at V is $\boldsymbol{M}^{\min}$; that is,

$$E_x\Big(\int_0^{\sigma_V} e^{-\alpha t}f(x_t)dt\Big)=G_\alpha^0 f(x) \qquad (\alpha>0),$$

$$E_x(e^{-\alpha\sigma_V}g(x_{\sigma_V}))=E_x^{\min}(e^{-\alpha\sigma_V}g(x_{\sigma_V}))\equiv H_\alpha g(x) \qquad (\alpha\geqq 0)$$

for any $f\in B(S)$, $g\in B(V)$ and $x\in S$.

Note that the following ($\boldsymbol{M}$.3′) is an immediate consequence of ($\boldsymbol{M}^{\min}$.2) and ($\boldsymbol{M}$.3).

($\boldsymbol{M}$.3′) every point of V is regular to V with respect to $\boldsymbol{M}$.

Then, M. Motoo [17] proved, fixing a positive number $\gamma>0$, that $\boldsymbol{M}$ can be decomposed into the minimal process $\boldsymbol{M}^{\min}$ and the boundary system $(\tilde{\boldsymbol{M}}, l, m, \boldsymbol{Q})$ and that $\boldsymbol{M}$ is uniquely determined by them, where $\tilde{\boldsymbol{M}}$ is a Markov process on V (U-process of $\boldsymbol{M}$) ([16]), $l, m\in B^+(V)$ and $\boldsymbol{Q}$ is a bounded kernel on $V\times D$; roughly speaking $l(\xi)$, $m(\xi)$ represent the (suitable weighted) proportions of sojourn on V, of reflection from ξ to D ($\xi\in V$) respectively, and the measure $\boldsymbol{Q}(\xi, \cdot)$ ($\cdot\subset D$) denotes the mode of jump from V to D averaged by $G_\gamma^0 1$, under the following additional conditions ($\boldsymbol{M}^{\min}$.4), ($\boldsymbol{M}^{\min}$.5) and ($\boldsymbol{M}^{\min}$.6).

($\boldsymbol{M}^{\min}$.4) $$G_\alpha^0(C(S))\subset C(S) \qquad \text{for } \alpha>0,$$

$$H_\alpha(C(V))\subset C(S) \qquad \text{for } \alpha\geqq 0.$$

($\boldsymbol{M}^{\min}$.5) $$\hat{H}_\alpha f=\frac{G_\alpha^0 f}{G_\gamma^0 1}\in C(S) \qquad \text{if } \alpha>0 \text{ and } f\in C(S).$$

($\boldsymbol{M}^{\min}$.6) $$\{\hat{H}_\alpha f;\ f\in C(S)\} \qquad \text{is dense in } C(S).$$

One of the open problems is to replace the above conditions ($\boldsymbol{M}^{\min}$.4), ($\boldsymbol{M}^{\min}$.5) and ($\boldsymbol{M}^{\min}$.6), in particular ($\boldsymbol{M}^{\min}$.5) and ($\boldsymbol{M}^{\min}$.6), by deeper (probabilistic) and more general ones. As an example which does not satisfy ($\boldsymbol{M}^{\min}$.5), let S be the closed interval $[-1, 1]$, V be $\{-1, 0, 1\}$ and $\boldsymbol{M}^{\min}$ be the Brownian motion stopped at V. Then, ($\boldsymbol{M}^{\min}$.5) is not satisfied near 0, since 0 consists of two entrance points (see § 7). In general cases, M. Motoo [17] and T. Ueno [20] suggest that this problem shall be reduced to obtain the exit and entrance boundaries for a given minimal process, and to make suitable identification of certain parts of these boundaries, and then to determine the class of all consistent boundary systems on the boundary thus constructed.

In this paper, we prove, without the conditions ($\boldsymbol{M}^{\min}$.4), ($\boldsymbol{M}^{\min}$.5) and ($\boldsymbol{M}^{\min}$.6) and without introducing the entrance boundary, that $\boldsymbol{M}$ satisfying ($\boldsymbol{M}$.1), ($\boldsymbol{M}$.2) and ($\boldsymbol{M}$.3) is decomposed and uniquely determined by given minimal process $\boldsymbol{M}^{\min}$ and the boundary system $(\tilde{\boldsymbol{M}}, l, \boldsymbol{P}, \boldsymbol{Q})$——*the problem* *stated is resolved completely and generally*——, where $\tilde{\boldsymbol{M}}$ is the U-process of $\boldsymbol{M}$, l and $\boldsymbol{Q}$ are the same as those of M. Motoo [17], and $\boldsymbol{P}$ is a bounded linear positive operator from $\bar{\mathcal{M}}$[1] $\left(\mathcal{M}=\left\{\frac{G_\alpha^0 f}{G_\gamma^0 1};\ \alpha>0,\ f\in B(D)\right\}\right)$ into $L^\infty(V, \boldsymbol{B}(V), \nu)$ (ν is the canonical measure of the γ-order sweeping out Φ to the boundary V of the time additive functional $t\wedge\zeta(w)$ for $\boldsymbol{M}$), that is, we shall prove the following Theorems 3.2, 4.1 and Proposition 4.1 in §3 and §4.

THEOREM 3.2. (*Feller-Ueno decomposition*)

$$G_\alpha f(x)=G_\alpha^0 f(x)+H_\alpha K^\alpha\left\{lf+(\boldsymbol{P}+\boldsymbol{Q})\left(\frac{G_\alpha^0 f}{G_\gamma^0 1}\right)\right\}(x)$$

for any $\alpha>0$, $f\in B(S)$ *and* $x\in S$, *where* K^α *is the* 0-*order resolvent of the* α-*order* U-*process of* $\boldsymbol{M}$ ([12], [17]).

PROPOSITION 4.1. *For any* $\alpha>0$ *and* $f, h\in B^+(D)$,

$$l\cdot\Phi\approx\chi_V{}^{2)}\cdot dt\,,$$

$$(\boldsymbol{P}+\boldsymbol{Q})\left(\frac{G_\alpha^0 f}{G_\gamma^0 1}\right)\cdot\Phi\approx\widetilde{(\chi_D f\cdot dt)}_\alpha\,,$$

$$\boldsymbol{Q}h\cdot\Phi\approx\chi_V P_D(hG_\gamma^0 1)\cdot L\,,$$

where a system (P, L) *is the Lévy system of* $\boldsymbol{M}$ *and* $P_D g(x)=\int_D P(x, dy)g(y)$ *for* $g\in B(D)$ *and* $x\in S$.

We shall call the system $(\tilde{\boldsymbol{M}}, l, \boldsymbol{P}, \boldsymbol{Q})$ *the boundary system of* $\boldsymbol{M}$, where $\tilde{\boldsymbol{M}}$ is the U-process of $\boldsymbol{M}$, $l\in B(V)^+$, $\boldsymbol{P}$ is a bounded positive linear operator from $\bar{\mathcal{M}}$ into $L^\infty(V, \boldsymbol{B}(V), \nu)$ and $\boldsymbol{Q}$ is a bounded kernel on $V\times D$ satisfying the properties in Proposition 4.1, since $\tilde{\boldsymbol{M}}$ is uniquely determined by $\boldsymbol{M}$ and the system $(l, \boldsymbol{P}, \boldsymbol{Q})$ is uniquely determined by $\boldsymbol{M}$ up to equivalence with respect to ν.

THEOREM 4.2. *The process* $\boldsymbol{M}$ *is uniquely determined by the boundary system* $(\tilde{\boldsymbol{M}}, l, \boldsymbol{P}, \boldsymbol{Q})$.

Next, we shall prove the following theorem which gives the characterization of $\boldsymbol{P}$ in §3.

THEOREM 3.3 (*Local character of* $\boldsymbol{P}$). *Let* $\varphi\in\bar{\mathcal{M}}$, $\xi_0\in V$ *and* $\varepsilon>0$. *If there exists an open neighbourhood* U *of* ξ_0 *in* S *such that* $|\varphi(x)|\leqq\varepsilon$ *for any* $x\in U\cap D$, *then*

$$|\boldsymbol{P}\varphi(\xi)|\leqq\varepsilon\,,\qquad \nu\text{-a.e. }\xi\in U\cap V\,.$$

In particular, for any $f\in C_\infty(D)\cap\bar{\mathcal{M}}$, $\boldsymbol{P}f=0$.

Finally, for the purpose of revealing the operator $\boldsymbol{P}$, in §5, we shall

1) $\bar{\mathcal{M}}$ is the uniform closure of $\mathcal{M}$ in $B(D)$.

2) χ_V is the indicator function of V.

introduce the entrance boundary, which is obtained under the following condition.

($\boldsymbol{M}^{\min}.4^*$) $$G_\alpha^0(C_\infty(D)) \subset C_\infty(D) \quad \text{and}$$
$$G_\alpha^0 1 \in C_\infty(D) \quad \text{for any } \alpha > 0 .$$

Note that this is equivalent to $G_\alpha^0(C(D)) \subset C_\infty(D)$ for any $\alpha > 0$. Consider the following superharmonic transformation of $\{G_\alpha^0 ; \alpha > 0\}$ by $G_\gamma^0 1$;

$$G_\alpha^1 f(x) = \frac{G_\alpha^0(fG_\gamma^0 1)(x)}{G_\gamma^0 1(x)} \qquad \text{for } \alpha > 0 \text{ and } f \in B(D) .$$

Then, noting that $\mathcal{R} = G_\alpha^1(C(D)) \subset C(D)$ and $\mathcal{R}$ separates the points of D, we can obtain the space D^{**} such that

(i) D^{**} is a compact metrizable space,

(ii) there exists a continuous and injective mapping i^{**} from D into D^{**} with $i^{**}(D)$ a dense Borel subset of D^{**},

(iii) each element $\varphi \in \mathcal{R}$ can be extended to a continuous function φ^{**} on the space D^{**}, that is, $\varphi^{**} \circ i^{**} = \varphi$,

(iv) $\mathcal{R}^{**} = \{\varphi^{**} ; \varphi \in \mathcal{R}\}$ separates the points of D^{**}. The existence and uniqueness (up to homeomorphism) of such a D^{**} will be proved in § 6 in a general setup. In our case, the existence of such a D^{**} can be easily proved as follows; let D^* be an $\mathcal{R}$-compactification of D ([**3**]), and φ^* be a continuous extension of $\varphi \in \mathcal{R}$. Next, let D^{**} be the quotient space of D^* by the following equivalence relation: two points x and y in D^* are said to be equivalent if $\varphi^*(x) = \varphi^*(y)$ for any $\varphi \in \mathcal{R}$. Then, we see that D^{**} is our desired space (see § 5). We shall call the space $V^{**} = D^{**} - i^{**}(D)^0$ the entrance boundary of $\{G_\alpha^0 ; \alpha > 0\}$. Note that there is a Hunt process $\boldsymbol{M}^0$ associated with $\{G_\alpha^0 ; \alpha > 0\}$ its resolvent (§ 4). For any $f \in C(D^{**})$ and $\alpha > 0$, let

$$G_\alpha^{**} f = (G_\alpha^1(f \circ i^{**}))^{**} .$$

Then, we see that $\{G_\alpha^{**} ; \alpha > 0\}$ is a sub-Markov resolvent satisfying Ray's hypothesis ([**11**], [**18**]). Denote the set of its branching points by D_b^{**}. Then, we shall prove the representation theorem for $\boldsymbol{P}$.

Theorem 5.1. *For each point $\xi \in V$, there exists a sub-stochastic measure $\mu(\xi, dx)$ on D^{**} such that*

(i) $$\boldsymbol{P}\Big(\frac{G_\alpha^0 f}{G_\gamma^0 1}\Big)(\xi) = \int_{D^{**}} \mu(\xi, dx) \hat{H}_\alpha f(x) ,$$

ν-a. e. $\xi \in V$ for any $\alpha > 0$ and $f \in C(D)$,

where $\hat{H}_\alpha f = \lim_{\beta \to \infty} \Big(\beta G_\beta^1\Big(\frac{G_\alpha^0 f}{G_\gamma^0 1}\Big)\Big)^{**}$ *for any $\alpha > 0$ and $f \in C(D)$,*

(ii) $$\boldsymbol{P}1(\xi) = \mu(\xi, D^{**}), \ \nu\text{-a. e. } \xi \in V ,$$

(iii) $$\mu(\xi, D_b^{**})=0 \qquad \textit{for any } \xi \in V.$$

Moreover, such a $\mu(\xi, dx)$ *is uniquely determined up to equivalence with respect to* ν.

Set $S_\xi=\bigcap_{\varepsilon>0} \overline{i^{**}\{x \in D\,;\, \mathrm{dis}(x, \xi)<\varepsilon\}}$ for $\xi \in V$. Then, we shall have the following interesting theorem.

THEOREM 5.3. *Under the hypothesis* $D_b^{**}=\emptyset$,

$$\mu(\xi, D^{**}-S_\xi)=0 \qquad \textit{for any } \xi \in V.$$

See M. Brelot [**1**] as for S_ξ, where he calls any point $\eta \in S_\xi$ the point associated with ξ $(\xi \in V)$.

The author does not know whether $S_{\xi_1} \cap S_{\xi_2}=\emptyset$ for $\xi_1 \neq \xi_2$.[3] A study of this interesting and significant question would be important and useful in the future.

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REMARK. We shall list some notations and definitions which will be used repeatedly in this paper.

Let S be a topological space and $S^*=S \cup \{\partial\}$ where ∂ is an isolated point if S is compact, or an adjoined point of one point compactification S^* of S if not. $\boldsymbol{B}(S)$ is the Borel field generated by all the closed sets in S. We introduce several spaces of functions: $B(S)$ is the space of real-valued bounded $\boldsymbol{B}(S)$-measurable functions on S, $C(S)$ is the space of real-valued bounded continuous functions on S, $C_\infty(S)$ is the subset of $C(S)$ consisting of the functions which for any $\varepsilon>0$, $\{x \in S\,;\, |f(x)| \geqq \varepsilon\}$ is compact, and $C_0(S)$ is the subset of $C(S)$ consisting of the functions with compact supports. For any function space $\mathcal{L}$ on S, $\mathcal{L}^+$ means the subset of $\mathcal{L}$ consisting of the functions which are nonnegative. Sometimes, we consider f in $B(E)$ $(E \in \boldsymbol{B}(S))$ as a function on S^*, setting $f(x)=0$, $x \notin E$. For $E, F \in \boldsymbol{B}(S)$, $K(x, A)$ $(x \in E, A \in \boldsymbol{B}(F))$ is called a kernel on $E \times F$ if $K(\cdot, A)$ is Borel measurable on E and $K(x, \cdot)$ is a measure on F. If $K(x, A)$ is a kernel, we write $Kf(x)=\int_F K(x, dy)f(y)$ and $K_A f(x)=\int_A K(x, dy)f(y)$ $(A \in \boldsymbol{B}(F))$.

Now we shall prepare the notations of Markov processes. The path space W that we shall consider is the set of all mappings w from $[0, \infty]$ into S^* which satisfies the following properties:

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3) See the second proof of Theorem 5.5 in § 5.

(W.1) $w(t)$ is right continuous and has a left limit in $[0, \infty)$.

(W.2) $w(\infty)=\partial$.

(W.3) There exists $\zeta=\zeta(w)$ (which is called the *life time*) such that

$$w(t)\in S \quad \text{if} \quad t<\zeta,$$
$$w(t)=\partial \quad \text{if} \quad t\geqq\zeta.$$

The Markov process $\boldsymbol{M}=(W, P_x ; x\in S^*)$ on S that we shall consider is a Hunt process, that is, it has the strong Markov property and the quasi-left continuity of sample functions ([**16**]). Furthermore we shall assume the following hypothesis: let G_α be the Green operator of $\boldsymbol{M}$, then there exists a measure m on S such that for any $E\in \boldsymbol{B}(S)$, $m(E)=0$ is equivalent to $G_\alpha(x, E)=0$ for any $\alpha>0$ and $x\in S$.

The additive functionals that we shall use in this paper are nonnegative continuous additive functionals, unless otherwise stated (see [**14**]). Two additive functionals A and B are said to be equivalent if and only if

$$P_x(A(t, w)=B(t, w) \quad \text{for all } t)=1 \quad \text{for all } x\in S,$$

and in this case we use the notation $A\approx B$. We also write $A\ll B$ if $P_x(A(t, w)\leqq B(t, w)$ for all $t)=1$ for all $x\in S$. For any $f\in B(S)^+$, we define

$$f\cdot A(t, w)=\int_{(0,t]} f(x_s(w))dA(s, w).$$

Then, $f\cdot A$ is an additive functional.

Next, we shall define the sweeping-out of additive functionals. From now on, assume that S is a locally compact Hausdorff space with the axiom of second countability. Let V be an element of $\boldsymbol{B}(S)$ such that every point of V is regular to V with respect to $\boldsymbol{M}$, that is, denoting the hitting time to V by σ_V, $P_\xi(\sigma_V=0)=1$ for all $\xi\in V$. Then, for any $\alpha>0$ and any additive functional A, such that $E_x\left(\int_0^\infty e^{-\alpha t}dA\right)<\infty$, there exists a unique additive functional $\tilde{A}_\alpha$ such that

$$E_x\left(\int_0^\infty e^{-\alpha t}d\tilde{A}_\alpha\right)=E_x\left(\int_{\sigma_V}^\infty e^{-\alpha t}dA\right)$$

(see [**16**]). We shall call $\tilde{A}_\alpha$ the *α-th order sweeping-out* of A. In general, $\tilde{A}_\alpha$ depends on α.

Finally, we shall define the canonical measure of additive functionals. For any additive functional A, such that $E_x(A(t))<\infty$, there exists a unique (up to absolutely continuity) measure ν on $\boldsymbol{B}(S)$ such that for $E\in \boldsymbol{B}(S)$,

$$\chi_E\cdot A\approx 0 \text{ is equivalent to } \nu(E)=0$$

(see [**14**], pp. 146). Such a ν is called a *canonical measure* of A.

§ 2. The multilinear functionals: $\mathcal{E}^c_{\alpha,\beta}$ and $\mathcal{E}^d_{\alpha,\beta}$.

In this section, we prepare a series of lemmas which can be considered to be the continuities and precisions of the results in § 5 in M. Motoo's paper [17]: '*Decomposition of the resolvent. The boundary system*'. Those can be proved by the deep analysis in § 4 in [17]: '*Properties of the excursion at V*'.

In the sequel, we fix a positive number $\gamma>0$. Let Φ be the γ-th order sweeping-out to V of the time additive functional $t\wedge\zeta(w)$ for $\boldsymbol{M}$ (ζ is the killing time of $\boldsymbol{M}$), τ be its right continuous inverse function, that is, $\tau(s, w)=\sup\{t\,;\,\Phi(t, w)\leqq s\}$, ν be the canonical measure of Φ and (P, L) be the Lévy system of $\boldsymbol{M}$ ([15], [21]).

For any Markov process $\boldsymbol{M}$ satisfying ($\boldsymbol{M}$.1), ($\boldsymbol{M}$.2) and ($\boldsymbol{M}$.3'), M. Motoo [17] proved that the following fact holds.

LEMMA 2.1 ([17], p. 88, (31)). *There exist two functions l, $m\in B^+(V)$ and a bounded kernel $\boldsymbol{Q}$ on $V\times D$ such that*

(1) $\quad l\cdot\Phi\approx\chi_V\cdot dt\,,$

(2) $\quad E_x\Big(\int_0^\infty e^{-\gamma t}m(x_t)\,d\Phi\Big)=E_x\Big(\sum_{s\in T_c}\int_{\tau(s-)}^{\tau(s)}e^{-\gamma t}l(x_t)\,dt\Big),\ x\in S\,,$

(3) $\quad \boldsymbol{Q}h\cdot\Phi\approx\chi_V P_D(hG^0_\gamma 1)\cdot L,\ h\in B^+(D)\,,$

(4) $\quad l(\xi)+m(\xi)+\boldsymbol{Q}(\xi, D)=1,\ \nu\text{-}a.\,e.\ \xi\in V\,,$

where

$$T(w)=\{s>0\,;\,\tau(s-, w)<\tau(s, w)\wedge\zeta(w)\}\,,$$

$$T_c(w)=\{s\in T(w)\,;\,x_{\tau(s-)-}(w)=x_{\tau(s-)}(w)\}\,,$$

$$T_d(w)=\{s\in T(w)\,;\,x_{\tau(s-)-}(w)\neq x_{\tau(s-)}(w)\}\,.$$

DEFINITION 2.1. For any $\alpha>0$, $\beta>0$ and $x\in S$, the following multilinear functionals on $B(S)\times B(S)\times B(S)$ or $B(S)\times B(S)$ are defined:

$$\mathcal{E}^c_{\alpha,\beta}(f, g, h)(x)=E_x\Big(\sum_{s\in T_c}e^{-\alpha\tau(s-)}f(x_{\tau(s-)-})g(x_{\tau(s-)})\int_{\tau(s-)}^{\tau(s)}e^{-\beta(t-\tau(s-))}h(x_t)dt\Big)\,,$$

$$\mathcal{E}^d_{\alpha,\beta}(f, g, h)(x)=E_x\Big(\sum_{s\in T_d}e^{-\alpha\tau(s-)}f(x_{\tau(s-)-})g(x_{\tau(s-)})\int_{\tau(s-)}^{\tau(s)}e^{-\beta(t-\tau(s-))}h(x_t)dt\Big)\,,$$

$$\mathcal{E}_{\alpha,\beta}(f, g, h)(x)=E_x\Big(\sum_{s\in T}e^{-\alpha\tau(s-)}f(x_{\tau(s-)-})g(x_{\tau(s-)})\int_{\tau(s-)}^{\tau(s)}e^{-\beta(t-\tau(s-))}h(x_t)dt\Big)\,,$$

$$\mathcal{E}^c_{\alpha,\beta}(f, g)(x)=\mathcal{E}^c_{\alpha,\beta}(f, 1, g)(x)\,,$$

$$\mathcal{E}^d_{\alpha,\beta}(f, g)(x)=\mathcal{E}^d_{\alpha,\beta}(f, 1, g)(x)\,,$$

$$\mathcal{E}_{\alpha,\beta}(f, g)(x)=\mathcal{E}_{\alpha,\beta}(f, 1, g)(x)\,.$$

LEMMA 2.2.

(1) $|\mathcal{E}^{c}_{\alpha,\beta}(f, g, h)(x)|,\ |\mathcal{E}^{d}_{\alpha,\beta}(f, g, h)(x)|,\ |\mathcal{E}_{\alpha,\beta}(f, g, h)(x)| \leqq \frac{\|f\|^{4)}\cdot\|g\|\cdot\|h\|}{\alpha\wedge\beta}.$

(2) $\mathcal{E}_{\alpha,\beta}(f, g, h)(x)=\mathcal{E}^{c}_{\alpha,\beta}(f, g, h)(x)+\mathcal{E}^{d}_{\alpha,\beta}(f, g, h)(x).$

(3) $\mathcal{E}^{c}_{\alpha,\beta}(f, g, h)(x)=\mathcal{E}^{c}_{\alpha,\beta}(f\cdot g, h)(x).$

(4) $\mathcal{E}^{c}_{\alpha,\beta}(f, g)(x)=\mathcal{E}^{c}_{\alpha,\beta}(f, \chi_V, g)(x).$

(5) $\mathcal{E}^{d}_{\alpha,\beta}(f, g)(x)=\mathcal{E}^{d}_{\alpha,\beta}(f, \chi_D, g)(x).$

(6) $\mathcal{E}^{c}_{\alpha,\beta}(\chi_{D\cup\partial}, g)(x)=\mathcal{E}^{d}_{\alpha,\beta}(\chi_{D\cup\partial}, g)(x)=0.$

(7) $\mathcal{E}^{c}_{\alpha,\beta}(f, \chi_V)(x)=\mathcal{E}^{d}_{\alpha,\beta}(f, \chi_V)(x)=0.$

(8) *For any* $f, g, h\in B^{+}(S)$, $\mathcal{E}^{c}_{\alpha,\beta}(f, g)(x)\geqq 0$, $\mathcal{E}^{d}_{\alpha,\beta}(f, g, h)(x)\geqq 0.$

(9) *Let* $(f_n)_{n=0}^{+\infty}$, $(g_n)_{n=0}^{+\infty}$ *and* $(h_n)_{n=0}^{+\infty}$ *be uniformly bounded in* n *such that* $\lim_{n\to\infty} f_n(x)=f_0(x)$ *on* V,

$$\lim_{n\to\infty} g_n(x)=g_0(x) \text{ in } D \text{ and } \lim_{n\to\infty} h_n(x)=h_0(x) \text{ in } D.$$

Then, we have, for any $\alpha>0$, $\beta>0$ *and* $x\in S$,

$$\lim_{n\to\infty}\mathcal{E}^{c}_{\alpha,\beta}(f_n, g_n)(x)=\mathcal{E}^{c}_{\alpha,\beta}(f_0, g_0)(x),$$

$$\lim_{n\to\infty}\mathcal{E}^{d}_{\alpha,\beta}(f_n, g_n, h_n)(x)=\mathcal{E}^{d}_{\alpha,\beta}(f_0, g_0, h_0)(x).$$

PROOF. (1) follows from the following inequality:

$$\left|e^{-\alpha\tau(s-)}f(x_{\tau(s-)-})g(x_{\tau(s-)})\int_{\tau(s-)}^{\tau(s)} e^{-\beta(t-\tau(s-))}h(x_t)dt\right|$$

$$\leqq \|f\|\cdot\|g\|\cdot\|h\|e^{-\alpha\tau(s-)}\int_{\tau(s-)}^{\tau(s)} e^{-\beta(t-\tau(s-))}dt$$

$$\leqq \|f\|\cdot\|g\|\cdot\|h\|e^{-(\alpha\wedge\beta)\tau(s-)}\int_{\tau(s-)}^{\tau(s)} e^{-(\alpha\wedge\beta)(t-\tau(s-))}dt$$

$$=\|f\|\cdot\|g\|\cdot\|h\|\int_{\tau(s-)}^{\tau(s)} e^{-(\alpha\wedge\beta)t}dt.$$

(2) follows from the fact that $T(w)=T_c(w)\cup T_d(w)$ and $T_c(w)\cap T_d(w)=\emptyset$. (3) follows from the definition of $T_c(w)$. (4), (5), (6) and (7) follow from the fact that (i) for any $s>0$, $x_t(w)\notin V$ if $t\in(\tau(s-), \tau(s))$, (ii) for any $s>0$, $x_{\tau(s-)-}\in V$ and (iii) for any $s\in T(w)$, $x_{\tau(s-)-}(w)=x_{\tau(s-)}(w)$ if $x_{\tau(s-)}(w)\in V$ ([17]; pp. 82, [4.3], (1) and pp. 83, [4.11]). (8) is clear. (9) can be proved by applying the Lebesgue's dominated convergence theorem to the inequality obtained in the proof of (1).

4) $\|f\|$ is the supremum norm.

LEMMA 2.3. *For any* $f, g \in C(S)$, $h \in B(S)$, $\alpha > 0$, $\beta > 0$ *and* $x \in S$,

$$\lim_{k\to\infty} E_x\Big(\sum_{n=1}^{\infty} e^{-\alpha\rho_n(k)}\chi_D f(x_{\rho_n(k)-})g\cdot G^0_\beta h(x_{\rho_n(k)})\Big) = \mathcal{E}^{c}_{\alpha,\beta}(f, g, h)(x)\,,$$

where $\{\rho_n(k)\}$ *is a sequence of Markov times defined by* (2) *in* ([**17**], pp. 87).

PROOF. By (**M.3**), strong Markov property and Definition 4.2 in [**17**], we have, for any k,

$$(2.1)\qquad E_x\Big(\sum_{n=1}^{\infty} e^{-\alpha\rho_n(k)}\chi_D f(x_{\rho_n(k)-})g G^0_\beta h(x_{\rho_n(k)})\Big)$$

$$= E_x\Big(\sum_{n=1}^{\infty} e^{-\alpha\rho_n(k)}\chi_D f(x_{\rho_n(k)-})\chi_D g(x_{\rho_n(k)})$$

$$\cdot \int_{\rho_n(k)}^{\sigma_{n+1}(k)} e^{-\beta(t-\rho_n(k))}h(x_t)dt\Big)$$

$$= E_x\Big(\sum_{s\in T}\chi(s\in T_k)e^{-\alpha\rho(k,s)}\chi_D f(x_{\rho(k,s)-})g(x_{\rho(k,s)})$$

$$\cdot \int_{\rho(k,s)}^{\sigma(k,s)} e^{-\beta(t-\rho(k,s))}h(x_t)dt\Big)\,.$$

Put

$$\varphi_k(s, w) = \chi(s\in T_k)e^{-\alpha\rho(k,s)}\chi_D f(x_{\rho(k,s)-})g(x_{\rho(k,s)})$$

$$\cdot \int_{\rho(k,s)}^{\sigma(k,s)} e^{-\beta(t-\rho(k,s))}h(x_t)dt$$

and $\varphi(s, w) = \|f\|\cdot\|g\|\cdot\|h\|\int_{\rho(k,s)}^{\sigma(k,s)} e^{-(\alpha\wedge\beta)t}dt$. Then, we have

$$(2.2)\qquad |\varphi_k(s, w)| \leqq \varphi(s, w)$$

and

$$\sum_{s\in T}\varphi(s, w) \leqq \frac{\|f\|\cdot\|g\|\cdot\|h\|}{\alpha\wedge\beta}\,.$$

Since for any $s\in T_d$, $x_{\rho(k,s)-}\in V$ for sufficiently large k by [4.13], [4.15] and [4.16] in [**17**], we have

$$(2.3)\qquad \lim_{k\to\infty}\varphi_k(s, w) = 0 \qquad \text{for any } s\in T_d(w)\,.$$

On the other hand, since for any $s\in T_c$, $T_k\to T$, $\rho(k, s)\to\tau(s)$, $x_{\rho(k,s)-}\to x_{\tau(s-)-}$ and $x_{\rho(k,s)}\to x_{\tau(s-)} = x_{\tau(s-)-}$ by [4.7], [4.8] and [4.9] in [**17**], we have, for any $s\in T_c(w)$,

$$(2.4)\qquad \lim_{k\to\infty}\varphi_k(s, w) = e^{-\alpha\tau(s-)}f\cdot g(x_{\tau(s-)-})\int_{\tau(s-)}^{\tau(s)} e^{-\beta(t-\tau(s-))}h(x_t)dt\,.$$

Therefore, (2.2), (2.3) and (2.4) complete the proof of Lemma 2.3 by applying the dominated convergence theorem in (2.1).

