J. Math. Soc. Japan
Vol. 22, No. 1, 1970

On principal normal vector fields of submanifolds in a Riemannian manifold of constant curvature

By Tominosuke OTSUKI*)

(Received March 27, 1969)

Introduction.

In [9], the author proved the complete integrability of the distribution of principal tangent vector spaces of a hypersurface in a Riemannian manifold of constant curvature and applied it to the investigation of minimal hypersurfaces in a sphere with some special properties. The principal tangent vector space at a point of a hypersurface in a Riemannian manifold is the linear subspace of vectors which are eigenvectors corresponding to a fixed eigenvalue of the 2nd fundamental form at the point. If we consider this eigenvalue as the length of a normal vector to the hypersurface, we can generalize the above consideration to any submanifold in a Riemannian manifold of constant curvature.

In the present paper, he will prove a more generalized theorem than Theorem 2 in [9] and study the properties of the integral submanifolds of the distribution of principal tangent vector spaces corresponding to a principal normal vector field.

§ 1. Preliminary.

For any C^∞ vector bundle $E \to M$ over a C^∞ differentiable manifold M, we denote the set of C^∞ cross sections by $\Gamma(E, M)$.

Let $\bar{M} = \bar{M}^{n+p}$ be an $(n+p)$-dimensional C^∞ Riemannian manifold and $M = M^n$ be an n-dimensional immersed C^∞ submanifold in $\bar{M}$ by an immersion $\psi: M \to \bar{M}$. Let $P: \psi^* T(\bar{M}) \to T(\bar{M})$ be the projection defined by the orthogonal decomposition:

$$T_{\psi(x)}(\bar{M}) = \psi_*(T_x(M)) + N_x, \qquad x \in M \text{ and put } P^\perp = 1 - P.$$

We denote the normal bundle of M in $\bar{M}$ by the immersion ψ by $N(M, \bar{M})$ whose total space is $\bigcup_{x \in M} x \times N_x \subset M \times T(\bar{M})$. Then

*) Work done under partial support from NSF grant GP-8623.

$$\phi^* T(\bar{M}) = T(M) \oplus N(M, \bar{M}).$$

In the following, we denote simply $\mathfrak{X}(M) = \Gamma(T(M), M)$ and $\mathfrak{X}^{\perp}(M) = \Gamma(N(M, \bar{M}), M)$. We denote the covariant differentiation operators for $\bar{M}$ and M by $\bar{\nabla}$ and ∇ respectively. For the vector bundle $N(M, \bar{M})$, we have the naturally induced metric connection from the one of $\bar{M}$ and denote the corresponding covariant differential operator by $\nabla^{\perp}$. According to Martz [2] for an $X \in \mathfrak{X}(M)$, we have the following decomposition of $\bar{\nabla}_X$.

$$\bar{\nabla}_X = \nabla_X + T_X \quad \text{on} \quad \mathfrak{X}(M), \tag{1.1}$$

where

$$\nabla_X = P\bar{\nabla}_X \quad \text{and} \quad T_X = P^{\perp}\bar{\nabla}_X$$

and

$$\bar{\nabla}_X = T_X + \nabla_X^{\perp} \quad \text{on} \quad \mathfrak{X}^{\perp}(M), \tag{1.2}$$

where

$$T_X = P\bar{\nabla}_X \quad \text{and} \quad \nabla_X^{\perp} = P^{\perp}\bar{\nabla}_X.$$

T_X is called the shape operator of M in $\bar{M}$ due to O'Neill.

Applying P and $P^{\perp}$ to the equation of definition of curvature tensor of $\bar{M}$,

$$\bar{R}_{X,Y} = [\bar{\nabla}_X, \bar{\nabla}_Y] - \bar{\nabla}_{[X,Y]}$$

considered only for $X, Y \in \mathfrak{X}(M)$, and substituting the decompositions (1.1) and (1.2) of $\bar{\nabla}_X$, $\bar{\nabla}_Y$, we have the following formulas:

(i) On $\mathfrak{X}(M)$

$$P\bar{R}_{X,Y} = R_{X,Y} + [T_X, T_Y] \qquad \text{(the Gauss equation)} \tag{1.3}$$

$$P^{\perp}\bar{R}_{X,Y} = T_X\nabla_Y - T_Y\nabla_X + \nabla_X^{\perp}T_Y - \nabla_Y^{\perp}T_X - T_{[X,Y]} \tag{1.4}$$

(the second Codazzi-Mainardi equation),

where $R_{X,Y} = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$ is the curvature tensor of M and $[T_X, T_Y] = T_XT_Y - T_YT_X$.

