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Galois cohomology and birational invariant of algebraic varieties

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Introduction.

This paper was written motivated by Manin's recent paper [**4**] in which he suggested the importance of Tamagawa number of the dual torus of the Neron-Severi group of a rational surface, in connection with the ζ-function of the surface. But in this paper we shall deal with arbitrary dimensional algebraic varieties without the restriction of the rationality and define some birational invariant of them. When we consider only the rational varieties, we can define the birational invariant using only the Neron-Severi groups of them but for arbitrary algebraic varieties we must take into account the contributions of the Albanese varieties of them.

Since we use the arguments developed in T. Ono's paper [**6**], we have to restrict the basic field k to a field of dimension one.

Let k be a field of dimension one i.e. either a finite algebraic number field or an algebraic function field of one variable over a finite field. Let V be a complete non-singular algebraic variety defined over k. Let $N^0(V)$ be the torsion free part of the Neron-Severi group of V (i.e. $N^0(V)=D(V)/D_t(V)$; $D(V)$ is the group of all divisors on V and D_t is the group of torsion divisors), and A be the Albanese variety of V defined over k. Let $\mathrm{Hom}(A, \hat{A})$ be the finite type Z-free module of all the rational homomorphisms from A to $\hat{A}$. Then the birational invariant $\mu_k(V)$ of V over k will be defined by

$$\mu_k(V)=\boldsymbol{h}_k(V)/\boldsymbol{i}_k(V),$$

$$\boldsymbol{h}_k=\frac{h_k^1(N^0(V))}{h_k^1(\mathrm{Hom}(A, \hat{A}))^{1/2}}, \qquad \boldsymbol{i}_k(V)=\frac{i_k(N^0(V))}{i_k(\mathrm{Hom}(A, \hat{A}))^{1/2}}$$

where h_k^1 and i_k are the notations used in [**5**] (see § 3). When V is a rational variety, the Albanese variety A vanishes and $\mu_k(V)$ depends only on $N^0(V)$. In § 2 and § 3 we show that $h_k^1(N^0(V))$ and $i_k(N^0(V))$ are birational invariant over k. (See Theorem 2 and Proposition 6). Since the Albanese variety A attaches to V birational-invariantly, we see that $\mu_k(V)$ is a birational invariant of V over k. The reason why we have considered the contributions of the

Albanese variety (i. e. $h_k^1(\mathrm{Hom}(A, \hat{A}))$ and $i_k(\mathrm{Hom}(A, \hat{A}))$) is not to make μ_k birational invariant, but to make it fulfil the fundamental equalities which are also true for the Tamagawa number of algebraic groups, namely

$$\mu_k(V\times V')=\mu_k(V)\cdot\mu_k(V') \quad\text{and}\quad \mu_{k_0}(R_{k/k_0}(V))=\mu_k(V)\,,$$

where V and V' are complete non-singular algebraic varieties defined over k, k/k_0 is a finite separable algebraic extension and R_{k/k_0} is the k/k_0-trace of V (see § 4). Under the same situation we shall also prove

$$\boldsymbol{h}_k(V\times V')=\boldsymbol{h}_k(V)\cdot\boldsymbol{h}_k(V')\,,\qquad \boldsymbol{h}_{k_0}(R_{k/k_0}(V))=\boldsymbol{h}_k(V)$$

and

$$\boldsymbol{i}_k(V\times V')=\boldsymbol{i}_k(V)\cdot\boldsymbol{i}_k(V')\,,\qquad \boldsymbol{i}_{k_0}(R_{k/k_0}(V))=\boldsymbol{i}_k(V)\,.$$

(See Theorem 3 and 4).

§ 1. Notations and preliminaries.

In this section we define some notations which we use all through this paper and recall some facts about the Galois modules.

Let k be a field (which will be restricted to a field of dimension one in § 3 and § 5) and V be an algebraic variety defined over k. For the divisor groups of V we use the following notations.

$D(V)$; the group of all divisors on V.

$D_l(V)$; the group of all divisors on V which are linearly equivalent to zero, i. e. the divisors of rational functions on V.

$D_a(V)$; the group of divisors on V which are algebraically equivalent to zero.

$D_t(V)$; the group of divisors on V which are torsion equivalent to zero, i. e. the divisors whose some integer multiples are in $D_a(V)$.

This group can be identified with the numerical equivalence group if V is complete non-singular projective variety. (See p. 329, (C) [**11**]).

$N(V)=D(V)/D_a(V)$; Neron-Severi group.

$N^0(V)=D(V)/D_t(V)$; torsion free part of Neron-Severi group.

Let Ω be the algebraic closure of the field k and K be an intermediate field between k and Ω (i. e. $k\subset K\subset\Omega$). We denote the Galois group of Ω over K by $g(\Omega/K)$. By $D(V)_\Omega$ (resp. $D_l(V)_\Omega$, $D_a(V)_\Omega$, $D_t(V)_\Omega$) we denote the subgroup of $D(V)$ (resp. $D_l(V)$, $D_a(V)$, $D_t(V)$) consisting of those elements which are rational over Ω. Then obviously $g(\Omega/K)$ operates on $D(V)_\Omega$, $D_l(V)_\Omega$, $D_a(V)_\Omega$ and $D_t(V)_\Omega$ making them $g(\Omega/K)$-modules. We know that $N(V)=D(V)_\Omega/D_a(V)_\Omega$ and $N^0(V)=D(V)_\Omega/D_t(V)_\Omega$ and $g(\Omega/K)$ also acts on them, making them $g(\Omega/K)$-modules. By $D(V)_k$ (resp. $D_l(V)_K$, $D_a(V)_K$, $D_t(V)_K$, $N(V)_K$, $N^0(V)_K$) we denote the subgroup of $D(V)_\Omega$ (resp. $d_l(V)_\Omega$, $D_a(V)_\Omega$, $D_t(V)_\Omega$,

$N(V)$, $N^0(V)$) consisting of $g(\Omega/K)$ invariant elements. If K is a normal extension of k, then the Galois group $g(K/k)$ of the extension K/k operates on $D(V)_K$, $D_l(V)_K$, $D_a(V)_K$, $D_t(V)_K$, $N(V)_K$ and $N^0(V)_K$ making them $g(K/k)$-modules.

Let P be the Picard variety of V defined over the field k. Then we have an isomorphism $\varphi: D_a(V)/D_l(V) \xrightarrow{\sim} P$, which has the following properties. 1) If a divisor X in $D_a(V)$ is rational over a field k' containing k, then the image $\varphi(Cl(X))$ of the class $Cl(X)$ of X in $D_a(V)/D_l(V)$ is a rational point of P over k'. 2) If a point a of P is rational over a field k' containing k, then there exists a divisor X in $D_a(V)_{k'}$ such that $\varphi(Cl(X))=a$. 3) φ is compatible with the specialization over k. We call the isomorphism with these properties the canonical isomorphism.

Here we recall some facts about the Galois modules and its cohomology groups. Let G be a finite group and H be its subgroup. Let E be a H-module (i. e. an abelian group on which the group H acts as group of automorphisms). Then we define the G-module $M_G^H(E)$ as the submodule of $\mathrm{Hom}_Z(Z[G], E)$ such that $f(gh)=h^{-1}f(g)$ for $h\in H$, $g\in G$, $f\in \mathrm{Hom}_Z(Z[G], E)$. The structure of G-module on $M_G^H(E)$ is defined by $(gf)(g')=f(g^{-1}g')$ for $g, g'\in G$. If we associate to $f\in M_G^H(E)$ its value at the neutral element of G, we get a homomorphism $\theta: M_G^H\to E$ which is compatible with the natural injection of H in G (i. e. θ is an H-homomorphism, because $\theta^h(f)=h\cdot\theta(h^{-1}f)=h((h^{-1}f)(e))$ $=h(f(h))=f(e)=\theta(f)$).

LEMMA 1. *Using the above notations, we have the isomorphism of G-modules*

$$Z[G]\otimes_{Z[H]}E\cong M_G^H(E).$$

PROOF. To an element $g\otimes m$ in left side we associate an element f in right side such that $f(g)=m$, $f(gh)=h^{-1}\cdot m$ and $f(g')=0$ for $h\in H$, $g'\notin gH$. By this mapping the isomorphism of Lemma is given. Q. E. D.