LEMMA 2.4. *For any $f, g, h \in B(S)$, $\alpha > 0$, $\beta > 0$ and $x \in S$,*

$$\lim_{k\to\infty} E_x\Big(\sum_{n=1}^{\infty} e^{-\alpha\rho_n(k)}\chi_V f(x_{\rho_n(k)-}) g\cdot G_\beta^0 h(x_{\rho_n(k)})\Big) = \mathcal{E}_{\alpha,\beta}^d(f, g, h)(x),$$

where $\rho_n(k)$ is the same as in Lemma 2.3.

PROOF. By (**M**.3), strong Markov property, Definitions 4.2, 4.4 in [**17**] and [4.16] in [**17**], we have, for any k,

$$(2.5) \qquad E_x\Big(\sum_{n=1}^{\infty} e^{-\alpha\rho_n(k)}\chi_V f(x_{\rho_n(k)-}) g\cdot G_\beta^0 h(x_{\rho_n(k)})\Big)$$
$$= E_x\Big(\sum_{n=1}^{\infty} e^{-\alpha\rho_n(k)}\chi_V f(x_{\rho_n(k)-})\chi_D g(x_{\rho_n(k)}) \cdot \int_{\rho_n(k)}^{\sigma_{n+1}(k)} e^{-\beta(t-\rho_n(k))} h(x_t)dt\Big)$$
$$= E_x\Big(\sum_{s\in T_{d,k}} e^{-\alpha\rho(k,s)}\chi_V f(x_{\rho(k,s)-})\chi_D g(x_{\rho(k,s)}) \cdot \int_{\rho(k,s)}^{\sigma(k,s)} e^{-\beta(t-\rho(k,s))} h(x_t)dt\Big)$$
$$= E_x\Big(\sum_{s\in T}\chi(s\in T_{d,k}) e^{-\alpha\tau(s-)}\chi_V f(x_{\tau(s-)-}) g(x_{\tau(s-)}) \cdot \int_{\tau(s-)}^{\tau(s)} e^{-\beta(t-\tau(s-))} h(x_t)dt\Big).$$

Since each term in the last member is dominated by $\|f\|\cdot\|g\|\cdot\|h\|\int_{\tau(s-)}^{\tau(s)} e^{-(\alpha\wedge\beta)t}dt$, which is independent of k, and

$$\sum_{s\in T}\|f\|\cdot\|g\|\cdot\|h\|\int_{\tau(s-)}^{\tau(s)} e^{-(\alpha\wedge\beta)t}dt \leqq \frac{\|f\|\cdot\|g\|\cdot\|h\|}{\alpha\wedge\beta},$$

and $T_{d,k}\to T_d$ (as $k\to\infty$) ([4.15] of [**17**]), Lemma 2.4 holds by applying the dominated convergence theorem in (2.5).

LEMMA 2.5. *For any $f\in C(S)$, $g\in B^+(S)$, $\alpha>0$, $\beta>0$ and $x\in S$,*

$$\lim_{k\to\infty} E_x\Big(\sum_{n=1}^{\infty} e^{-\alpha\overline{\rho_n(k)}} f(x_{\overline{\rho_n(k)}-}) G_\beta^0 g(x_{\overline{\rho_n(k)}})\Big)$$
$$= E_x\Big(\int_0^{\infty} e^{-\alpha t} f(x_t) d(\widetilde{\chi_D g\cdot dt})_\beta\Big),$$

where $\{\overline{\rho_n(k)}\}$ is a sequence of Markov times defined by (21) *in* [**17**], pp. 87.

PROOF. By (**M**.3), $G_\beta^0 g(x) = E_x\Big(\int_0^{\sigma_V} e^{-\beta t} d(\chi_D g\cdot dt)\Big)$. Noting that $\chi_D g\cdot dt\in\mathfrak{C}$ (in the sense of [**16**]) and $\chi_V\cdot(\chi_D g\cdot dt)\approx 0$, Lemma 2.5 follows from the Theorem 4.4 in [**16**].

The next Lemma 2.6 can be found in [**17**], pp. 86, [5.4].

LEMMA 2.6. *For any $f, g \in B^+(S)$, $\alpha > 0$ and $x \in S$,*

$$\lim_{k\to\infty} E_x\Big(\sum_{n=1}^{\infty} e^{-\alpha\rho_n(k)}\chi_V f(x_{\rho_n(k)-})\chi_D g(x_{\rho_n(k)})\Big)$$

$$= E_x\Big(\int_0^{\infty} e^{-\alpha t}\chi_V f P_D g(x_t) dL(t)\Big)\,,$$

where $\{\rho_n(k)\}$ is the same as in Lemma 2.3.

LEMMA 2.7. *For any $f \in B(S)$, $g \in B^+(S)$, $\alpha > 0$, $\beta > 0$ and $x \in S$,*

$$E_x\Big(\int_0^{\infty} e^{-\alpha t} f(x_t) d(\widetilde{\chi_D g \cdot dt})_\beta\Big)$$

$$= \mathcal{E}^c_{\alpha,\beta}(f, g)(x) + E_x\Big(\int_0^{\infty} e^{-\alpha t}\chi_V f P_D(G^0_\beta g)(x_t) dL(t)\Big).$$

PROOF. By Lemma 2.2 (9), we can assume that $f \in C(S)$. By Lemma 2.5 and (4) in [17], pp. 76,

$$E_x\Big(\int_0^{\infty} e^{-\alpha t} f(x_t) d(\widetilde{\chi_D g \cdot dt})_\beta\Big)$$

$$= \lim_{k\to\infty} E_x\Big(\sum_{n=1}^{\infty} e^{-\alpha\overline{\rho_n(k)}} f(x_{\overline{\rho_n(k)}-}) G^0_\beta g(x_{\overline{\rho_n(k)}})\Big)$$

$$= \lim_{k\to\infty} \Big\{E_x\Big(\sum_{n=1}^{\infty} e^{-\alpha\overline{\rho_n(k)}}\chi_D f(x_{\overline{\rho_n(k)}-}) G^0_\beta g(x_{\overline{\rho_n(k)}})\Big)$$

$$+ E_x\Big(\sum_{n=1}^{\infty} e^{-\alpha\overline{\rho_n(k)}}\chi_V f(x_{\overline{\rho_n(k)}-})\chi_D G^0_\beta g(x_{\overline{\rho_n(k)}})\Big)\Big\}$$

Therefore, Lemma 2.7 follows from Lemma 2.3 and Lemma 2.6.

LEMMA 2.8. *For any $f, g, h \in B(S)$, $\alpha > 0$, $\beta > 0$ and $x \in S$,*

$$\mathcal{E}^d_{\alpha,\beta}(f, g, h)(x) = E_x\Big(\int_0^{\infty} e^{-\alpha t}\chi_V f P_D(g G^0_\beta h)(x_t) dL(t)\Big)$$

$$= H_\alpha K^\alpha\Big\{f\boldsymbol{Q}\Big(g \cdot \frac{G^0_\beta h}{G^0_\gamma 1}\Big)\Big\}(x)\,,$$

where K^α is the operator on $B(V)$ defined by

$$K^\alpha \varphi(\xi) = E_\xi\Big(\int_0^{\infty} e^{-\alpha t}\varphi(x_t) d\Phi(t)\Big)\,.$$

PROOF. The first equality follows from Lemma 2.4 and Lemma 2.6. The second equality follows from Lemma 2.1 (3), the definition of K^α and [5.8] of [17].

LEMMA 2.9. *For any $f \in B(S)$, $\alpha > 0$ and $x \in S$,*

$$G_\alpha(f \cdot \chi_V)(x) = H_\alpha K^\alpha(l \cdot f)(x)\,.$$

PROOF. By Lemma 2.1(1) and the definition of K^{α},

$$G_{\alpha}(f\cdot\chi_V)(x)=E_x\Big(\int_0^{\infty}e^{-\alpha t}f\chi_V(x_t)dt\Big)$$

$$=E_x\Big(\int_0^{\infty}e^{-\alpha t}f(x_t)d(\chi_V\cdot dt)\Big)$$

$$=E_x\Big(\int_0^{\infty}e^{-\alpha t}f\cdot l(x_t)d\Phi(t)\Big)$$

$$=H_{\alpha}K^{\alpha}(l\cdot f)(x)\,.$$

LEMMA 2.10. *For any* $f\in B(S)$, $g\in B^{+}(S)$, $\alpha>0$, $\beta>0$ *and* $x\in S$,

$$E_x\Big(\int_0^{\infty}e^{-\alpha t}f(x_t)d\widetilde{(\chi_D g\cdot dt)}_{\beta}\Big)=\mathcal{E}_{\alpha,\beta}(f,g)(x)\,.$$

This Lemma follows immediately from Lemma 2.2, Lemma 2.7 and Lemma 2.8.

LEMMA 2.11. *For any* f, $g\in C(S)$, $h\in B(S)$, $\alpha>0$, $\beta>0$ *and* $x\in S$,

$$\lim_{k\to\infty}E_x\Big(\sum_{n=1}^{\infty}e^{-\alpha\rho_n(k)}f(x_{\rho_n(k)-})g\cdot G_{\beta}^{0}h(x_{\rho_n(k)})\Big)=\mathcal{E}_{\alpha,\beta}(f,g,h)(x)\,,$$

where $\{\rho_n(k)\}$ *is the same as in Lemma* 2.3.

This is an immediate consequence of Lemma 2.2, Lemma 2.3 and Lemma 2.4.

LEMMA 2.12. *For any* f, g, $h\in B(S)$, $\alpha>0$, $\beta>0$, $\delta>0$ *and* $x\in S$,

$$\mathcal{E}_{\alpha,\beta}^{c}(f,g,h)(x)-\mathcal{E}_{\alpha,\delta}^{c}(f,g,h)(x)+(\beta-\delta)\mathcal{E}_{\alpha,\beta}^{c}(f,g,G_{\delta}^{0}h)(x)=0\,.$$

The same kind of equations hold also for $\mathcal{E}_{\alpha,\beta}^{d}$ *and* $\mathcal{E}_{\alpha,\beta}$.

PROOF. We can assume that f, $g\in C(S)$ (Definition 2.1 and Lemma 2.2). Then, from Lemma 2.3 and (2) in [17], p. 76, the assertion about $\mathcal{E}_{\alpha,\beta}^{c}$ follows. As for $\mathcal{E}_{\alpha,\beta}^{d}$, Lemma 2.4 assures. Therefore, from Lemma 2.2, the assertion about $\mathcal{E}_{\alpha,\beta}$ follows.

LEMMA 2.13. *For any* $f\in B(S)$, $\alpha>0$ *and* $x\in S$, $\xi\in V$,

(1) $\mathcal{E}_{\alpha,\gamma}^{c}(f,1)(x)=H_{\alpha}K^{\alpha}(mf)(x)\,,$

(2) $K^{\alpha}f(\xi)=\mathcal{E}_{\alpha,\gamma}(f,1)(\xi)+G_{\alpha}(f\cdot\chi_V)(\xi).$

PROOF. By Lemma 2.7, the definition of Φ and Lemma 2.1(3),

$$\mathcal{E}_{\alpha,\gamma}^{c}(f,1)(x)=E_x\Big(\int_0^{\infty}e^{-\alpha t}f(x_t)d\widetilde{(\chi_D\cdot dt)}_{\gamma}\Big)-E_x\Big(\int_0^{\infty}e^{-\alpha t}\chi_V fP_D(G_{\gamma}^{0}1)(x_t)dL\Big)$$

$$=E_x\Big(\int_0^{\infty}e^{-\alpha t}f(x_t)d\Phi\Big)-E_x\Big(\int_0^{\infty}e^{-\alpha t}f(x_t)d\widetilde{(\chi_V\cdot dt)}_{\gamma}\Big)$$

$$-E_x\Big(\int_0^{\infty}e^{-\alpha t}f\cdot Q1(x_t)d\Phi\Big)\,.$$

Noting that $\widetilde{(\chi_V \cdot dt)}_r \approx \chi_V \cdot dt$ ([**16**]), by Lemma 2.1 (1), (4),

$$\begin{aligned}\mathcal{E}^c_{\alpha,r}(f,1)(x) &= E_x\Big(\int_0^\infty e^{-\alpha t} f(x_t) d\Phi\Big) - E_x\Big(\int_0^\infty e^{-\alpha t} f\cdot l(x_t) d\Phi\Big) \\ &\qquad - E_x\Big(\int_0^\infty e^{-\alpha t} f\cdot \boldsymbol{Q}1(x_t) d\Phi\Big) \\ &= E_x\Big(\int_0^\infty e^{-\alpha t} fm(x_t) d\Phi\Big) \\ &= H_\alpha K^\alpha(mf)(x).\end{aligned}$$

The second assertion follows from Lemma 2.1 (4), Lemma 2.8, Lemma 2.9 and the above assertion just proved.

§ 3. The excursion at the boundary V and the operator $\boldsymbol{P}$.

In this section, we have to cross the first pass in this paper. This pass is an inevitable one that results from removing the conditions ($\boldsymbol{M}^{\min}$.4), ($\boldsymbol{M}^{\min}$.5) and ($\boldsymbol{M}^{\min}$.6). Though $\tilde{\boldsymbol{M}}$, l, m and $\boldsymbol{Q}$ are already defined, $\boldsymbol{M}$ can not be necessarily determined by them without M. Motoo's additional conditions ($\boldsymbol{M}^{\min}$.4), ($\boldsymbol{M}^{\min}$.5) and ($\boldsymbol{M}^{\min}$.6). Considering the probabilistic meaning of m and $\boldsymbol{Q}$, the first work which we have to do is to find a refinement of m and this is done by introducing the operator $\boldsymbol{P}$ from $\bar{\mathcal{M}}$ into $L^\infty(V, \boldsymbol{B}(V), \nu)$, where $\mathcal{M} = \Big\{\dfrac{G^0_\alpha f}{G^0_r 1}\,;\ \alpha > 0,\ f \in B(D)\Big\}$. This operator has the same probabilistic meaning as m; in fact $\boldsymbol{P}1 = m$. Our next purpose in this section is to characterize this operator $\boldsymbol{P}$. The important way in introducing the operator $\boldsymbol{P}$ is to use a theorem for additive functionals ([**12**], Theorem 1.7) analogous to the Radon-Nykodym theorem for measures. K. Sato [**19**] used the latter in introducing his $\hat{H}_\alpha$ and proved that $\hat{H}_\alpha$ depends only upon the minimal process, but our $\boldsymbol{P}$ is not necessarily determined only by the minimal process $\boldsymbol{M}^{\min}$ as will be seen in § 5.

LEMMA 3.1. *For each $\beta > 0$, there exists a bounded linear positive operator $\hat{\hat{H}}_\beta$ from $B(S)$ into $L^\infty(V, \boldsymbol{B}(V), \nu)$ such that for any $f, g \in B(S)$, $\alpha > 0$, $\beta > 0$ and $x \in S$,*

$$\mathcal{E}^c_{\alpha,\beta}(f,g)(x) = H_\alpha K^\alpha(f\cdot \hat{\hat{H}}_\beta g)(x).$$

PROOF. Fix any $g \in B^+(S)$. Noting that $\widetilde{(\chi_D g\cdot dt)}_r \ll \|g\|\widetilde{(dt)}_r = \|g\|\Phi$, by Theorem 1.7 in [**12**], there exists an $\tilde{H}_r g \in L^\infty(V, \boldsymbol{B}(V), \nu)$ such that

(3.1) $$0 \leqq \tilde{H}_r g \leqq \|g\| \quad \text{and}$$

(3.2) $$\widetilde{(\chi_D g\cdot dt)}_r \approx \tilde{H}_r g\cdot \Phi.$$

$$\text{(3.3)}\qquad \text{Put}\quad \hat{\hat{H}}_\gamma g=\tilde{H}_\gamma g-\boldsymbol{Q}\Big(\frac{G_\gamma^0 g}{G_\gamma^0 1}\Big).$$

Then, by Lemma 2.8, Lemma 2.10 and (3.2), we have, for any $f\in B(S)$, $\alpha>0$ and $x\in S$,

$$\begin{aligned}\mathcal{E}^c_{\alpha,\gamma}(f,g)(x)&=\mathcal{E}_{\alpha,\gamma}(f,g)(x)-\mathcal{E}^d_{\alpha,\gamma}(f,g)(x)\\&=E_x\Big(\int_0^\infty e^{-\alpha t}f(x_t)\tilde{H}_\gamma g(x_t)d\Phi\Big)\\&\quad-E_x\Big(\int_0^\infty e^{-\alpha t}f(x_t)\boldsymbol{Q}\Big(\frac{G_\gamma^0 g}{G_\gamma^0 1}\Big)(x_t)\,d\Phi\Big)\\&=E_x\Big(\int_0^\infty e^{-\alpha t}f(x_t)\hat{\hat{H}}_\gamma g(x_t)d\Phi\Big)\\&=H_\alpha K^\alpha(f\hat{\hat{H}}_\gamma g)(x).\end{aligned}$$

Therefore, for any $g\in B(S)$, setting

$$\text{(3.4)}\qquad \hat{H}_\gamma g=\hat{\hat{H}}_\gamma g^+-\hat{\hat{H}}_\gamma g^-,$$

we have

$$\text{(3.5)}\qquad \mathcal{E}^c_{\alpha,\gamma}(f,g)(x)=H_\alpha K^\alpha(f\hat{H}_\gamma g)(x)\quad\text{for any } f\in B(S),\ \alpha>0 \text{ and } x\in S.$$

For any $g\in B(S)$ and $\beta>0$, put

$$\text{(3.6)}\qquad \hat{H}_\beta g=\hat{H}_\gamma(g+(\gamma-\beta)G_\beta^0 g).$$

Then, by Lemma 2.12 and (3.5), we have, for any $f\in B(S)$ and $x\in S$,

$$\begin{aligned}\text{(3.7)}\qquad \mathcal{E}^c_{\alpha,\beta}(f,g)(x)&=\mathcal{E}^c_{\alpha,\gamma}(f,g+(\gamma-\beta)G_\beta^0 g)(x)\\&=H_\alpha K^\alpha(f\cdot\hat{H}_\gamma(g+(\gamma-\beta)G_\beta^0 g))(x)\\&=H_\alpha K^\alpha(f\cdot\hat{H}_\beta g)(x).\end{aligned}$$

From Lemma 2.1 (4), (3.1), (3.3), (3.4) and (3.6), the boundedness of $\hat{H}_\beta$ as an operator from $B(S)$ into $L^\infty(V,\boldsymbol{B}(V),\nu)$ follows. Definition 2.1 and (3.7) show that $\hat{H}_\beta$ is a linear positive operator. This completes the proof of Lemma 3.1.

LEMMA 3.2. *For any $f, g\in B(S)$ and $\alpha>0$, $\beta>0$, if $\frac{G_\alpha^0 f}{G_\gamma^0 1}(x)=\frac{G_\beta^0 g}{G_\gamma^0 1}(x)$ holds for any $x\in D$, then*

$$\hat{H}_\alpha f=\hat{H}_\beta g.$$

PROOF. Fix any $h\in C(S)$. By Lemma 2.3, under the hypothesis of Lemma 3.2, $\mathcal{E}^c_{\gamma,\alpha}(h,f)(x)=\mathcal{E}^c_{\gamma,\beta}(h,g)(x)$ holds for any $x\in S$. Therefore, by Lemma 3.1,

$$H_\gamma K^\gamma(h\hat{H}_\alpha f)(x)=H_\gamma K^\gamma(h\hat{H}_\beta g)(x)$$

holds for any $x\in S$. Lemma 3.2 follows from this relation.

Put $\mathscr{M}=\left\{\varphi \in B(D)\,;\ \text{there exist } \alpha>0 \text{ and } f\in B(D) \text{ such that } \varphi=\frac{G_\alpha^0 f}{G_\gamma^0 1}\right\}$. Then, by the resolvent equation for $\{G_\alpha^0\,;\ \alpha>0\}$, $\mathscr{M}$ is a linear subspace of $B(D)$ and contains constant functions.

THEOREM 3.1. *There exists a bounded linear positive operator* $\boldsymbol{P}$ *from* $\bar{\mathscr{M}}$ *into* $L^\infty(V, \boldsymbol{B}(V), \nu)$ *such that*

(1) $\quad \boldsymbol{P}\left(\frac{G_\alpha^0 f}{G_\gamma^0 1}\right)=\hat{\hat{H}}_\alpha f \quad$ *for* $f\in B(S)$ *and* $\alpha>0$,

(2) $\quad \boldsymbol{P}1=m$.

PROOF. Set $\boldsymbol{P}\left(\frac{G_\alpha^0 f}{G_\gamma^0 1}\right)=\hat{\hat{H}}_\alpha f$, then $\boldsymbol{P}$ is well-defined by Lemma 3.2. The linearity of $\boldsymbol{P}$ in $\mathscr{M}$ follows from the linearity of $\hat{\hat{H}}_\alpha$ and the well-definedness of $\boldsymbol{P}$. The positivity of $\boldsymbol{P}$ in $\mathscr{M}$ can be proved as follows; for any $h\in C^+(S)$, by Lemma 2.3 if $\frac{G_\beta^0 f}{G_\gamma^0 1}\geqq 0$, then $\mathscr{E}_{\alpha,\beta}^c(h, f)(x)\geqq 0$ for any $x\in S$. Therefore, by Lemma 3.1 and the definition of $\boldsymbol{P}$, $H_\alpha K^\alpha\left(h\boldsymbol{P}\left(\frac{G_\beta^0 f}{G_\gamma^0 1}\right)\right)(x)\geqq 0$ for any $x\in S$. Noting that this inequality holds also for any $h\in B^+(S)$, we have $\boldsymbol{P}\left(\frac{G_\beta^0 f}{G_\gamma^0 1}\right)\geqq 0$ if $G_\beta^0 f\geqq 0$. The boundedness of $\boldsymbol{P}$ in $\mathscr{M}$ can be proved as follows; since we have seen that $\boldsymbol{P}$ is linear and positive, it is sufficient to prove $\boldsymbol{P}1=m$, while this equality $\boldsymbol{P}1=m$ follows from Lemma 2.13 and the definition of $\boldsymbol{P}$. This completes the proof of Theorem 3.1.

COROLLARY 3.1. *For any* $f, g, h\in B(S)$, $\alpha>0$, $\beta>0$ *and* $x\in S$,

$$\mathscr{E}_{\alpha,\beta}^c(f, g, h)(x)=H_\alpha K^\alpha\left\{fg\boldsymbol{P}\left(\frac{G_\beta^0 h}{G_\gamma^0 1}\right)\right\}(x).$$

Comparing this corollary with Lemma 2.8, we see a difference between $\boldsymbol{P}$ and $\boldsymbol{Q}$. But, this corollary does not exhibit completely the characterization of $\boldsymbol{P}$.

LEMMA 3.3. *For any* $f\in B^+(S)$ *and* $\alpha>0$,

$$\widetilde{(\chi_D f\cdot dt)}_\alpha\approx(\boldsymbol{P}+\boldsymbol{Q})\left(\frac{G_\alpha^0 f}{G_\gamma^0 1}\right)\cdot\Phi.$$

This lemma follows from Lemma 2.8, Lemma 2.10 and Corollary 3.1. Now, we shall prove the decomposition theorem for the Green kernel of $\boldsymbol{M}$.

THEOREM 3.2. (*Feller-Ueno decomposition theorem*)
For any $f\in B(S)$, $\alpha>0$ *and* $x\in S$,

$$G_\alpha f(x)=G_\alpha^0 f(x)+H_\alpha K^\alpha\left\{l\cdot f+(\boldsymbol{P}+\boldsymbol{Q})\left(\frac{G_\alpha^0 f}{G_\gamma^0 1}\right)\right\}(x).$$

PROOF. We can assume that $f\in B^+(S)$. By ($\boldsymbol{M}$.3),

$$G_\alpha f(x)-G_\alpha^0 f(x)=E_x\Big(\int_{\sigma_V}^\infty e^{-\alpha t}fdt\Big)$$

$$=E_x\Big(\int_{\sigma_V}^\infty e^{-\alpha t}\chi_V fdt\Big)+E_x\Big(\int_{\sigma_V}^\infty e^{-\alpha t}\chi_D fdt\Big)$$

$$=E_x\Big(\int_0^\infty e^{-\alpha t}\chi_V fdt\Big)+E_x\Big(\int_0^\infty e^{-\alpha t}d(\widetilde{\chi_D f\cdot dt})_\alpha\Big).$$

Therefore, Theorem 3.2 follows from Lemma 2.9 and Lemma 3.3.

LEMMA 3.4. *Fix* $\alpha>0$. *Let* $(f_m)_{m=0}^\infty$ *be a sequence in* $B(D)$ *such that*

(1) $\|f_m\|\leqq M$ $(m=0,1,2,3,\cdots)$,

(2) $f_m\to f_0$ *as* $m\to\infty$ *(pointwise convergence in D),*

(3) *there exists a* $\lim_{m\to\infty}\boldsymbol{P}\Big(\frac{G_\alpha^0 f_m}{G_\gamma^0 1}\Big)(\xi)$ (ν-*a. e.* $\xi\in V$).

Then, $\lim_{m\to\infty}\boldsymbol{P}\Big(\frac{G_\alpha^0 f_m}{G_\gamma^0 1}\Big)(\xi)=\boldsymbol{P}\Big(\frac{G_\alpha^0 f_0}{G_\gamma^0 1}\Big)(\xi)$ (ν-*a. e.* $\xi\in V$).

PROOF. By Corollary 3.1, for any $g\in B(S)$,

$$H_\alpha K^\alpha\Big(g\boldsymbol{P}\Big(\frac{G_\alpha^0 f_m}{G_\gamma^0 1}\Big)\Big)(x) \tag{3.8}$$

$$=\mathcal{E}_{\alpha,\alpha}^c(g,f_m)(x)$$

$$=E_x\Big(\sum_{s\in T_c}e^{-\alpha\tau(s-)}g(x_{\tau(s-)-})\cdot\int_{\tau(s-)}^{\tau(s)}e^{-\alpha(t-\tau(s-))}f_m(x_t)\,dt\Big)$$

$$=E_x\Big(\sum_{s\in T_c}g(x_{\tau(s-)})\cdot\int_{\tau(s-)}^{\tau(s)}e^{-\alpha t}f_m(x_t)\,dt\Big).$$

Put $\varphi_{m,s}(w)=g(x_{\tau(s-)-}^{(w)})\cdot\int_{\tau(s-)}^{\tau(s)}e^{-\alpha t}f_m(x_t(w))dt$ and $\varphi_s(w)=M\|g\|\int_{\tau(s-)}^{\tau(s)}e^{-\alpha t}dt$. Then,

$$|\varphi_{m,s}(w)|\leqq\varphi_s(w),\qquad \sum_{s\in T_c}\varphi_s(w)\leqq\frac{M\|g\|}{\alpha}, \tag{3.9}$$

and

$$\varphi_{m,s}(w)\xrightarrow[m\to\infty]{} g(x_{\tau(s-)-})\int_{\tau(s-)}^{\tau(s)}e^{-\alpha t}f_0(x_t)dt$$

for any w and $s\in T_c(w)$, since $x_t(w)\in D$ for $t\in(\tau(s-),\tau(s))$ by [**17**], [4.3]. Therefore, by the dominated convergence theorem, (3.8) and (3.9), we have

$$H_\alpha K^\alpha\Big(g\boldsymbol{P}\Big(\frac{G_\alpha^0 f_m}{G_\gamma^0 1}\Big)\Big)(x)\to E_x\Big(\sum_{s\in T_c}g(x_{\tau(s-)-})\int_{\tau(s-)}^{\tau(s)}e^{-\alpha t}f_0(x_t)\,dt\Big)\text{ as } m\to\infty. \tag{3.10}$$

This limit function equals to $\mathcal{E}_{\alpha,\alpha}^c(g,f_0)(x)=H_\alpha K^\alpha\Big(g\boldsymbol{P}\Big(\frac{G_\alpha^0 f_0}{G_\gamma^0 1}\Big)\Big)(x)$ by Definition 2.1 and Corollary 3.1.