(ii) On $\mathfrak{X}^{\perp}(M)$

$$P\bar{R}_{X,Y} = \nabla_X T_Y - \nabla_Y T_X + T_X\nabla_Y^{\perp} - T_Y\nabla_X^{\perp} - T_{[X,Y]} \tag{1.5}$$

(the first Codazzi-Mainardi equation),

$$P^{\perp}\bar{R}_{X,Y} = R_{X,Y}^{\perp} + [T_XT_Y] \qquad \text{(the Ricci equation)}, \tag{1.6}$$

where $R_{X,Y}^{\perp} = \nabla_X^{\perp}\nabla_Y^{\perp} - \nabla_Y^{\perp}\nabla_X^{\perp} - \nabla_{[X,Y]}^{\perp}$ is the curvature tensor of $N(M, \bar{M})$.

Now, take a fixed point $x \in M$ and a normal vector $v \in N_x$. A non zero tangent vector $u \in M_x = T_x(M)$ is called a *principal tangent vector* for v if

$$T_u z = \langle u, z\rangle v \qquad \text{for any } z \in M_x, \tag{1.7}$$

where $\langle\, ,\rangle$ denotes the inner product of M. And v is called a *principal normal vector* of M in $\bar{M}$ at the point. If v is a principal normal vector at $x\in M$, the set of all principal tangent vectors for v and the zero vector is clearly a linear subspace of M_x which we call the *principal tangent vector space* for v and denote it by $E(x, v)$. On the principal tangent vector spaces, we have the following.

LEMMA 1. *If v_1 and v_2 are principal normal vectors at $x\in M$ and $v_1\neq v_2$, then $E(x, v_1)\perp E(x, v_2)$.*

PROOF. Let $u_1\in E(x, v_1)$ and $u_2\in E(x, v_2)$. By means of (1.7), we get

$$T_{u_1}u_2=\langle u_1, u_2\rangle v_1 \quad \text{and} \quad T_{u_2}u_1=\langle u_2, u_1\rangle v_2\,.$$

Since the operator T is self adjoint, we have $T_{u_1}u_2=T_{u_2}u_1$, hence $\langle u_1, u_2\rangle(v_1-v_2)=0$, which follows $\langle u_1, u_2\rangle=0$. q. e. d.

Now, let $F(\bar{M})$ and $F(M)$ be the orthonormal frame bundles over $\bar{M}$ and M respectively, here we consider M has the induced Riemannian metric from $\bar{M}$ through ψ. Let B be the set of elements $b=(x, e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+p})$, where $(x, e_1, \cdots, e_n)\in F(M)$ and $(\psi(x), \psi_*e_1, \cdots, \psi_*e_n, e_{n+1}, \cdots, e_{n+p})\in F(\bar{M})$. B is considered naturally C^∞ manifold and $B\rightarrow M$ is a principal fibre bundle over M. We denote the basic forms and connection forms on $F(\bar{M})$ of $\bar{M}$ by

$$\bar{\omega}_A,\ \bar{\omega}_{AB}=-\bar{\omega}_{BA}\,, \qquad A, B=1, 2, \cdots, n+p$$

and the induced forms on B through the natural mapping $B\rightarrow F(\bar{M})$ by the notations omitted bars. We have the structure equations for $\bar{M}$:

$$\text{(1.8)} \qquad \begin{cases} d\bar{\omega}_A=\sum_B \bar{\omega}_B\wedge\bar{\omega}_{BA}\,, \\ d\bar{\omega}_{AB}=\sum\bar{\omega}_{AC}\wedge\bar{\omega}_{CB}+\bar{\Omega}_{AB} \end{cases}$$

and on B

$$\text{(1.9)} \qquad \begin{cases} \omega_\alpha=0\,, \qquad \alpha=n+1, \cdots, n+p \\ \omega_{i\alpha}=\sum_j A_{\alpha ij}\omega_j\,, \qquad i=1, 2, \cdots, n\,, \end{cases}$$

where[1)]

$$\text{(1.10)} \qquad A_{\alpha ij}=A_{\alpha ji}\,.$$

Then using the frame $b=(x, e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+p})$, the shape operator T_X will be expressed as follows:

1) In the following, the indices run as follows:
$i, j, k, \cdots=1, 2, \cdots, n$;
$\alpha, \beta, \gamma, \cdots=n+1, \cdots, n+p$;
$A, B, C, \cdots=1, 2, \cdots, n+p$
with some exceptions.