LEMMA 2. *The homomorphism θ induces an isomorphism*

$$H^q(G, M_G^H(E))\cong H^q(H, E) \qquad q\geqq 0.$$

PROOF. First we notice that M_G^H is an exact functor from the category of H-modules to the category of G-modules. Since Hom is a left exact functor, we have only to show that a surjective homomorphism $p: E\to E'$ of H-modules induces a surjective homomorphism $p^*: M_G^H(E)\to M_G^H(E')$. For an element f of $M_G^H(E')$ and the left representatives $g_1, \cdots, g_n$ of the right cosets G/H (i. e. $G=g_1H+\cdots+g_nH$), we select elements e_i $(i=1, \cdots, n)$ of E such that $p(e_i)$ $=f'(g_i)$ $(i=1, \cdots, n)$. If we put $f(g_i)=e_i$ and $f(g_ih)=h^{-1}e_i$ $(i=1, \cdots, n)$, then f defines an element of $M_G^H(E)$ and we have $p^*(f)=f'$. Thus p^* is surjective. On the other hand we show that θ induces an isomorphism $\theta^*: \mathrm{Hom}^G(B, M_G^H(E))$ $\xrightarrow{\sim}\mathrm{Hom}^H(B, E)$ for a G-module B. For an element α of $\mathrm{Hom}^G(B, M_G^H(E))$ and

h in H, we have $[(\alpha\cdot\theta)^h](b)=h[(\alpha\cdot\theta)(h^{-1}b)]=h[(\alpha(h^{-1}b))(e)]=h[(\alpha(b))(h)]$ $=[\alpha(b)(e)]=(\alpha\cdot\theta)(b)$. Hence we have $\alpha\cdot\theta\in \mathrm{Hom}^H(B, E)$. For an element β in $\mathrm{Hom}^H(B, E)$, put $(\alpha(b))(e)=\beta(b)$ and $(\alpha(b))(g)=(\alpha(g^{-1}b))(e)$ for all $b\in B$ and $g\in G$. Then α defines an element of $\mathrm{Hom}^G(B, M_G^H(E))$ because we have $(\alpha^g(b))(g')=(g\cdot\alpha(g^{-1}b))(g')=(\alpha(g^{-1}b))(g^{-1}g')=(\alpha(b))(g')$, and we have $\theta^*(\alpha)$ $=\beta$. Therefore θ^* is surjective. The injectivity of θ^* is clear. Therefore by standard comparison theorem we get the Lemma. Q. E. D.

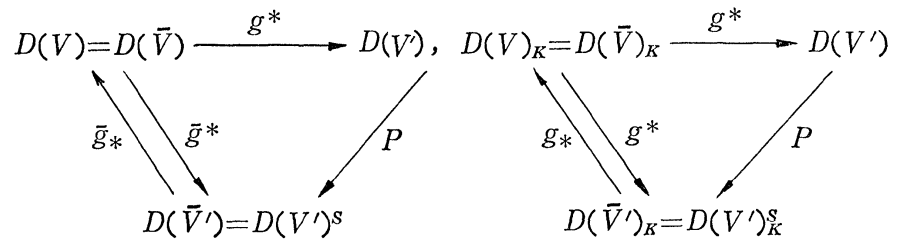
The divisor group $D(V)_K$ can be written as the direct sum $\coprod_{\xi}\coprod_{X\in\xi} Z\cdot X$, where ξ ranges over all the prime rational divisors on V over k and X's are the prime rational components of ξ over K. When K is a finite Galois extension of k and G is its Galois group, we denote by G_ξ the subgroup of G consisting of those elements which make invariant one fixed components of ξ. (If we replace the fixed component of ξ by another one, then G_ξ will be replaced by a conjugate subgroup of G_ξ in G). If we have $G=g_1G_\xi+g_2G_\xi$ $+\cdots+g_nG_\xi$, we have the isomorphism of G-modules $\coprod_{X\in\xi} Z\cdot X=\sum_{i=1}^{n} Z\cdot(X)^{g_i}$ $\cong Z[G/G_\xi]\cong M_G^{G_\xi}(Z)$, where Z is the additive group of rational integers on which G acts trivially. The isomorphism is given by $(X)^{g_i}\leftrightarrow a_i=g_iG_\xi\leftrightarrow a_i^*$; $a_i^*(g_ih)=1$, $a_i^*(g_jh)=0$ $(i\neq j,\ h\in H)$. Therefore by Lemma 1, we have $H^1(G, \coprod_{X\in\xi} Z\cdot X)=H^1(G_\xi, Z)=0$ and $H^1(G, D(V)_K)=0$.

§ 2. The birational invariance of the first cohomology of divisor groups.

In this section we consider the Galois cohomology of the divisor groups defined in section 1 and prove the birational invariance of the first cohomology of them.

Let V and V' be complete non-singular algebraic varieties defined over the field k and g be a birational morphism from V' to V defined over k. The assumption "non-singular" is necessary for g to induce a natural isomorphism of the Picard groups of V and V'. (See the last paragraph of p. 152 of [2]). Let S be the set of all prime rational divisors on V over k which vanish under the morphism g. Namely, S consists of all prime rational divisors on V' over k such that the set theoretic images of them by g have codimensions larger than 1. Let K be a finite normal extension of k and Σ_K be the Z-free module generated by all the rational prime components of the divisors of S over K (i. e. $\Sigma_K=\coprod_{\xi\in S}\coprod_{X\in\xi} Z\cdot X$). We consider the homomorphisms between $D(V')$ and $D(V)$ induced by the birational mapping g. Let Γ_g be the graph of g on $V'\times V$. For a divisor X in $D(V)$, we define a divisor $g^*(X)$ in $D(V')$ by $g^*(X)=pr_{V'}[\Gamma_g\cdot(V'\times X)]$. Since g is a birational morphism and V' is complete non-singular, $g^*(X)$ is defined for every X in $D(V)$ and we get a homomorphism $g^*: D(V)\to D(V')$. Since g is defined over k, g^* induces a homomorphism $g^*: D(V)_K\to D(V')_K$ which commutes with the action of the Galois group G of the extension K/k. Let $D(V')^S$ be the subgroup of divisors in $D(V')$ which are free from the components (over Ω) of the prime divisors in S and $P: D(V')\to D(V')^S$ be the natural projection. Let $\bar{V}$ be the subset of V on which the birational mapping g^{-1} is everywhere defined. Then $\bar{V}$ is Zariski k-open subvariety of V. Since $\bar{V}$ is non-singular, g^{-1} is defined along the subvarieties of codimension one. Therefore $V-\bar{V}$ is k-closed subset with codimension larger than one and we have $D(\bar{V})=D(V)$ by natural injection. If we put $\bar{V}'=g^{-1}(\bar{V})$, then g is everywhere biregular on $\bar{V}'$ and $\bar{V}'$ is a k-open subvariety of V'. By natural identification we have $D(\bar{V}')\subset D(V')$. Since g is not biregular along the components of the prime divisors in S, we also have $D(\bar{V}')\subset D(V')^S$. Since S contains of all the vanishing prime divisors, we get $D(\bar{V}')=D(V')^S$. If we denote by $\bar{g}$ the birational and biregular morphism from $\bar{V}'$ to $\bar{V}$, $\bar{g}$ induces the isomorphism $\bar{g}^*: D(\bar{V})\xrightarrow{\sim} D(\bar{V}')=D(V')^S$, which commutes with the projection P. Thus we get the commutative diagrams

(1)



where $\bar{g}^*(\bar{X})=pr_{\bar{V}'}[\Gamma_{\bar{g}}\cdot(\bar{V}'\times\bar{X})]$ for $\bar{X}\in D(\bar{V})$ and $\bar{g}_*(\bar{Y})=pr_{\bar{V}}[\Gamma_{\bar{g}}\cdot(\bar{Y}\times\bar{X})]$ for $\bar{Y}\in D(\bar{V}')$. Clearly g^* is an injection. It is also clear that for a rational function f on V and $f\cdot g$ on V' we have $g^*((f))=(f\cdot g)$, where (*) means the divisor of function *. We define the divisor groups of V' restricted to $D(V')^S$ by $D(V')^S_K=D(V')_K\cap D(V')^S$ and $D_l(V')^S_K=D_l(V')_K\cap D(V')^S$. And also we define a restricted group $D_l(V)^S_K$ in $D_l(V)_K$ by $D(V)^S_K=\{(f)\mid f\in K(V),\ (f\cdot g)=g^*(f)\in D_l(V')^S_K\}$. Then we get the homomorphisms of divisor groups which commute with the action of G.

The natural injection $D(V')^S_K\to D(V')_K$ induces an isomorphism

(2) $$I: D(V')^S_K/D_l(V')^S_K \xrightarrow{\sim} D(V')_K/D_l(V')_K\,.$$

The injection $D(V)_K\to D(V')_K$ induces the isomorphisms

(3) $$J: D(V)_K/D_l(V)_K \longrightarrow D(V')_K/D_l(V')_K\,,$$

(4) $$J^S: D(V)_K/D_l(V)^S_K \longrightarrow D(V')_K/D_l(V')^S_K\,.$$

The isomorphism $\bar{g}^*: D(V)_K\xrightarrow{\sim} D(V')^S_K$ induces an isomorphism

(5) $$I^S: D(V)_K/D_l(V)^S_K \xrightarrow{\sim} D(V')^S_K/D_l(V')^S_K\,.$$

REMARK. For the natural surjection $h: D(V)_K/D_l(V)^S_K \to D(V)_K/D_l(V)_K$ $J \cdot h$ and $I \cdot I^S$ are not necessarily coincides, because the kernel of the homomorphism $D(V')_K \to D(V')^S_K/D_l(V')^S_K$ defined by the projection P is $D_l(V')^S_K \otimes \Sigma_K$ and not $D_l(V')_K$.