On the other hand, noting that $\Big|\boldsymbol{P}\Big(\frac{G_\alpha^0 f_m}{G_\gamma^0 1}\Big)(\xi)\Big|\leqq M\Big(1+\frac{|\gamma-\alpha|}{\alpha}\Big)$ by Theorem 3.1, by the hypothesis of Lemma 3.4 and the bounded convergence theorem, (3.11) $$H_\alpha K^\alpha\Big(g\boldsymbol{P}\Big(\frac{G_\alpha^0 f_m}{G_\gamma^0 1}\Big)\Big)(x) \longrightarrow H_\alpha K^\alpha\Big(g\lim_{m\to\infty}\boldsymbol{P}\Big(\frac{G_\alpha^0 f_m}{G_\gamma^0 1}\Big)\Big)(x) \text{ as } m\to\infty.$$

Therefore, by (3.10) and (3.11), we have

$$H_\alpha K^\alpha\Big(g\boldsymbol{P}\Big(\frac{G_\alpha^0 f_0}{G_\gamma^0 1}\Big)\Big)(x) = H_\alpha K^\alpha\Big(g\lim_{m\to\infty}\boldsymbol{P}\Big(\frac{G_\alpha^0 f_m}{G_\gamma^0 1}\Big)\Big)(x).$$

Lemma 3.4 follows from this relation.

LEMMA 3.5. *Fix* $f\in C(S)$, $\varphi\in\bar{\mathcal{M}}$, $\alpha>0$ *and* $\varepsilon>0$. *If there exists an open neighbourhood* U *of* ξ *in* S *for any* $\xi\in\{\xi\in V; f(\xi)\neq 0\}$ *such that for any* $x\in D\cap U$, $|\varphi(x)|\leqq\varepsilon$, *then*

$$|H_\alpha K^\alpha(fg\boldsymbol{P}\varphi)(x)|\leqq\varepsilon H_\alpha K^\alpha(|f|\cdot|g|\cdot\boldsymbol{P}1)(x) \quad \textit{for any } x\in S \textit{ and } g\in C(S).$$

PROOF. It is sufficient to prove the result for $\varphi=\dfrac{G_\beta^0 h}{G_\gamma^0 1}$ for $\beta>0$ and $h\in B(S)$. Then, by Lemma 2.3 and Corollary 3.1,

(3.12) $$H_\alpha K^\alpha\Big(fg\boldsymbol{P}\Big(\frac{G_\beta^0 h}{G_\gamma^0 1}\Big)\Big)(x)$$
$$=\lim_{k\to\infty}E_x\Big(\sum_{n=1}^{\infty}e^{-\alpha\rho_n(k)}\chi_D f(x_{\rho_n(k)-})g(x_{\rho_n(k)})G_\beta^0 h(x_{\rho_n(k)})\Big).$$

By ($\boldsymbol{M}$.3) and strong Markov property of $\boldsymbol{M}$,

(3.13) $$E_x\Big(\sum_{n=1}^{\infty}e^{-\alpha\rho_n(k)}\chi_D f(x_{\rho_n(k)-})g\cdot G_\beta^0 h(x_{\rho_n(k)})\Big)$$
$$=E_x\Big(\sum_{n=1}^{\infty}e^{-\alpha\rho_n(k)}\chi_D f(x_{\rho_n(k)-})\chi_D g\varphi(x_{\rho_n(k)})\cdot G_\gamma^0 1(x_{\rho_n(k)})\Big)$$
$$=E_x\Big(\sum_{n=1}^{\infty}e^{-\alpha\rho_n(k)}\chi_D f(x_{\rho_n(k)-})\chi_D g\varphi(x_{\rho_n(k)})E_{x_{\rho_n(k)}}\Big(\int_0^{\sigma_V}e^{-\gamma t}1(x_t)\,dt\Big)\Big)$$
$$=E_x\Big(\sum_{n=1}^{\infty}e^{-\alpha\rho_n(k)}\chi_D f(x_{\rho_n(k)-})\chi_D g\varphi(x_{\rho_n(k)})\cdot\int_{\rho_n(k)}^{\sigma_{n+1}(k)}e^{-\gamma(t-\rho_n(k))}1(x_t)\,dt\Big)$$
$$=E_x\Big(\sum_{s\in T}\chi(s\in T_k)e^{-\alpha\rho(k,s)}\chi_D f(x_{\rho(k,s)-})\chi_D g\varphi(x_{\rho(k,s)})\cdot\int_{\rho(k,s)}^{\tau(s)}e^{-\gamma(t-\rho(k,s))}1(x_t)\,dt\Big)$$

by the Definition 4.2 of [**17**].

Put

$$I_k=\sum_{s\in T_c}\chi(s\in T_k)e^{-\alpha\rho(k,s)}\chi_D f(x_{\rho(k,s)-})\chi_D g\varphi(x_{\rho(k,s)})\cdot\int_{\rho(k,s)}^{\tau(s)}e^{-\gamma(t-\rho(k,s))}1(x_t)dt$$

and

$$II_k=\sum_{s\in T_d}\chi(s\in T_k)e^{-\alpha\rho(k,s)}\chi_D f(x_{\rho(k,s)-})\chi_D g\varphi(x_{\rho(k,s)})$$
$$\cdot\int_{\rho(k,s)}^{\tau(s)} e^{-\gamma(t-\rho(k,s))}1(x_t)dt\,.$$

Then, each term in I_k is dominated by $\|f\|\cdot\|g\|\cdot\|\varphi\|\int_{\tau(s-)}^{\tau(s)} e^{-(\alpha\wedge\gamma)t}dt$ which is independent of k, and

$$\sum_{s\in T_c}\|f\|\cdot\|g\|\cdot\|\varphi\|\int_{\tau(s-)}^{\tau(s)} e^{-(\alpha\wedge\gamma)t}dt\leqq\frac{\|f\|\cdot\|g\|\cdot\|\varphi\|}{\alpha\wedge\gamma}.$$

Therefore, by Fatou's lemma,

$$\overline{\lim_{k\to\infty}}\,|I_k|\leqq\sum_{s\in T_c}\Big(\overline{\lim_{k\to\infty}}\,\chi(s\in T_k)e^{-\alpha\rho(k,s)}\chi_D f(x_{\rho(k,s)-})$$
$$\cdot\chi_D g\cdot\varphi(x_{\rho(k,s)})\int_{\rho(k,s)}^{\tau(s)} e^{-\gamma(t-\rho(k,s))}1(x_t)dt\Big)\,. \tag{3.14}$$

Noting that for $s\in T_c(w)$

$$\rho(k,s)\longrightarrow\tau(s-)\,, \tag{3.15}$$

$$x_{\rho(k,s)-}\longrightarrow x_{\tau(s-)-}\,, \tag{3.16}$$

$$x_{\rho(k,s)}\longrightarrow x_{\tau(s-)}=x_{\tau(s-)-} \tag{3.17}$$

and

$$T_k\longrightarrow T \tag{3.18}$$

by [4.6], [4.8] and [4.9] of [**17**], for $s\in T_c(w)$ satisfying $f(x_{\tau(s-)-}(w))\neq 0$, using the hypothesis of Lemma 3.5 to $\xi=x_{\tau(s-)-}(w)$, there exists an open subset $U\ni\xi$ such that $|\varphi(y)|\leqq\varepsilon$ for any $y\in U\cap D$. Therefore, by (3.16), for sufficiently large k, $|\varphi(x_{\rho(k,s)})|\leqq\varepsilon$. Then, by (3.14), (3.15), (3.17), (3.18) and the continuity of f, g,

$$\overline{\lim_{k\to\infty}}\,|I_k|\leqq\sum_{s\in T_c}e^{-\alpha\tau(s-)}|f(x_{\tau(s-)-})|\,|g(x_{\tau(s-)})|\cdot\int_{\tau(s-)}^{\tau(s)} e^{-\gamma(t-\tau(s-))}1(x_t)dt\,. \tag{3.19}$$

By [4.15] of [**17**], the same method as in I_k shows that

$$\overline{\lim_{k\to\infty}}\,|II_k|=0\,. \tag{3.20}$$

Therefore, noting that $|I_k|$, $|II_k|\leqq\frac{\|f\|\cdot\|g\|\cdot\|\varphi\|}{\alpha\wedge\gamma}$, by (3.12), (3.13), (3.19), (3.20), Fatou's lemma and Corollary 3.1, we have

$$|H_\alpha K^\alpha(fg\boldsymbol{P}\varphi)(x)|\leqq\varepsilon E_x\Big(\sum_{s\in T_c}e^{-\alpha\tau(s-)}|f(x_{\tau(s-)-})|\,|g(x_{\tau(s-)})|\int_{\tau(s-)}^{\tau(s)}e^{-\gamma(t-\tau(s-))}1(x_t)dt\Big)$$
$$=\varepsilon\mathcal{E}^c_{\alpha,\gamma}(|f|,\,|g|,\,1)(x)$$
$$=\varepsilon H_\alpha K^\alpha(|f|\cdot|g|\cdot\boldsymbol{P}1)(x)\,.$$

This completes the proof of Lemma 3.5.

Finally, we prove the following theorem which gives the characterization of $\boldsymbol{P}$, that is, the local character near the boundary V.

THEOREM 3.3 (*Local character of* $\boldsymbol{P}$). *Let* $\varphi \in \mathscr{M}$, $\xi_0 \in V$ *and* $\varepsilon > 0$. *If there exists an open neighbourhood* U *of* ξ_0 *in* S *such that* $|\varphi(x)| \leqq \varepsilon$ *for any* $x \in U \cap D$, *then,*

$$|\boldsymbol{P}\varphi(\xi)| \leqq \varepsilon \boldsymbol{P}1(\xi), \qquad \nu\text{-a.e. } \xi \in U \cap V.$$

In particular, $\boldsymbol{P}\varphi = 0$ *for any* $\varphi \in C_\infty(D) \cap \mathscr{M}$.

PROOF. Choose $f_n \in C_0^+(U)$ $(n=1, 2, 3, \cdots)$ such that $f_n \uparrow 1$ in U. Then, for each n, there exists $\tilde{f}_n \in C^+(S)$ such that $\tilde{f}_n = f_n$ in U and $\tilde{f}_n = 0$ in $S-U$. Fix any n. Since $\{\xi \in V ; \tilde{f}_n(\xi) \neq 0\} \subset U$, we have, by Lemma 3.5, for any $g \in C^+(S)$ and any $x \in S$,

$$|H_\gamma K^\gamma(g\tilde{f}_n \boldsymbol{P}\varphi)(x)| \leqq \varepsilon H_\gamma K^\gamma(g f_n \boldsymbol{P}1)(x).$$

Since this inequality holds for any $g \in B^+(S)$, we have

$$|\tilde{f}_n \boldsymbol{P}\varphi(\xi)| \leqq \varepsilon \tilde{f}_n \boldsymbol{P}1(\xi), \quad \nu\text{-a.e. } \xi \in V.$$

Therefore, setting $B_n = \{\xi \in V ; |\tilde{f}_n \boldsymbol{P}\varphi(\xi)| > \varepsilon \tilde{f}_n \boldsymbol{P}1(\xi)\}$, we have $B_n \subset U$ and $\nu(B_n) = 0$. So, putting $B = \bigcup_{n=1}^{\infty} B_n$, we have $B \subset U$ and $\nu(B) = 0$. Fix any $\xi \in U-B$. Then, $|f_n \boldsymbol{P}\varphi(\xi)| \leqq \varepsilon f_n \boldsymbol{P}1(\xi)$ for any n. Therefore, letting n tends to infinity, we have $|\boldsymbol{P}\varphi(\xi)| \leqq \varepsilon \boldsymbol{P}1(\xi)$. This completes the proof of Theorem 3.3.

COROLLARY 3.2. *Let* φ *be any element of* $\mathscr{M}$. *If for any* $\varepsilon > 0$ *there exists an open neighbourhood* U *of* V *in* S *such that* $|\varphi(x)| \leqq \varepsilon$ *for any* $x \in U \cap D$, *then* $\boldsymbol{P}\varphi = 0$.

COROLLARY 3.3. *Let* $\varphi, \psi \in \mathscr{M}$ *and* $\xi_0 \in V$. *If there exists an open neighbourhood* U *of* ξ_0 *in* S *such that* $\varphi(x) = \psi(x)$ *for any* $x \in U \cap D$, *then* $\boldsymbol{P}\varphi(\xi) = \boldsymbol{P}\psi(\xi)$, ν-*a.e.* $\xi \in U \cap V$.

§ 4. The boundary system and $\mathscr{D}(\boldsymbol{P})$ $(= \mathscr{M})$.

We have introduced $l \in B(V)^+$ and a bounded kernel $\boldsymbol{Q}$ on $V \times D$ in § 2 (Lemma 2.1), and a bounded linear positive operator $\boldsymbol{P}$ from $\mathscr{M}$ into $L^\infty(V, \boldsymbol{B}(V), \nu)$ in § 3 (Theorem 3.1). In this section, we shall see that the process $\boldsymbol{M}$ is uniquely determined by the U-process $\tilde{\boldsymbol{M}}$ and l, $\boldsymbol{P}$ and $\boldsymbol{Q}$ (Theorem 4.1). This fact implies that we have solved the problem to characterize the process $\boldsymbol{M}$ by its U-process (on the boundary) and certain auxiliary factors (cf. Sato [19]). But, naturally, the following question arises: What is the character revealing the difference between $\boldsymbol{P}$ and $\boldsymbol{Q}$? We may consider that Theorem 3.3 answers to this question. However, unless $\{\varphi \in \mathscr{M}$; for any $\xi_0 \in V$ and $\varepsilon > 0$, there exists an open subset U of ξ_0 in S such that $|\varphi(x)| \leqq \varepsilon$

for any $x \in U \cap D\}$ contains sufficiently many functions, Theorem 3.3 has not any significance. For that purpose, next we shall investigate $\mathcal{M}$.

Let $\tilde{\boldsymbol{M}}$ be a Markov process on V. Let $l \in B^+(V)$, $\boldsymbol{P}$ be a bounded linear positive operator from $\mathcal{M}$ into $L^\infty(V, \boldsymbol{B}(V), \nu)$ and $\boldsymbol{Q}$ be a bounded kernel on $V \times D$.

DEFINITION 4.1. The system $(\tilde{\boldsymbol{M}}, l, \boldsymbol{P}, \boldsymbol{Q})$ is called the boundary system of $\boldsymbol{M}$ if and only if $\tilde{\boldsymbol{M}}$ is the U-process of $\boldsymbol{M}$ ([16]), and l, $\boldsymbol{P}$ and $\boldsymbol{Q}$ satisfy the following conditions;

$$l \cdot \Phi \approx \chi_V \cdot dt$$

$$(4.1) \qquad (\boldsymbol{P}+\boldsymbol{Q})\Big(\frac{G_\alpha^0 f}{G_\gamma^0 1}\Big) \cdot \Phi \approx (\widetilde{\chi_D f \cdot dt})_\alpha \qquad \text{for any } f \in B^+(D) \text{ and } \alpha > 0,$$

$$\boldsymbol{Q}h \cdot \Phi \approx \chi_V P_D(hG_\gamma^0 1) \cdot L \qquad \text{for any } h \in B^+(D).$$

From Lemma 2.1 and Lemma 3.3, we have

PROPOSITION 4.1. *For $\boldsymbol{M}$ satisfying ($\boldsymbol{M}$.1), ($\boldsymbol{M}$.2) and ($\boldsymbol{M}$.3′) there exists a boundary system. For a fixed $\gamma > 0$, $\tilde{\boldsymbol{M}}$ is uniquely determined by $\boldsymbol{M}$, and l, $\boldsymbol{P}\varphi$ and $\boldsymbol{Q}h$ (for any $\varphi \in \mathcal{M}$ and $h \in B(D)$) are uniquely determined by $\boldsymbol{M}$ except for a set of ν measure zero, where ν is a canonical measure of Φ.*

Let $(\tilde{\boldsymbol{M}}, l, \boldsymbol{P}, \boldsymbol{Q})$ be a boundary system. Then, by Lemma 2.1, Theorem 3.1 and Lemma 3.4, we have the following two lemmas.

LEMMA 4.1. *$l(\xi)+\boldsymbol{P}1(\xi)+\boldsymbol{Q}(\xi, D)=1$, ν-a. e. $\xi \in V$.*

LEMMA 4.2. *Fix $\alpha > 0$. Let $(f_m)_{m=0}^\infty$ be a sequence of $B(D)$ such that*

(i) $\sup_m \|f_m\| < +\infty$,

(ii) $\lim_{m\to\infty} f_m(x) = f_0(x)$ *for any $x \in D$, and*

(iii) *there exists a* $\lim_{m\to\infty} \boldsymbol{P}\Big(\frac{G_\alpha^0 f_m}{G_\gamma^0 1}\Big)(\xi)$ (ν-*a. e.* $\xi \in V$).

Then, we have

$$\lim_{m\to\infty} \boldsymbol{P}\Big(\frac{G_\alpha^0 f_m}{G_\gamma^0 1}\Big)(\xi) = \boldsymbol{P}\Big(\frac{G_\alpha^0 f_0}{G_\gamma^0 1}\Big)(\xi) \qquad (\nu\text{-a. e. } \xi \in V).$$

Noting that Proposition 2 of M. Motoo [17] holds for $\boldsymbol{M}$ satisfying ($\boldsymbol{M}$.1), ($\boldsymbol{M}$.2) and ($\boldsymbol{M}$.3′), by Lemma 2.1 and Theorem 3.1 we have

PROPOSITION 4.2. (i) *The paths of $\boldsymbol{M}$ have no sojourn on V a. e. if and only if $l=0$, a. e. ν.*

(ii) *The paths of $\boldsymbol{M}$ have no excursion which starts from V (that is, T_c is empty) a. e. if and only if $\boldsymbol{P}1=0$, a. e. ν.*

(iii) *The paths of $\boldsymbol{M}$ have no jump from V to D a. e. if and only if $\boldsymbol{Q}(\xi, D) = 0$, a. e. ν.*

Now, by making use of our boundary system $(\tilde{\boldsymbol{M}}, l, \boldsymbol{P}, \boldsymbol{Q})$, we replace C_α and U_α in 5.3 "*Correspondence of $\boldsymbol{M}$ and its boundary system*" of M. Motoo [17] by $C_\alpha f = l \cdot [f]_V + (\boldsymbol{P}+\boldsymbol{Q})\Big(\frac{G_\alpha^0 f}{G_\gamma^0 1}\Big)$ and $U_\alpha = \alpha C_\alpha H$, respectively. Then, noting that C_α and U_α are bounded linear positive operators from $B(S)$ into $L^\infty(V, \boldsymbol{B}(V), \nu)$ and from $B(V)$ into $L^\infty(V, \boldsymbol{B}(V), \nu)$, by taking the same procedure as in 5.3 of M. Motoo [17], we can prove the uniqueness theorem—— one-to-one correspondence between the processes $\boldsymbol{M}$ and their boundary systems $(\tilde{\boldsymbol{M}}, l, \boldsymbol{P}, \boldsymbol{Q})$—from Theorem 3.2 and Proposition 4.1.

THEOREM 4.1. *Let $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ be processes on S satisfying the conditions* ($\boldsymbol{M}$.1), ($\boldsymbol{M}$.2) *and* ($\boldsymbol{M}$.3). *Let $(\tilde{\boldsymbol{M}}_1, l_1, \boldsymbol{P}_1, \boldsymbol{Q}_1)$ and $(\tilde{\boldsymbol{M}}_2, l_2, \boldsymbol{P}_2, \boldsymbol{Q}_2)$ be their boundary systems respectively. Then, $\boldsymbol{M}_1=\boldsymbol{M}_2$ if and only if $\tilde{\boldsymbol{M}}_1=\tilde{\boldsymbol{M}}_2$ and $l_1=l_2$, $\boldsymbol{P}_1\varphi=\boldsymbol{P}_2\varphi$ and $\boldsymbol{Q}_1h=\boldsymbol{Q}_2h$ (for any $\varphi\in\mathcal{M}$ and $h\in B(D)$), a. e. ν_1, where ν_1 is the canonical measure of $\widetilde{(dt)}_r$ for $\boldsymbol{M}_1$* (Note that in either 'if' part or 'only if' part, ν_1 and ν_2 associated with $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ respectively are absolutely continuous each other.).

For future use, we prepare the following proposition, which corresponds to M. Motoo's [5.14] ([17]).

PROPOSITION 4.3. *Let $l_1, l_2\in B^+(V)$, let $\boldsymbol{P}_1$, $\boldsymbol{P}_2$ be bounded linear operators from $\mathcal{M}$ into $L^\infty(V, \boldsymbol{B}(V), \nu)$ such that $\boldsymbol{P}_1\varphi=\boldsymbol{P}_2\varphi=0$ for any $\varphi\in\mathcal{M}\cap C_\infty(D)$, where ν is a measure on V, and let $\boldsymbol{Q}_1$, $\boldsymbol{Q}_2$ be bounded kernels on $V\times D$. Under the hypothesis*

$$(*)\qquad C_\infty(D)\subset\mathcal{M},$$

the following fact holds; for some $\alpha>0$, if

$$(4.2)\qquad l_1(\xi)f(\xi)+(\boldsymbol{P}_1+\boldsymbol{Q}_1)\Big(\frac{G_\alpha^0 f}{G_\gamma^0 1}\Big)(\xi)$$

$$=l_2(\xi)f(\xi)+(\boldsymbol{P}_2+\boldsymbol{Q}_2)\Big(\frac{G_\alpha^0 f}{G_\gamma^0 1}\Big)(\xi),\ \nu\text{-a. e. } \xi\in V,$$

for any $f\in B(S)$, then $l_1(\xi)=l_2(\xi)$, $\boldsymbol{P}_1\varphi(\xi)=\boldsymbol{P}_2\varphi(\xi)$ and $\boldsymbol{Q}_1h(\xi)=\boldsymbol{Q}_2h(\xi)$, ν-a. e. $\xi\in V$ for any $\varphi\in\mathcal{M}$ and $h\in B(D)$.

PROOF. Substituting $f=\chi_V$ in (4.2), since $G_\alpha^0(\chi_V)=0$, we have $l_1(\xi)=l_2(\xi)$, ν-a. e. $\xi\in V$. Therefore, for any $f\in B(S)$,

$$(\boldsymbol{P}_1+\boldsymbol{Q}_1)\Big(\frac{G_\alpha^0 f}{G_\gamma^0 1}\Big)(\xi)=(\boldsymbol{P}_2+\boldsymbol{Q}_2)\Big(\frac{G_\alpha^0 f}{G_\gamma^0 1}\Big)(\xi),\quad \nu\text{-a. e. } \xi\in V.$$

Noting that $\mathcal{M}=\Big\{\frac{G_\alpha^0 f}{G_\gamma^0 1};f\in B(D)\Big\}$, since $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ are bounded operators and $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ are bounded kernels, we have, for any $\varphi\in\mathcal{M}$,

$$(4.3)\qquad (\boldsymbol{P}_1+\boldsymbol{Q}_1)(\varphi)(\xi)=(\boldsymbol{P}_2+\boldsymbol{Q}_2)(\varphi)(\xi),\quad \nu\text{-a. e. } \xi\in V.$$

Since $\boldsymbol{P}_1\varphi=\boldsymbol{P}_2\varphi=0$ for any $\varphi\in C_\infty(D)$ by the hypothesis and the quasi-local character of $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$, substituting $\varphi\in C_\infty(D)$ in (4.3), we have

$$(4.4)\qquad \boldsymbol{Q}_1\varphi(\xi)=\boldsymbol{Q}_2\varphi(\xi),\quad \nu\text{-a. e. } \xi\in V.$$

Since $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ are bounded kernels, (4.4) holds also for any $\varphi \in B(D)$. Therefore, by (4.3), we have, for any $\varphi \in \mathcal{M}$, $\boldsymbol{P}_1\varphi(\xi) = \boldsymbol{P}_2\varphi(\xi)$, ν-a. e. $\xi \in V$. This completes the proof of Proposition 4.3.

Now, we prove another uniqueness theorem, which gives the characterization of (the boundary system) $(l, \boldsymbol{P}, \boldsymbol{Q})$ apart from $\boldsymbol{M}$.

Note that the definition of the boundary system depends upon $\boldsymbol{M}$. Let $\boldsymbol{M}$ be a process on S satisfying the conditions ($\boldsymbol{M}$.1), ($\boldsymbol{M}$.2) and ($\boldsymbol{M}$.3), $K^\alpha(\alpha > 0)$ be the 0-th order resolvent of the α-th order U-process of $\boldsymbol{M}$ ([**12**], [**17**]), and ν be the canonical measure of the γ-th order sweeping-out Φ to V of time additive functional $t \wedge \zeta(\omega)$ for $\boldsymbol{M}$. Then we can prove the uniqueness theorem, which will be important in the future.

THEOREM 4.2. *Let $l_1, l_2 \in B^+(V)$, let $\boldsymbol{P}_1, \boldsymbol{P}_2$ be bounded linear positive operators from $\mathcal{M}$ into $L^\infty(V, \boldsymbol{B}(V), \nu)$ with the quasi-local character near the boundary V, and $\boldsymbol{Q}_1, \boldsymbol{Q}_2$ be bounded kernels on $V \times D$. And the following Feller-Ueno decomposition holds:*

$$G_\alpha f = G_\alpha^0 f + H_\alpha K^\alpha \Big\{ l_1 f + (\boldsymbol{P}_1 + \boldsymbol{Q}_1)\Big(\frac{G_\alpha^0 f}{G_\gamma^0 1}\Big)\Big\} \tag{4.5}$$

$$= G_\alpha^0 f + H_\alpha K^\alpha \Big\{ l_2 f + (\boldsymbol{P}_2 + \boldsymbol{Q}_2)\Big(\frac{G_\alpha^0 f}{G_\gamma^0 1}\Big)\Big\}$$

for some $\alpha > 0$ and any $f \in B(S)$.

If $C_\infty(D) \subset \mathcal{M}$, then we have $l_1(\xi) = l_2(\xi)$, $\boldsymbol{P}_1\varphi(\xi) = \boldsymbol{P}_2\varphi(\xi)$ and $\boldsymbol{Q}_1 h(\xi) = \boldsymbol{Q}_2 h(\xi)$, ν-a. e. $\xi \in V$ for any $\varphi \in \mathcal{M}$ and $h \in B(D)$.

PROOF. From (4.5), $H_\alpha K^\alpha \Big\{ l_1 f + (\boldsymbol{P}_1 + \boldsymbol{Q}_1)\Big(\frac{G_\alpha^0 f}{G_\gamma^0 1}\Big)\Big\} = H_\alpha K^\alpha \Big\{ l_2 f + (\boldsymbol{P}_2 + \boldsymbol{Q}_2)\Big(\frac{G_\alpha^0 f}{G_\gamma^0 1}\Big)\Big\}$ for any $f \in B(S)$. Since $H_\alpha K^\alpha \varphi(x) = E_x\Big(\int_0^\infty e^{-\alpha t}\varphi(x_t)\, d\Phi(t)\Big)$, and ν is the canonical measure of Φ, by Lemma 3.5 in [**14**], we have

$$l_1(\xi) f(\xi) + (\boldsymbol{P}_1 + \boldsymbol{Q}_1)\Big(\frac{G_\alpha^0 f}{G_\gamma^0 1}\Big)(\xi)$$

$$= l_2(\xi) f(\xi) + (\boldsymbol{P}_2 + \boldsymbol{Q}_2)\Big(\frac{G_\alpha^0 f}{G_\gamma^0 1}\Big)(\xi), \quad \nu\text{-a. e. } \xi \in V$$

for any $f \in B(S)$. Therefore, by Proposition 4.3, we have Theorem 4.2.

Finally, we answer the following question: "*Under what condition, can the hypothesis* (*) *be satisfied*?". To pursue this, from now on, we take the general formulation apart from previous sections. Let D be a locally compact Hausdorff space with the axiom of second countability, and let $\{G_\alpha^0 ; \alpha > 0\}$ be a family of linear positive operators from $B(D)$ into $B(D)$ such that

$$\alpha G_\alpha^0 1 \leqq 1 \quad \text{for any } \alpha > 0 \quad \text{(sub-Markov)}, \tag{4.6}$$

$$G_\alpha^0 - G_\beta^0 + (\alpha - \beta) G_\alpha^0 G_\beta^0 = 0 \quad \text{(resolvent equation)}, \tag{4.7}$$

(4.8) $$\lim_{\alpha\to\infty} \alpha G_\alpha^0 f(x) = f(x) \quad \text{for any } f \in C_\infty(D) \text{ and } x \in D.$$

Before answering the above question, we prepare two lemmas.