(1.11) $$T_{e_i}(e_j)=\sum_\alpha \omega_{j\alpha}(e_i)e_\alpha=\sum_\alpha A_{\alpha ij}e_\alpha$$

and

(1.12) $$T_{e_i}(e_\alpha)=\sum_j \omega_{\alpha j}(e_i)e_j=-\sum_j A_{\alpha ij}e_j$$

and the bilinearity of $T_u(v)$ in u and v.

§ 2. Principal normal vector fields.

Now, we suppose that a principal normal vector field $V\in\Gamma(N(M,\bar{M}))$ is given. Then, $\dim E(x, V(x))$, $x\in M$, is clearly an upper semi-continuous positive integer valued function on M. Hence for its minimal value $m(>0)$, the point set M_0 of x such that $m=\dim E(x, V(x))$ is an open subset of M.

LEMMA 2. *If V is a C^∞ principal normal vector field of M in $\bar{M}$, then $E(M, V)=\bigcup_{x\in M_0} E(x, V(x))$ makes a C^∞ m-dimensional distribution on M_0.*

PROOF. Take any point $x_0\in M_0$ and a local cross section of the bundle $B\to M$ about x_0. Then, any vector $u\in E(x, V(x))$ is a solution of the equations

(2.1) $$\sum_j \{A_{\alpha ij}(x)-v_\alpha(x)\delta_{ij}\}u_j=0\,,\qquad i=1, 2, \cdots, n\,;\ \alpha=n+1, \cdots, n+p\,,$$

where $u=\sum_i u_ie_i$ and $V(x)=\sum_\alpha v_\alpha(x)e_\alpha(x)$, because we have

$$T_uz=\sum_\alpha A_{\alpha ij}u_iz_je_\alpha\,,\qquad z=\sum_j z_je_j$$

by (1.11). Since the system of linear equations in $u_1, \cdots, u_n$ has rank $n-m$, the distribution $E(x, V(x))$ is C^∞ about x_0. q. e. d.

LEMMA 3. *Let M be an n-dimensional C^∞ submanifold immersed in an $(n+p)$-dimensional C^∞ Riemannian manifold $\bar{M}$ of constant curvature $\bar{c}$ with a C^∞ principal normal vector field V such that $E(M, V)$ has constant dimension $m>1$. Then, the m-dimensional distribution $E(M, V)$ is completely integrable if and only if $\nabla^\perp_u V=0$ for any $u\in E(M, V)$.*

PROOF. By Lemma 2, $E(M, V)$ is an m-dimensional C^∞ distribution on M. We take any two tangent vector fields $X, Y\in\mathfrak{X}(M)$ such that $X(x), Y(x)\in E(x, V(x))$, $x\in M$, which we denote simply by $X, Y\subset E(M, V)$. For any $Z\in\mathfrak{X}(M)$, we have

(2.2) $$T_XZ=\langle X, Z\rangle V\,,\qquad T_YZ=\langle Y, Z\rangle V\,.$$

By means of the second Codazzi-Mainardi equation (1.4), from (2.2) we have

$$\begin{aligned}(T_{[X,Y]}+P^\perp\bar{R}_{X,Y})Z&=T_X\nabla_YZ-T_Y\nabla_XZ+\nabla^\perp_XT_YZ-\nabla^\perp_YT_XZ\\&=\{\langle X, \nabla_YZ\rangle-\langle Y, \nabla_XZ\rangle\}V+P^\perp(\bar{\nabla}_X(\langle Y, Z\rangle V))-P^\perp(\bar{\nabla}_Y(\langle X, Z\rangle V))\\&=\langle\nabla_XY-\nabla_YX, Z\rangle V+\langle Y, Z\rangle\nabla^\perp_XV-\langle X, Z\rangle\nabla^\perp_YV\,,\end{aligned}$$

that is

$$T_{[X,Y]}Z=\langle[X, Y], Z\rangle V+\langle Y, Z\rangle\nabla_X^{\perp}V-\langle X, Z\rangle\nabla_Y^{\perp}V-P^{\perp}(\bar{R}_{X,Y}Z). \tag{2.3}$$