By the relation $D(V')^S_K \cap D_l(V')_K = D_l(V')^S_K$ and (2) we get the commutative diagram of G-modules with the exact rows and columns;

(6)
$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & D(V')^S_K/D_l(V')^S_K & \longrightarrow & D(V')^S_K/D_l(V')^S_K & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D_l(V')_K/D_l(V')^S_K & \longrightarrow & D(V')_K/D_l(V')^S_K & \longrightarrow & D(V')_K/D_l(V')_K & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D_l(V')_K/D_l(V')^S_K & \longrightarrow & D(V')_K/D_l(V')^S_K & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Taking the cohomology groups of the diagram (6), we get the commutative diagram with the exact rows and columns;

(7)
$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & H^n(G, D(V')^S_K/D_l(V')^S_K) & \to & H^n(G, D(V')^S_K/D_l(V')^S_K) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^n(G, D_l(V')_K/D_l(V')^S_K) & \to & H^n(G, D(V')_K/D_l(V')^S_K) & \to & H^n(G, D(V')_K/D_l(V')_K) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^n(G, D_l(V')_K/D_l(V')^S_K) & \to & H^n(G, D(V')_K/D(V')^S_K) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where the exactness of the central row and column follows from the isomorphism on the sides.

PROPOSITION 1.

$$H^1(G, D_l(V)_K/D_l(V)^S_K) = H^1(G, D_l(V')/D_l(V')^S_K) = H^1(G, \Sigma_K) = 0\,.$$

PROOF. By the diagram (6) and the canonical isomorphisms $D_l(V)_K = D_l(V)_K \cong D_l(V')_K$ and $D(V')_K \cong D(V')^S_K \oplus \Sigma_K$, we obtain the isomorphisms of G-modules $D_l(V)_K/D_l(V)^S_K = D_l(V')_K/D_l(V')^S_K = D(V')_K/D(V')^S_K = \Sigma_K$. Since we have $\Sigma_K = \coprod_{\xi \in S} \coprod_{X \in \xi} Z \cdot X$ and $\coprod_{X \in \xi} Z \cdot X = M^{G_\xi}_G(Z)$ (see the last paragraph of § 1), we get $H^1(G, \Sigma_K) = 0$. Thus Proposition is proved. Q. E. D.

LEMMA 3. *Let us have a commutative diagram of modules with the exact rows*

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow{\wr\, a} & & \downarrow{b} & & \downarrow{c} & & \downarrow{\wr\, d} & & \downarrow{e} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

where a and d are isomorphisms. i) *If b is injective, then c is injective.* ii) *If c is surjective, then b is surjective.* iii) *If b is surjective and e is injective, then c is surjective.*

PROPOSITION 2. *The homomorphisms j_n: $H^n(G, D(V)_K/D_l(V)_K) \to H^n(G, D(V')_K/D_l(V')_K)$ and j_n^S: $H^n(G, D(V)_K/D_l(V)_{K}^{S}) \to H^n(G, D(V')_K/D_l(V')_{K}^{S})$, which are induced by J and J^S respectively, are injective for every $n \geqq 0$.*

PROOF. We have the commutative diagram of G-modules

$$(8)\qquad \begin{array}{ccc} D(V)_K/D_1(V)_K^S & \xrightarrow[\sim]{\;I^S\;} & D(V')_K^S/D_1(V')_K^S \\ & {\searrow}^{J^S} \qquad {\nearrow}^{P} & \\ & D(V')_K/D_1(V')_K^S & \end{array}$$

and get the commutative diagram

$$(9)\qquad \begin{array}{ccc} H^n(G, D(V)_K/D_1(V)_K^S) & \xrightarrow{\;i_n^S\;} & H^n(G, D(V')_K^S/D_1(V')_K^S) \\ & {\searrow}^{j_n^S} \qquad {\nearrow}^{P_n} & \\ & H^n(G, D(V')_K/D_1(V')_K^S) & \end{array}$$

Since i_n^S is an isomorphism, j_n^S is injective for every $n \geqq 0$. On the other hand by the commutative diagram with the exact rows

$$(10)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & D_l(V)_K/D_l(V)_K^S & \longrightarrow & D(V)_K/D_l(V)_K^S & \longrightarrow & D(V)_K/D_l(V)_K & \longrightarrow & 0 \\ & & \wr\downarrow A & & \downarrow J^S & & \downarrow J & & \\ 0 & \longrightarrow & D_l(V')_K/D_l(V')_K^S & \longrightarrow & D(V')_K/D_l(V')_K^S & \longrightarrow & D(V')_K/D_l(V')_K & \longrightarrow & 0, \end{array}$$

we deduce the commutative diagram with the exact rows

$$\begin{array}{ccccc} H^n(G, D_l(V)_K/D_l(V)_K^S) & \longrightarrow & H^n(G, D(V)_K/D_l(V)_K^S) & \longrightarrow & H^n(G, D(V)_K/D_l(V)_K) \\ \downarrow a_n & & \downarrow j_n^S & & \downarrow j_n \\ H^n(G, D_l(V')_K/D_l(V')_K^S) & \longrightarrow & H^n(G, D(V')_K/D_l(V')_K^S) & \longrightarrow & H^n(G, D(V')_K/D_l(V')_K) \end{array}$$

$$\begin{array}{ccc} \longrightarrow H^{n+1}(G, D_l(V)_K/D_l(V)_K^S) & \longrightarrow & H^{n+1}(G, D(V)_K/D_l(V)_K^S) \\ \downarrow a_{n+1} & & \downarrow j_{n+1}^S \\ \longrightarrow H^{n+1}(G, D_l(V')_K/D_l(V')_K) & \longrightarrow & H^{n+1}(G, D(V')_K/D_l(V')_K^S) \end{array}$$

Since j_n^S is injective for every $n \geqq 0$, j_n is injective for every $n \geqq 0$ by Lemma 3. Q. E. D.

PROPOSITION 3. *The following conditions are equivalent; a) j_n^S is an isomorphism, b) j_n is an isomorphism, c) $H^n(G, \Sigma_K)=0$.*

PROOF. Since we have $D_l(V')_K^S \cap \Sigma_K = 0$, we have the isomorphisms $D(V')_K/D_l(V')_K^S = [D(V')_K^S \oplus \Sigma_K]/D_l(V')_K^S = [D(V')_K^S/D_l(V')_K^S] \oplus \Sigma_K$ and $H^n(G, D(V')_K/D_l(V')_K^S) = H^n(G, D(V')_K^S/D_l(V')_K^S) \oplus H^n(G, \Sigma_K)$. Therefore by the diagram (9) the conditions a) and c) are equivalent. The equivalence of a) and b) follows from Lemma 3, Proposition 2 and diagram (13). Q. E. D.

Thus we get

THEOREM 1 (l).

$$j_1 : H^1(G, D(V)_K/D_l(V)_K) \longrightarrow H^1(G, D(V')_K/D_l(V')_K)$$

is an isomorphism.

This follows immediately from Proposition 1 and 3.

Since we assumed that V and V' are non-singular, the injection J: $D(V)_K/D_l(V)_K \rightarrow D(V')_K/D_l(V')_K$ induces the isomorphism of the Picard groups of V and V', (See p. 152 of [2]). Namely we have the isomorphism

$$J^a : D_a(V)_K/D_l(V)_K \xrightarrow{\sim} D_a(V')_K/D_l(V')_K \tag{12}$$

and an injection

$$J_a : D(V)_K/D_a(V)_K \longrightarrow D(V')_K/D_a(V')_K . \tag{13}$$

PROPOSITION 4. *The homomorphism j_{an}: $H^n(G, D(V)_K/D_a(V)_K) \rightarrow H^n(G, D(V')_K/D_a(V')_K)$, induced by J_a, is injective for every $n \geqq 0$ and is an isomorphism if and only if we have $H^n(G, \Sigma_K)=0$.*

PROOF. By the commutative diagram with the exact rows

$$\begin{array}{ccccccccc} 0 \longrightarrow & D_a(V)_K/D_l(V)_K & \longrightarrow & D(V)_K/D_l(V)_K & \longrightarrow & D(V)_K/D_a(V)_K & \longrightarrow 0 \\ & \wr\downarrow J^a & & \downarrow J & & \downarrow J_a & \\ 0 \longrightarrow & D_a(V')_K/D_l(V')_K & \longrightarrow & D(V')_K/D_l(V')_K & \longrightarrow & D(V')_K/D_a(V')_K & \longrightarrow 0 \end{array} \tag{14}$$

we deduce the commutative diagram with the exact rows

$$\begin{array}{ccccc} H^n(G, D_a(V)_K/D_l(V)_K) & \longrightarrow & H^n(G, D(V)_K/D_l(V)_K) & \longrightarrow & H^n(G, D(V)_K/D_a(V)_K) \\ \downarrow j_n^a & & \downarrow j_n & & \downarrow j_{an} \\ H^n(G, D_a(V')_K/D_l(V')_K) & \longrightarrow & H^n(G, D(V')_K/D_l(V')_K) & \longrightarrow & H^n(G, D(V')_K/D_a(V')_K) \end{array}$$

$$\begin{array}{ccc} \longrightarrow H^{n+1}(G, D_a(V)_K/D_l(V)_K) & \longrightarrow & H^{n+1}(G, D(V)_K/D_l(V)_K) \\ \downarrow j_{n+1}^a & & \downarrow j_{n+1} \\ \longrightarrow H^{n+1}(G, D_a(V')_K/D_l(V')_K) & \longrightarrow & H^{n+1}(G, D(V')_K/D_l(V')_K) . \end{array}$$

Since j_n is injective for every $n \geqq 0$, j_{an} is injective for every $n \geqq 0$ by Lemma 3.