LEMMA 4.3. *For some* $\alpha > 0$, *if*

(4.9) $$G_\alpha^0(C_\infty(D)) \subset C_\infty(D)$$

then

(4.10) $$\overline{(G_\beta^0(C_\infty(D)))} = C_\infty(D) \qquad \textit{for any } \beta > 0.$$

PROOF. By (4.7) and (4.9), we see that $\mathfrak{R} = G_\beta^0(C_\infty(D))$ is a subspace of Banach space $C_\infty(D)$ and is independent of β. Assume that $\bar{\mathfrak{R}} \subsetneqq C(D)$. Then, by the Hahn-Banach theorem ([22]), there exists a bounded linear functional T on $C_\infty(D)$ such that $T \neq 0$ and $T\mathfrak{R} = 0$. Furthermore, by the Riesz-Markov-Kakutani theorem, there exists a bounded Borel signed measure μ such that

(4.11) $$Tf = \int_D f(x) d\mu(x) \qquad \text{for any } f \in C_\infty(D).$$

Therefore, for any $f \in C_\infty(D)$ and $\beta > 0$,

(4.12) $$\int_D \beta G_\beta^0 f(x) d\mu(x) = 0.$$

Since $\|\beta G_\beta^0 f\| \leqq \|f\|$ by (4.6), (4.8) and the bounded convergence theorem imply that for any $f \in C_\infty(D)$,

$$\int_D f(x) d\mu(x) = \lim_{\beta\to\infty} \int_D \beta G_\beta^0 f(x) d\mu(x)$$
$$= 0, \qquad \text{by (4.12)}.$$

Therefore, by (4.11), $T = 0$. This contradicts with $T \neq 0$. This completes the proof of Lemma 4.3.

LEMMA 4.4. *If* (4.9) *is satisfied for some* $\alpha > 0$, *then*

(4.13) $$\lim_{\beta\to\infty} \|\beta G_\beta^0 f - f\| = 0 \qquad \textit{for any } f \in C_\infty(D).$$

PROOF. Fix any $f \in C_\infty(D)$. For any $\varepsilon > 0$, by Lemma 4.3, there exists a $g \in C_\infty(D)$ such that $\|G_\gamma^0 g - f\| < \varepsilon$. Then, by (4.6) and (4.7),

$$\|\alpha G_{\alpha+\gamma}^0 f - f\| \leqq \|\alpha G_{\alpha+\gamma}^0 (f - G_\gamma^0 g)\| + \|\alpha G_{\alpha+\gamma}^0 G_\gamma^0 g - G_\gamma^0 g\| + \|G_\gamma^0 g - f\|$$

$$\leqq 2\|f - G_\gamma^0 g\| + \|G_{\alpha+\gamma}^0 g\| \leqq 2\varepsilon + \frac{\|g\|}{\alpha+\gamma}.$$

Therefore, we have $\|\alpha G_\alpha^0 f - f\| \to 0$ $(\alpha \to \infty)$.

Now, we answer the question mentioned before. Set

$$\mathcal{M}_\infty = \left\{ \frac{G_\alpha^0 f}{G_\gamma^0 1} \,;\ \alpha > 0,\ f \in C_\infty(D) \right\}.$$

THEOREM 4.3. *Let* $\{G_\alpha^0 ; \alpha > 0\}$ *be a family of resolvents corresponding to a diffusion process* $\boldsymbol{M}^0$ *(strong Markov and the paths are continuous in D). If* $G_\alpha^0(C_\infty(D)) \subset C_\infty(D)$ $(\alpha > 0)$ *and*

$$G_\alpha^0 1 \in C(D) \quad \text{for any } \alpha > 0, \text{ then} \tag{4.14}$$

$$C_\infty(D) \subset \mathcal{M}_\infty (\subset C(D)) . \tag{4.15}$$

PROOF. Since $C_0(D)$ is dense in $C_\infty(D)$, it suffices to prove $C_0(D) \subset \mathcal{M}_\infty$. Fix any $f \in C_0(D)$. Put $K = \operatorname{supp} f$. Then, there exists an open subset U with compact closure such that $K \subset U \subset \bar{U}$[5] $\subset D$. Noting that $f \cdot G_\gamma^0 1 \in C_\infty(D)$ by (4.14), by Lemma 4.4, we have

$$\|\alpha G_\alpha^0(f \cdot G_\gamma^0 1) - f \cdot G_\gamma^0 1\|_D \longrightarrow 0 \qquad \text{as } \alpha \to \infty . \tag{4.16}$$

Therefore, noting that $\inf_{x \in \bar{U}} G_\gamma^0 1(x) > 0$, we have

$$\left\| \frac{\alpha G_\alpha^0(f G_\gamma^0 1)}{G_\gamma^0 1} - f \right\|_{\bar{U}} \longrightarrow 0 \qquad \text{as } \alpha \to \infty . \tag{4.17}$$

Next, fix any $x \in D - \bar{U}$. Noting that $\operatorname{supp} f \subset U$, by Dynkin's formula,

$$G_\alpha^0(f G_\gamma^0 1)(x) = E_x^0(e^{-\alpha \sigma_{\bar{U}}} G_\alpha^0(f G_\gamma^0 1)(x_{\sigma_{\bar{U}}})) . \tag{4.18}$$

Noting that $P_x^0(x_{\sigma_{\bar{U}}} \notin \partial \bar{U})$[6] $= 0$ by the continuity of $\boldsymbol{M}^0$ and $\bar{U} \subset D$, by (4.18), we have

$$| G_\alpha^0(f G_\gamma^0 1)(x) | \leqq \sup_{y \in \partial \bar{U}} \left| \frac{G_\alpha^0(f G_\gamma^0 1)}{G_\gamma^0 1}(y) \right| E_x^0(e^{-\alpha \sigma_{\bar{U}}} G_\gamma^0 1(x_{\sigma_{\bar{U}}})) . \tag{4.19}$$

Since $E_x^0(e^{-\alpha \sigma_{\bar{U}}} G_\gamma^0 1(x_{\sigma_{\bar{U}}})) \leqq G_\gamma^0 1(x)$ for $\alpha \geqq \gamma$ by Dynkin's formula, by (4.19),

$$\sup_{x \in D - \bar{U}} \left\| \frac{\alpha G_\alpha^0(f G_\gamma^0 1)(x)}{G_\gamma^0 1(x)} \right\| \leqq \sup_{y \in \partial \bar{U}} \left\| \frac{\alpha G_\alpha^0(f G_\gamma^0 1)(y)}{G_\gamma^0 1(y)} \right\| . \tag{4.20}$$

Since $\lim_{\alpha \to \infty} \sup_{y \in \partial \bar{U}} \left\| \frac{\alpha G_\alpha^0(f G_\gamma^0 1)(y)}{G_\gamma^0 1(y)} \right\| = 0$ by $\operatorname{supp} f \subset U \subset \bar{U} \subset D$, $\inf_{y \in \partial \bar{U}} G_\gamma^0 1(y) > 0$ and (4.16), by (4.17) and (4.20), we have

$$\lim_{\alpha \to \infty} \left\| \frac{\alpha G_\alpha^0(f G_\gamma^0 1)}{G_\gamma^0 1} - f \right\|_D = 0 . \tag{4.21}$$

This shows $f \in \mathcal{M}_\infty$. This completes the proof of $C_0(D) \subset \mathcal{M}_\infty$.

§ 5. The entrance boundary and the expression of $\boldsymbol{P}$.

We have introduced the operator $\boldsymbol{P}$ of local character (§ 3, Theorem 3.1, Theorem 3.3), characterized the Markov process $\boldsymbol{M}$ satisfying ($\boldsymbol{M}$.1), ($\boldsymbol{M}$.2) and

5) $\bar{U}$ is the closure of U in S.
6) $\partial \bar{U}$ is the boundary of $\bar{U}$.

($\boldsymbol{M}$.3) by the boundary system ($\tilde{\boldsymbol{M}}, l, \boldsymbol{P}, \boldsymbol{Q}$) (§ 4, Theorem 4.1) and characterized the system ($l, \boldsymbol{P}, \boldsymbol{Q}$) under the following assumption

$$(*) \qquad C_\infty(D) \subset \bar{\mathcal{M}}$$

where $\mathcal{M}=\left\{\frac{G_\alpha^0 f}{G_\gamma^0 1} ; \alpha>0, f \in B(D)\right\}$ (§ 4, Theorem 4.2). In doing these, the important key was to use the Feller-Ueno decomposition theorem 3.2:

$$G_\alpha f=G_\alpha^0 f+H_\alpha K^\alpha\left\{l \cdot f+(\boldsymbol{P}+\boldsymbol{Q})\left(\frac{G_\alpha^0 f}{G_\gamma^0 1}\right)\right\}.$$

M. Motoo, under the additional conditions ($\boldsymbol{M}^{\min}$.4), ($\boldsymbol{M}^{\min}$.5) and ($\boldsymbol{M}^{\min}$.6), had used also the Feller-Ueno decomposition theorem in characterizing the process $\boldsymbol{M}$. M. Motoo's Feller-Ueno decomposition theorem is the following:

$$G_\alpha f=G_\alpha^0 f+H_\alpha K^\alpha\left\{l \cdot f+m \cdot \hat{H}_\alpha f+\boldsymbol{Q}\left(\frac{G_\alpha^0 f}{G_\gamma^0 1}\right)\right\}.$$

So, noting that the assumptions ($\boldsymbol{M}^{\min}$.4), ($\boldsymbol{M}^{\min}$.5) and ($\boldsymbol{M}^{\min}$.6) imply the assumption (*), the following question arises naturally: " *What is the mechanics that* $\boldsymbol{P}\left(\frac{G_\alpha^0 f}{G_\gamma^0 1}\right)(\xi)$ *equals to* $m(\xi)\hat{H}_\alpha f(\xi)$ *under the additional conditions* ($\boldsymbol{M}^{\min}$.4), ($\boldsymbol{M}^{\min}$.5) *and* ($\boldsymbol{M}^{\min}$.6)? " Our aim in this section is to introduce the entrance boundary, to reveal the structure of the operator $\boldsymbol{P}$ and to solve the above question. Since a series of lemmas before Theorem 5.1 do not relate to the process $\boldsymbol{M}$, we formulate generally apart from the process $\boldsymbol{M}$. From Theorem 5.1 on, we return to the process $\boldsymbol{M}$ again.

Let D be a locally compact Hausdorff topological space with the axiom of second countability, let $\{G_\alpha^0 ; \alpha>0\}$ be a family of linear positive operators from $B(D)$ into $B(D)$ such that

$$(5.1) \qquad \alpha G_\alpha^0 1 \leqq 1 \quad \text{for any } \alpha>0,$$

$$(5.2) \qquad G_\alpha^0-G_\beta^0+(\alpha-\beta) G_\alpha^0 G_\beta^0=0 \quad \text{for any } \alpha, \beta>0,$$

$$(5.3) \qquad \lim_{\alpha \rightarrow \infty} \alpha G_\alpha^0 f(x)=f(x) \quad \text{for any } f \in C_\infty(D) \text{ and } x \in D,$$

$$(5.4) \qquad G_\alpha^0(C_\infty(D)) \subset C_\infty(D) \quad \text{for any } \alpha>0,$$

$$(5.5) \qquad G_\alpha^0 1 \in C_\infty(D) \text{ and } G_\alpha^0 1(x)>0 \text{ for any } \alpha>0 \text{ and } x \in D.$$

Fix a positive number $\gamma>0$. Put $\mathcal{M}=\left\{\frac{G_\alpha^0 f}{G_\gamma^0 1} ; \alpha>0, f \in B(D)\right\}$ and $\mathcal{M}_\infty=\left\{\frac{G_\alpha^0 f}{G_\gamma^0 1} ; \alpha>0, f \in C_\infty(D)\right\}$. Then, we can see easily that the following lemma holds.

LEMMA 5.1. (1) *$\mathcal{M}$ is a subspace of $B(D)$ and contains* 1.

(2) *$\mathcal{M}_\infty$ is a subspace of $\mathcal{M} \cap C(D)$ and separates the points of D.*

Next lemma can be found in ([9], pp. 296, Lemma 2).

LEMMA 5.2. *There exists at most countable subset C' of $C_0(D)$ such that*

(1) $af+bg\in C'$ *for any* $f, g\in C'$ *and rational numbers* a, b,

(2) C' *is dense in* $C_0(D)$ *with supremum norm.*

LEMMA 5.3. *There exists at most countable subset* $\mathcal{M}'$ *of* $\mathcal{M}_\infty$ *such that*

(1) $a\varphi+b\phi\in\mathcal{M}'$ *for any* φ, $\phi\in\mathcal{M}'$ *and rational numbers* a, b,

(2) $\mathcal{M}'$ *is dense in* $\mathcal{M}_\infty$ *with supremum norm.*

PROOF. Taking C' in Lemma 5.2, put $\mathcal{M}'=\left\{\frac{G_\gamma^0 f}{G_\gamma^0 1}\,;\,f\in C'\right\}$. (Q. E. D.)

Let D^* be an $\mathcal{M}_\infty$-compactification of D, that is,

(i) D is imbedded homeomorphically in D^* as an open dense subset,

(ii) each element $f\in\mathcal{M}_\infty$ can be extended to a continuous function f^* on the space D^*,

(iii) $\{f^*\,;\,f\in\mathcal{M}_\infty\}$ separates the points of D^*-D.

The existence and uniqueness up to homeomorphism of such a D^* can be found in ([3], pp. 96). In our case, D^* is metrizable by Lemma 5.3 (2). Next, consider the following superharmonic transformation of $\{G_\alpha^0\,;\,\alpha>0\}$ by $G_\gamma^0 1$:

$$G_\alpha^1 f(x)=\frac{G_\alpha^0(fG_\gamma^0 1)(x)}{G_\gamma^0 1(x)} \qquad \text{for } f\in B(D),\ \alpha>0 \text{ and } x\in D\,.$$

Then, the space D^* is the same as $\{G_\alpha^1 f;\ \alpha>0,\ f\in C(D)\}$-compactification of D as will be seen in next Lemma 5.4.

LEMMA 5.4. (1) $\{G_\alpha^1\,;\,\alpha>0\}$ *is a family of linear positive operators from* $B(D)$ *into* $B(D)$,

(2) $\alpha G_\alpha^1 1\leqq 1$ *for any* $\alpha>0$,

(3) $G_\alpha^1-G_\beta^1+(\alpha-\beta)G_\alpha^1 G_\beta^1=0$ *for any* α, $\beta>0$,

(4) $G_\alpha^1(C(D))\subset C(D)$ *for any* $\alpha>0$,

(5) *for any* $f\in C(D)$ *and* $x\in D$, $\lim_{\alpha\to\infty}\alpha G_\alpha^1 f(x)=f(x)$,

(6) $\mathcal{R}=\{G_\alpha^1 f;\ \alpha>0,\ f\in C(D)\}$ *separates the points of* D,

(7) $\mathcal{R}$ *is a subspace of* $\mathcal{M}_\infty$,

(8) $\bar{\mathcal{R}}=\bar{\mathcal{M}}_\infty$.

PROOF. (1), (2): Linearity and positivity are clear. Since

$$\alpha G_\alpha^1 1(x)=\frac{\alpha G_\alpha^0(G_\gamma^0 1)(x)}{G_\gamma^0 1(x)}=\frac{G_\gamma^0(\alpha G_\alpha^0 1)(x)}{G_\gamma^0 1(x)}\leqq\frac{G_\gamma^0 1(x)}{G_\gamma^0 1(x)}=1 \text{ by (5.2)},$$

we have (2) and $G_\alpha^1(B(D))\subset B(D)$ for any $\alpha>0$. (3) can be proved as follows:

$$G_\alpha^1 f(x)-G_\beta^1 f(x)=\frac{G_\alpha^0(fG_\gamma^0 1)(x)-G_\beta^0(fG_\gamma^0 1)(x)}{G_\gamma^0 1(x)}$$

$$=\frac{(\beta-\alpha)G_\alpha^0 G_\beta^0(fG_\gamma^0 1)(x)}{G_\gamma^0 1(x)}$$

$$=\frac{(\beta-\alpha)G_\alpha^0\Big(\frac{G_\beta^0(fG_\gamma^0 1)}{G_\gamma^0 1}\cdot G_\gamma^0 1\Big)(x)}{G_\gamma^0 1(x)}$$

$$=(\beta-\alpha)\frac{G_\alpha^0(G_\beta^1 f\cdot G_\gamma^0 1)(x)}{G_\gamma^0 1(x)}$$

$$=(\beta-\alpha)G_\alpha^1 G_\beta^1 f.$$

(4) follows from (5.4) and (5.5), and (5) follows from (5.3). (5) implies (6). (7) follows from (5.5) and Lemma 5.4 (1), (3). (8) can be proved as follows: for any $f\in C_\infty(D)$, by Lemma 4.4, $\|\alpha G_\alpha^0 f-f\|\to 0$ as $\alpha\to\infty$. Therefore, noting that

$$\left\|\frac{G_\gamma^0\Big(\frac{\alpha G_\alpha^0 f}{G_\gamma^0 1}\cdot G_\gamma^0 1\Big)}{G_\gamma^0 1}-\frac{G_\gamma^0 f}{G_\gamma^0 1}\right\|$$

$$=\left\|\frac{G_\gamma^0(\alpha G_\alpha^0 f-f)}{G_\gamma^0 1}\right\|\leqq\|\alpha G_\alpha^0 f-f\|,$$

$\frac{G_\gamma^0 f}{G_\gamma^0 1}\in\bar{\mathfrak{R}}$. This implies $\mathcal{M}_\infty\subset\bar{\mathfrak{R}}$. Combining this with (7), we have (8).

(Q. E. D.)

For any $f\in C(D^*)$ and $\alpha>0$, put $G_\alpha^* f=[G_\alpha^1(f|_D)]^*$, where $[\cdot]^*$ is continuous extension of $\cdot$. Then, Lemma 5.4 implies

LEMMA 5.5. (1) *$\{G_\alpha^*;\alpha>0\}$ is a family of linear positive operators from $C(D^*)$ into $C(D^*)$.*

(2) *$\alpha G_\alpha^* 1\leqq 1$ for any $\alpha>0$,*

(3) *$G_\alpha^*-G_\beta^*+(\alpha-\beta)G_\alpha^* G_\beta^*=0$ for any α, $\beta>0$,*

(4) *for any $f\in C(D^*)$ and $x\in D$, $\lim_{\alpha\to\infty}\alpha G_\alpha^* f(x)=f(x)$.*

Put $\mathfrak{R}^*=\{g^*;g\in\mathfrak{R}\}$ and $\mathcal{M}_\infty^*=\{\varphi^*;\varphi\in\mathcal{M}_\infty\}$. Then, the following lemma holds.

LEMMA 5.6. (1) *$\mathfrak{R}(G_\alpha^*)\subset\mathfrak{R}^*\subset\mathcal{M}_\infty^*$,*

(2) *$\overline{\mathcal{M}_\infty^*}=(\overline{\mathcal{M}_\infty})^*$ and $\overline{\mathfrak{R}^*}=(\bar{\mathfrak{R}})^*$,*

(3) *$\overline{\mathfrak{R}(G_\alpha^*)}=\overline{\mathfrak{R}^*}=\overline{\mathcal{M}_\infty^*}$.*

PROOF. (1) follows from the definition of G_α^* and Lemma 5.4 (7). (2) is clear. (3) can be proved as follows: by Lemma 5.4 (8) and Lemma 5.6 (2), it suffices to prove $\mathfrak{R}^*\subset\overline{\mathfrak{R}(G_\alpha^*)}$. Fix any $g^*\in\mathfrak{R}^*$; $g^*=(G_1^1 f)^*$ for some $f\in C(D)$. Then,

$$\alpha G_\alpha^* g^*=\alpha[G_\alpha^1(g^*|_D)]^*$$

$$=\alpha[G_\alpha^1 G_1^1 f]^*.$$

Noting that $\|\alpha G_\alpha^1 G_1^1 f-G_1^1 f\|\to 0$ by Lemma 5.4 (1), (2) and (3), we have

$$\|\alpha G_\alpha^* g^*-g^*\|\longrightarrow 0\qquad\text{as }\alpha\to\infty.$$

Therefore, $g^*\in\overline{\mathfrak{R}(G_\alpha^*)}$. This implies $\mathfrak{R}^*\subset\overline{\mathfrak{R}(G_\alpha^*)}$. (Q. E. D.)

The definition of D^*, Lemma 5.5 (4) and Lemma 5.6 imply

LEMMA 5.7. *$\mathcal{R}(G_\alpha^*)$ separates the points of D and separates the points of D^*-D.*

Generally speaking, $\mathcal{R}(G_\alpha^*)$ does not always separate the points of D^*. Therefore, we introduce the following equivalence relation: for each x and $y \in D^*$, $x \sim y$ if and only if for any $g \in \mathcal{R}(G_\alpha^*)$ $g(x)=g(y)$. Let D^{**} be the quotient space of D^* by this equivalence relation, π be the projection from D^* onto D^{**}, i^{**} be the restriction of π to D. Then, the next lemma holds.

LEMMA 5.8. (1) *D^{**} is a compact metrizable space.*

(2) *i^{**} is continuous and injective from D into D^{**} and $i^{**}(D)$ is a dense Borel subset of D^{**}.*

(3) *i^{**-1} is Borel measurable from $i^{**}(D)$ onto D.*

(4) *Each element $\varphi \in \mathcal{M}_\infty$ can be extended to a continuous function φ^{**} on D^{**} such that $\varphi^{**} \circ \pi = \varphi^*$.*

(5) *$\{\varphi^{**}; \varphi \in \mathcal{M}_\infty\}$ separates the points of D^{**}.*

PROOF. We shall prove Lemma 5.8 bundling up the statements (1)~(5). By the definition of D^{**}, π is continuous from D^* onto D^{**} and a function f on D^{**} is continuous if and only if $f \circ \pi$ is continuous on D^*. Hence D^{**} is compact and i^{**} is continuous. The denseness of $i^{**}(D)$ in D^{**} follows from the denseness of D in D^*, the surjectivity and continuity of π and the definition of i^{**}. The injectivity of i^{**} follows from Lemma 5.7. Next, noting that for each $x, y \in D^*$, $x \sim y$ if and only if for any $\varphi^* \in \mathcal{M}_\infty^*$ $\varphi^*(x) = \varphi^*(y)$ by Lemma 5.6 (3), and each element $\varphi \in \mathcal{M}_\infty$ can be extended to a continuous function φ^* on D^*, Lemma 5.8 (4) holds. (5) can be proved as follows: for two points $y_1, y_2 \in D^{**}$; $y_1=\pi(x_1)$, $y_2=\pi(x_2)$ for some $x_1, x_2 \in D^*$, assume that for any $\varphi \in \mathcal{M}_\infty$, $\varphi^{**}(y_1)=\varphi^{**}(y_2)$. Then, by (4) just proved above, $\varphi^*(x_1)=\varphi^*(x_2)$ for any $\varphi \in \mathcal{M}_\infty$. Therefore, $x_1 \sim x_2$, that is, $y_1=\pi(x_1) = \pi(x_2)=y_2$. By (4) and (5) just proved, D^{**} is Hausdorff. Therefore, noting that D is σ-compact and i^{**} is continuous, $i^{**}(D)$ is a F_σ-set, hence $i^{**}(D)$ is a Borel subset of D^{**}. Consequently, (2) is proved. (3) can be proved as follows: for any open subset G of D, there exists a sequence K_n $(n=1, 2, 3, \cdots)$ of compact subsets of D such that $G=\bigcup_{n=1}^{\infty} K_n$. Therefore, noting that $i^{**}(G) = \bigcup_{n=1}^{\infty} i^{**}(K_n)$ and D^{**} is Hausdorff, $i^{**}(G)$ is a F_σ-set, hence, a Borel subset of D^{**}. Hence $i^{**}(\boldsymbol{B}(D)) \subset \boldsymbol{B}(D^{**})$ ([**6**]). This implies (3). Finally, the metrizability of D^{**} remains to be proved. This can be proved as follows: Put $\mathcal{L}=\{\varphi_1^{(1)}+\cdots+\varphi_{n_1}^{(1)}+\cdots+\varphi_1^{(p)}+\cdots+\varphi_{n_p}^{(p)}+a\,;\, a \in \boldsymbol{R},\ \varphi_j^{(i)},\ i=1, 2, \cdots, p,\ 1 \leqq j \leqq n_i\}$. Then, $\mathcal{L}$ is a subalgebra of $C(D)$, contains 1 and has countable dense subfamily by Lemma 5.1 and Lemma 5.3. And each element $\varphi \in \mathcal{L}$ can be extended to a continuous function φ^{**} on D^{**} by (4). Therefore, (5) and the

Stone-Weierstrass theorem imply $\overline{\mathcal{L}^{**}}=C(D^{**})$. From this, the metrizability of D^{**} follows. (Q. E. D.)

For any $f\in C(D^{**})$ and $\alpha>0$, put $G_\alpha^{**}f=(G_\alpha^1(f\circ i^{**}))^{**}$. This is well-defined by Lemma 5.4 (6), (7) and Lemma 5.8 (4). Then, by Lemma 5.8 (4), $G_\alpha^{**}f\circ\pi=G_\alpha^*(f\circ\pi)$ for any $f\in C(D^{**})$ and $\alpha>0$. Also, by the definition of D^{**}, for any $g\in C(D^*)$ and $\alpha>0$, $G_\alpha^* g$ can be extended to a continuous function $(G_\alpha^* g)^*$ such that $(G_\alpha^* g)^*\circ\pi=G_\alpha^* g$. Then, the following lemma holds.

LEMMA 5.9. (1) *$\{G_\alpha^{**}\,;\alpha>0\}$ is a family of linear positive operators from $C(D^{**})$ into $C(D^{**})$.*

(2) *$\alpha G_\alpha^{**}1\leqq 1$ for any $\alpha>0$,*

(3) *$G_\alpha^{**}-G_\beta^{**}+(\alpha-\beta)G_\alpha^{**}G_\beta^{**}=0$ for any α, $\beta>0$,*

(4) *for any $f\in C(D^{**})$ and $x\in D$, $\lim_{\alpha\to\infty}\alpha G_\alpha^{**}f(i^{**}(x))=f(i^{**}(x))$,*

(5) *$\mathfrak{R}(G_\alpha^{**})$ separates the points of D^{**}.*

PROOF. (1) follows from the definition of G_α^{**}. (2) follows from Lemma 5.5 (2). (3) can be proved as follows: $G_\alpha^{**}f(\pi(\xi))-G_\beta^{**}f(\pi(\xi))=G_\alpha^*(f\circ\pi)(\xi)-G_\beta^*(f\circ\pi)(\xi)=(\beta-\alpha)G_\alpha^*(G_\beta^*(f\circ\pi))(\xi)$ (by Lemma 5.5 (3)) $=(\beta-\alpha)G_\alpha^*(G_\beta^{**}f\circ\pi)(\xi)=(\beta-\alpha)G_\alpha^{**}G_\beta^{**}f(\pi(\xi))$. (4) follows from Lemma 5.5 (4). For the proof of (5), consider two points $\pi(\xi)$, $\pi(\eta)\in D^{**}$ such that $G_\alpha^{**}f(\pi(\xi))=G_\alpha^{**}f(\pi(\eta))$ for any $f\in C(D^{**})$ and $\alpha>0$. Fix any $g\in C(D^*)$ and $\beta>0$. Then, noting that $G_\alpha^{**}f\circ\pi=G_\alpha^*(f\circ\pi)$ and $(G_\beta^* g)^*\circ\pi=G_\beta^* g$ by the paragraph before Lemma 5.9, we have, $\alpha G_\alpha^* G_\beta^* g(\xi)=\alpha G_\alpha^* G_\beta^* g(\eta)$ for any $\alpha>0$. Letting $\alpha\to\infty$, $G_\beta^* g(\xi)=G_\beta^* g(\eta)$ by Lemma 5.5 (4). Therefore, by the definition of π, $\pi(\xi)=\pi(\eta)$. This completes the proof of (5). (Q. E. D.)

By Lemma 5.9 and the Ray's theorem ([**11**], [**18**]), we have the following lemma.

LEMMA 5.10. (1) *For any $x\in D^{**}$, there exists uniquely a sub-stochastic measure $\mu_1(x,dy)$ on D^{**} such that*

$$\lim_{\alpha\to\infty}\alpha G_\alpha^{**}f(x)=\int_{D^{**}}\mu_1(x,dy)f(y)\qquad \textit{for any } f\in C(D^{**}).$$

(2) *For any $f\in C(D^{**})$, $\alpha>0$ and $x\in D^{**}$,*

$$G_\alpha^{**}f(x)=\int_{D^{**}}\mu_1(x,dy)G_\alpha^{**}f(y).$$

(3) *For any Borel subset E of D^{**},*

$$\mu_1(\cdot,E)\ \textit{is Borel measurable in } D^{**}.$$

(4) *Put $D_b^{**}=\{x\in D^{**}\,;\mu_1(x,dy)\neq\delta_x(dy)\}$. Then,*

$$D_b^{**}\ \textit{is } F_\sigma\textit{-set and for any } x\in D^{**},$$

$$\mu_1(x,D_b^{**})=0.$$

(5) $D_b^{**} \subset D^{**} - i^{**}(D)$.

Before proving the representation theorem for $\boldsymbol{P}$ introduced in § 3, we shall pick up the properties of $\boldsymbol{P}$ which do not relate to the process $\boldsymbol{M}$, and replace the boundary V of D, the topological Borel field $\boldsymbol{B}(V)$ and the canonical measure ν of Φ by any measure space $(V, \mathcal{F}, \nu)$, and we shall prepare the next lemma, which would be useful in the future.[7] Set $\mathcal{L} = \{\varphi + a\,;\ \varphi \in \mathcal{M}_\infty,\ a \in \boldsymbol{R}\}$.