Since $\bar{M}$ is of constant curvature $\bar{c}$, we have

$$\bar{R}_{X,Y}Z=\bar{c}(\langle Y, Z\rangle X-\langle X, Z\rangle Y) \tag{2.4}$$

in general. Thus, (2.3) is equivalent to

$$T_{[X,Y]}Z=\langle[X, Y], Z\rangle V+\langle Y, Z\rangle\nabla_X^{\perp}V-\langle X, Z\rangle\nabla_Y^{\perp}V. \tag{2.5}$$

Therefore $[X, Y]\subset E(M, V)$ if and only if

$$\langle Y, Z\rangle\nabla_X^{\perp}V=\langle X, Z\rangle\nabla_Y^{\perp}V \tag{2.6}$$

for any $Z\in\mathfrak{X}(M)$. This condition is dependent only on $X(x)$, $Y(x)$ at each $x\in M$. Since $m>1$, we can take $Z(x)$ such that $\langle Y(x), Z(x)\rangle\neq 0$, $\langle X(x), Z(x)\rangle=0$ if $X(x)\wedge Y(x)\neq 0$. Then (2.6) implies

$$\nabla_{X(x)}^{\perp}V=0.$$

Therefore, if $E(M, V)$ is completely integrable, then $\nabla_u^{\perp}V=0$ for any $u\in E(M, V)$. Conversely, if this condition is satisfied, then $E(M, V)$ is completely integrable from (2.5). q. e. d.

LEMMA 4. *Let M, $\bar{M}$ and V be supposed as in Lemma 3. Then, the principal normal vector field V satisfies $\nabla_u^{\perp}V=0$ for any $u\in E(M, V)$.*

PROOF. For a fixed point $x_0\in M$, we can take a sufficiently small neighborhood of x_0 and a local cross section $(x, e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+p})$ of the bundle $B\rightarrow M$ defined on this neighborhood such that $e_1, \cdots, e_m\subset E(M, V)$. Then, putting $V=\sum_\alpha v_\alpha e_\alpha$, by (1.11) and (1.7) we have

$$T_{e_a}(e_j)=\sum_\alpha A_{\alpha aj}e_\alpha=\delta_{aj}\sum_\alpha v_\alpha e_\alpha, \qquad a=1, 2, \cdots, m \tag{2.7}$$

or

$$A_\alpha=((A_{\alpha ij}))=\left(\begin{pmatrix} v_\alpha & & 0 \\ & \ddots & \\ 0 & & v_\alpha \end{pmatrix} \begin{matrix} 0 \\ \\ \end{matrix} \atop \begin{matrix} 0 & \bar{A}_\alpha \end{matrix}\right), \qquad \bar{A}_\alpha=((A_{\alpha rt})),$$

$$r, t=m+1, \cdots, n; \qquad \alpha=n+1, \cdots, n+p.^{2)}$$

On the other hand, we get from $\omega_{i\alpha}=\sum_j A_{\alpha ij}\omega_j$ by means of exterior derivative and the structure equation (1.8) replaced with

$$\bar{\Omega}_{AB}=-\bar{c}\bar{\omega}_A\wedge\bar{\omega}_B, \tag{2.8}$$

2) In the following, $a, b, c, \cdots$ run from 1 to m and $r, s, t, \cdots$ run from $m+1$ to n.

$$d\omega_{i\alpha}=d(\sum_j A_{\alpha ij}\omega_j)=\sum_j dA_{\alpha ij}\wedge\omega_j+\sum_j A_{\alpha ij}d\omega_j$$
$$=\sum_j dA_{\alpha ij}\wedge\omega_j+\sum_{k,j} A_{\alpha ik}\omega_{kj}\wedge\omega_j$$

and

$$d\omega_{i\alpha}=\sum_j \omega_{ij}\wedge\omega_{j\alpha}+\sum_\beta \omega_{i\beta}\wedge\omega_{\beta\alpha}-\bar{c}\omega_i\wedge\omega_\alpha$$
$$=\sum_{k,j}\omega_{ik}A_{\alpha kj}\wedge\omega_j+\sum_{j,\beta}\omega_{\alpha\beta}A_{\beta ij}\wedge\omega_j\,,$$

hence

$$\sum_j\{dA_{\alpha ij}+\sum_k A_{\alpha kj}\omega_{ki}+\sum_k A_{\alpha ik}\omega_{kj}+\sum_\beta A_{\beta ij}\omega_{\beta\alpha}\}\wedge\omega_j=0\,.$$