The second assertion follows from Lemma 3 and Proposition 3 with the diagram (15).

Thus we get

THEOREM 1 (a).

$$j_{a1}\colon H^1(G, D(V)_K/D_a(V)_K) \xrightarrow{\sim} H^1(G, D(V')_K/D_a(V')_K)$$

is an isomorphism.

LEMMA 4. *The injection* g^* *induces an isomorphism* $J^t\colon D_t(V)_K/D_a(V)_K \to D_t(V')_K/D_a(V')_K$ *and an injection* $J_t\colon D(V)_K/D_t(V)_K \to D(V')_K/D_t(V')_K$.

PROOF. The injection $g^*\colon D(V)_K \to D(V')_K$ induces the isomorphisms $D_t(V)_K \xrightarrow{\sim} D_t(V')_K$ and $D_a(V)_K \xrightarrow{\sim} D_a(V')_K$. Therefore we have $D_t^* = g^*(D_t(V)_K) \subset D_t(V')_K$. If we have a divisor Y in $g^*(D(V)_K) \cap D_t(V')_K$, there exists a divisor X in $D(V)_K$ such that $g^*(X) = Y$ and $g^*(mX) = mY \in D_a(V')_K$ for some integer m. Hence we have $mX \in D_a(V)_K$ and $X \in D_t(V)_K$. Therefore we obtain $g^*(D(V)_K) \cap D_t(V')_K = g^*(D_t(V)_K)$. The natural injection $D_t(V')_K \to D(V')_K$ induces an injection $i\colon D_t(V')_K/g^*(D_t(V)_K) \to D(V')_K/g^*(D_t(V)_K)$. On the other hand, by the diagram (1) we have $g^*(D(V)_K) \cap \Sigma_K = 0$ and $g^*(D(V)_K) \oplus \Sigma_K = D(V')_K$. Therefore $D(V')_K/g^*(D(V)_K) = \Sigma_K$ is a torsion free module but $D_t(V')_K/g^*(D_t(V)_K)$ is a torsion module. Thus we must have $g^*(D_t(V)_K) = D_t(V')_K$. This proves Lemma 4. Q. E. D.

PROPOSITION 5. *The homomorphism* $j_{tn}\colon H^n(G, D(V)_K/D_t(V)_K) \xrightarrow{\sim} H^n(G, D(V')_K/D_t(V')_K)$, *induced by* J_t *is injective for every* $n \geqq 0$ *and is an isomorphism if and only if we have* $H^n(G, \Sigma_K) = 0$.

PROOF. By the commutative diagram with the exact rows

$$\text{(16)}\quad \begin{array}{ccccccccc} 0 & \longrightarrow & D_t(V)_K/D_a(V)_K & \longrightarrow & D(V)_K/D_a(V)_K & \longrightarrow & D(V)_K/D_t(V)_K & \longrightarrow & 0 \\ & & \downarrow J^t & & \downarrow J_a & & \downarrow J_t & & \\ 0 & \longrightarrow & D_t(V')_K/D_a(V')_K & \longrightarrow & D(V')_K/D_a(V')_K & \longrightarrow & D(V')_K/D_t(V')_K & \longrightarrow & 0 \end{array}$$

we deduce the commutative diagram with the exact rows.

(17)

$$\begin{array}{ccccc} H^n(G, D_t(V)_K/D_a(V)_K) & \longrightarrow & H^n(G, D(V)_K/D_a(V)_K) & \longrightarrow & H^n(G, D(V)_K/D_t(V)_K) \\ \downarrow j_n^t & & \downarrow j_{an}^t & & \downarrow j_{an}^t \\ H^n(G, D_t(V')_K/D_a(V')_K) & \longrightarrow & H^n(G, D(V')_K/D_a(V')_K) & \longrightarrow & H^n(G, D(V')_K/D_t(V')_K) \end{array}$$

$$\begin{array}{ccccc} \longrightarrow & H^{n+1}(G, D_t(V)_K/D_a(V)_K) & \longrightarrow & H^{n+1}(G, D(V)_K/D_a(V)_K) \\ & \downarrow j_{n+1}^t & & \downarrow j_{a,n+1} \\ \longrightarrow & H^{n+1}(G, D_t(V')_K/D_a(V')_K) & \longrightarrow & H^{n+1}(G, D(V')_K/D_a(V')_K)\,. \end{array}$$

Since j_{an} is injective for every $n \geqq 0$, j_{tn} is injective for every $n \geqq 0$ by Lemma 3. The second assertion follows from Lemma 3 and Proposition 4 with the diagram (17).

THEOREM 1 (*t*).

$$j_{t1}: H^1(G, D(V)_K/D_t(V)_K) \xrightarrow{\sim} H^1(G, D(V')_K/D_t(V')_K)$$

is an isomorphism.

Now we consider the following conditions for a normal extension K of the field k.

(s. *l*.) $\qquad [D(V)_\Omega/D_l(V)_\Omega]_K = D(V)_K/D_l(V)_K$.

(s. *a*.) $\qquad N(V) = D(V)_\Omega/D_a(V)_\Omega = D(V)_K/D_a(V)_K$.

(s. *t*.) $\qquad N^0(V) = D(V)_\Omega/D_t(V)_\Omega = D(V)_K/D_t(V)_K$.

By the commutative diagram with the exact rows

(18)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D_a(V)_K/D_l(V)_K & \longrightarrow & D(V)_K/D_l(V)_K & \longrightarrow & D(V)_K/D_a(V)_K & \longrightarrow & 0 \\ & & \downarrow\wr & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & [D_a(V)_\Omega/D_l(V)_\Omega]_K & \longrightarrow & [D(V)_\Omega/D_l(V)_\Omega]_K & \longrightarrow & [D(V)_\Omega/D_a(V)_\Omega]_K & & \end{array}$$

(where the vertical arrow on the left side of the diagram is an isomorphism by the theory of Picard variety), we deduce that the condition (s. *a*.) implies the condition (s. *l*.). Since $N(V)$ and $N^0(V)$ are the finitely generated modules, there exists a finite normal extension of k which satisfies the conditions (s. *a*.) and (s. *t*.).

THEOREM 2. *Let k be a field and K be a finite normal extension of k. Let V and V' be the complete non-singular algebraic varieties defined over k and g be a birational morphism from V' to V defined over k. If K satisfies the condition* (s. *a*.) *for V and V', then g induces the isomorphisms;*

(19) $\qquad H^1(G, [D(V)_\Omega/D_l(V)_\Omega]_K) \cong H^1(G, [D(V')_\Omega/D_l(V')_\Omega]_K)$,

(20) $\qquad H^1(G, N(V)) \cong H^1(G, N(V'))$.

If K satisfies the condition (s. *t*.) *for V and V', then g induces the isomorphism*

(21) $\qquad H^1(G, N^0(V)) \cong H^1(G, N^0(V'))$.

§ 3. Definition of the birational invariant μ_k.

Let us define the birational invariant μ_k of algebraic variety. Since we have to use the theory of Tamagawa number of algebraic torus, which is developed by T. Ono [**6**], we restrict the basic field k to a field of dimension one i. e. either a finite algebraic number field or an algebraic function field of one variable over a finite field. First we define two symbols h_k^1 and i_k after the paper [**6**]. Let K be a finite normal extension of k and G be its Galois group. Let M be a finite type Z-free G-module. Let I_K be the group of ideals of K and C_K be its class group. Then we have the exact sequences,

(22) $$0 \longrightarrow K^* \longrightarrow I_K \longrightarrow C_K \longrightarrow 0.$$

(23) $$0 \longrightarrow \mathrm{Hom}_Z(M, K^*) \longrightarrow \mathrm{Hom}_Z(M, I_K) \longrightarrow \mathrm{Hom}_Z(M, C_K) \longrightarrow 0.$$

Taking the cohomology groups of (23), we get the exact sequence

(24) $$\mathrm{Hom}^G(M, I_K) \xrightarrow{a} \mathrm{Hom}^G(M, C_K) \longrightarrow H^1(G, \mathrm{Hom}_Z(M, K^*)) \xrightarrow{b} H^1(G, \mathrm{Hom}_Z(M, I_K)).$$

We define the notations h_k^1 and i_k by

(25) $$h_k^1(M) = [H^1(G, M)],$$

(26) $$i_k(M) = [\mathrm{Ker}(b)] = [\mathrm{Cok}(a)],$$

where [*] means the cardinal number of *, which is assured to be finite by Nakayama's duality. (See p. 53 of [**6**]).