LEMMA 5.11. *Let* $(V, \mathcal{F}, \nu)$ *be a measure space and* $\boldsymbol{P}$ *be a bounded linear positive operator from* $\bar{\mathcal{L}}$ *into* $L^\infty(V, \mathcal{F}, \nu)$ *such that* $\boldsymbol{P}1 \leqq 1$. *Then, for each* $\xi \in V$, *there exists a sub-stochastic measure* $\mu(\xi, dy)$ *on* D^{**} *such that*

(1) $\boldsymbol{P}\varphi(\xi) = \int_{D^{**}} \mu(\xi, dy)\varphi^{**}(y)$, ν-*a. e.* $\xi \in V$, *for any* $\varphi \in \bar{\mathcal{M}}_\infty$,

(2) $\boldsymbol{P}1(\xi) \geqq \mu(\xi, D^{**})$, ν-*a. e.* $\xi \in V$,

(3) $\mu(\xi, D_b^{**}) = 0$ *for any* $\xi \in V$.

Moreover, such a measure $\mu(\xi, dx)$ *is uniquely determined except for a set of* $\xi \in V$ *with* ν-*measure zero.*

PROOF. (I) Taking $\mathcal{M}'$ in Lemma 5.3, put $\mathcal{M}'' = \{\varphi + a\,;\ \varphi \in \mathcal{M}',\ a \in \boldsymbol{Q}\}$. Then, by Lemma 5.3, we have

$$\mathcal{M}'' \ni 1\,. \tag{5.6}$$

$$a\varphi + b\phi \in \mathcal{M}'' \quad \text{for any } \varphi, \phi \in \mathcal{M}'' \text{ and } a, b \in \boldsymbol{Q}\,, \tag{5.7}$$

$$\mathcal{M}'' \text{ is dense in } \mathcal{L}\,, \tag{5.8}$$

$$\sharp\mathcal{M}'' \leqq \infty\,. \tag{5.9}$$

Therefore, there exists a measurable set $V_0 \in \mathcal{F}$ such that

$$\nu(V - V_0) = 0 \tag{5.10}$$

and for any $\xi \in V_0$,

$$\boldsymbol{P}(a\varphi + b\phi)(\xi) = a\boldsymbol{P}\varphi(\xi) + b\boldsymbol{P}\phi(\xi) \quad \text{for any } \varphi, \phi \in \mathcal{M}'' \text{ and } a, b \in \boldsymbol{Q}\,, \tag{5.11}$$

$$\boldsymbol{P}\varphi(\xi) \geqq 0 \quad \text{for any } \varphi \in (\mathcal{M}'')^+\,, \tag{5.12}$$

$$\boldsymbol{P}1(\xi) \leqq 1\,, \tag{5.13}$$

$$|\boldsymbol{P}\varphi(\xi)| \leqq \|\varphi\| \quad \text{for } \varphi \in \mathcal{M}''\,, \tag{5.14}$$

$$\boldsymbol{P}\varphi(\xi) \leqq \sup\{\varphi(x)\,;\ x \in D\} \quad \text{for } \varphi \in \mathcal{M}''\,. \tag{5.15}$$

By Lemma 5.8, for any $\varphi \in \mathcal{L}$, there exists $\varphi^{**} \in C(D^{**})$ such that $\varphi^{**} \circ i^{**} = \varphi$. Set $\mathcal{L}^{**} = \{\varphi^{**}\,;\ \varphi \in \mathcal{L}\}$ and $(\mathcal{M}'')^{**} = \{\varphi^{**}\,;\ \varphi \in \mathcal{M}''\}$. Fix $\xi \in V_0$. We define $T_1\varphi^{**} = \boldsymbol{P}\varphi(\xi)$ for $\varphi \in \mathcal{M}''$. Then, by (5.11)~(5.15), we have

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7) For example, by using this Lemma 5.11, we can get the representation theorem for r-excessive measure for $\{G_\alpha^0\,;\ \alpha > 0\}$ ([**11**]).

(5.16) $\qquad T_1(a\varphi^{**}+b\phi^{**})=aT_1\varphi^{**}+bT_1\phi^{**}$ for any $\varphi, \phi \in \mathcal{M}''$ and $a, b \in \boldsymbol{Q}$,

(5.17) $\qquad T_1\varphi^{**} \geqq 0$ for any $\varphi^{**} \in ((\mathcal{M}'')^{**})^+$,

(5.18) $\qquad T_1 1^{**} \leqq 1$,

(5.19) $\qquad |T_1\varphi^{**}| \leqq \|\varphi^{**}\|$ for $\varphi \in \mathcal{M}''$,

(5.20) $\qquad T_1\varphi^{**} \leqq \sup\{\varphi^{**}(x) ; x \in D^{**}\}$ for $\varphi \in \mathcal{M}''$.

Since $(\mathcal{M}'')^{**}$ is dense in $\mathcal{L}^{**}$ by (5.8), from (5.16) and (5.19), we can easily see that T_1 can be extended to an functional T_2 on $\mathcal{L}^{**}$. Thus, by the definition of T_2 and (5.16)$\sim$(5.20),

(5.21) $\qquad T_2(a\varphi^{**}+b\phi^{**})=aT_2\varphi^{**}+bT_2\phi^{**}$ for any $\varphi, \phi \in \mathcal{L}$ and $a, b \in \boldsymbol{R}$,

(5.22) $\qquad T_2\varphi^{**} \geqq 0$ for any $\varphi^{**} \in (\mathcal{L}^{**})^+$,

(5.23) $\qquad T_2 1^{**} \leqq 1$,

(5.24) $\qquad |T_2\varphi^{**}| \leqq \|\varphi^{**}\|$ for any $\varphi \in \mathcal{L}$,

(5.25) $\qquad T_2\varphi^{**} \leqq \sup\{\varphi^{**}(x) ; x \in D^{**}\}$ for any $\varphi \in \mathcal{L}$.

By (5.21), (5.25) and the Hahn-Banach theorem for a linear space $C(D^{**})$ ([22]), the linear functional T_2 in $\mathcal{L}^{**}$ can be extended to a linear functional T_3 on $C(D^{**})$ such that

(5.26) $\qquad T_3\varphi \leqq \sup\{\varphi(x) ; x \in D^{**}\}$ for any $\varphi \in C(D^{**})$.

Fix $\varphi \in C(D^{**})$. $-T_3\varphi = T_3(-\varphi) \leqq \sup\{-\varphi(x) ; x \in D^{**}\}$ by (5.26). Therefore

(5.27) $\qquad T_3\varphi \geqq \inf\{\varphi(x) ; x \in D^{**}\}$ for any $\varphi \in C(D^{**})$.

Thus, by (5.26) and (5.27), we can easily see that

$$|T_3\varphi| \leqq \|\varphi\| \quad \text{and } T_3 \text{ is positive.}$$

Therefore, by the Riesz-Markov-Kakutani theorem, there exists a bounded measure $\mu_2(\xi, dy)$ on D^{**} such that

(5.28) $\qquad T_3\varphi = \int_{D^{**}} \mu_2(\xi, dy)\varphi(y)$ for any $\varphi \in C(D^{**})$.

Since $1 \in \mathcal{M}''$, we have, by (5.28),

(5.29) $\qquad \boldsymbol{P}1(\xi) = T_1 1^{**}(\xi) = T_2 1^{**}(\xi) = T_3 1^{**}(\xi) = \mu_2(\xi, D^{**})$.

Next, for any $\xi \in V - V_0$, let

$$\mu_2(\xi, \cdot) = \delta_{\eta_0}(\cdot) \qquad (\text{Fix any point } \eta_0 \in D^{**}).$$

Take any $\varphi \in \bar{\mathcal{L}}$. Then, by (5.8), there exists a sequence $(\varphi_n)_{n=0}^{\infty}$ in $\mathcal{M}''$ such that $\lim_{n\to\infty} \|\varphi_n - \varphi\| = 0$. Therefore, by the boundedness of $\boldsymbol{P}$, there exists a measurable set $V_1 \in \mathcal{F}$ such that

(5.30) $$\nu(V - V_1) = 0$$

and

(5.31) $$\boldsymbol{P}\varphi_n(\xi) \longrightarrow \boldsymbol{P}\varphi(\xi) \quad \text{as } n \to \infty \text{ for any } \xi \in V_1 .$$

By (5.10) and (5.30), we have

(5.32) $$\nu(V - V_0 \cap V_1) = 0 .$$

Fix any $\xi \in V_0 \cap V_1$. By (5.28) and (5.31)

(5.33) $$\begin{aligned}\int_{D^{**}} \mu_2(\xi, dy)\varphi^{**}(y) &= T_3\varphi^{**} \\ &= \lim_{n\to\infty} T_1\varphi_n^{**} \\ &= \lim_{n\to\infty} \boldsymbol{P}\varphi_n(\xi) \\ &= \boldsymbol{P}\varphi(\xi) .\end{aligned}$$

(II) Using two measures $\mu_1(x, dy)$ and $\mu_2(\xi, dx)$ in Lemma 5.10 and (I) respectively, for any $\xi \in V$ and $E \in \boldsymbol{B}(D^{**})$, let

(5.34) $$\mu(\xi, E) = \int_{D^{**}} \mu_2(\xi, dx)\mu_1(x, E) .$$

This integral is well-defined by Lemma 5.10 (3). By Lemma 5.10 (4) and (5.34)

(5.35) $$\mu(\xi, D_b^{**}) = 0 \qquad \text{for any } \xi \in V .$$

And, by Lemma 5.10 and (5.29),

(5.36) $$\mu(\xi, D^{**}) \leqq \boldsymbol{P}1(\xi) \leqq 1, \qquad \nu\text{-a. e. } \xi \in V .$$

Next, for any $f \in C(D^{**})$ and $\alpha > 0$, by Lemma 5.10 (2) and (5.34),

(5.37) $$\begin{aligned}&\int_{D^{**}} \mu(\xi, dx) G_\alpha^{**} f(x) \\ &= \int_{D^{**}} \mu_2(\xi, dx)\Big(\int_{D^{**}} \mu_1(x, dy) G_\alpha^{**} f(y)\Big) \\ &= \int_{D^{**}} \mu_2(\xi, dx) G_\alpha^{**} f(x) .\end{aligned}$$

For the proof of Lemma 5.11 (1), take any $f \in C_\infty(D)$. Noting that

$$\begin{aligned}\Big(\frac{G_\beta^0 G_\gamma^0 f}{G_\gamma^0 1}\Big)^{**} &= \Big(G_\beta^1\Big(\frac{G_\gamma^0 f}{G_\gamma^0 1}\Big)\Big)^{**} \\ &= G_\beta^{**}\Big(\Big[\frac{G_\gamma^0 f}{G_\gamma^0 1}\Big]^{**}\Big)^{**},\end{aligned}$$

by (5.32), (5.33) and (5.37), we have

$$\boldsymbol{P}\Big(\frac{\beta G_\beta^0 G_\gamma^0 f}{G_\gamma^0 1}\Big)(\xi) = \int_{D^{**}} \mu(\xi, dx)\beta G_\beta^{**}\Big(\Big[\frac{G_\gamma^0 f}{G_\gamma^0 1}\Big]^{**}\Big)(x), \quad \nu\text{-a. e. } \xi \in V. \tag{5.38}$$

Since $\Big\|\beta\frac{G_\beta^0 G_\gamma^0 f}{G_\gamma^0 1}-\frac{G_\gamma^0 f}{G_\gamma^0 1}\Big\| \leqq \|\beta G_\beta^0 f - f\| \to 0$ as $\beta \to \infty$ by Lemma 4.4, by the boundedness of $\boldsymbol{P}$, we have

$$\Big\|\boldsymbol{P}\Big(\frac{\beta G_\beta^0 G_\gamma^0 f}{G_\gamma^0 1}\Big)-\boldsymbol{P}\Big(\frac{G_\gamma^0 f}{G_\gamma^0 1}\Big)\Big\| \longrightarrow 0 \quad \text{as } \beta \to \infty. \tag{5.39}$$

On the other hand, since $\Big\|\beta G_\beta^{**}\Big(\Big[\frac{G_\gamma^0 f}{G_\gamma^0 1}\Big]^{**}\Big)\Big\| \leqq \|f\|$ and for any $x \in D^{**}-D_b^{**}$, $\lim_{\beta\to\infty} \beta G_\beta^{**}\Big(\Big[\frac{G_\gamma^0 f}{G_\gamma^0 1}\Big]^{**}\Big)(x) = \Big(\frac{G_\gamma^0 f}{G_\gamma^0 1}\Big)^{**}(x)$ by Lemma 5.9 (2) and Lemma 5.10, letting $\beta \to \infty$ in (5.38), (5.35), (5.36) and (5.39) imply that

$$\boldsymbol{P}\Big(\frac{G_\gamma^0 f}{G_\gamma^0 1}\Big)(\xi) = \int_{D^{**}} \mu(\xi, dx)\Big(\frac{G_\gamma^0 f}{G_\gamma^0 1}\Big)^{**}(x), \quad \nu\text{-a. e. } \xi \in V. \tag{5.40}$$

(5.35), (5.36) and (5.40) completes the proof of the first part of Lemma 5.11. The second part of Lemma 5.11 can be proved as follows: let $\mu^{(1)}(\ , dx)$ and $\mu^{(2)}(\ , dx)$ be measures satisfying (1), (2) and (3). Take any $f \in C(D^{**})$. Since $\|\alpha G_\alpha^{**} f\| \leqq \|f\|$ and $\lim_{\alpha\to\infty} \alpha G_\alpha^{**} f(x) = f(x)$ for any $x \in D^{**}-D_b^{**}$ by Lemma 5.9 (2) and Lemma 5.10, the bounded convergence theorem implies that

$$\begin{aligned}\int_{D^{**}} \mu^{(1)}(\xi, dx) f(x) &= \lim_{\alpha\to\infty} \int_{D^{**}} \mu^{(1)}(\xi, dx)\alpha G_\alpha^{**} f(x) \\ &= \lim_{\alpha\to\infty} \boldsymbol{P}\Big(\frac{\alpha G_\alpha^0 (f \circ i^{**} G_\gamma^0 1)}{G_\gamma^0 1}\Big)(\xi) \\ &= \lim_{\alpha\to\infty} \int_{D^{**}} \mu^{(2)}(\xi, dx)\alpha G_\alpha^{**} f(x) \\ &= \int_{D^{**}} \mu^{(2)}(\xi, dx) f(x), \quad \nu\text{-a. e. } \xi \in V.\end{aligned}$$

Therefore, since D^{**} is compact metrizable by Lemma 5.8, we have $\mu^{(1)}(\xi, \cdot) = \mu^{(2)}(\xi, \cdot)$. This completes the proof of Lemma 5.11. (Q. E. D.)

Now, we shall return to the process $\boldsymbol{M}$. Our aim was to get the representation theorem for $\boldsymbol{P}$ introduced in §3. Lemma 5.11 does not necessarily satisfy us, because from Lemma 5.11 we can only prove that $\boldsymbol{P}\Big(\frac{G_\alpha^0 f}{G_\gamma^0 1}\Big)(\xi) = \int_{D^{**}} \mu(\xi, dx)\Big(\frac{G_\alpha^0 f}{G_\gamma^0 1}\Big)^{**}(x)$ for $f \in C_\infty(D)$ and so $l \cdot f$ in the Feller-Ueno decomposition theorem (Theorem 3.2) disappears in replacing the operator $\boldsymbol{P}$ in

Theorem 3.2 by measure $\mu(\xi, dx)$ in Lemma 5.11 for $f\in C_\infty(D)$. For the purpose of overcoming this difficulty, we shall prepare some lemmas.

Since $\beta G^0_{\alpha+\beta}G^0_\alpha f$ is increasing in β for any $\alpha>0$ and $f\in B(D)^+$, we can easily see that there exists a

$$\lim_{\beta\to\infty}\Big(\beta G^1_{\alpha+\beta}\Big(\frac{G^0_\alpha f}{G^0_\gamma 1}\Big)\Big)^{**}=\lim_{\beta\to\infty}\Big(\beta G^1_\beta\Big(\frac{G^0_\alpha f}{G^0_\gamma 1}\Big)\Big)^{**}\equiv\hat{H}_\alpha f, \tag{5.41}$$

for any $\alpha>0$ and $f\in C(D)$.

Then, the following lemma follows from Lemma 5.4.

LEMMA 5.12. (1) *$\hat{H}_\alpha$ is a bounded linear positive operator from $C(D)$ into $B(D^{**})$.*

(2) $\|\hat{H}_\alpha f\|\leqq\Big\|\frac{G^0_\alpha f}{G^0_\gamma 1}\Big\|$ *for $\alpha>0$ and $f\in C(D)$.*

(3) $\hat{H}_\alpha f\circ i^{**}=\frac{G^0_\alpha f}{G^0_\gamma 1}$ *for $\alpha>0$ and $f\in C(D)$.*

(4) *$\hat{H}_\alpha f$ is lower semi-continuous for $\alpha>0$ and $f\in C(D)^+$.*

LEMMA 5.13. *For any $\alpha>0$ and $f\in C(D)$ such that $\frac{G^0_\alpha f}{G^0_\gamma 1}$ can be extended to a continuous function $\Big(\frac{G^0_\alpha f}{G^0_\gamma 1}\Big)^{**}$ on D^{**},*

$$\hat{H}_\alpha f(\eta)=\Big(\frac{G^0_\alpha f}{G^0_\gamma 1}\Big)^{**}(\eta)\qquad\text{for any }\eta\in D^{**}-D^{**}_b.$$

PROOF. Since $\Big(\beta G^1_{\alpha+\beta}\Big(\frac{G^0_\alpha f}{G^0_\gamma 1}\Big)\Big)^{**}=\beta G^{**}_{\alpha+\beta}\Big(\Big(\frac{G^0_\alpha f}{G^0_\gamma 1}\Big)^{**}\Big)$, Lemma 5.13 follows from Lemma 5.10 and (5.41). (Q. E. D.)

LEMMA 5.14. *For any α, $\beta>0$ and $f\in C(D)$,*

$$\hat{H}_\alpha f-\hat{H}_\beta f+(\alpha-\beta)\hat{H}_\alpha G^0_\beta f=0.$$

PROOF.

$$\begin{aligned}\hat{H}_\alpha f-\hat{H}_\beta f&=\lim_{\delta\to\infty}\delta\Big(G^1_\delta\Big(\frac{G^0_\alpha f}{G^0_\gamma 1}\Big)\Big)^{**}-\lim_{\delta\to\infty}\delta\Big(G^1_\delta\Big(\frac{G^0_\beta f}{G^0_\gamma 1}\Big)\Big)^{**}\\&=\lim_{\delta\to\infty}\delta\Big(G^1_\delta\Big(\frac{G^0_\alpha f-G^0_\beta f}{G^0_\gamma 1}\Big)\Big)^{**}\\&=(\beta-\alpha)\lim_{\delta\to\infty}\delta\Big(G^1_\delta\Big(\frac{G^0_\alpha G^0_\beta f}{G^0_\gamma 1}\Big)\Big)^{**}\\&=(\beta-\alpha)\hat{H}_\alpha G^0_\beta f.\end{aligned}$$

(Q. E. D.)

LEMMA 5.15. *$\hat{H}_\alpha f$ is continuous on $D^{**}-D^{**}_b$ for $\alpha>0$ and $f\in C(D)$.*

PROOF. By Lemma 5.12 (1) and Lemma 5.14, we have only to prove the result in case $\alpha=\gamma$ and $0\leqq f\leqq 1$. By Lemma 5.13, we have

$$\hat{H}_\gamma 1(\eta)=1\qquad\text{for any }\eta\in D^{**}-D^{**}_b. \tag{5.42}$$

Since $\hat{H}_\gamma f=\hat{H}_\gamma 1-\hat{H}_\gamma(1-f)$, by Lemma 5.12 (4) and (5.42), we have Lemma 5.15. (Q. E. D.)

LEMMA 5.16. *If $D_b^{**}=\emptyset$, then*

(1) $\hat{H}_\alpha f\in C(D^{**})$ *for any* $\alpha>0$ *and* $f\in C(D)$,

(2) $\lim_{\beta\to\infty}\|\beta\hat{H}_{\alpha+\beta}G_\alpha^0 f-\hat{H}_\alpha f\|=0$ *for any* $\alpha>0$ *and* $f\in C(D)$.

PROOF. (1) follows immediately from Lemma 5.15. Since

$$\beta\hat{H}_{\alpha+\beta}G_\alpha^0 f=\beta\Big(\frac{G_{\alpha+\beta}^0 G_\alpha^0 f}{G_\gamma^0 1}\Big)^{**} \quad \text{by Lemma 5.13}$$

$$=\beta(G_{\alpha+\beta}^1(\hat{H}_\alpha f\circ i^{**}))^{**} \quad \text{by Lemma 5.12 (3),}$$

we have, by (1),

$$\beta\hat{H}_{\alpha+\beta}G_\alpha^0 f=\beta G_{\alpha+\beta}^{**}\hat{H}_\alpha f.$$

Therefore, by $D_b^{**}=\emptyset$ and (1), we have (2). (Q. E. D.)

In fact, we have the following general lemma.

LEMMA 5.17. *For any $\alpha>0$ and $f\in C(D)$,*

(i) $\lim_{\beta\to\infty}\beta\hat{H}_\beta G_\alpha^0 f=\hat{H}_\alpha f$ *on* $D^{**}-D_b^{**}$

(ii) $\lim_{\beta\to\infty}\beta\hat{H}_\alpha G_\beta^0 f=\hat{H}_\alpha f$ *on* $D^{**}-D_b^{**}$.

PROOF. (i): By Lemma 5.13, we have

$$\beta\hat{H}_\beta G_\alpha^0 f=\beta\Big(\frac{G_\beta^0 G_\alpha^0 f}{G_\gamma^0 1}\Big)^{**}=\beta\Big(G_\beta^1\Big(\frac{G_\alpha^0 f}{G_\gamma^0 1}\Big)\Big)^{**} \quad \text{on } D^{**}-D_b^{**}.$$

Therefore, Lemma 5.17 (i) follows from (5.41).

(ii): this follows immediately from Lemma 5.14 and Lemma 5.17 (i). (Q.E.D.)

Now, we are able to prove the following representation theorem for $\boldsymbol{P}$ introduced in § 3.

THEOREM 5.1. *For each point $\xi\in V$, there exists a sub-stochastic measure $\mu(\xi, dx)$ on D^{**} such that*

(i) $\boldsymbol{P}\Big(\frac{G_\alpha^0 f}{G_\gamma^0 1}\Big)(\xi)=\int_{D^{**}}\mu(\xi, dx)\hat{H}_\alpha f(x)$, ν-*a. e.* $\xi\in V$ *for any* $\alpha>0$ *and* $f\in C(D)$,

(ii) $\boldsymbol{P}1(\xi)=\mu(\xi, D^{**})$, ν-*a. e.* $\xi\in V$,

(iii) $\mu(\xi, D_b^{**})=0$ *for any* $\xi\in V$.

Moreover, such a measure $\mu(\xi, dx)$ is uniquely determined up to equivalence with respect to ν.

PROOF. We shall show that measures $\mu(\xi, dx)$ gained in Lemma 5.11 are ones that we want. Fix any $f\in C(D)$. Since $G_\beta^0 g(\beta>0,\ g\in C_\infty(D))$ is the β-th order Green operator of a Hunt process $\boldsymbol{M}^0$ (Lemma 4.4), we have, for any

$\beta>0$ and $g\in C_{\infty}(D)$,

(5.43) $$G_{\beta}^{0}g(x)=E_{x}^{0}\left(\int_{0}^{\infty}e^{-\beta t}g(x_{t})dt\right)\qquad(x\in D).$$

By the definition of G_{β}^{0}, (5.43) holds for any $\beta>0$ and $g\in C(D)$. Therefore, noting that for any $g\in C(D)$ and $y\in D$

(5.44) $$\lim_{\beta\to\infty}\beta E_{y}^{0}\left(\int_{0}^{\infty}e^{-\beta t}g(x_{t})dt\right)=g(y),$$

we have,

(5.45) $$\lim_{\beta\to\infty}\beta G_{\beta}^{0}f(y)=f(y)\qquad\text{for any } y\in D.$$

Since $\|\beta G_{\beta}^{0}f\|\leqq\|f\|$ by (5.1), we have, by Lemma 3.4 and (5.45),

(5.46) $$\lim_{\beta\to\infty}\boldsymbol{P}\Big(\frac{\beta G_{\beta}^{0}G_{\alpha}^{0}f}{G_{\gamma}^{0}1}\Big)(\xi)=\boldsymbol{P}\Big(\frac{G_{\alpha}^{0}f}{G_{\gamma}^{0}1}\Big)(\xi),\quad \nu\text{-a. e. } \xi\in V.$$

On the other hand, by Lemma 5.11 (1),

(5.47) $$\boldsymbol{P}\Big(\frac{\beta G_{\beta}^{0}G_{\alpha}^{0}f}{G_{\gamma}^{0}1}\Big)(\xi)=\int_{D^{**}}\mu(\xi,dx)\Big(\beta\frac{G_{\beta}^{0}G_{\alpha}^{0}f}{G_{\gamma}^{0}1}\Big)^{**}(x)$$
$$=\int_{D^{**}}\mu(\xi,dx)\Big(\beta G_{\beta}^{1}\Big(\frac{G_{\alpha}^{0}f}{G_{\gamma}^{0}1}\Big)\Big)^{**}(x).$$

Noting that $\Big\|\Big(\beta G_{\beta}^{1}\Big(\frac{G_{\alpha}^{0}f}{G_{\gamma}^{0}1}\Big)\Big)^{**}\Big\|\leqq\Big\|\frac{G_{\alpha}^{0}f}{G_{\gamma}^{0}1}\Big\|$, by Lemma 5.11 (2) and (5.41), letting $\beta\to\infty$ in (5.47), we have

(5.48) $$\lim_{\beta\to\infty}\boldsymbol{P}\Big(\frac{\beta G_{\beta}^{0}G_{\alpha}^{0}f}{G_{\gamma}^{0}1}\Big)(\xi)=\int_{D^{**}}\mu(\xi,dx)\hat{H}_{\alpha}f(x).$$

Therefore, (5.46) and (5.48) completes the proof of Theorem 5.1 (i). (iii) has been proved in Lemma 5.11 (3). (ii) follows from (i), (iii) and Lemma 5.13 by putting $\alpha=\gamma$ and $f=1$. This completes the proof. (Q. E. D.)

The next theorem is an immediate consequence of Theorem 3.2 and Theorem 5.1, which is an another form of the Feller-Ueno decomposition.

THEOREM 5.2. (*Feller-Ueno decomposition theorem*). *For any $\alpha>0$, $x\in S$ and $f\in C(S)$,*

$$G_{\alpha}f(x)=G_{\alpha}^{0}f(x)+H_{\alpha}K^{\alpha}\Big\{l\cdot f+\int_{D^{**}}\mu(\cdot,dx)\hat{H}_{\alpha}f(x)+\int_{D}\boldsymbol{Q}(\cdot,dx)\frac{G_{\alpha}^{0}f}{G_{\gamma}^{0}1}(x)\Big\}.$$

Put, for each $\xi\in V$,

$$S_{\xi}=\{\eta\in D^{**};\ \text{for any neighbourhood } U \text{ of } \xi \text{ in } S,\ \eta\in\overline{i^{**}(U\cap D)}\}.$$

Then, we have,

(5.49) $S_\xi=\{\eta\in D^{**}$; there exists a sequence $(x_n)_{n=1}^\infty$ in D such that

$$\lim_{n\to\infty} x_n=\xi \quad \text{and} \quad \lim_{n\to\infty} i^{**}(x_n)=\eta\} .$$

Note that the sets S_ξ $(\xi\in V)$ is contained in $D^{**}-i^{**}(D)$ if the process $\boldsymbol{M}^0$ associated with $\{G_\alpha^0;\ \alpha>0\}$ has continuous paths, or more generally if $C_\infty(D)\subset\bar{\mathcal{M}}_\infty$.[8)]

Now, we shall prove Theorem 5.3, which realizes the profound fact that Lemma 3.5 states and is more precise than Theorem 3.3. For that purpose, we prepare the following lemma. Let $\mathcal{L}_+=\{f\in C(D^{**})^+;$ for any $\alpha>0$, $\alpha G_{\alpha+1}^{**}f\leqq f\}$ and $\mathcal{L}=\{f_1-f_2;\ f_1,f_2\in\mathcal{L}_+\}$.

LEMMA 5.18. (1) $\{G_1^{**}f;\ f\in C(D^{**})\}\subset\mathcal{L}\subset C(D^{**})$,

(2) *af, $f+g$, $f\wedge g\in\mathcal{L}_+$ for any $f, g\in\mathcal{L}_+$ and $a\geqq 0$.*

(3) $\mathcal{L}_+\ni 1$.