By E. Cartan's lemma, we can put

$$dA_{\alpha ij}+\sum_k A_{\alpha kj}\omega_{ki}+\sum_k A_{\alpha ik}\omega_{kj}+\sum_\beta A_{\beta ij}\omega_{\beta\alpha}=\sum_k B_{\alpha ijk}\omega_k \tag{2.9}$$

and we see that

(2.10) $B_{\alpha ijk}$ is symmetric with respect to i, j, k

by using (1.10) in addition. Putting $i=j=a$ in (2.9) we get

$$dv_\alpha+\sum_{\beta=n+1}^{n+p} v_\beta\omega_{\beta\alpha}=\sum_{k=1}^{n} B_{\alpha aak}\omega_k\,, \tag{2.11}$$

and putting $i=a$, $j=b$, $a\neq b$, in (2.9) we get

$$\sum_{k=1}^{n} B_{\alpha abk}\omega_k=0\,,$$

hence

$$B_{\alpha abk}=0\,,\qquad a\neq b\,. \tag{2.12}$$

Making use of (2.10) and (2.12), the right hand side of (2.11) can be written as

$$dv_\alpha+\sum_{\beta=n+1}^{n+p} v_\beta\omega_{\beta\alpha}=B_{\alpha aaa}\omega_a+\sum_{r=m+1}^{n} B_{\alpha aar}\omega_r\,,$$
$$a=1,2,\cdots,m;\qquad \alpha=n+1,\cdots,n+p\,.$$

Since $m>1$, it must be

$$\begin{cases} B_{\alpha aaa}=0 \\ B_{\alpha 11r}=B_{\alpha 22r}=\cdots=B_{\alpha mmr}(=B_{\alpha r})\,. \end{cases} \tag{2.13}$$

Hence, we get

$$dv_\alpha+\sum_\beta v_\beta\omega_{\beta\alpha}=\sum_{r=m+1}^{n} B_{\alpha r}\omega_r\,. \tag{2.14}$$

This equation implies that for any $u\in E(M,V)$

$$\nabla_u^{\perp} V = \sum_{r=m+1}^{n} B_{\alpha r}\omega_r(u)e_\alpha = 0,$$

since $e_a \in E(M, V)$. q. e. d.

Thus, we get the following

THEOREM 1. *Let M be an n-dimensional C^∞ submanifold immersed in an $(n+p)$-dimensional C^∞ Riemannian manifold $\bar{M}$ of constant curvature $\bar{c}$ with a C^∞ principal normal vector field V. Let M_0 be the open subset of points x of M such that* $\dim E(x, V(x)) =$ minimum. *Then, $E(x, V(x))$, $x \in M_0$, is a completely integrable distribution on M_0.*

REMARK. When $p=1$, this theorem contains Theorem 2 in [9], in which we supposed that the multiplicities of principal curvatures are all constant.

§ 3. Integral submanifolds of $E(M, V)$.

In this section, we consider only the case in Theorem 1 with $M_0 = M$ and $\dim E(M, V) > 1$, i. e.

(α) $\begin{cases} M \text{ is an } n\text{-dimensional } C^\infty \text{ submanifold immersed in an} \\ (n+p)\text{-dimensional } C^\infty \text{ Riemannian manifold } \bar{M} \text{ of constant} \\ \text{curvature } \bar{c} \text{ with a } C^\infty \text{ principal normal vector field } V \neq 0 \\ \text{such that } E(M, V) \text{ is an } m\text{-dimensional distribution and } m > 1 \end{cases}$

$E(M, V)$ is C^∞ by Lemma 2 and completely integrable by Theorem 1. We denote the integral submanifold of $E(M, V)$ through $x \in M$ by $M^m(x) = M^m$.