Let A and B be the abelian varieties defined over k and $\mathrm{Hom}(A, B)$ be the set of all the rational homomorphisms from A to B. Then it is well known that $\mathrm{Hom}(A, B)$ is a finite type Z-free module and there exists a finite normal extension K of k over which all the elements of $\mathrm{Hom}(A, B)$ are defined. In this case we say that $\mathrm{Hom}(A, B)$ splits over K or K is a splitting field of $\mathrm{Hom}(A, B)$. (When the normal extension K/k satisfies the condition (s. t.) of § 2 we say also that $N^0(V)$ splits over K or K is a splitting field of $N^0(V)$). Then the Galois group G of the extension K/k operates on $\mathrm{Hom}(A, B)$ so that $\mathrm{Hom}(A, B)$ becomes a finite type Z-free G-module and we can define $h_k^1(\mathrm{Hom}(A, B))$ and $i_k(\mathrm{Hom}(A, B))$. In the following we denote by $\hat{A}$ the dual abelian variety of A.

Let V be a complete non-singular algebraic variety defined over k and $A(V)$ be the Albanese variety defined over k. (We can select an Albanese variety of V defined over k. (See Remark of p. 52 [**2**]).) Let K be a finite normal extension of k satisfying the following condition.

(S) $N^0(V)$ and $\mathrm{Hom}(A(V), \hat{A}(V))$ split over K.

We define the number μ_k by

(27) $$\mu_k(V) = \frac{h_k^1(N^0(V))}{i_k(N^0(V))} \cdot \frac{i_k(\mathrm{Hom}(A(V), \hat{A}(V)))^{1/2}}{h_k^1(\mathrm{Hom}(A(V), \hat{A}(V)))^{1/2}}$$

or

$$\mu_k(V) = \boldsymbol{h}_k(V)/\boldsymbol{i}_k(V),$$

where

$$\boldsymbol{h}_k(V) = \frac{h_k^1(N^0(V))}{h_k^1(\mathrm{Hom}(A(V), \hat{A}(V)))^{1/2}}, \qquad \boldsymbol{i}_k(V) = \frac{i_k(N^0(V))}{i_k(\mathrm{Hom}(A(V), \hat{A}(V)))^{1/2}}.$$

REMARK. As far as we consider the finite normal extension satisfying the condition (S), h_k^1, $\boldsymbol{h}_k$, i_k, $\boldsymbol{i}_k$ and μ_k are independent of the choice of the field K. (See pp. 56-58 [6]).

LEMMA 5. *Let K be a finite normal extension of k and k' be an intermediate field between k and K. We denote the Galois groups of the extension K/k and K/k' by G and H respectively. Let E be a finite type Z-free H-module. Then we have $i_k(M_G^H(E))=i_k(E)$.*

PROOF. We have the natural isomorphism, by Lemma 1,

$$\begin{aligned} H^1(G, \mathrm{Hom}_Z(M_G^H(E), K^*)) &= H^1(G, \mathrm{Hom}_Z(Z[G]\otimes_{Z[H]}E, K^*)) \\ &= H^1(G, \mathrm{Hom}_{Z[H]}(Z[G], \mathrm{Hom}_Z(E, K^*))) \\ &= H^1(G, \mathrm{Hom}_Z(E, K^*)) . \end{aligned}$$

Similarly we have

$$H^1(G, \mathrm{Hom}_Z(M_G^H(E), I_K)) = H^1(G, \mathrm{Hom}_Z(E, I_K)) .$$

From these isomorphisms Lemma follows. Q. E. D.

PROPOSITION 6. *The notations being as in Theorem 2, we have*

$$i_k(N^0(V)) = i_k(N^0(V')) .$$

PROOF. By Lemma 3, we have $N^0(V')=N^0(V)\oplus\Sigma_K$. Since we have seen $\Sigma_K=\prod_{\xi\in S}M_G^{G_\xi}(Z)$, we have $i_k(\Sigma_K)=\prod_{\xi\in S}i_{k_\xi}(Z)=1$, where k_ξ is the fixed subfield of K under G_ξ. Therefore we get $i_k(N^0(V))=i_k(N^0(V'))$. Q. E. D.

THEOREM 3. *Let V and V' be the complete nonsingular algebraic varieties defined over k with birational morphism from V' to V defined over k. Then we have $\mu_k(V)=\mu_k(V')$.*

§4. Divisorial correspondences of product varieties.

In this section we consider the divisorial correspondences of product varieties which we need to prove the important properties, of μ_k, similar to the case of the Tamagawa numbers. Therefore in this section we take an arbitrary basic field k dropping the assumption of dimension one.

Let V_1 and V_2 be the complete non-singular algebraic varieties defined over k and A_1, A_2 respectively be the Albanese varieties defined over k. Then we have the exact sequence (see p. 155 [2])

$$(28)\qquad 0\longrightarrow D(V_1)/D_l(V_1)\times D(V_2)/D_l(V_2)\xrightarrow{\ i\ } D(V_1\times V_2)/D_l(V_1\times V_2) \xrightarrow{\ p\ } \mathrm{Hom}(A_1, \hat{A}_2)\longrightarrow 0 ,$$

where i is the injection induced by the projections $p_i\colon V_1\times V_2\to V_i$ $(i=1, 2)$

and p is a homomorphism defined as follows. Let $\bar{\varphi}_1$ be the canonical mapping from V_1 into its Albanese variety A_1 and φ_2 a canonical isomorphism $D_a(V_2)/D_l(V_2) \xrightarrow{\sim} \hat{A}_2$. For a divisor X in $D(V_1 \times V_2)$, the image $\lambda_X = p(Cl(X))$ of the class $Cl(X)$ in $D(V_1 \times V_2)/D_l(V_1 \times V_2)$ is defined by

$$(29) \qquad \lambda_X(\bar{\varphi}_1(P) - \bar{\varphi}_1(Q)) = \varphi_2(Cl[X(P) - X(Q)]) \qquad \text{for} \quad P, Q \in V_1,$$

where $X \cdot (P \times V_2) = P \times X(P)$ and $X \cdot (Q \times V_2) = Q \times X(Q)$ will be defined for some divisor in $Cl(X)$. Since i induces the isomorphism $D_a(V_1)/D_l(V_1) \times D_a(V_2)/D_l(V_2) \xrightarrow{\sim} D_a(V_1 \times V_2)/D_l(V_1 \times V_2)$ and $\mathrm{Hom}\,(A_1, A_2)$ is a torsion free module, i induces also an isomorphism $D_t(V_1)/D_l(V_1) \times D_t(V_2)/D_l(V_2) \xrightarrow{\sim} D_t(V_1 \times V_2)/D_l(V_1 \times V_2)$. Therefore we get the exact sequence

$$(30) \qquad 0 \longrightarrow D(V_1)/D_t(V_1) \times D(V_2)/D_t(V_2) \longrightarrow D(V_1 \times V_2)/D_t(V_1 \times V_2) \longrightarrow \mathrm{Hom}\,(A_1, \hat{A}_2) \longrightarrow 0,$$

or

$$(31) \qquad 0 \longrightarrow N^0(V_1) \times N^0(V_2) \xrightarrow{\ i\ } N^0(V_1 \times V_2) \xrightarrow{\ p\ } \mathrm{Hom}\,(A_1, \hat{A}_2) \longrightarrow 0.$$

Let K be a finite normal extension of k over which $N^0(V_1)$, $N^0(V_2)$, $N^0(V_1 \times V_2)$ and $\mathrm{Hom}\,(A_1, \hat{A}_2)$ split and moreover V_1 and V_2 have rational points P_1 and P_2 respectively. We denote the Galois group of the extension K/k by G. We show that the exact sequence (31) splits as a sequence of G-modules. The injection i clearly commutes with the action of G. For the surjection p, we select a representative X, of a class $C(X)$ of $N^0(V_1 \times V_2)$, in $D(V_1 \times V_2)_K$ and a pair of independent generic points P and Q of V_1 over K. Then $X(P)$ and $X(Q)$ are defined. Since the mapping $(P, Q) \rightarrow (\bar{\varphi}_1(P) - \bar{\varphi}_1(Q))$ can be assumed to be defined over k and φ_2 commutes with the action of G, we get the following equation by acting an element g of G on the equation (29).

$$(32) \qquad \lambda_X^g(\bar{\varphi}_1(P) - \bar{\varphi}_1(Q)) = \varphi_2(Cl[X^g(P) - X^g(Q)]).$$

Therefore we get $\lambda_X^g = \lambda_{X^g}$. This shows the commutativity of p with the action of G. Let us define a projection P from $N^0(V_1 \times V_2)$ onto $N^0(V_1) \times N^0(V_2)$ such that $P \cdot i =$ indentity on $N^0(V_1) \times N^0(V_2)$ and P commutes with the action of G. For a class $C(X)$ in $N^0(V_1 \times V_2)$ represented by a divisor X in $D(V_1 \times V_2)_K$ such that $X \cdot (P_1 \times V_2) = P_1 \times X(P_1)$ and $X \cdot (V_1 \times P_2) = X(P_2) \times P_2$ are defined, we put $P(C(X)) = (C(X(P_2)), C(X(P_1)))$, where $C(X(P_1))$ (resp. $C(X(P_2))$) is the class in $N^0(V_2)$ (resp. $N^0(V_1)$) represented by $X(P_1)$ (resp. $X(P_2)$). Then for an element g of G, we have $P(C(X))^g = (C(X^g(P_2^g)), C(X^g(P_1^g)))$ and $P(C(X^g)) = (C(X^g(P_2)), C(X^g(P_1)))$. Since $X^g(P_i)$ and $X^g(P_i^g)$ are algebraically equivalent, we have $C(X^g(P_i^g)) = C(X^g(P_i))$ $(i = 1, 2)$ and we get $P(C(X))^g = P(C(X)^g)$. Hence P commutes with the action of G. The relation $P \cdot i =$ identity is trivial. Thus we have proved the assertion on the splitting of the sequence (31).