(4) *$\mathcal{L}$ is a vector lattice of $C(D^{**})$.*

(5) *$\mathcal{L}$ separates the points of D^{**}.*

(6) *$\mathcal{L}$ is dense in $C(D^{**})$.*

PROOF. (1) and (3) follow from Lemma 5.9 (3) and Lemma 5.9 (2), respectively. Clearly, (2) holds. (4) follows from the following relation: $(f_1-f_2)\wedge(g_1-g_2)=(f_1+g_2)\wedge(f_2+g_1)-(f_2+g_2)$. (5) follows from (1) and Lemma 5.9 (5). By (1), (3), (4) and (5), the Stone-Weierstrass theorem implies that (6) holds. This completes the proof of Lemma 5.18. (Q. E. D.)

Under the hypothesis $D_b^{**}=\emptyset$, we prove the following interesting theorem, which solves the problem mentioned in the first paragraph in § 5.

THEOREM 5.3 (*support of $\mu(\xi, dx)$*). *If $D_b^{**}=\emptyset$, then*

$$\mu(\xi, D^{**}-S_\xi)=0 \qquad \textit{for any } \xi\in V .$$

PROOF. Since V is a compact metrizable space, there exists a countable dense subset $\{\xi_1,\xi_2,\cdots,\xi_k,\cdots\}$ in V. For any $\xi\in V$ and $\varepsilon>0$, put

$U(\xi,\varepsilon)=\{x\in S;\ \mathrm{dis}(x,\xi)<\varepsilon\}$ and $U(k,\xi)=U(\xi_k,\varepsilon)$ $(k=1,2,3,\cdots)$.

Then,

$$V=\bigcup_{k\geqq 1}(U(k,\varepsilon)\cap V), \tag{5.50}$$

$$S_\xi=\bigcap_{Q\ni\varepsilon>0}\overline{i^{**}(U(\xi,\varepsilon)\cap D)} \qquad (\text{for } \xi\in V). \tag{5.51}$$

Fix any k and $\varepsilon>0$, and then fix any $f\in C(D^{**})$ such that for any $x\in\overline{i^{**}(U(k,\varepsilon)\cap D)}$ $f(x)=0$. By the same method in Lemma 4.3 and Lemma 4.4 we can see that the hypothesis $D_b^{**}=\emptyset$ implies that

$$\|\alpha G_\alpha^{**}f-f\|\longrightarrow 0 \qquad (\alpha\to\infty). \tag{5.52}$$

8) See example 2 in § 7. Though the process $\boldsymbol{M}^0$ in this example has no continuous paths, the fact that $C_\infty(D)\subset\bar{\mathcal{M}}_\infty$ holds.

Thus, for any positive number $\delta>0$, there exists $\alpha_0(>0)$ such that for any $\alpha\geqq\alpha_0$ and $x\in\overline{i^{**}(U(k,\varepsilon)\cap D)}$, $|\alpha G_\alpha^{**}f(x)|\leqq\delta$. Since this implies that for any $x\in U(k,\varepsilon)\cap D$ and $\alpha\geqq\alpha_0$,

$$\left|\frac{\alpha G_\alpha^0(f\circ i^{**}G_\gamma^0 1)}{G_\gamma^0 1}(x)\right|\leqq\delta,$$

we have, by Theorem 3.3 (local character of $\boldsymbol{P}$),

$$\left|\boldsymbol{P}\left(\frac{\alpha G_\alpha^0(f\circ i^{**}G_\gamma^0 1)}{G_\gamma^0 1}\right)(\xi)\right|\leqq\delta\boldsymbol{P}1(\xi),\quad \nu\text{-a. e. } \xi\in U(k,\varepsilon)\cap V. \tag{5.53}$$

On the other hand, since by Theorem 5.1

$$\boldsymbol{P}\left(\frac{\alpha G_\alpha^0(f\circ i^{**}G_\gamma^0 1)}{G_\gamma^0 1}\right)(\xi)=\int_{D^{**}}\mu(\xi,dx)\alpha G_\alpha^{**}f(x),\quad \nu\text{-a. e. } \xi\in V,$$

we have, by (5.53),

$$\left|\int_{D^{**}}\mu(\xi,dx)\alpha G_\alpha^{**}f(x)\right|\leqq\delta,\quad \nu\text{-a. e. } \xi\in U(k,\varepsilon)\cap V. \tag{5.54}$$

Therefore, by (5.52) and the bounded convergence theorem, letting $\alpha\to\infty$ and then $\delta\to 0$ in (5.54), we have

$$\int_{D^{**}}\mu(\xi,dx)f(x)=0,\quad \nu\text{-a. e. } \xi\in U(k,\varepsilon)\cap V. \tag{5.55}$$

Next, fix any $f\in C_0^+(D^{**}-\overline{i^{**}(U(k,\varepsilon)\cap D)})$. Then, since f can be extended to D^{**} continuously as $f(x)=0$, $x\in\overline{i^{**}(U(k,\varepsilon)\cap D)}$, we have, by (5.55),

$$\int_{D^{**}-\overline{i^{**}(U(k,\varepsilon)\cap D)}}\mu(\xi,dx)f(x)=0,\quad \nu\text{-a. e. } \xi\in U(k,\varepsilon)\cap V. \tag{5.56}$$

Since D^{**} is a compact metrizable space, (5.56) implies that

$$\mu(\xi, D^{**}-\overline{i^{**}(U(k,\varepsilon)\cap D)})=0,\quad \nu\text{-a. e. } \xi\in U(k,\varepsilon)\cap V. \tag{5.57}$$

Therefore, by (5.57), there exists a Borel subset B_k^ε of $U(k,\varepsilon)\cap V$ such that

$$\nu(B_k^\varepsilon)=0 \quad\text{and} \tag{5.58}$$

$$\mu(\xi, D^{**}-\overline{i^{**}(U(k,\varepsilon)\cap D)})=0 \quad \text{for any } \xi\in U(k,\varepsilon)\cap V-B_k^\varepsilon.$$

Set

$$B=\bigcup_{\substack{k\geqq 1\\ \boldsymbol{Q}\ni\varepsilon>0}}B_k^\varepsilon. \tag{5.59}$$

Then, by (5.58),

$$B\subset V \quad\text{and} \tag{5.60}$$

$$\nu(B)=0.$$

Next, fix any $\xi \in V-B$. Since for any rational number $\varepsilon>0$ there exists k_0 such that $\xi \in U(k_0, \varepsilon/2)$ and so $U(k_0, \varepsilon/2) \subset U(\xi, \varepsilon)$, noting that

$$D^{**}-\overline{i^{**}(U(\xi, \varepsilon) \cap D)} \subset D^{**}-\overline{i^{**}(U(k_0, \varepsilon/2))}\,, \qquad \xi \in U(k_0, \varepsilon/2) \cap V-B_{k_0}^{\varepsilon/2}\,, \tag{5.61}$$

we have, by (5.58) and (5.61),

$$\mu(\xi, D^{**}-\overline{i^{**}(U(\xi, \varepsilon) \cap D)})=0\,. \tag{5.62}$$

Consequently, by (5.51) and (5.62), we have

$$\mu(\xi, D^{**}-S_{\xi})=0 \qquad \text{for any } \xi \in V-B\,. \tag{5.63}$$

(5.60) and (5.63) complete the proof of Theorem 5.3. (Q. E. D.)

We are interested in Theorem 5.3, for it states that the supports of the measures $\mu(\xi, \cdot)$ $(\xi \in V)$ which depend on our process $\boldsymbol{M}$ to characterize are contained in the sets $S_{\xi}(\xi \in V)$ which depend only on the minimal process $\boldsymbol{M}^{\min}$ or the process $\boldsymbol{M}^0$ associated with $\{G_\alpha^0\,;\, \alpha>0\}$, under the condition $D_b^{**}=\emptyset$ (this condition depends also on $\boldsymbol{M}^0$), even if we don't suppose that the path functions of $\boldsymbol{M}$ are continuous. Next, the question arises whether the supports of the measures $\mu(\xi, \cdot)$ $(\xi \in V)$ are contained in $D^{**}-i^{**}(D)$. Since the set $S_{\xi}(\xi \in V)$ is not necessarily contained in $D^{**}-i^{**}(D)$ even under the condition $D_b^{**}=\emptyset$, Theorem 5.3 does not answer the above question. The answer is found in

THEOREM 5.4. *If the following condition:*

$$C_\infty(D) \subset \mathfrak{M}_\infty \tag{**}$$

is satisfied, then

(1) i^{**} *is a homeomorphism from* D *onto* $i^{**}(D)$ *and* $i^{**}(D)$ *is open in* D^{**},

(2) $\mu(\xi, i^{**}(D))=0$ *for any* $\xi \in V$.

PROOF. Since (1) will be proved in § 6 (Theorem 6.3) in a general setup, we omit its proof here. (2) can be proved as follows: by Theorem 3.1 (the boundedness of $\boldsymbol{P}$) and Lemma 5.11, we have,

$$\boldsymbol{P}\varphi(\xi)=\int_{D^{**}} \mu(\xi, dx)\varphi^{**}(x), \ \nu\text{-a. e. } \xi \in V \quad \text{for any } \varphi \in \mathfrak{M}_\infty\,. \tag{5.64}$$

On the other hand, by Theorem 3.3 (local character) and the assumption (**), we have,

$$\boldsymbol{P}\varphi=0 \qquad \text{for any } \varphi \in C_\infty(D)\,. \tag{5.65}$$

Since the continuous extension φ^{**} of $\varphi \in C_\infty(D)$ is zero on $D^{**}-i^{**}(D)$ by (1), we have, by (5.64) and (5.65),

$$\int_{i^{**}(D)} \mu(\xi, dx)\varphi^{**}(x)=0, \ \nu\text{-a. e. } \xi \in V \qquad \text{for any } \varphi \in C_\infty(D)\,. \tag{5.66}$$

Therefore, the assertion (1) and (5.66) imply that $\mu(\xi, i^{**}(D))=0$ for ν-a.e. $\xi\in V$. This completes the proof of Theorem 5.4 (2). (Q.E.D.)

Finally, we shall resolve the question stated in the first paragraph of § 5. Though we give the answer in such a form as the following Theorem 5.5 for making our results complete, the mechanics that M. Motoo's Feller-Ueno decomposition theorem becomes equal to our Feller-Ueno decomposition theorem under the conditions ($\boldsymbol{M}^{\min}.4$), ($\boldsymbol{M}^{\min}.5$) and ($\boldsymbol{M}^{\min}.6$) will be revealed in the second proof of Theorem 5.5 under the more additional conditions. To revote the confusions in notations, let M. Motoo's $\hat{H}_\alpha$ in ($\boldsymbol{M}^{\min}.5$) and our $\hat{H}_\alpha$ be $\hat{H}_\alpha^{(1)}$ and $\hat{H}_\alpha^{(2)}$, respectively.

THEOREM 5.5. *If the additional conditions* ($\boldsymbol{M}^{\min}.4$) *and* ($\boldsymbol{M}^{\min}.5$) *are satisfied, then*

$$m(\xi)\hat{H}_\alpha^{(1)}f(\xi)=\int_{D^{**}}\mu(\xi, dx)\hat{H}_\alpha^{(2)}[f]_D(x), \quad \nu\text{-}a.e.\ \xi\in V$$

$$\text{for any } \alpha>0 \text{ and } f\in C(S).$$

FIRST PROOF. By Theorem 3.2 (Feller-Ueno decomposition) and M. Motoo's Feller-Ueno decomposition theorem ([17], Theorem 3), we have

$$G_\alpha^0 f(x)+H_\alpha K^\alpha\Big\{l\cdot f+(\boldsymbol{P}+\boldsymbol{Q})\Big(\frac{G_\alpha^0 f}{G_\gamma^0 1}\Big)\Big\}(x)$$

$$=G_\alpha^0 f(x)+H_\alpha K^\alpha\Big\{l\cdot f+m\hat{H}_\alpha^{(1)}f+\boldsymbol{Q}\Big(\frac{G_\alpha^0 f}{G_\gamma^0 1}\Big)\Big\}(x)$$

for any $\alpha>0$, $f\in C(S)$ and $x\in S$. Therefore, we have

$$H_\alpha K^\alpha\Big\{\boldsymbol{P}\Big(\frac{G_\alpha^0 f}{G_\gamma^0 1}\Big)\Big\}(x)=H_\alpha K^\alpha\{m\hat{H}_\alpha^{(1)}f\}(x) \tag{5.67}$$

for any $\alpha>0$, $f\in C(S)$ and $x\in S$. Noting that $\boldsymbol{P}\Big(\frac{G_\alpha^0 f}{G_\gamma^0 1}\Big)$ and $m\hat{H}_\alpha^{(1)}f$ are bounded, by applying Lemma 3.5 in [14] in (5.67), we have

$$\boldsymbol{P}\Big(\frac{G_\alpha^0 f}{G_\gamma^0 1}\Big)(\xi)=m(\xi)\hat{H}_\alpha^{(1)}f(\xi), \quad \nu\text{-a.e. } \xi\in V \tag{5.68}$$

for any $\alpha>0$ and $f\in C(S)$. Thus, by (5.68) and Theorem 5.1, we have

$$\int_{D^{**}}\mu(\xi, dx)\hat{H}_\alpha^{(2)}[f]_D(x)=m(\xi)\hat{H}_\alpha^{(1)}f(\xi), \quad \nu\text{-a.e. } \xi\in V$$

for any $\alpha>0$ and $f\in C(S)$.

SECOND PROOF. Moreover, we shall assume the condition ($\boldsymbol{M}^{\min}.6$) and the following condition:

$$S \text{ is compact and } \hat{H}_\alpha\chi_V=0 \quad \text{for any } \alpha>0. \tag{***}$$

Before giving the second proof, we prepare some lemmas.

LEMMA 5.19. $\{\hat{H}_\alpha^{(1)}f;\ \alpha>0,\ f\in C_\infty(D)\}$ *is dense in* $C(S)$.

PROOF. Fix any $f\in C(S)$. No generality is lost in assuming $f\in C(S)^+$. Choosing a sequence $(f_n)_{n=1}^\infty$ in $C_\infty(D)$ such that f_n increases to $\frac{f}{G_\gamma^0 1}$ in D, by the hypothesis (***) and the monotone convergence theorem, we have

$$0\leqq \hat{H}_\alpha^{(1)}(f_n G_\gamma^0 1)\underset{(n)}{\uparrow}\hat{H}_\alpha^{(1)}f \qquad \text{for any } \alpha>0 \text{ and } x\in S.$$

Therefore, by the Dini theorem, this convergence is uniform on S. This fact and ($\boldsymbol{M}^{\min}$.6) complete the proof. (Q. E. D.)

LEMMA 5.20. i^{**} *is a homeomorphism from* D *onto* $i^{**}(D)$.

PROOF. Assume that Lemma 5.20 is false. Then, by the compactness of S, there exist a sequence $(x_n)_{n=1}^\infty$ in D and two points $x\in D$, $\xi\in V$ such that $x_n\to\xi$ and $i^{**}(x_n)\to i^{**}(x)$. For any $f\in C_\infty(D)$ and $\alpha>0$,

$$\begin{aligned}\hat{H}_\alpha^{(1)}f(x)&=\frac{G_\alpha^0 f}{G_\gamma^0 1}(x)\\&=\Big(\frac{G_\alpha^0 f}{G_\gamma^0 1}\Big)^{**}(i^{**}(x))\\&=\lim_{n\to\infty}\Big(\frac{G_\alpha^0 f}{G_\gamma^0 1}\Big)^{**}(i^{**}(x_n))\\&=\lim_{n\to\infty}\frac{G_\alpha^0 f}{G_\gamma^0 1}(x_n)=\hat{H}_\alpha^{(1)}f(\xi).\end{aligned}$$

Therefore, by Lemma 5.19, we have $x=\xi$. This is absurd. (Q.E.D.)

LEMMA 5.21. *There exists a continuous mapping* Φ *from* D^{**} *onto* S *such that* $\Phi(i^{**}(x))=x$ *for any* $x\in D$ *and* $\Phi^{-1}(\{\xi\})=S_\xi$ *for any* $\xi\in V$.

PROOF. We claim that $S_{\xi_1}\cap S_{\xi_2}=\emptyset$ for any $\xi_1\neq\xi_2$. Assume that there exists a point $\eta\in S_{\xi_1}\cap S_{\xi_2}$ for some $\xi_1\neq\xi_2$. Then, there exist two sequences $(x_n^{(i)})_{n=1}^\infty$ in D $(i=1,2)$ such that $x_n^{(i)}\to\xi_i$ and $i^{**}(x_n^{(i)})\to\eta$ as $n\to\infty$. For any $f\in C_\infty(D)$ and $\alpha>0$,

$$\begin{aligned}\hat{H}_\alpha^{(1)}f(\xi_i)&=\lim_{n\to\infty}\hat{H}_\alpha^{(1)}f(x_n^{(i)})\\&=\lim_{n\to\infty}\Big(\frac{G_\alpha^0 f}{G_\gamma^0 1}\Big)(x_n^{(i)})\\&=\lim_{n\to\infty}\Big(\frac{G_\alpha^0 f}{G_\gamma^0 1}\Big)^{**}(i^{**}(x_n^{(i)}))=\Big(\frac{G_\alpha^0 f}{G_\gamma^0 1}\Big)^{**}(\eta)\ \ (i=1,2).\end{aligned}$$

Therefore, by Lemma 5.19, $\xi_1=\xi_2$. This is absurd. Since a family $((S_\xi)\,;\,\xi\in V)$ is a partition of $D^{**}-i^{**}(D)$ by Lemma 5.20 and the fact proved above, we can define a mapping Φ from D^{**} into S by setting $\Phi(i^{**}(x))=x$ and $\Phi(\eta)=\xi$ for $\eta\in S_\xi$. We can see easily that this Φ is our desiring mapping. (Q.E.D.)

LEMMA 5.22. $\# S_\xi = 1$ *for any* $\xi \in V$.

PROOF. Assume that there exist two points $\eta_1 \neq \eta_2$ in S_ξ for some $\xi \in V$. Then, there exist two sequences $(x_n^{(i)})_{n=1}^\infty$ in D $(i=1,2)$ such that $x_n^{(i)} \to \xi$ and $i^{**}(x_n^{(i)}) \to \eta_i$ as $n \to \infty$.

For any $\alpha > 0$ and $f \in C_\infty(D)$,

$$\left(\frac{G_\alpha^0 f}{G_\gamma^0 1}\right)^{**}(\eta_i) = \lim_{n\to\infty}\left(\frac{G_\alpha^0 f}{G_\gamma^0 1}\right)^{**}(i^{**}(x_n^{(i)}))$$
$$= \lim_{n\to\infty}\frac{G_\alpha^0 f}{G_\gamma^0 1}(x_n^{(i)})$$
$$= \hat{H}_\alpha^{(1)} f(\xi) \qquad (i=1,2).$$

Therefore, by Lemma 5.8 (5), we have $\eta_1 = \eta_2$. This is absurd. Thus, noting that $\# S_\xi \geqq 1$, the proof of Lemma 5.22 is completed. (Q. E. D.)

LEMMA 5.23. Φ *in Lemma* 5.21 *is a homeomorphism from* D^{**} *onto* S.

PROOF. By Lemma 5.22, Φ is injective. Therefore, noting that D^{**} is compact and S is Hausdorff, by Lemma 5.21, we have Lemma 5.23. (Q. E. D.)

LEMMA 5.24. *For any* $f \in C(S)$, $\lim_{\alpha\to\infty}\left\|\alpha\frac{G_\alpha^0(fG_\gamma^0 1)}{G_\gamma^0 1} - f\right\|_D = 0$.

PROOF. Fix any positive number $\varepsilon > 0$. By Lemma 5.4 (8) and Lemma 5.19, we can find $g \in C(D)$ such that $\|f - G_\gamma^1 g\| < \varepsilon$. Therefore, by Lemma 5.4, we have

$$\left\|\alpha\frac{G_{\alpha+\gamma}^0(fG_\gamma^0 1)}{G_\gamma^0 1} - f\right\|_D \tag{5.69}$$
$$= \|\alpha G_{\alpha+\gamma}^1([f]_D) - f\|_D$$
$$\leqq \|\alpha G_{\alpha+\gamma}^1([f]_D - G_\gamma^1 g)\|_D + \|\alpha G_{\alpha+\gamma}^1 G_\gamma^1 g - G_\gamma^1 g\|_D + \|G_\gamma^1 g - f\|_D$$
$$\leqq 2\|f - G_\gamma^1 g\|_D + \|G_{\alpha+\gamma}^1 g\|$$
$$\leqq 2\varepsilon + \frac{\|g\|}{\alpha+\gamma}.$$

Thus, letting $\alpha \to \infty$ and $\varepsilon \to 0$ in (5.69), we have Lemma 5.24. (Q. E. D.)

LEMMA 5.25. $$D_b^{**} = \emptyset.$$

PROOF. Noting that $i^{**}(x) = \Phi^{-1}(x)$ for any $x \in D$ by Lemma 5.21 and Lemma 5.23, we have, for any $f \in C(D^{**})$ and $\alpha > 0$,

$$\|\alpha G_\alpha^{**} f - f\| = \left\|\alpha\frac{G_\alpha^0(f\circ i^{**} G_\gamma^0 1)}{G_\gamma^0 1} - f\circ i^{**}\right\|_D \tag{5.70}$$
$$= \left\|\alpha\frac{G_\alpha^0(f\circ \Phi^{-1} G_\gamma^0 1)}{G_\gamma^0 1} - f\circ \Phi^{-1}\right\|_D.$$

Since $f\circ\Phi^{-1} \in C(S)$ for any $f \in C(D^{**})$ by Lemma 5.23, therefore, we have, by applying Lemma 5.24 in (5.70), $\lim_{\alpha\to\infty}\|\alpha G_\alpha^{**}f-f\|=0$. This implies that $D_\delta^{**}=\emptyset$. (Q. E. D.)

Now, we are able to give the second proof of Theorem 5.5, under the additional conditions ($\boldsymbol{M}^{\min}$.6) and (***), in the following way: Fix any $f\in C(S)$ and $\alpha>0$. Since $D_\delta^{**}=\emptyset$ by Lemma 5.25, we have, by Lemma 5.16 and Theorem 5.3,

$$\int_{D^{**}}\mu(\xi,dx)\hat{H}_\alpha^{(2)}[f]_D(x)=\int_{S_\xi}\mu(\xi,dx)\hat{H}_\alpha^{(2)}[f]_D(x). \tag{5.71}$$

Since $\hat{H}_\alpha^{(2)}[f]_D(x)=\hat{H}_\alpha^{(1)}f(\xi)$ for any $x\in S_\xi$ and $\mu(\xi,D^{**})=m(\xi)$ by Theorem 3.1 (3) and Theorem 5.1 (ii), we have, by (5.71),

$$\int_{D^{**}}\mu(\xi,dx)\hat{H}_\alpha^{(2)}[f]_D(x)=m(\xi)\hat{H}_\alpha^{(1)}f(\xi),\quad \nu\text{-a. e. } \xi\in V.$$

(Q. E. D.)

§ 6. Compactification and Completion; The characterization of D^{**}.

In order to reveal the operator $\boldsymbol{P}$, in § 5, we have introduced the space D^{**} as the field which represents $\boldsymbol{P}$ (Theorem 5.1). In fact, D^{**} was the quotient space of an $\mathcal{M}_\infty$-compactification D^* of D by the following equivalence relation——an $\mathcal{M}_\infty^*$-equivalence relation: for $x, y\in D^*$, $x\sim y$ if and only if $\varphi^*(x)=\varphi^*(y)$ for any $\varphi\in\mathcal{M}_\infty$. And by Lemma 5.8, the space D^{**} satisfies the following properties:

(6.1) D^{**} is a compact Hausdorff space.

(6.2) There exists a continuous and injective mapping i^{**} from D into D^{**} such that $i^{**}(D)$ is dense in D^{**}.

(6.3) Each element $\varphi\in\mathcal{M}_\infty$ can be extended to a continuous function φ^{**} on D^{**}, that is, $\varphi^{**}\circ i^{**}=\varphi$.

(6.4) $\mathcal{M}_\infty^{**}=\{\varphi^{**};\ \varphi\in\mathcal{M}_\infty\}$ separates the points of D^{**}.

In this section, we shall prove the converse——the characterization of the space D^{**}, that is, any space D^{**} satisfying the above (6.1)~(6.4) is the quotient space of an $\mathcal{M}_\infty$-compactification D^* of D by an $\mathcal{M}_\infty^*$-equivalence relation (Theorem 6.2). This is an immediate consequence of the following uniqueness theorem (Theorem 6.1): the space D^{**} satisfying (6.1)~(6.4) is unique up to homeomorphism.

Since the essence of the problems stated above does not relate to a minimal process $\boldsymbol{M}^{\min}$, we shall take the general formulation. The following lemma (it had been used in § 5) can be found in [3].

LEMMA 6.1. *Let D be a locally compact Hausdorff space and Q be a subset* *of* $C(D)$. *Then, there exists a topological space* D^* *such that*

(6.5) D^* *is a compact Hausdorff space,*

(6.6) *there exists a continuous and injective mapping* i^* *from* D *into* D^* *such that* $i^*(D)$ *is open and dense in* D^* *and* i^* *is a homeomorphism from* D *onto* $i^*(D)$,

(6.7) *each element* $f \in Q$ *can be extended to a continuous function* f^* *on* D^*, *that is,* $f^* \circ i^* = f$,

(6.8) $Q^* = \{f^*; f \in Q\}$ *separates the points of* $D^* - i^*(D)$.

Moreover, such a D^* *is unique in a following sense. Let* (D_1^*, i_1^*) *and* (D_2^*, i_2^*) *be systems satisfying* (6.5)$\sim$(6.8). *Then, there exists a homeomorphism* Φ *from* D_1^* *onto* D_2^* *such that* $\Phi \circ i_1^* = i_2^*$.

DEFINITION 6.1. We shall call (D^*, i^*) a *Q-compactification* of D.

REMARK 6.1. Q^* does not always separate the points of D^* (Theorem 6.3).

Next, we shall introduce the notion of a Q-completion of D. Such a notion has not been used explicitly ([**4**], [**5**], [**11**], [**13**]). Let D be an abstract space ($\neq \emptyset$) and Q be a function space on D such that separates the points of D. Then, introduce the uniform structure on D by Q in the following way. Set a parameter set $A = \{\alpha = \langle f_1, \cdots, f_n; \varepsilon \rangle; f_k \in Q, \varepsilon > 0, k = 1, 2, \cdots, n\}$ and $U_\alpha(x) = \{y \in D; |f_k(y) - f_k(x)| < \varepsilon, k=1, 2, 3, \cdots, n\}$ for $\alpha \in A$ and $x \in D$. Then, we can easily see that $\mathfrak{U}_Q = \{U_\alpha(x); \alpha \in A, x \in D\}$ satisfies the axiom of uniform structure ([**10**]). Thus, $(D, \mathfrak{U}_Q)$ becomes a uniform space with the above structure. Therefore, there exists a complete uniform space D^{**} such that there exists a uniformly continuous and injective mapping i^{**} from D into D^{**} with $\overline{i^{**}(D)} = D^{**}$ and i^{**} is a uniform homeomorphism from D onto $i^{**}(D)$. Therefore, noting that $f \circ i^{**-1}$ is uniformly continuous in $i^{**}(D)$ for any $f \in Q$, any $f \in Q$ can be extended to a uniformly continuous function f^{**} on D^{**}, that is, $f^{**} \circ i^{**} = f$. Then, the function space $Q^{**} = \{f^{**}; f \in Q\}$ separates the points of D^{**}. This can be proved as follows. Consider $\eta_1, \eta_2 \in D^{**}$, such that $f^{**}(\eta_1) = f^{**}(\eta_2)$ for any $f \in Q$. Since $i^{**}(D)$ is dense in D^{**}, there exist two nets $(x_\lambda)_{\lambda \in \Lambda}$ and $(y_\mu)_{\mu \in M}$ such that $\lim_{\lambda \in \Lambda} i^{**}(x_\lambda) = \eta_1$ and $\lim_{\mu \in M} i^{**}(y_\mu) = \eta_2$. Therefore, $\lim_{\lambda \in \Lambda} f(x_\lambda) = \lim_{\lambda \in \Lambda} f^{**}(i^{**}(x_\lambda)) = f^{**}(\eta_1) = f^{**}(\eta_2) = \lim_{\mu \in M} f^{**}(i^{**}(y_\mu)) = \lim_{\mu \in M} f(y_\mu)$. Thus, by the definition of $\mathfrak{U}_Q$, $(x_\lambda)_{\lambda \in \Lambda}$ and $(y_\mu)_{\mu \in M}$ are equivalent ([**10**]). Since i^{**} is uniformly continuous, $(i^{**}(x_\lambda))_{\lambda \in \Lambda}$ and $(i^{**}(y_\mu))_{\mu \in M}$ are equivalent. Therefore, noting that $\lim_{\lambda \in \Lambda} i^{**}(x_\lambda) = \eta_1$ and $\lim_{\mu \in M} i^{**}(y_\mu) = \eta_2$, we have $\eta_1 = \eta_2$. Moreover, we can prove that if any element $f \in Q$ is bounded, $(D, \mathfrak{U}_Q)$ is totally bounded and so D^{**} is compact. Thus, we have the first part of the following theorem.