THEOREM 2. *Any integral submanifold M^m of the distribution $E(M, V)$ under the condition (α) is of constant curvature and totally umbilic in M^n and $\bar{M}^{n+p}$.*

PROOF. About any point $x_0 \in M$, we use $b = (x, e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+p}) \in B$ such that $e_1, \cdots, e_m \subset E(M, V)$ and $V = \lambda e_{n+1}$. By the way analogous to (2.7) and (2.14) we have

$$A_{n+1aj} = \lambda\delta_{aj} \quad \text{or} \quad \omega_{an+1} = \lambda\omega_a, \tag{3.1}$$

$$A_{\beta aj} = 0 \quad \text{or} \quad \omega_{a\beta} = 0, \ \beta = n+2, \cdots, n+p \tag{3.2}$$

and

$$d\lambda = \sum_{r=m+1}^{n} B_{n+1r}\omega_r, \tag{3.3}$$

$$\lambda\omega_{n+1\beta} = \sum_{r=m+1}^{n} B_{\beta r}\omega_r, \qquad \beta = n+2, \cdots, n+p. \tag{3.4}$$

From (3.1) and (3.2), we get

$$d\omega_{an+1}=\sum_{b=1}^{m}\omega_{ab}\wedge\omega_{bn+1}+\sum_{r=m+1}^{n}\omega_{ar}\wedge\omega_{rn+1}$$

$$+\sum_{\beta=n+2}^{n+p}\omega_{a\beta}\wedge\omega_{\beta n+1}-\bar{c}\omega_a\wedge\omega_{n+1}$$

$$=\lambda\sum_{b}\omega_{ab}\wedge\omega_b+\sum_{r}\omega_{ar}\wedge\omega_{rn+1},$$

$$d(\lambda\omega_a)=d\lambda\wedge\omega_a+\lambda\sum_{j=1}^{n}\omega_j\wedge\omega_{ja},$$

hence

$$d\lambda\wedge\omega_a+\lambda\sum_{r=m+1}^{n}\omega_r\wedge\omega_{ra}-\sum_{r=m+1}^{n}\omega_{ar}\wedge\omega_{rn+1}=0. \tag{3.5}$$

Analogously, from $d\omega_{a\beta}=0$ we get

$$\lambda\omega_a\wedge\omega_{n+1\beta}+\sum_{r=m+1}^{n}\omega_{ar}\wedge\omega_{r\beta}=0, \qquad \beta=n+2, \cdots, n+p. \tag{3.6}$$

From (3.3) and (3.5), we get

$$\sum_{r}(B_{n+1r}\omega_a-\lambda\omega_{ar}+\sum_{t}\omega_{at}A_{n+1tr})\wedge\omega_r=0.$$

From (3.4) and (3.6), we get

$$\sum_{r}(B_{\beta r}\omega_a+\sum_{t}\omega_{at}A_{\beta tr})\wedge\omega_r=0.$$

By means of E. Cartan's lemma, we get from these

$$\sum_{t}\omega_{at}(A_{n+1tr}-\lambda\delta_{tr})+B_{n+1r}\omega_a=\sum_{t}C_{art}\omega_t,$$

$$\sum_{t}\omega_{at}A_{\beta tr}+B_{\beta r}\omega_a=\sum_{t}C_{\beta art}\omega_t, \qquad \beta=n+2, \cdots, n+p,$$

where $C_{art}=C_{atr}$ and $C_{\beta art}=C_{\beta atr}$. From the assumption ($\alpha$), the solution of the following equations in $u_{m+1}, \cdots, u_n$:

$$\sum_{t}u_t(A_{n+1tr}-\lambda\delta_{tr})=0,$$

$$\sum_{t}u_tA_{\beta tr}=0$$

is $u_{m+1}=u_{m+2}=\cdots=u_n=0$. Therefore, ω_{ar} can be written as

$$\omega_{ar}=\rho_r\omega_a+\sum_{t=m+1}^{n}\Gamma_{art}\omega_t \tag{3.7}$$

where ρ_t, Γ_{art} are C^∞ functions on the submanifold of B whose points satisfy the conditions in the beginning of the proof.