Using the induction on the number of the factors in the product, we can prove

PROPOSITION 7. *Let V_i $(i=1, 2, \cdots, n)$ be a finite number of complete non-singular algebraic varieties defined over a field k and A_i $(i=1, 2, \cdots, n)$ be the Albanese varieties defined over k respectively. Let K be a finite normal extension of k over which $N^0(V_i)$ $(i=1, \cdots, n)$, $N^0\left(\prod_{j=1}^{k} V_{i_j}\right)$ $(1 \leqq i_j \leqq n,\ 1 \leqq k \leqq n)$ and $\mathrm{Hom}(A_i, \hat{A}_j)$ $(i<j)$ split and moreover V_i $(i=1, \cdots, n)$ have rational points P_i $(i=1, \cdots, n)$ respectively. We denote the Galois group of the extension K/k by G. Then the following exact sequence of G-modules splits*

$$0 \longrightarrow \bigoplus_{i=1}^{n} N^0(V_i) \xrightarrow{\ i\ } N^0\left(\prod_{i=1}^{n} V_i\right) \xrightarrow{\ p\ } \bigoplus_{i<j} \mathrm{Hom}(A_i, \hat{A}_j) \longrightarrow 0. \tag{33}$$

The surjection p is defined as follows. Let $\bar{\varphi}_i$: $V_i \to A_i$ be the canonical mapping and φ_j: $D_a(V_j)/D_l(V_j) \xrightarrow{\sim} \hat{A}_j$ be the canonical isomorphism. Let $C(X)$ be a class in $N^0\left(\prod_{i=1}^{n} V_i\right)$ represented by a divisor X in $D\left(\prod_{i=1}^{n} V_i\right)$ and P, Q be points of V_i. We may assume that $X \cdot \left(\prod_{k \neq i,j} P_k \times V_i \times V_j\right) = \prod_{k \neq i,j} P_k \times X_{ij}$ and $X_{ij} \cdot (P \times V_j) = P \times X_{ij}(P)$, $X_{ij} \cdot (Q \times V_j) = Q \times X_{ij}(Q)$ are defined. Then we put

$$P(C(X)) = (\cdots, \lambda_X^{ij}, \cdots), \tag{34}$$
$$\lambda_X^{ij}(\bar{\varphi}_i(P) - \bar{\varphi}_i(Q)) = \varphi_j(Cl[X_{ij}(P) - X_{ij}(Q)]),$$

and we have the isomorphism of G-modules,

$$N^0\left(\prod_{i=1}^{n} V_i\right) \cong \left[\bigoplus_{i=1}^{n} N^0(V_i)\right] \oplus \left[\bigoplus_{i<j} \mathrm{Hom}(A_i, \hat{A}_j)\right]. \tag{35}$$

REMARK. In Proposition 7 if we replace K by a splitting field of $N\left(\prod_{j=1}^{n} V_{i_j}\right)$ $(1 \leqq i_j \leqq n,\ 1 \leqq k \leqq n)$ and $\mathrm{Hom}(A_i, \hat{A}_j)$, $(i<j)$, then the following exact sequence splits.

$$0 \longrightarrow \bigoplus_{i=1}^{n} N(V_i) \longrightarrow N\left(\prod_{i=1}^{n} V_i\right) \longrightarrow \bigoplus_{i<j} \mathrm{Hom}(A_i, \hat{A}_j) \longrightarrow 0.$$

If moreover V_i $(i=1, \cdots, n)$ have rational points over the basic field k, then the following exact sequence of G-modules splits.

$$0 \longrightarrow \bigoplus_{i=1}^{n} D(V_i)_K / D_l(V_i)_K \longrightarrow D\left(\prod_{i=1}^{n} V_i\right)_K \Big/ D_l\left(\prod_{i=1}^{n} V_i\right)_K \longrightarrow \bigoplus_{i<j} \mathrm{Hom}(A_i, \hat{A}_j) \longrightarrow 0.$$

Next we deal with the k/k_0-trace of algebraic varieties. Let us start with the definition. Let k_0 be a field and k be a finite separable extension of k_0. For an algebraic variety V defined over k, the k/k_0-trace $R_{k/k_0}(V)$ is defined as follows. Let $\{\sigma_1 = id., \sigma_2, \cdots, \sigma_d\}$ be the set of all distinct isomorphisms of k over k_0. $R_{k/k_0}(V) = W$ is an algebraic variety defined over k_0 with a morphism $p: W \to V$ defined over k such that the morphism

$$P=p^{\sigma_1}\times\cdots\times p^{\sigma_d}: W\longrightarrow V^{\sigma_1}\times\cdots\times V^{\sigma_d} \tag{38}$$

is an isomorphism. The existence and the uniqueness (up to isomorphisms over k_0) of $R_{k/k_0}(V)$ are assured in [13]. $R_{k/k_0}(V)=W$ has the universal mapping property. Namely, if there exists an algebraic variety X and a morphism $r: X\to V$ defined over k then there exists a unique morphism $b: X\to W$ such that $r=p\cdot b$. Let A and B be the Albanese varieties of V and W defined over k and k_0 respectively. Then there exists a unique morphism $q: B\to A$ defined over k and we have the commutative diagram

$$\begin{array}{ccccc} W & \xrightarrow{p} & V, & W & \xrightarrow{p^{\sigma_1}\times\cdots\times p^{\sigma_d}} & V^{\sigma_1}\times\cdots\times V^{\sigma_d} \\ \bar{\phi}\downarrow & & \downarrow\bar{\varphi} & \downarrow q^{\sigma_1}\times\cdots\times q^{\sigma_d} & & \downarrow \bar{\varphi}^{\sigma_1}\times\cdots\times\bar{\varphi}^{\sigma_d} \\ B & \xrightarrow{q} & A & B & \longrightarrow & A^{\sigma_1}\times\cdots\times A^{\sigma_d} \end{array} \tag{39}$$

where $\bar{\phi}$ and $\bar{\varphi}$ are canonical mappings. Since $p^{\sigma_1}\times\cdots\times p^{\sigma_d}$ is an isomorphism, $q^{\sigma_1}\times\cdots\times q^{\sigma_d}$ is also isomorphism (birational biregular mapping). Therefore we have $B=R_{k/k_0}(A)$.

REMARK. Adding some constant we can assume that q is a homomorphism. If there is an abelian variety C defined over k_0 and a homomorphism $s: C\to A$ defined over k, then we have a unique homomorphism $c: C\to B$ defined over k (put $c=(q^{\sigma_1}\times\cdots\times q^{\sigma_d})^{-1}\cdot(s^{\sigma_1}\times\cdots\times s^{\sigma_d})$) such that we have $q\cdot c=s$. Therefore we can consider B as the k/k_0-trace of A in the similar sense of Chow.

We prove a Lemma for the later use.

LEMMA. *The notations being as above, we have the G-isomorphism*

$$\mathrm{Hom}(B, \hat{B})=\bigoplus_{i,j=1}^{d}\mathrm{Hom}(A^{\sigma_i}, \hat{A}^{\sigma_j}).$$

PROOF. Since we have the isomorphism $Q=(q^{\sigma_1}\times\cdots\times q^{\sigma_d}): B\to A^{\sigma_1}\times\cdots\times A^{\sigma_d}$ and $\hat{Q}=(\hat{q}^{\sigma_1}\times\cdots\times\hat{q}^{\sigma_k}): \hat{A}^{\sigma_1}\times\cdots\times\hat{A}^{\sigma_d}\xrightarrow{\sim}\hat{B}$, we can define an isomorphism of modules w: $\mathrm{Hom}\left(\prod_{i=1}^{d}A^{\sigma_i}, \prod_{j=1}^{d}\hat{A}^{\sigma_j}\right)\xrightarrow{\sim}\mathrm{Hom}(B, \hat{B})$ by $w(\lambda)=\hat{Q}\cdot\lambda\cdot Q$. Since the permutations of the factors of $\prod_{i=1}^{d}A^{\sigma_i}$ and $\prod_{i=1}^{d}\hat{A}^{\sigma_i}$ are identified, Q and $\hat{Q}$ commute with the action of G. Therefore we have $[w(\lambda)]^g=[\hat{Q}\cdot\lambda\cdot Q]^g=\hat{Q}\cdot\lambda^g\cdot Q=w(\lambda^g)$ for every g in G and hence w is a G-isomorphism. Q. E. D.