THEOREM 6.1. *Let D be a topological space ($\neq\emptyset$) and Q be a subset of $C(D)$ that separates the points of D. Then, there exists a topological space D^{**} such that*

(6.9) D^{**} *is a compact Hausdorff space,*

(6.10) *there exists a continuous and injective mapping* i^{**} *from* D *into* D^{**} *with* $\overline{i^{**}(D)}=D^{**}$,

(6.11) *each element* $f\in Q$ *can be extended to a continuous function* f^{**} *on* D^{**}, *that is,* $f^{**}\circ i^{**}=f$,

(6.12) $Q^{**}=\{f^{**}\,;\,f\in Q\}$ *separates the points of* D^{**}.

Moreover, such a D^{**} *is unique in the same sense as in Lemma* 6.1.

DEFINITION 6.2. We shall call $D^{**}((D^{**}, i^{**}))$ a *Q-completion* of D.

For completing the proof of Theorem 6.1, we have only to prove the uniqueness of such a D^{**}——the essential part of this theorem. Our aim is to find a homeomorphism Φ from D_1^{**} onto D_2^{**} such that

(*) $\Phi\circ i_1^{**}=i_2^{**}$, given systems (D_1^{**}, i_1^{**}) and (D_2^{**}, i_2^{**})

satisfying (6.9)~(6.12). Before proving (*), we shall prepare the next lemma.

LEMMA 6.2. *Any space* $D^{**}((D^{**}, i^{**}))$ *satisfying* (6.9)~(6.12) *has the following property:*

Let $(x_\lambda)_{\lambda\in\Lambda}$ *be a net in* D *such that there exists a*

(6.13) $$\lim_{\lambda\in\Lambda} f(x_\lambda)\ (\in \boldsymbol{R}) \qquad \text{for any } f\in Q\,.$$

Then, there exists uniquely a point $\eta\in D^{**}$ *such that* $\lim_{\lambda\in\Lambda} i^{**}(x_\lambda)=\eta$.

PROOF. Let η_1 and η_2 be sub-convergent points of $(i^{**}(x_\lambda))_{\lambda\in\Lambda}$ (Since D^{**} is compact, there exists at least such a point ([**10**])). We shall write $i^{**}(x_\lambda)\dashrightarrow\eta_1$ and $i^{**}(x_\lambda)\dashrightarrow\eta_2$. Fix any $f\in Q$. Since $f^{**}\in C(D^{**})$ and $f^{**}\circ i^{**}=f$, we have

$$f(x_\lambda)\dashrightarrow f^{**}(\eta_1) \quad\text{and}\quad f(x_\lambda)\dashrightarrow f^{**}(\eta_2)\,.$$

Therefore, by the hypothesis, $f^{**}(\eta_1)=f^{**}(\eta_2)$. Thus, (6.12) implies that $\eta_1=\eta_2$. This shows that there exists only a subconvergent point of $(i^{**}(x_\lambda))_{\lambda\in\Lambda}$. By the compactness of D^{**}, this completes the proof of Lemma 6.2.

Now, we shall prove (*). Denote Q^{**} in D_1^{**} and D_2^{**} by Q_1^{**} and Q_2^{**} respectively. Take any $\eta_1\in D_1^{**}$. Moreover, take any net $(x_\lambda)_{\lambda\in\Lambda}$ in D such that $\lim_{\lambda\in\Lambda} i_1^{**}(x_\lambda)=\eta_1$. (By (6.10), there exists at least such a net.). Then, for any $f\in Q$, by (6.11), $\lim_{\lambda\in\Lambda} f(x_\lambda)=f_1^{**}(\eta_1)$. Therefore, by Lemma 6.2, we can find a point $\eta_2\in D_2^{**}$ such that $\lim_{\lambda\in\Lambda} i_2^{**}(x_\lambda)=\eta_2$. Such a point η_2 does not depend upon the choice of a net $(x_\lambda)_{\lambda\in\Lambda}$. This can be proved as follows. Consider another net $(y_\mu)_{\mu\in M}$ in D such that $\lim_{\mu\in M} i_1^{**}(y_\mu)=\eta_1$ and $\lim_{\mu\in M} i_2^{**}(y_\mu)=\tilde{\eta}_2$. Then, for any $f\in Q$, by (6.11), $f_2^{**}(\tilde{\eta}_2)=\lim_{\mu\in M} f_2^{**}(i_2^{**}(y_\mu))=\lim_{\mu\in M} f(y_\mu)=\lim_{\mu\in M} f_1^{**}(i_1^{**}(y_\mu))$ $=f_1^{**}(\eta_1)$. In the same way, $f_2^{**}(\eta_2)=f_1^{**}(\eta_1)$. Therefore, $f_2^{**}(\eta_2)=f_2^{**}(\tilde{\eta}_2)$ for any $f\in Q$. (6.12) implies that $\eta_2=\tilde{\eta}_2$. Thus, we can define a mapping Φ from D_1^{**} into D_2^{**} as $\Phi(\eta_1)=\eta_2$. Then, by the definition of Φ, we can easily see that

$$\Phi\circ i_1^{**}=i_2^{**} \tag{6.14}$$

and

$$f_2^{**}\circ\Phi=f_1^{**} \qquad \text{for any } f\in Q\,. \tag{6.15}$$

From (6.12) and (6.15), the injectiveness of Φ follows. For the proof of the surjectiveness of Φ, take any $\eta_2\in D_2^{**}$. By (6.10), there exists a net $(x_\lambda)_{\lambda\in\Lambda}$ in D such that $\lim_{\lambda\in\Lambda} i_2^{**}(x_\lambda)=\eta_2$. Then, for any $f\in Q$, by (6.11), $\lim_{\lambda\in\Lambda} f(x_\lambda)$ $=\lim_{\lambda\in\Lambda} f_2^{**}(i_2(x_\lambda))=f_2^{**}(\eta_2)$. Therefore, by Lemma 6.2, there exists a point $\eta_1\in D_1^{**}$ such that $\lim_{\lambda\in\Lambda} i_1^{**}(x_\lambda)=\eta_1$. Thus, by the definition of Φ, $\Phi(\eta_1)=\eta_2$. Finally, for the proof of the continuity of Φ, consider any net $(\eta_\lambda)_{\lambda\in\Lambda}$ and any point η_1 in D_1^{**} such that $\lim_{\lambda\in\Lambda}\eta_\lambda=\eta_1$. Then, by (6.15), for any $f\in Q$, we have

$$\lim_{\lambda\in\Lambda} f_2^{**}(\Phi(\eta_\lambda))=\lim_{\lambda\in\Lambda} f_1^{**}(\eta_\lambda)=f_1^{**}(\eta_1)=f_2^{**}(\Phi(\eta_1))\,. \tag{6.16}$$

Let $\eta_2\in D_2^{**}$ be any subconvergent point of $(\Phi(\eta_\lambda))_{\lambda\in\Lambda}$. That is, $\Phi(\eta_\lambda)\dashrightarrow\eta_2$. Then, by (6.11),

$$f_2^{**}(\Phi(\eta_\lambda))\dashrightarrow f_2^{**}(\eta_2)\,.$$

Therefore, by (6.16), $f_2^{**}(\Phi(\eta_1))=f_2^{**}(\eta_2)$. Thus, by (6.12), $\Phi(\eta_1)=\eta_2$. Since D_2^{**} is compact, we have $\lim_{\lambda\in\Lambda}\Phi(\eta_\lambda)=\Phi(\eta_1)$. This completes the proof of Theorem 6.1.

REMARK 6.2. For a Q-completion (D^{**}, i^{**}), i^{**} is not always a homeomorphism from D onto $i^{**}(D)$ (Theorem 6.3).

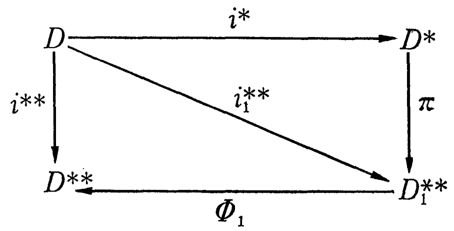
Finally, we shall examine the relation between a Q-compactification and a Q-completion. Let D be a locally compact Hausdorff space and Q be a subset of $C(D)$ that separates the points of D. Let (D^*, i^*) and (D^{**}, i^{**}) be a Q-compactification of D and a Q-completion respectively. Then, we are now able to prove the following main theorem in § 6.

THEOREM 6.2. *There uniquely exists a continuous mapping Φ from D^* onto D^{**} such that $\Phi\circ i^*=i^{**}$. Moreover, Φ has the following properties:*

(i) *Φ is injective as a mapping from $i^*(D)$ into D^{**} or from $D^*-i^*(D)$ into D^{**}.*

(ii) $f^{**} \circ \Phi = f^*$ *for any* $f \in Q$.

PROOF. Introduce a Q^*-equivalence relation on D^* and let D_1^{**} be the quotient space of D^* by this equivalence relation and π be a canonical projection from D^* onto D_1^{**}. Set $i_1^{**} = \pi \circ i^*$. Then, we can easily see that (D_1^{**}, i_1^{**}) satisfies (6.9)$\sim$(6.12). Therefore, by Theorem 6.1, there exists a



homeomorphism Φ_1 from D_1^{**} onto D^{**} such that $\Phi_1 \circ i_1^{**} = i^{**}$. Put $\Phi = \Phi_1 \circ \pi$. Then, $\Phi \circ i^* = \Phi_1 \circ \pi \circ i^* = \Phi_1 \circ i_1^* = i^{**}$. Clearly Φ is continuous from D^* onto D^{**}. This completes the proof of the first part of Theorem 6.2. The second part can be proved as follows. The first part of (i) follows from the injectiveness of i^{**} and $\Phi \circ i^* = i^{**}$. For the proof of the second part of (i), by (6.8), we have only to prove (ii). (ii) can be proved as follows. Fix any $f \in Q$. By (6.7) and (6.11), $(f^{**} \circ \Phi) \circ i^* = f^{**} \circ (\Phi \circ i^*) = f^{**} \circ i^{**} = f$ and $f^* \circ i^* = f$. Since $f^{**} \circ \Phi$, $f^* \in C(D^*)$ and $i^*(D)$ is dense in D^*, we have $f^{**} \circ \Phi = f^*$. This completes the proof of Theorem 6.2.

The following theorem answers Remark 6.1 and Remark 6.2.

THEOREM 6.3. *The following propositions are equivalent.*

(i) Q^* *separates the points of* D^*.

(ii) i^{**} *is a homeomorphism from* D *onto* $i^{**}(D)$.

Then, D^* *is homeomorphic to* D^{**}.

PROOF. In the proof of Theorem 6.2, we have seen that $\Phi = \Phi_1 \circ \pi$, where Φ_1 is a homeomorphism from D_1^{**} onto D^{**} and π is a canonical projection from D^* onto D_1^{**}. (i) $\Rightarrow$ (ii): Then, π is injective and is a homeomorphism from D^* onto D_1^{**}. Therefore, Φ is a homeomorphism from D^* onto D^{**}. Thus, by $i^{**} = \Phi \circ i^*$ and (6.6), we have (ii). (ii) $\Rightarrow$ (i): Since Q^* separates the points of $i^*(D)$ by (6.7) and separates the points of $D^* - i^*(D)$ by (6.8), it suffices to prove that for any $x \in D$ and $\xi \in D^* - i^*(D)$, there exists $f \in Q$ such that $f(x) \neq f^*(\xi)$. Assume the contrary. Then, there exist $x \in D$ and $\xi \in D^* - i^*(D)$, such that $\pi(i^*(x)) = \pi(\xi)$. By (6.6), there exists a net $(x_\lambda)_{\lambda \in A}$ in D such that $\lim_{\lambda \in A} i^*(x_\lambda) = \xi$. Thus, $\pi(i^*(x)) = \lim_{\lambda \in A} \pi(i^*(x_\lambda))$. Since Φ_1 is continuous and $\Phi_1 \circ \pi \circ i^* = \Phi \circ i^* = i^{**}$, we have $i^{**}(x) = \lim_{\lambda \in A} i^{**}(x_\lambda)$. Since i^{**} is a homeomorphism from D onto $i^{**}(D)$ by the hypothesis (ii), we have $x = \lim_{\lambda \in A} x_\lambda$. Therefore, $i^*(x) = \lim_{\lambda \in A} i^*(x_\lambda) = \xi$. This contradicts $\xi \notin i^*(D)$. This completes the proof of Theorem 6.3.

§ 7. Examples.

We shall consider particular examples of the Markov processes $\boldsymbol{M}^{\min}$ which do not satisfy ($\boldsymbol{M}^{\min}$.5) and investigate completely how D^{**}, D_b^{**} and S_ξ can be.

Example 1. Let $S=[-1, 1]$, $D=(-1, 0)\cup(0, 1)$ and $V=\{-1, 0, 1\}$. Let $\{G_\alpha^0\,;\,\alpha>0\}$ be the resolvent operators of the Brownian motion absorbed at V. Since $G_\alpha^0 f$ $(\alpha>0, f\in B(D))$ is the unique solution of the following equation,

$$(7.1)\qquad \begin{gathered} u(x)-\frac{1}{2}u''(x)=f(x)\,, \qquad x\in D \\ u(-1)=u(0)=u(1)=0 \end{gathered}$$

we can see by the Sturm-Liouvill theory that

$$(7.2)\qquad G_\alpha^0 f(x)=\frac{1}{\sqrt{2\alpha}\,(1-e^{2\sqrt{2\alpha}})}\Big\{(e^{\sqrt{2\alpha}x}-e^{-\sqrt{2\alpha}x})\int_{-1}^{x}(e^{2\sqrt{2\alpha}}e^{\sqrt{2\alpha}y}-e^{-\sqrt{2\alpha}y})f(y)dy$$
$$+(e^{2\sqrt{2\alpha}}e^{\sqrt{2\alpha}x}-e^{-\sqrt{2\alpha}x})\int_{x}^{0}(e^{\sqrt{2\alpha}y}-e^{-\sqrt{2\alpha}y})f(y)dy\Big\} \quad \text{for } x\in(-1, 0)\,,$$

and

$$(7.3)\qquad G_\alpha^0 f(x)=\frac{1}{\sqrt{2\alpha}(e^{-2\sqrt{2\alpha}}-1)}\Big\{(e^{-2\sqrt{2\alpha}}e^{\sqrt{2\alpha}x}-e^{-\sqrt{2\alpha}x})\int_{0}^{x}(e^{\sqrt{2\alpha}y}-e^{-\sqrt{2\alpha}y})f(y)dy$$
$$+(e^{\sqrt{2\alpha}x}-e^{-\sqrt{2\alpha}x})\int_{x}^{1}(e^{-2\sqrt{2\alpha}}e^{\sqrt{2\alpha}y}-e^{-\sqrt{2\alpha}y})f(y)dy\Big\} \quad \text{for } x\in(0, 1)\,.$$

Since the Brownian motion absorbed at V has the continuous path functions, the strong Markov property, and by (7.2) and (7.3),

$$(7.4)\qquad G_\alpha^0(B(D))\subset C_\infty(D)\,,$$

by Theorem 4.3, we have

$$(7.5)\qquad C_\infty(D)\subset\bar{\mathcal{M}}_\infty\,,$$

where $\mathcal{M}_\infty=\Big\{\dfrac{G_\alpha^0 f}{G_\gamma^0 1}\,;\ \alpha>0,\ f\in C_\infty(D)\Big\}$. Therefore, by Theorem 6.3,

$$(7.6)\qquad \mathcal{M}_\infty\text{-compactification } D^* \text{ of } D=\mathcal{M}_\infty\text{-completion } D^{**} \text{ of } D\,.$$

By (7.2) and (7.3), we can prove that for any $\alpha>0$ and $f\in B(D)$,

$$(7.7)\qquad \lim_{x\to0-}\frac{G_\alpha^0 f}{G_\gamma^0 1}(x)=\frac{\sqrt{2\gamma}(1-e^{2\sqrt{2\gamma}})}{(1-e^{2\sqrt{2\alpha}})\cdot(e^{\sqrt{2\gamma}}-1)^2}\int_{-1}^{0}(e^{2\sqrt{2\alpha}}e^{\sqrt{2\alpha}y}-e^{-\sqrt{2\alpha}y})f(y)dy\,,$$

$$(7.8)\qquad \lim_{x\to-1+0}\frac{G_\alpha^0 f}{G_\gamma^0 1}(x)=-\frac{\sqrt{2\gamma}(1-e^{2\sqrt{2\gamma}})e^{\sqrt{2\alpha}}}{(1-e^{2\sqrt{2\alpha}})\,(e^{\sqrt{2\gamma}}-1)^2}\int_{-1}^{0}(e^{\sqrt{2\alpha}y}-e^{-\sqrt{2\alpha}y})f(y)dy\,,$$

$$(7.9)\qquad \lim_{x\to0+}\frac{G_\alpha^0 f}{G_\gamma^0 1}(x)=-\frac{\sqrt{2\gamma}(1-e^{-2\sqrt{2\gamma}})}{(1-e^{-2\sqrt{2\alpha}})\cdot(1-e^{-\sqrt{2\gamma}})^2}\int_{0}^{1}(e^{-2\sqrt{2\alpha}}e^{\sqrt{2\alpha}y}-e^{-\sqrt{2\alpha}y})f(y)dy\,,$$

$$(7.10)\qquad \lim_{x\to1-0}\frac{G_\alpha^0 f}{G_\gamma^0 1}(x)=\frac{\sqrt{2\gamma}(1-e^{-2\sqrt{2\gamma}})e^{-\sqrt{2\alpha}}}{(1-e^{-2\sqrt{2\alpha}})\cdot(1-e^{-\sqrt{2\gamma}})^2}\int_{0}^{1}(e^{\sqrt{2\alpha}y}-e^{-\sqrt{2\alpha}y})f(y)dy\,.$$

PROPOSITION 7.1. (1) $S_1=\{p_1\}$ *and* $S_{-1}=\{p_{-1}\}$,

(2) $S_1 \cap S_{-1}=\emptyset$.

PROOF. (i) Fix any $p \in S_1$. There exists a sequence $(x_n)_{n=1}^{\infty}$ in D such that $x_n \to 1$ and $i^*(x_n) \to p$. By (7.10), for any $f \in C_\infty(D)$,

$$\left(\frac{G_\alpha^0 f}{G_\gamma^0 1}\right)^*(p)=\lim_{n\to\infty}\left(\frac{G_\alpha^0 f}{G_\gamma^0 1}\right)^*(i^*(x_n))=\lim_{n\to\infty}\frac{G_\alpha^0 f}{G_\gamma^0 1}(x_n)=\lim_{x\to 1-0}\frac{G_\alpha^0 f}{G_\gamma^0 1}(x).$$

Therefore, for any $p, q \in S_1$, $\left(\frac{G_\alpha^0 f}{G_\gamma^0 1}\right)^*(p)=\left(\frac{G_\alpha^0 f}{G_\gamma^0 1}\right)^*(q)$. Since $p, q \in V^* = D^*-i^*(D)$, we have, by (6.8), $p=q$. Consequently, $\sharp S_1=1$. Similarly, we have $\sharp S_{-1}=1$. For the proof of (2), take $f \in C_\infty(D)$ such that $f(x)=0$ for $x \in (-1, 0)$ and $f(x)>0$ for $x \in (0, 1)$. Then, by (7.8) and (7.10), $\left(\frac{G_\alpha^0 f}{G_\gamma^0 1}\right)^*(p_1)>0$ and $\left(\frac{G_\alpha^0 f}{G_\gamma^0 1}\right)^*(p_{-1})=0$. Therefore, $p_1 \neq p_{-1}$. (Q. E. D.)

PROPOSITION 7.2. (1) $S_0=\{p_{0-}, p_{0+}\}$,

(2) $S_1 \cap S_0=\emptyset$ *and* $S_{-1} \cap S_0=\emptyset$.

PROOF. (1) Let p_{0-} be a point such that there exists a sequence $(x_n)_{n=1}^{\infty}$ in $(-1, 0)$ such that $\lim_{n\to\infty} x_n=0$ and $\lim_{n\to\infty} i^*(x_n)=p_{0-}$, and let p_{0+} be a point such that $\lim_{n\to\infty} y_n=0$ and $\lim_{n\to\infty} i^*(y_n)=p_{0+}$. We shall prove that $p_{0-} \neq p_{0+}$. Take $f \in C_\infty(D)$ such that $f(x)=0$ for $x \in (-1, 0)$ and $f(x)>0$ for $x \in (0, 1)$. Then, by (7.7) and (7.9), $\left(\frac{G_\alpha^0 f}{G_\gamma^0 1}\right)^*(p_{0-})=0$ and $\left(\frac{G_\alpha^0 f}{G_\gamma^0 1}\right)^*(p_{0+})>0$. Therefore, $p_{0-} \neq p_{0+}$. Now, we shall prove that $\sharp S_0=2$. Assume that there exist different three points ξ_1, ξ_2 and ξ_3. Then, there exist sequences $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$ and $(z_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty} x_n=\lim_{n\to\infty} y_n=\lim_{n\to\infty} z_n=0$, and $\lim_{n\to\infty} i^*(x_n)=\xi_1$, $\lim_{n\to\infty} i^*(y_n)=\xi_2$ and $\lim_{n\to\infty} i^*(z_n)=\xi_3$. Therefore, for infinitely many n, at least one of the following cases hold: (i) $x_n, y_n \in (0, 1)$, (ii) $x_n, z_n \in (0, 1)$, (iii) $y_n, z_n \in (0, 1)$, (iv) $x_n, y_n \in (-1, 0)$, (v) $x_n, z_n \in (-1, 0)$, (vi) $y_n, z_n \in (-1, 0)$. For example, consider the case (iii). Then, for any $f \in C_\infty(D)$, by (7.9), we have $\left(\frac{G_\alpha^0 f}{G_\gamma^0 1}\right)^*(\xi_2)=\left(\frac{G_\alpha^0 f}{G_\gamma^0 1}\right)^*(\xi_3)$. Since $\xi_2, \xi_3 \in V^*$, we have, by (6.8), $\xi_2=\xi_3$. This is absurd. Other cases are similar. Therefore, $\sharp S_0=2$. For the proof of (2), take $f \in C_\infty(D)$ such that $f(x)>0$ for $x \in (-1, 0)$ and $f(x)=0$ for $x \in (0, 1)$. Then, by (7.8) and (7.9), $\left(\frac{G_\alpha^0 f}{G_\gamma^0 1}\right)^*(p_{-1})<0$ and $\left(\frac{G_\alpha^0 f}{G_\gamma^0 1}\right)^*(p_{0+})=0$. Therefore, $p_{-1} \neq p_{0+}$. Next, assume that $p_{-1}=p_{0-}$. Then, by (7.7) and (7.8), for any $f \in B(D)$,

$$\int_{-1}^{0}(e^{2\sqrt{2\alpha}}e^{\sqrt{2\alpha}y}-e^{-\sqrt{2\alpha}y})f(y)dy=-e^{\sqrt{2\alpha}}\int_{-1}^{0}(e^{\sqrt{2\alpha}y}-e^{-\sqrt{2\alpha}y})f(y)dy.$$

Therefore,

$$\int_{-1}^{0}(e^{2\sqrt{2\alpha}}+e^{\sqrt{2\alpha}})e^{\sqrt{2\alpha}y}f(y)=\int_{-1}^{0}(e^{\sqrt{2\alpha}}+1)f(y)dy.$$

In particular, taking $\alpha=\frac{1}{2}$ and $f(y)=e^{\sqrt{2\alpha}y}$, we have, $e^2-2e-1=0$. This is absurd. Consequently, $p_{-1}\neq p_{0-}$. This completes the proof of $S_{-1}\cap S_0=\emptyset$. Similarly, $S_1\cap S_0=\emptyset$. (Q. E. D.)

PROPOSITION 7.3. $D_0^*=\emptyset$.

PROOF. We shall prove that $\lim_{\alpha\to\infty}\alpha G_\alpha^* f(p_{0-})=f(p_{0-})$ for any $f\in C(D^*)$. Other cases are similar as will be seen below. By (7.7) and (7.2),

$$
\begin{aligned}
(7.11)\qquad \alpha G_\alpha^* f(p_{0-}) &= \alpha\Big(\frac{G_\alpha^0(fG_\gamma^0 1)}{G_\gamma^0 1}\Big)^*(p_{0-}) \\
&= \alpha\frac{\sqrt{2\gamma}(1-e^{2\sqrt{2\gamma}})}{(1-e^{2\sqrt{2\alpha}})(e^{\sqrt{2\gamma}}-1)^2}\int_{-1}^0 (e^{2\sqrt{2\alpha}}e^{\sqrt{2\alpha}y}-e^{-\sqrt{2\alpha}y})f(y)G_\gamma^0 1(y)dy \\
&= \frac{\sqrt{2}}{\sqrt{\gamma}(e^{\sqrt{2\gamma}}-1)^2}\frac{\alpha}{1-e^{2\sqrt{2\alpha}}}\int_{-1}^0 (e^{\sqrt{2\alpha}}e^{\sqrt{2\alpha}y}-e^{-\sqrt{2\alpha}y}) \\
&\qquad \Big\{(e^{2\sqrt{2\gamma}}-e^{\sqrt{2\gamma}})e^{\sqrt{2\gamma}y}+(e^{\sqrt{2\gamma}}-1)e^{-\sqrt{2\gamma}y}+(1-e^{2\sqrt{2\gamma}})\Big\}f(y)dy \\
&= \frac{1}{\sqrt{\gamma}(e^{\sqrt{2\gamma}}-1)^2}\frac{\sqrt{\alpha}}{e^{-2\sqrt{2\alpha}}-1}\int_{-\sqrt{2\alpha}}^0 (e^z-e^{-z-2\sqrt{2\alpha}}) \\
&\qquad \Big\{(e^{2\sqrt{2\gamma}}-e^{\sqrt{2\gamma}})e^{\frac{\sqrt{\gamma}z}{\sqrt{\alpha}}}+(e^{\sqrt{2\gamma}}-1)e^{-\frac{\sqrt{\gamma}z}{\sqrt{\alpha}}}+(1-e^{2\sqrt{2\gamma}})\Big\}f\Big(\frac{z}{\sqrt{2\alpha}}\Big)dz \\
&= \frac{1}{\sqrt{\gamma}(e^{\sqrt{2\gamma}}-1)^2}\frac{1}{e^{-2\sqrt{2\alpha}}-1}\int_{-\sqrt{2\alpha}}^0 (e^z-e^{-z-2\sqrt{2\alpha}})\varphi_\alpha(z)f\Big(\frac{z}{\sqrt{2\alpha}}\Big)dz,
\end{aligned}
$$

where

$$
(7.12)\qquad \varphi_\alpha(z)=\frac{e^{\sqrt{2\gamma}}(e^{\sqrt{2\gamma}}-1)e^{\frac{\sqrt{\gamma}z}{\sqrt{\alpha}}}+(e^{\sqrt{2\gamma}}-1)e^{-\frac{\sqrt{\gamma}z}{\sqrt{\alpha}}}+(1-e^{2\sqrt{2\gamma}})}{\frac{1}{\sqrt{\alpha}}}.
$$

We can prove that for any $z\in(-\infty,0)$

$$
(7.13)\qquad \lim_{\alpha\to\infty}\varphi_\alpha(z)=\sqrt{\gamma}(e^{\sqrt{2\gamma}}-1)^2 z.
$$

Put $t=\frac{1}{\sqrt{\alpha}}$ and $g_t(z)=e^{\sqrt{2\gamma}}(e^{\sqrt{2\gamma}}-1)e^{\sqrt{\gamma}tz}+(e^{\sqrt{2\gamma}}-1)e^{-\sqrt{\gamma}tz}$. Then,

$$
(7.14)\qquad \psi_t(z)=\varphi_\alpha(z)=\frac{g_t(z)-g_0(z)}{t}.
$$

Fix any $z\in(-\sqrt{2\alpha},0)$. By the mean value theorem, there exists $s\in(0,t)$ such that $\psi_t(z)=g_s'(z)=e^{\sqrt{2\gamma}}(e^{\sqrt{2\gamma}}-1)\sqrt{\gamma}e^{\sqrt{\gamma}sz}z-(e^{\sqrt{2\gamma}}-1)\sqrt{\gamma}e^{-\sqrt{\gamma}sz}z$. Since $sz\in(-\sqrt{2},0)$, there exists a constant $c>0$ such that

$$
(7.15)\qquad |\varphi_\alpha(z)|\leqq -cz.
$$

Since $e^z-e^{-z-2\sqrt{2\alpha}}>0$ for $z\in(-\sqrt{2\alpha},0)$, by (7.15), (7.16) $$\chi(-\sqrt{2\alpha}<z<0)(e^{z}-e^{-z-2\sqrt{2\alpha}})\varphi_{\alpha}(z)f\Big(\frac{z}{\sqrt{2\alpha}}\Big)\Big|\leqq -c\|f\|ze^{z}.$$

Noting that $\int_{-\infty}^{0}-ze^{z}dz=1$ and $\lim_{\alpha\to\infty}f\Big(\frac{z}{\sqrt{2\alpha}}\Big)=f(p_{0-})$, applying the Lebesgue's dominated convergence theorem, by (7.11), (7.12), (7.13) and (7.16), we have

$$\lim_{\alpha\to\infty}\alpha G_{\alpha}^{*}f(p_{0-})=-\frac{1}{\sqrt{\gamma}(e^{\sqrt{2\gamma}}-1)^{2}}\int_{-\infty}^{0}e^{z}\sqrt{\gamma}(e^{\sqrt{2\gamma}}-1)^{2}zf(p_{0-})dz$$
$$=f(p_{0-}).$$

This completes the proof of Proposition 7.3. (Q. E. D.)