From (3.7), (3.1), (3.2) and the structure equations, we get the following equalities

$$d\omega_{ar} \equiv d\rho_r \wedge \omega_a + \rho_r d\omega_a + \sum_t \Gamma_{art} d\omega_t$$

$$\equiv d\rho_r \wedge \omega_a + \rho_r \sum_b \omega_b \wedge \omega_{ba} + \sum_{t,b} \Gamma_{art} \omega_b \wedge \omega_{bt}$$

$$\equiv d\rho_r \wedge \omega_a + \rho_r \sum_b \omega_b \wedge \omega_{ba}$$

$$(\mathrm{mod}\ \omega_{m+1}, \cdots, \omega_n),$$

$$d\omega_{ar} = \sum_b \omega_{ab} \wedge \omega_{br} + \sum_t \omega_{at} \wedge \omega_{tr} + \omega_{an+1} \wedge \omega_{n+1r} - \bar{c}\omega_a \wedge \omega_r$$

$$\equiv \sum_b \rho_r \omega_{ab} \wedge \omega_b + \sum \rho_t \omega_a \wedge \omega_{tr}$$

$$(\mathrm{mod}\ \omega_{m+1}, \cdots, \omega_n),$$

hence $(d\rho_r + \sum_t \rho_t \omega_{tr}) \wedge \omega_a \equiv 0$, that is

$$d\rho_r + \sum_t \rho_t \omega_{tr} \equiv 0 \qquad (\mathrm{mod}\ \omega_{m+1}, \cdots, \omega_n). \tag{3.8}$$

This follows that

$$d(\sum_r \rho_r \rho_r) \equiv 0 \qquad (\mathrm{mod}\ \omega_{m+1}, \cdots, \omega_n). \tag{3.9}$$

Now, we consider any integral submanifold M^m of the distribution $E(M, V)$. On M^m, we have

$$\omega_{m+1} = \cdots = \omega_n = 0$$

and

$$\omega_{ar} = \rho_r \omega_a . \tag{3.10}$$

Therefore, the curvature forms of M^m are given by

$$d\omega_{ab} - \sum_{c=1}^{m} \omega_{ac} \wedge \omega_{cb} = d\omega_{ab} - \sum_{C=1}^{n+p} \omega_{aC} \wedge \omega_{Cb} \tag{3.11}$$

$$+ \sum_{r=m+1}^{n} \omega_{ar} \wedge \omega_{rb} + \omega_{an+1} \wedge \omega_{n+1b}$$

$$= -(\bar{c} + \sum_r \rho_r^2 + \lambda^2) \omega_a \wedge \omega_b .$$

By means of Lemma 3, λ is constant on M^m and $\sum_r \rho_r^2$ is also constant on it by (3.9). Therefore M^m is of constant curvature $\bar{c} + \sum_r \rho_r^2 + \lambda^2$. The relations

$$\omega_{ar} = \rho_r \omega_a , \quad \omega_{an+1} = \lambda \omega_a , \quad \omega_{a\beta} = 0$$

show that M^m is totally umbilic in $\bar{M}^{n+p}$ and the first relations show that M^m is also totally umbilic in M^n. q. e. d.

COROLLARY. *Let M^n be a C^∞ submanifold of a sphere $S^{n+p} \subset R^{n+p+1}$ with a C^∞ principal normal vector field $V \neq 0$ such that $E(M, V)$ is an m-dimensional* *distribution* ($m>1$). *Then* $E(M, V)$ *is completely integrable and any one of its integral submanifolds is contained in an m-dimensional sphere which is the intersection of* S^{n+p} *and an* $(m+1)$*-dimensional linear subspace in* R^{n+p+1}.

By Theorem 2, the tangent vector field $U=\sum_{r=m+1}^{n}\rho_r e_r$ of M^n is a principal normal vector field on each integral submanifold M^m of $E(M, V)$ whose principal tangent vector space is the tangent space to M^m. We call U the *induced principal normal vector field* of M^m in M^n from V. (3.8) gives us immediately the following

THEOREM 3. *The induced principal normal vector field* U *of any integral submanifold* M^m *of the distribution* $E(M, V)$ *under* (α) *is parallel in the normal vector bundle* $N(M^m, M^n)$ *with the induced connection from* M^n.

THEOREM 4. *For any integral submanifold* M^m *of* $E(M, V)$ *under* (α), *there exists a totally geodesic submanifold* $\bar{M}^{m+2}(\bar{M}^{m+1})$ *of* $\bar{M}^{n+p}$ *in which* M^m *is immersed, if* U *does not vanish (or vanishes near* M^m *in* M^n).