Now we assume that V is a complete non-singular algebraic variety defined over k. Let K_0 be the Galois closure of k over k_0 and K be a finite normal extension of k_0 containing K_0 over which $N^0(W)$, $N^0\left(\prod_{j=1}^{k}V^{\sigma_{i_j}}\right)$ $(1\leq i_j\leq d,\ 1\leq k\leq d)$ and $\mathrm{Hom}(A^{\sigma_i}, \hat{A}^{\sigma_j})$ $(1\leq i<j\leq d)$ split and moreover V has a rational point P. We denote the Galois group of the extension K/k_0 by G, that of K/k by H. We have the exact sequence of Z-modules

$$
(40) \qquad 0 \longrightarrow \bigoplus_{i=1}^{d} N^0(V^{\sigma_i}) \overset{i}{\longrightarrow} N^0\Big(\prod_{i=1}^{d} V^{\sigma_i}\Big) \overset{p}{\longrightarrow} \bigoplus_{i<j} \mathrm{Hom}\,(A^{\sigma_i}, \hat{A}^{\sigma_j}) \longrightarrow 0 .
$$
$$
N^0\Big(\prod_{i=1}^{d} V^{\sigma_i}\Big) \underset{\wr}{\downarrow} P^* \quad N^0(W)
$$

We consider the structure of G-module in $N^0\Big(\prod_{i=1}^{d} V^{\sigma_i}\Big)$ as follows. Extending σ_i $(i=1, 2, \cdots, d)$ to the automorphisms of K/k_0 we consider σ_i's as the elements of G. Then we have $G=\sigma_1 H+\cdots+\sigma_d H$. The multiplication of an element g in G, on the left side, induces a permutation of the coset space G/H i.e. $g\cdot\sigma_i H=\sigma_{g(i)}H$. For a divisor X in $D\Big(\prod_{i=1}^{d} V^{\sigma_i}\Big)_K$ we have $(X)^g \in D\Big(\prod_{i=1}^{d} V^{\sigma_{g(i)}}\Big)_K$. Identifying the permutation of the factors of the product, we have $D\Big(\prod_{i=1}^{d} V^{\sigma_i}\Big)_K=D\Big(\prod_{i=1}^{d} V^{\sigma_{g(i)}}\Big)_K$ and $(X)^g\in D\Big(\prod_{i=1}^{d} V^{\sigma_i}\Big)_K$. We can also consider the structure of G-modules in $\bigoplus_{i=1}^{d} N^0(V^{\sigma_i})$ and $\bigoplus_{i<j}\mathrm{Hom}\,(A^{\sigma_i}, \hat{A}^{\sigma_j})$ by natural way.

REMARK. In $\bigoplus_{i<j}\mathrm{Hom}\,(A^{\sigma_i}, \hat{A}^{\sigma_j})$ we have identified $\mathrm{Hom}\,(A^{\sigma_i}, \hat{A}^{\sigma_j})$ and $\mathrm{Hom}\,(A^{\sigma_j}, \hat{A}^{\sigma_i})$ by the isomorphism $\lambda\to{}^t\lambda$ (where ${}^t\lambda$ means the transpose of λ).

$N^0(W)$ has a natural G-module structure. First we show that by these structures of G-modules the homomorphisms in the sequence (40) are all G-homomorphisms. Since we indentified the permutations of the factors in the product $\prod_{i=1}^{d} V^{\sigma_i}$, we have $P^*=(P^*)^g$ for all g in G. The commutativity of i with the action of G is clear. Therefore we consider the homomorphism p. By (34) p is defined, for a class $C(X)$ of $N^0\Big(\prod_{i=1}^{d} V^{\sigma_i}\Big)$ represented by a divisor X in $D\Big(\prod_{i=1}^{d} V^{\sigma_i}\Big)_K$ and points P_i, Q_i of V_i, by

$$
(41) \qquad p(C(X))=(\cdots, \lambda_X^{ij}, \cdots),
$$
$$
X_{ij}(\bar{\varphi}^{\sigma_i}(P_i)-\bar{\varphi}^{\sigma_i}(Q_i))=\varphi^{\sigma_j}(Cl[X_{ij}(P_i)-X_{ij}(Q_i)]),
$$

where $\bar{\varphi}: V\to A$ is the canonical mapping, $\varphi: D_a(V)/D_l(V)\xrightarrow{\sim}\hat{A}$ is the canonical isomorphism ($\varphi^{\sigma_j}: D_a(V^{\sigma_j})/D_l(V^{\sigma_j})\xrightarrow{\sim}\hat{A}^{\sigma_j}$ is also a canonical isomorphism) and X_{ij} is defined by $X\cdot(\prod_{k\neq i,j} P^{\sigma_k}\times V_i\times V_j)=\prod_{k\neq i,j} P^{\sigma_k}\times X_{ij}$. It is no loss of generality to assume P_i's, Q_i's to be the independent generaric points of V_i's over k. Put

$$
(42) \qquad \bar{\boldsymbol{\Phi}}=(\bar{\varphi}^{\sigma_1}, \cdots, \bar{\varphi}^{\sigma_d}), \qquad \boldsymbol{\Phi}=(\varphi^{\sigma_1}, \cdots, \varphi^{\sigma_d}),
$$
$$
\boldsymbol{P}=(P_1, \cdots, P_d), \qquad \boldsymbol{Q}=(Q_1, \cdots, Q_d),
$$
$$
\boldsymbol{X}_j(X)=\bigoplus_{i=1}^{d} X_{ij} \qquad \boldsymbol{X}(X)=(X_1(X), \cdots, X_d(X)).
$$

Then we can write the formulas (41) in a united form

(43) $$p(C(X))(\bar{\Phi}(\boldsymbol{P})-\bar{\Phi}(\boldsymbol{Q}))=\Phi(Cl[\boldsymbol{X}(X)(\boldsymbol{P})-\boldsymbol{X}(X)(\boldsymbol{Q})]).$$

Since we have identified the permutations of the factors of the product, we have $\bar{\Phi}^g=\bar{\Phi}$, $\Phi^g=\Phi$ for g in G and we may also assume $\boldsymbol{P}^g=\boldsymbol{P}$, $\boldsymbol{Q}^g=\boldsymbol{Q}$. Operating an element g in G on the equality (43), we get

(44) $$[p(C(X))]^g(\bar{\Phi}(\boldsymbol{P})-\bar{\Phi}(\boldsymbol{Q}))=\Phi(Cl[\boldsymbol{X}(X)^g(\boldsymbol{P})-\boldsymbol{X}(X)^g(\boldsymbol{Q})]).$$

On the other hand we have

(45) $$p(C(X)^g)(\bar{\Phi}(\boldsymbol{P})-\bar{\Phi}(\boldsymbol{Q}))=\Phi(Cl[\boldsymbol{X}(X^g)(P)-\boldsymbol{X}(X^g)(\boldsymbol{Q})]).$$

In order to see that p commutes with the action of G, it is sufficient for us to show that the component $(X_{ij})^g$ of $\boldsymbol{X}(X)^g$ and the component $(X^g)_{g(i)g(j)}$ of $\boldsymbol{X}(X^g)$ are algebraically equivalent.

i. e.
$$\begin{aligned}(X^g)_{g(i)g(j)}&=pr_{V^{\sigma_{g(i)}}\times V^{\sigma_{g(j)}}}[X^g\cdot(\prod_{k\neq g(i)g(j)}P^{\sigma_k}V^{\sigma_{g(i)}}\times V^{\sigma_{g(j)}})]\\&=pr_{V^{\sigma_{g(i)}}\times V^{\sigma_{g(j)}}}[X^g\cdot(\prod_{k\neq i,j}P^{\sigma_{g(k)}}\times V^{\sigma_{g(i)}}\times V^{\sigma_{g(j)}})]\\&=pr_{V^{\sigma_{g(i)}}\times V^{\sigma_{g(j)}}}[X^g\cdot(\prod_{k\neq i,j}P^{h_k\sigma_k g}\times V^{\sigma_{g(i)}}\times V^{\sigma_{g(j)}})]\\&=(pr_{V^{\sigma_i}\times V^{\sigma_j}}[X\cdot(\prod_{k\neq i,j}P^{h_k\sigma_k}\times V^{\sigma_i}\times V^{\sigma_j})])^g\\&\equiv(pr_{V^{\sigma_i}\times V^{\sigma_j}}[X\cdot(\prod_{k\neq i,j}P^{\sigma_k}\times V^{\sigma_i}\times V^{\sigma_j})])^g\\&=(X_{ij})^g\end{aligned}$$

where $g\sigma_i=\sigma_{g(i)}h_i^{-1}$ and $\equiv$ means the algebraic equivalence. Next we show that the exact sequence (41) applies as a sequence of G-modules. We define a projection $P: N^0\left(\prod_{i=1}^{d}V^{\sigma_i}\right)\to\bigoplus_{i=1}^{d}N^0(V^{\sigma_i})$ such that $P\cdot i=$ identity on $\bigoplus_{i=1}^{d}N^0(V^{\sigma_i})$, as follows,