Example 2. The spaces S, D and V are the same as in Example 1. Let $\boldsymbol{M}^{0}$ be the process whose path functions behave according to the Brownian motion $\{\bar{G}_{\alpha};\alpha>0\}$ absorbed at $\{-1,0\}$ in $(-1,0)$ and behaviors in $(0,1)$ are the ones of the uniform motions moving from the left to the right such that are killed by $\frac{c\,dt}{\xi}$ (c is a constant and $0<c<1$) for the time interval of dt and then jump to a point $-\xi$. Then, we can see that the resolvent $\{G_{\alpha}^{0};\alpha>0\}$ associated with $\boldsymbol{M}^{0}$ is the following:

(7.17) $$G_{\alpha}^{0}f(x)=\int_{0}^{1-x}\frac{cx^{c}}{(x+s)^{c+1}}\Big\{\int_{0}^{s}e^{-\alpha t}f(x+t)dt+e^{-\alpha s}\bar{G}_{\alpha}([f]_{(-1,0)})(-x-s)\Big\}ds$$

for $x\in(0,1)$,

(7.18) $$G_{\alpha}^{0}f(x)=\bar{G}_{\alpha}([f]_{(-1,0)})(x)\qquad\text{for } x\in(-1,0).$$

Proposition 7.4. *$G_{\alpha}^{0}(C(D))\subset C_{\infty}(D)$ for any $\alpha>0$.*

Proof. By (7.4), (7.17) and (7.18), we have only to prove that $G_{\alpha}^{0}1(\xi)\to 0$ as $\xi\downarrow 0$, which can be proved as follows. Since

(7.19) $$\int_{0}^{1-\xi}\frac{c\xi^{c}}{(\xi+s)^{c+1}}\Big\{\int_{0}^{s}e^{-\alpha t}f(\xi+t)dt\Big\}ds$$
$$=\int_{0}^{\infty}e^{-\alpha t}\frac{\xi^{c}}{(\xi+t)^{c}}f\chi_{(0,1)}(\xi+t)dt$$

for any $\alpha>0$, $f\in B(D)$ and $\xi\in(0,1)$, we have

(7.20) $$\int_{0}^{1-\xi}\frac{c\xi^{c}}{(\xi+s)^{c+1}}\Big\{\int_{0}^{s}e^{-\alpha t}1(\xi+t)dt\Big\}\longrightarrow 0\qquad\text{as } \xi\downarrow 0.$$

Since by (7.2)

(7.21) $$\bar{G}_{\alpha}1(x)=\frac{g_{\alpha}(x)-g_{\alpha}(0)}{\alpha(1-e^{2\sqrt{2\alpha}})}\qquad\text{for } x\in(-1,0),$$

where

(7.22) $$g_{\alpha}(x)=e^{\sqrt{2\alpha}}(e^{\sqrt{2\alpha}}-1)e^{\sqrt{2\alpha}x}+(e^{\sqrt{2\alpha}}-1)e^{-\sqrt{2\alpha}x},$$

we have

(7.23)
$$\int_0^{1-\xi} \frac{c\xi^c}{(\xi+s)^{c+1}} e^{-\alpha s}\bar{G}_\alpha 1(-\xi-s)ds$$
$$= \frac{c\xi^c}{(1-e^{2\sqrt{2\alpha}})}\int_0^{1-\xi}\frac{e^{-\alpha s}}{(\xi+s)^c}\cdot\frac{g_\alpha(-\xi-s)-g_\alpha(0)}{\xi+s}ds.$$

By using the same method as in (7.14) and (7.15), we have

(7.24)
$$\left|\frac{g_\alpha(-\xi-s)-g_\alpha(0)}{\xi+s}\right| \leqq 2\sqrt{2\alpha}\, e^{\sqrt{2\alpha}}(e^{\sqrt{2\alpha}}-1).$$

Therefore, noting that

(7.25)
$$\int_0^\infty \frac{e^{-\alpha s}}{s^c}ds = \frac{\Gamma(1-c)}{1-c},$$

we have, by (7.23) and (7.24),

(7.26)
$$\int_0^{1-\xi}\frac{c\xi^c}{(\xi+s)^{c+1}}e^{-\alpha s}\bar{G}_\alpha 1(-\xi-s)ds \longrightarrow 0 \quad \text{as } \xi\downarrow 0.$$

Thus, by (7.17), (7.20) and (7.26), we have $\lim_{\xi\downarrow 0} G_\alpha^0 1(\xi)=0$. (Q. E. D.)

Noting that for any $\alpha>0$, $f\in B(D)$ and $\xi\in(0, 1)$,

(7.27)
$$\alpha G_\alpha^0 f(\xi) = \int_0^\infty \frac{c\xi^c}{(\xi+s)^{c+1}}\Big\{\int_0^{\alpha s} e^{-u}\chi_{(0,1)}f\Big(\xi+\frac{u}{\alpha}\Big)du$$
$$+e^{-\alpha s}\chi_{(-1,0)}\alpha\bar{G}_\alpha([f]_{(-1,0)})(-\xi-s)\Big\}ds$$

and $\int_0^\infty \frac{c\xi^c}{(\xi+s)^{c+1}}ds=1$, we have

PROPOSITION 7.5. *For any $f\in C(D)$ and $x\in D$,*
$$\lim_{\alpha\to\infty}\alpha G_\alpha^0 f(x) = f(x).$$

Next, we shall calculate the entrance boundary of D for $\{G_\alpha^0;\alpha>0\}$. Since by (7.17) and (7.19), for any $\xi\in(0, 1)$,

(7.28)
$$\frac{G_\alpha^0 f}{G_\gamma^0 1}(\xi) = \frac{\frac{1}{c}\int_0^{1-\xi}\frac{e^{-\alpha s}}{(\xi+s)^c}f(\xi+s)ds+\int_0^{1-\xi}\frac{e^{-\alpha s}}{(\xi+s)^c}\cdot\frac{\bar{G}_\alpha([f]_{(-1,0)})(-\xi-s)}{\xi+s}ds}{\frac{1}{c}\int_0^{1-\xi}\frac{e^{-\gamma s}}{(\xi+s)^c}ds+\int_0^{1-\xi}\frac{e^{-\gamma s}}{(\xi+s)^c}\,\frac{\bar{G}_\gamma 1(-\xi-s)}{\xi+s}ds},$$

noting (7.21), (7.24) and (7.25), we have

(7.29)
$$\lim_{\xi\downarrow 0}\frac{G_\alpha^0 f}{G_\gamma^0 1}(\xi) = \frac{\int_0^1\frac{e^{-\alpha s}}{s^c}f(s)ds+c\int_0^1\frac{e^{-\alpha s}}{s^c}\cdot\frac{\bar{G}_\alpha([f]_{(-1,0)})(-s)}{s}ds}{\int_0^1\frac{e^{-\gamma s}}{s^c}ds+c\int_0^1\frac{e^{-\gamma s}}{s^c}\cdot\frac{\bar{G}_\gamma 1(-s)}{s}ds}$$

for any $\alpha>0$ and $f\in C(D)$. Since by (7.28), for any $\xi\in(0, 1)$, (7.30) $$\frac{G_\alpha^0 f}{G_\gamma^0 1}(\xi)=\frac{e^{\alpha\xi}\Big(\int_\xi^1 \frac{e^{-\alpha t}}{t^c}f(t)dt+c\int_\xi^1 \frac{e^{-\alpha t}}{t^c}\cdot\frac{\bar{G}_\alpha([f]_{(-1,0)})(-t)}{t}dt\Big)}{e^{\gamma\xi}\Big(\int_\xi^1 \frac{e^{-\gamma t}}{t^c}dt+c\int_\xi^1 \frac{e^{-\gamma t}}{t^c}\cdot\frac{\bar{G}_\gamma 1(-t)}{t}dt\Big)},$$

noting (7.4), (7.21), (7.24) and (7.25), by multiplying (7.30) by $\frac{1}{1-\xi}$, we have, for any $\alpha>0$ and $f\in C((-1,0)\cup(0,1])$,

(7.31) $$\lim_{\xi\uparrow 1}\frac{G_\alpha^0 f}{G_\gamma^0 1}(\xi)=f(1)\,.$$

Let D^{**} be an $\mathcal{M}_\infty$-completion of D.

PROPOSITION 7.6. (1) $S_{-1}=\{p_{-1}\}$, $S_0=\{p_{0-},p_{0+}\}$, $S_1=\{p_1\}$

(2) $S_{-1}\cap S_0=\emptyset$, $S_{-1}\cap S_1=\emptyset$, $S_0\cap S_1=\emptyset$

(3) $D^{**}-i^{**}(D)=S_{-1}\cup S_0\cup S_1$.

PROOF. (1): By (7.18) and Proposition 7.1, we have $\#S_{-1}=1$. By (7.31), we have, for any $\alpha>0$ and $f\in C_\infty(D)$, $\left(\frac{G_\alpha^0 f}{G_\gamma^0 1}\right)^{**}(p)=0$ for any $p\in S_1$. Therefore, by the property (6.12) of the $\mathcal{M}_\infty$-completion, we have $\#S_1=1$. Noting (7.7) and (7.29), by using the same way as in Proposition 7.2 (1), we have $\#S_0=2$.

(2): Since $\left(\frac{G_\alpha^0 f}{G_\gamma^0 1}\right)^{**}(p_1)=0$ for any $\alpha>0$ and $f\in C_\infty(D)$ by (7.31), we can see, by (7.7), (7.8) and (7.29), that $S_{-1}\cap S_1=\emptyset$ and $S_0\cap S_1=\emptyset$. By Proposition 7.2 (2), $p_{-1}\neq p_{0-}$. By (7.8) and (7.29), $p_{-1}\neq p_{0+}$. Therefore, $S_{-1}\cap S_0=\emptyset$.

(3): Since S is compact, it is clear that $D^{**}-i^{**}(D)\subset S_{-1}\cup S_0\cup S_1$. Therefore, for the proof of (3), it is sufficient to prove that $p_{-1},p_{0-},p_{0+},p_1\notin i^{**}(D)$, which can be proved as follows. Since $\left\|\alpha\frac{G_\alpha^0(fG_\gamma^0 1)}{G_\gamma^0 1}-f\right\|_{(-1,0)}\to 0$ as $\alpha\to\infty$ for any $f\in C_\infty(D)$ by (7.18) and Theorem 4.3, we can see that $p_{-1},p_{0-}\notin i^{**}(D)$. Since $\left(\frac{G_\alpha^0 f}{G_\gamma^0 1}\right)^{**}(p_1)=0$ for any $\alpha>0$ and $f\in C_\infty(D)$ by (7.31), we have, by (7.2) and (7.28), $p_1\notin i^{**}(D)$. Finally, we shall prove that $p_{0+}\notin i^{**}(D)$. Assume that $p_{0+}=i^{**}(x)$ for some $x\in D$. Then, by (7.7) and (7.29), we can see that x should be contained in $(0,1)$. Take $f\in C_\infty(D)$ such that $f(y)=0$ for $y\in(-1,0)\cup[x,1)$ and $f(y)>0$ for $y\in(0,x)$. Then, by (7.28) and (7.29), $0=\left(\frac{G_\alpha^0 f}{G_\gamma^0 1}\right)(x)=\left(\frac{G_\alpha^0 f}{G_\gamma^0 1}\right)^{**}(p_{0+})>0$. This is absurd. (Q. E. D.)

PROPOSITION 7.7. *i^{**} is a homeomorphism from D onto $i^{**}(D)$.*

This follows from Proposition 7.6 (3).

PROPOSITION 7.8. $D_b^{**}=\{p_1\}$.

PROOF. Fix any $f\in C(D^{**})$. Since by (7.8) and (7.18),

$$\alpha G_\alpha^{**}f(p_{-1})=\lim_{x\to-1}\frac{\alpha G_\alpha^0(f\circ i^{**}G_\gamma^0 1)}{G_\gamma^0 1}(x)=\lim_{x\to-1}\frac{\alpha\bar{G}_\alpha([f\circ i^{**}]_{(-1,0)}\bar{G}_\gamma 1)}{\bar{G}_\gamma 1}(x)\,,$$

we have, by Proposition 7.3, $\lim_{\alpha\to\infty}\alpha G_\alpha^{**}f(p_{-1})=f(p_{-1})$. Therefore, $p_{-1}\notin D_b^{**}$. Similarly, $p_{0-}\notin D_b^{**}$. By (7.29),

(7.32) $$\alpha G_\alpha^{**}f(p_{0+}) = \frac{\alpha\int_0^1 \frac{e^{-\alpha s}}{s^c} f\circ i^{**}G_\gamma^0 1(s)ds + \alpha c\int_0^1 \frac{e^{-\alpha s}}{s^c}\, \frac{\bar G_\alpha([f\circ i^{**}]_{(-1,0)}\bar G_\gamma 1)(-s)}{s}ds}{\int_0^1 \frac{e^{-\gamma s}}{s^c}ds + c\int_0^1 \frac{e^{-\gamma s}}{s^c}\,\frac{\bar G_\gamma 1(-s)}{s}ds}.$$

Since $\left|\alpha\frac{\bar G_\alpha([f\circ i^{**}]_{(-1,0)}\bar G_\gamma 1)(-s)}{s}\right| \leqq \|f\|\frac{\bar G_\gamma 1(-s)}{s} \leqq M\cdot\|f\|$ by (7.21) and (7.24), noting (7.25), we have

(7.33) $$\lim_{\alpha\to\infty} c\alpha\int_0^1 \frac{e^{-\alpha s}}{s^c}\,\frac{\bar G_\alpha([f\circ i^{**}]_{(-1,0)}\bar G_\gamma 1)(-s)}{s}ds = 0.$$

By (7.17), since $\int_0^1 \frac{e^{-\alpha s}}{s^c} f\circ i^{**}G_\gamma^0 1(s)ds = \int_0^1 \frac{e^{-\gamma t}}{t^c}\left(\int_0^t e^{-(\alpha-\gamma)s}\cdot f(i^{**}(s))ds\right)dt$ $+c\int_0^1 \frac{e^{-\gamma t}}{t^c}\,\frac{\bar G_\gamma 1(-t)}{t}\left(\int_0^t e^{-(\alpha-\gamma)s} f(i^{**}(s))ds\right)dt$, noting (7.21), (7.24) and (7.25), we have,

(7.34) $$\lim_{\alpha\to\infty}\alpha\int_0^1 \frac{e^{-\alpha s}}{s^c} f\circ i^{**}G_\gamma^0 1(s)ds = f(p_{0+})\left\{\int_0^1 \frac{e^{-\gamma t}}{t^c}dt + c\int_0^1 \frac{e^{-\gamma s}}{s^c}\,\frac{\bar G_\gamma 1(-s)}{s}ds\right\}.$$

Therefore, by (7.32), (7.33) and (7.34), we have $\lim_{\alpha\to\infty}\alpha G_\alpha^{**}f(p_{0+})=f(p_{0+})$. Thus, $p_{0+}\notin D_b^{**}$. By (7.31), for any $\alpha>0$ and $f\in C(D^{**})$, $\alpha G_\alpha^{**}f(p_1)=0$. This implies that $p_1\in D_b^{**}$. Consequently, noting Proposition 7.5 and Proposition 7.6 (3), we have $D_b^{**}=\{p_1\}$. (Q. E. D.)

Finally, we shall investigate how large $\mathcal{M}_\infty$ can be. Since the path functions of $\boldsymbol{M}^0$ are not continuous, we can not apply Theorem 4.3 to our $\boldsymbol{M}^0$. But, also in this case, we have

PROPOSITION 7.9. $C_\infty(D)\subset\bar{\mathcal{M}}_\infty$.

PROOF. Since $C_0(D)$ is dense in $C_\infty(D)$, it suffices to prove that $C_0(D)\subset\bar{\mathcal{M}}_\infty$. Fix any $f\in C_0(D)$. By applying (4.21) in the proof of Theorem 4.3 to $\{\bar G_\alpha;\alpha>0\}$,

(7.35) $$\lim_{\alpha\to\infty}\left\|\frac{\alpha\bar G_\alpha([f]_{(-1,0)}\bar G_\gamma 1)}{\bar G_\gamma 1}-f\right\|_{(-1,0)}=0.$$

Therefore, by (7.18), we have only to prove that

(7.36) $$\lim_{\alpha\to\infty}\left\|\frac{\alpha G_\alpha^0(fG_\gamma^0 1)}{G_\gamma^0 1}-f\right\|_{(0,1)}=0.$$

By (7.17) and (7.18), for any $\xi\in(0,1)$,

(7.37) $$\frac{\alpha G_\alpha^0(fG_\gamma^0 1)}{G_\gamma^0 1}(\xi)$$
$$=\frac{\alpha\int_\xi^1 \frac{e^{-\alpha(t-\xi)}}{t^c} fG_\gamma^0 1(t)dt + c\alpha\int_\xi^1 \frac{e^{-\alpha(t-\xi)}}{t^c}\cdot\frac{\bar{G}_\alpha([f]_{(-1,0)}\bar{G}_\gamma 1)(-t)}{t}dt}{\int_\xi^1 \frac{e^{-\gamma(t-\xi)}}{t^c}dt + c\int_\xi^1 \frac{e^{-\gamma(t-\xi)}}{t^c}\cdot\frac{\bar{G}_\gamma 1(-t)}{t}dt},$$

(7.38) $$\alpha\int_\xi^1 \frac{e^{-\alpha(t-\xi)}}{t^c} fG_\gamma^0 1(t)dt$$
$$=\alpha\int_\xi^1 \frac{e^{-\alpha(t-\xi)}}{t^c} f(t)\left\{\int_0^{1-t} e^{-\gamma s}\frac{t^c}{(t+s)^c}ds + \int_0^{1-t}\frac{e^{-\gamma s}}{(t+s)^c}ct^c\frac{\bar{G}_\gamma 1(-t-s)}{t+s}ds\right\}dt$$
$$=\int_\xi^1\left(\frac{e^{-\gamma(t-\xi)}}{t^c} + c\frac{e^{-\gamma(t-\xi)}}{t^c}\cdot\frac{\bar{G}_\gamma 1(-t)}{t}\right)F_\alpha(t,\xi)dt,$$

where

(7.39) $$F_\alpha(t,\xi)=\int_0^{(\alpha-\gamma)(t-\xi)}\frac{\alpha}{\alpha-\gamma}e^{-v}f\left(\frac{v}{\alpha-\gamma}+\xi\right)dv.$$

Since $f\in C_0(D)$, there exists $a\in(0,1)$ such that

(7.40) $$f(x)=0 \qquad \text{for any } x\in(-1,-a]\cup[a,1).$$

For any $\varepsilon>0$, by (7.35), for sufficiently large $\alpha(>\gamma)$, $\left\|\frac{\alpha\bar{G}_\alpha([f]_{(-1,0)}\bar{G}_\gamma 1)}{\bar{G}_\gamma 1}-f\right\|_{(-1,0)}$ $<\varepsilon$. Therefore, for any $\xi\in(0,1)$, by (7.40),

(7.41) $$\frac{\alpha c\int_\xi^1 \frac{e^{-\alpha(t-\xi)}}{t^c}\cdot\frac{\bar{G}_\alpha([f]_{(-1,0)}\bar{G}_\gamma 1)(-t)}{t}dt}{\int_\xi^1 \frac{e^{-\gamma(t-\xi)}}{t^c}dt + c\int_\xi^1 \frac{e^{-\gamma(t-\xi)}}{t^c}\cdot\frac{\bar{G}_\gamma 1(-t)}{t}dt}$$
$$\leqq\frac{\int_\xi^1 \frac{e^{-\alpha(t-\xi)}}{t^c}\cdot c(f(-t)+\varepsilon)\cdot\frac{\bar{G}_\gamma 1(-t)}{t}dt}{\int_\xi^1 \frac{e^{-\gamma(t-\xi)}}{t^c}dt + c\int_\xi^1 \frac{e^{-\gamma(t-\xi)}}{t^c}\cdot\frac{\bar{G}_\gamma 1(-t)}{t}dt}$$
$$\leqq\frac{\int_\xi^1 \frac{e^{-\alpha(t-\xi)}}{t^c}\cdot|f(-t)|\cdot\frac{\bar{G}_\gamma 1(-t)}{t}dt}{\int_\xi^1 \frac{e^{-\gamma(t-\xi)}}{t^c}\cdot\frac{\bar{G}_\gamma 1(-t)}{t}dt}+\varepsilon$$
$$\leqq\frac{\int_\xi^1 \frac{e^{-\alpha(t-\xi)}}{t^c}\cdot|f(-t)|\cdot\frac{\bar{G}_\gamma 1(-t)}{t}dt}{\int_a^1 \frac{e^{-\gamma t}}{t^c}\cdot\frac{\bar{G}_\gamma 1(-t)}{t}dt}+\varepsilon$$
$$\leqq\frac{\|f\|}{\int_a^1 \frac{e^{-\gamma t}}{t^c}\cdot\frac{\bar{G}_\gamma 1(-t)}{t}dt}\cdot\int_0^{1-\xi}\frac{e^{-\alpha s}}{s^c}\cdot\frac{\bar{G}_\gamma 1(-\xi-s)}{\xi+s}ds+\varepsilon.$$

Similarly,

(7.42)
$$\frac{\alpha c\int_{\xi}^{1}\frac{e^{-\alpha(t-\xi)}}{t^{c}}\cdot\frac{\bar{G}_{\alpha}([f]_{(-1,0)}\bar{G}_{\gamma}1)(-t)}{t}dt}{\int_{\xi}^{1}\frac{e^{-\gamma(t-\xi)}}{t^{c}}dt+c\int_{\xi}^{1}\frac{e^{-\gamma(t-\xi)}}{t^{c}}\cdot\frac{\bar{G}_{\gamma}1(-t)}{t}dt}$$

$$\geqq-\frac{\|f\|}{\int_{a}^{1}\frac{e^{-\gamma t}}{t^{c}}\cdot\frac{\bar{G}_{\gamma}1(-t)}{t}dt}\cdot\int_{0}^{1-\xi}\frac{e^{-\alpha s}}{s^{c}}\cdot\frac{\bar{G}_{\gamma}1(-\xi-s)}{\xi+s}ds-\varepsilon.$$

Therefore, noting (7.21), (7.24) and (7.25), letting $\alpha\to\infty$ and then $\varepsilon\downarrow 0$ in (7.41) and (7.42), we have

(7.43)
$$\frac{\alpha c\int_{\xi}^{1}\frac{e^{-\alpha(t-\xi)}}{t^{c}}\cdot\frac{\bar{G}_{\alpha}([f]_{(-1,0)}\bar{G}_{\gamma}1)(-t)}{t}dt}{\int_{\xi}^{1}\frac{e^{-\gamma(t-\xi)}}{t^{c}}dt+c\int_{\xi}^{1}\frac{e^{-\gamma(t-\xi)}}{t^{c}}\cdot\frac{\bar{G}_{\gamma}1(-t)}{t}dt}\Rightarrow 0$$

as $\alpha\to\infty$ uniformly in $\xi\in(0,1)$. Put

(7.44)
$$G_{\alpha}(t,\xi)=\frac{e^{-\gamma(t-\xi)}}{t^{c}}+c\frac{e^{-\gamma(t-\xi)}}{t^{c}}\cdot\frac{\bar{G}_{\gamma}1(-t)}{t}.$$

Then, by (7.38),

(7.45)
$$\frac{\alpha\int_{\xi}^{1}\frac{e^{-\alpha(t-\xi)}}{t^{c}}fG_{\gamma}^{0}1(t)dt}{\int_{\xi}^{1}\frac{e^{-\gamma(t-\xi)}}{t^{c}}dt+c\int_{\xi}^{1}\frac{e^{-\gamma(t-\xi)}}{t^{c}}\cdot\frac{\bar{G}_{\gamma}1(-t)}{t}dt}-f(\xi)$$

$$=\frac{\int_{\xi}^{1}G_{\alpha}(t,\xi)(F_{\alpha}(t,\xi)-f(\xi))dt}{\int_{\xi}^{1}G_{\alpha}(t,\xi)dt}.$$

Since $f\in C_{0}(D)$, defining by $f(x)=0$ for $x\in R-D$, f becomes uniformly continuous in R and so for any $\varepsilon>0$, there exists $\delta>0$ such that $|f(x)-f(y)|<\varepsilon$ for $|x-y|<\delta$. Since

$$F_{\alpha}(t,\xi)-f(\xi)=\int_{0}^{\infty}\frac{\alpha}{\alpha-\gamma}e^{-v}\Big\{f\Big(\frac{v}{\alpha-\gamma}+\xi\Big)-f(\xi)\Big\}dv-\Big(1-\frac{\alpha}{\alpha-\gamma}\Big)f(\xi)$$
$$-\int_{(\alpha-\gamma)(t-\xi)}^{\infty}\frac{\alpha}{\alpha-\gamma}e^{-v}f\Big(\frac{v}{\alpha-\gamma}+\xi\Big)dv$$

and since

$$\Big|\int_{0}^{\infty}\frac{\alpha}{\alpha-\gamma}e^{-v}\Big\{f\Big(\frac{v}{\alpha-\gamma}+\xi\Big)-f(\xi)\Big\}dv\Big|$$
$$\leqq\int_{0}^{\delta}\alpha e^{-(\alpha-\gamma)u}|f(u+\xi)-f(\xi)|du+2\|f\|\int_{\delta}^{\infty}\alpha e^{-(\alpha-\gamma)u}du$$
$$\leqq\frac{\alpha}{\alpha-\gamma}\Big(\varepsilon+2\|f\|\int_{(\alpha-\gamma)\delta}^{\infty}e^{-v}dv\Big),$$

we have

$$(7.46)\qquad \left|\int_{\xi+\delta}^{1} G_\alpha(t,\xi)(F_\alpha(t,\xi)-f(\xi))dt\right|$$

$$\leqq \left\{\frac{\alpha}{\alpha-\gamma}\left(\varepsilon+3\|f\|\int_{(\alpha-\gamma)\delta}^{\infty} e^{-v}dv\right)+\left|1-\frac{\alpha}{\alpha-\gamma}\right|\|f\|\int_{\xi+\delta}^{1} G_\alpha(t,\xi)dt\right\}$$

$$\leqq e^{\gamma}\left(\int_0^1 \frac{e^{-\gamma t}}{t^c}dt+c\int_0^1 \frac{e^{-\gamma t}}{t^c}\,\frac{\bar{G}_\gamma 1(-t)}{t}dt\right)\left\{\frac{\alpha}{\alpha-\gamma}\left(\varepsilon+3\|f\|\int_{(\alpha-\gamma)\delta}^{\infty} e^{-v}dv\right)\right.$$

$$\left.+\left|1-\frac{\alpha}{\alpha-\gamma}\right|\|f\|\right\}.$$

On the other hand, noting that $|F_\alpha(t,\xi)|\leqq \frac{\alpha}{\alpha-\gamma}\|f\|$ from (7.39),

$$(7.47)\qquad \left|\int_{\xi}^{\xi+\delta} G_\alpha(t,\xi)(F_\alpha(t,\xi)-f(\xi))dt\right|$$

$$\leqq \left(1+\frac{\alpha}{\alpha-\gamma}\right)\|f\|\int_0^{\delta}\left(\frac{e^{-\gamma s}}{s^c}+\frac{e^{-\gamma s}}{s^c}\,\frac{\bar{G}_\gamma 1(-\xi-s)}{\xi+s}\right)ds.$$

Therefore, noting (7.21), (7.24) and (7.25), we have, by (7.46) and (7.47),

$$(7.48)\qquad \int_{\xi}^{1} G_\alpha(t,\xi)(F_\alpha(t,\xi)-f(\xi))dt \longrightarrow 0 \qquad \text{as } \alpha\rightarrow\infty$$

uniformly in ξ.

Moreover, since $F_\alpha(t,\xi)=f(\xi)=0$ for $t\geqq\xi\in[a,1)$ from (7.40),

$$(7.49)\qquad \left|\frac{\int_{\xi}^{1} G_\alpha(t,\xi)(F_\alpha(t,\xi)-f(\xi))dt}{\int_{\xi}^{1} G_\alpha(t,\xi)dt}\right|$$

$$\leqq \frac{\left|\int_{\xi}^{1} G_\alpha(t,\xi)(F_\alpha(t,\xi)-f(\xi))dt\right|}{\int_a^1 \frac{e^{-\gamma t}}{t^c}dt+c\int_a^1 \frac{e^{-\gamma t}}{t^c}\,\frac{\bar{G}_\gamma 1(-t)}{t}dt}.$$

Consequently, by (7.37), (7.43), (7.45), (7.48) and (7.49), we have (7.36).

(Q. E. D.)

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Added in proof.

In the course of drafting the author noticed that the same problem was dealt by E. B. Dynkin [8].