PROOF. Using the notation in the proof of Theorem 2, furthermore we may put

$$U=\rho e_{m+1} \qquad (\rho\neq 0) \tag{3.12}$$

locally, if $U\neq 0$. By Theorem 3, ρ is constant on M^m. Then (3.7) and (3.8) imply

$$\begin{cases} \omega_{am+1}=\rho\omega_a+\sum_{r=m+1}^{n}\Gamma_{am+1r}\omega_r\,, \\ \omega_{at}\;=\sum_{r=m+1}^{n}\Gamma_{atr}\omega_r\,, \qquad a=1, 2, \cdots, m\,, \\ \omega_{m+1t}=\sum_{r=m+1}^{n}\Pi_{tr}\omega_r\,, \qquad t=m+2, \cdots, n\,. \end{cases} \tag{3.13}$$

Hence, we have

$$\omega_{am+1}=\rho\omega_a,\ \omega_{at}=0,\ \omega_{m+1t}=0 \quad \text{on } M^m, \quad t=m+2, \cdots, n\,. \tag{3.14}$$

On the other hand, from (3.1), (3.2) and (3.4)

$$\begin{cases} \omega_{tn+1}=0\,, & t=m+2, \cdots, n\,, \\ \omega_{a\beta}\;=0\,, & a=1, 2, \cdots, m\,, \\ \omega_{n+1\beta}=0\,, & \beta=n+2, \cdots, n+p\,. \end{cases} \tag{3.15}$$

Finally, we consider the following exterior derivative on M^m

$$0=d\omega_{a\beta}=\omega_{am+1}\wedge\omega_{m+1\beta}=\rho\omega_a\wedge\omega_{m+1\beta}\,,$$

by means of (3.14), (3.15), (1.8) and (2.8). Hence, we get

$$\omega_{m+1\beta}=0\,,\quad \beta=n+2, \cdots, n+p\,, \qquad \text{on } M^m\,. \tag{3.16}$$

On the other hand, if we consider the following Pfaff equation system

$$
(3.17)\qquad \begin{cases} \bar{\omega}_t = 0, & \bar{\omega}_{at} = 0, \quad \bar{\omega}_{m+1t} = 0, \quad \bar{\omega}_{n+1t} = 0, \\ \bar{\omega}_\beta = 0, & \bar{\omega}_{a\beta} = 0, \quad \bar{\omega}_{m+1\beta} = 0, \quad \bar{\omega}_{n+1\beta} = 0. \end{cases}
$$

$$
a = 1, 2, \cdots, m;\ t = m+2, \cdots, n;\ \beta = n+2, \cdots, n+p \quad \text{on } F(\bar{M}),
$$

this is clearly completely integrable and its maximal integral submanifold in $F(\bar{M})$ gives an $(m+2)$-dimensional totally geodesic submanifold $\bar{M}^{m+2}$ in $\bar{M}^{n+p}$. (3.14), (3.15) and (3.16) show that M^m satisfies (3.17). Therefore we may consider that $\bar{M}^{m+2}$ contains M^m.

Next, we consider the case $U \equiv 0$ about a point of M^m in M^n. From (3.7), (3.2) and (3.4), we have

$$
(3.18)\qquad \omega_{ar} = 0, \quad \omega_{a\beta} = 0, \quad \omega_{n+1r} = 0, \quad \omega_{n+1\beta} = 0 \quad \text{on } M^m,
$$

$$
a = 1, 2, \cdots, m; \quad r = m+1, \cdots, n; \quad \beta = n+2, \cdots, n+p.
$$

On $F(\bar{M})$, the Pfaffian equation system

$$
(3.19)\qquad \begin{cases} \bar{\omega}_r = 0, & \bar{\omega}_{ar} = 0, \quad \bar{\omega}_{n+1r} = 0, \\ \bar{\omega}_\beta = 0, & \bar{\omega}_{a\beta} = 0, \quad \bar{\omega}_{n+1\beta} = 0 \end{cases}
$$

$$
a = 1, 2, \cdots, m; \quad r = m+1, \cdots, n; \quad \beta = n+2, \cdots, n+p,
$$

is completely integrable and its maximal integral submanifold in $F(\bar{M})$ gives an $(m+1)$-dimensional totally geodesic submanifold $\bar{M}^{m+1}$ in $\bar{M}^{n+p}$. (3.18) shows that we may consider that $\bar{M}^{m+1}$ contains M^m.

REMARK. $\bar{M}^{m+2}$ (or $\bar{M}^{m+1}$) is clearly the locus of geodesics tangent to the vector space spanned by V and U at each point of M^m.

University of California, Berkeley
Tokyo Institute of Technology

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