(46) $$P(C(X))=(\cdots,C(X_i),\cdots)$$

where $C(X)$ is a class in $N^0\left(\prod_{i=1}^{d}V^{\sigma_i}\right)$ represented by a divisor X in $D\left(\prod_{i=1}^{d}V^{\sigma_i}\right)$ such that $X\cdot(\prod_{k\neq i}P^{\sigma_k}\times V^{\sigma_i})=\prod_{k\neq i}P^{\sigma_k}\times X_i$ $(1\leqq i\leqq d)$ are defined and $C(X_i)$ is the class in $N^0(V^{\sigma_i})$ represented by X_i. Then the relation $P\cdot i=$ identity is trivial. On the other hand we have $(X^g)_i=pr_{V^{\sigma_i}}[X^g\cdot[\prod_{k\neq i}P^{\sigma_k}\times V^{\sigma_i}]]$ and $(X_{g^{-1}(i)})^g=(Pr_{V^{\sigma_{g^{-1}(i)}}}[X\cdot(\prod_{g^{-1}(k)\neq g^{-1}(i)}P^{\sigma_{g^{-1}(k)}}\times V^{\sigma_{g^{-1}(i)}})])^g=Pr_{V^{\sigma_{g^{-1}(i)}}}[X^g\cdot(\prod_{k\neq i}P^{h_k\sigma_k}\times V^{\sigma_i})]$ for $g\in G$, where $g^{-1}\sigma_k=\sigma_{g^{-1}(k)}h_k^{-1}$. Hence $(X^g)_i$ and $(X_{g^{-1}(i)})^g$ are algebraically equivalent and we get $C((X^g)_i)=C(X_{g^{-1}(i)})^g$, $P(C(X^g))=P(C(X))^g$. Thus we get

PROPOSITION 8. *Let k_0 be a field and k be a finite separable extension of k_0 with the Galois closure K_0. Let V be a complete non singular algebraic* *variety defined over k and* $W=R_{k/k_0}(V)$ *be the* k/k_0*-trace of V. Let A be the Albanese variety of V defined over k. Let* $\sigma_1, \cdots, \sigma_d$ *be the set of all distinct isomorphisms of k over* k_0 *and K be a finite normal extension of* k_0 *containing* K_0*, over which* $N^0\left(\prod_{j=1}^{k} V^{\sigma_{i_j}}\right)$ $(1 \leqq i_j \leqq d,\ 1 \leqq k \leqq d)$, $N^0(W)$ *and* $\mathrm{Hom}(A^{\sigma_i}, \hat{A}^{\sigma_j})$ $(1 \leqq i < j \leqq d)$ *split. We denote the Galois group of the extension* K/k_0 *by G and that of K/k by H. Then the following exact sequence of G-modules splits*

$$(47)\qquad 0 \longrightarrow \bigoplus_{i=1}^{d} N^0(V^{\sigma_i}) \longrightarrow N^0\left(\prod_{i=1}^{d} V^{\sigma_i}\right) \longrightarrow \bigoplus_{i<j} \mathrm{Hom}(A^{\sigma_i}, \hat{A}^{\sigma_j}) \longrightarrow 0$$

$$\begin{array}{c} N^0\left(\prod_{i=1}^{d} V^{\sigma_i}\right) \\ \downarrow \wr \\ N^0(W) \end{array}$$

and we have the isomorphisms

$$(48)\qquad \bigoplus_{i=1}^{d} N^0(V^{\sigma_i}) \cong M_G^H(N^0(V)), \qquad \bigoplus_{i=1}^{d} \mathrm{Hom}(A^{\sigma_i}, A^{\sigma_i}) \cong M_G^H(\mathrm{Hom}(A, \hat{A})).$$

PROOF. We have only to prove (48). By Lemma 1, it is sufficient to prove the isomorphisms a) $Z[G] \otimes_{Z[H]} N^0(V) \cong \bigoplus_{i=1}^{d} N^0(V^{\sigma_i})$ and b) $Z[G] \otimes_{Z[H]} \mathrm{Hom}(A, \hat{A}) \cong \bigoplus_{i=1}^{d} \mathrm{Hom}(A^{\sigma_i}, \hat{A}^{\sigma_i})$. For an element $g \otimes n$ in the left side of a), we put $f(g \otimes n) = n^g$ and for an element $g \otimes \lambda$ in the left side of b), we put $f'(g \otimes \lambda) = \lambda^g$. Then f and f' give the desired G-isomorphisms.

§ 5. Fundamental equality of μ_k.

In this section we prove the fundamental equalities of μ_k, $\boldsymbol{h}_k$ and $\boldsymbol{i}_k$. In order to do this we again restrict the basic field to a field of dimension one.

THEOREM 3. *Let* V_1 *and* V_2 *be complete non-singular algebraic varieties defined over k. Then we have the equalities.*

$$(49)\qquad \boldsymbol{h}_k(V_1 \times V_2) = \boldsymbol{h}_k(V_1) \cdot \boldsymbol{h}_k(V_2),$$

$$(50)\qquad \boldsymbol{i}_k(V_1 \times V_2) = \boldsymbol{i}_k(V_1) \cdot \boldsymbol{i}_k(V_2),$$

$$(51)\qquad \mu_k(V_1 \times V_2) = \mu_k(V_1) \cdot \mu_k(V_2).$$

PROOF. Let A_1 and A_2 be the Albanese varieties of V_1 and V_2 respectively defined over k. Then $A_1 \times A_2$ is the Albanese variety of $V_1 \times V_2$. By the equality $\mathrm{Hom}(A_1 \times A_2, \hat{A}_1 \times \hat{A}_2) = \bigoplus_{i,j=1}^{2} \mathrm{Hom}(A_i, \hat{A}_j)$ and Proposition 7 we have

$$\boldsymbol{h}_k(V_1 \times V_2) = \frac{h_k^1(N^0(V_1 \times V_2))}{h_k^1(\mathrm{Hom}(A_1 \times A_2, \hat{A}_1 \times \hat{A}_2))^{1/2}}$$

$$= \frac{h_k^1(N^0(V_1)) \cdot h_k^1(N^0(V_2)) \cdot h_k^1(\mathrm{Hom}(A_1, \hat{A}_2))}{h_k^1(\mathrm{Hom}(A_1, \hat{A}_1))^{1/2} \cdot h_k^1(\mathrm{Hom}(A_2, \hat{A}_2))^{1/2} \cdot h_k^1(\mathrm{Hom}(A_1, \hat{A}_2))}$$

$$=\boldsymbol{h}_k(V_1)\cdot\boldsymbol{h}_k(V_2).$$

For the equality (50), we notice that we have $\boldsymbol{i}_k(M\oplus N)=\boldsymbol{i}_k(M)\cdot\boldsymbol{i}_k(N)$ for finite type Z-free G-modules M and N. Then the equality (50) will be proved by the same way as the case of $\boldsymbol{h}_k$. Thence by the definition of μ_k the equality (51) is clear.

THEOREM 4. *Let k_0 be a field of dimension one and k a finite separable extension of k_0. Let V be a complete non-singular algebraic variety defined over k. Then we have the equalities*

$$\boldsymbol{h}_{k_0}(R_{k/k_0}(V))=\boldsymbol{h}_k(V), \tag{52}$$

$$\boldsymbol{i}_{k_0}(R_{k/k_0}(V))=\boldsymbol{i}_k(V), \tag{53}$$

$$\mu_{k_0}(R_{k/k_0}(V))=\mu_k(V). \tag{54}$$

PROOF. Let $W=R_{k/k_0}(V)$ and A, B be the Albanese varieties of V, W respectively defined over k, k_0. Then by the definition and Proposition 8 we have

$$\begin{aligned}\boldsymbol{h}_{k_0}(R_{k/k_0}(V))&=\boldsymbol{h}_{k_0}(W)\\ &=\frac{h^1_{k_0}(N^0(W))}{h^1_{k_0}(\mathrm{Hom}(B,B))^{1/2}}\\ &=\frac{h^1_{k_0}\Big(\bigoplus_{i=1}^{d}N^0(V^i)\Big)\cdot h^1_{k_0}\Big(\bigoplus_{i>j}\mathrm{Hom}(A^i,\hat{A}^j)\Big)}{h^1_{k_0}\Big(\bigoplus_{i=1}^{d}\mathrm{Hom}(A^i,\hat{A}^i)\Big)^{1/2}\cdot h^1_{k_0}\Big(\bigoplus_{i>j}\mathrm{Hom}(A^i,\hat{A}^j)\Big)}\\ &=\frac{h^1_{k_0}(M^H_G(N^0(V)))}{h^1_{k_0}(M^H_G(\mathrm{Hom}(A,\hat{A})))^{1/2}}\\ &=\frac{h^1_k(N^0(V))}{h^1_k(\mathrm{Hom}(A,\hat{A}))^{1/2}}\\ &=\boldsymbol{h}_k(V),\end{aligned}$$

where the notations G, H, σ_i's are taken over from Proposition 8. Since we have $i_{k_0}(M^H_G(E))=i_k(E)$ for a finite type Z-free H-module E, we can prove the equality (53) by the same way as the case of $\boldsymbol{h}_k$ and the equality (54) follows from the definition of μ_k. Q. E. D.

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