J. Math. Soc. Japan
Vol. 21, No. 4, 1969

On normal operations on models

By Nobuyoshi MOTOHASHI

(Received Aug. 1, 1968)

It is well known that special models and saturated models are nice tools in the theory of models. The technique of special models and saturated models is widely used in different problems, e.g. preservation theorems (see [3], [6], [7], [8], [12]), two cardinal problems (see [2]), theory of definition (see [1], [11]), and categoricity in powers (see [13]).

In this paper we shall consider some operations on models (i.e. functions whose domain and range are classes of models), which preserve special models and saturated models. An operation U on models which has the following property i.e. for any formula θ and any model $\mathfrak{A}$, the satisfaction of θ in $U(\mathfrak{A})$ by elements of $U(\mathfrak{A})$ is reducible to that of a formula corresponding to θ in $\mathfrak{A}$ by corresponding elements of $\mathfrak{A}$, will be called a *normal operation*. Then our main theorem says that normal operations preserve special models and saturated models. As an example of normal operations we can take the n-direct power operation and the reduct operation. Using this main theorem, we shall get some generalized forms of preservation theorems and interpolation theorems.

I would like to thank Prof. C.C. Chang and Prof. H.J. Keisler who read the original manuscript of this paper and gave me useful advices. I would also like to express my thanks to Prof. R. Lyndon who kindly encouraged me and to Prof. S. Maehara who suggested to me the possibility of a generalization of results of the original manuscript of this paper.

§ 0. Preliminaries.

We shall distinguish between classes and sets. Occasionally we shall consider collections of classes. We consider each ordinal number as coinciding with the set of smaller ordinal numbers. We use letters ξ, α to denote ordinal numbers and δ to denote limit ordinal numbers and n, m, l, j, k to denote natural numbers. We use Ω to denote the class of ordinal numbers. The cardinal numbers are identified with the corresponding ordinal numbers and denoted by κ. ω is the least infinite ordinal number, and κ^+ is the successor cardinal of κ.

If X, Y are sets we use the following ordinal notations, $\in$, $\{x_\xi\}_{\xi<\alpha}$, $\langle x_\xi\rangle_{\xi<\alpha}$, $\{x\colon \cdots x \cdots\}$, $\subseteqq$, $\cup$, $\cap$, X^Y, $\overline{\overline{X}}$, $f(\overbrace{*\cdots*}^{n})$, $D(f)$, $R(f)$, $f\restriction X$ and f^{-1}, respectively, membership, the class consisting of x_ξ, $\xi<\alpha$, the α-sequence whose ξ-th member is x_ξ, the class of x such that $\cdots x \cdots$, inclusion, union, intersection, the set of all functions from Y to X, the cardinality of X, n-ary function, domain of f, range of f, restriction of f to X and inverse of the one to one function f.

By a similarity type, or briefly type, we mean a function whose range is ω. If μ is a type, and $\alpha\in\Omega$, then $\mu\oplus\alpha$ is the type such that $D(\mu\oplus\alpha)=D(\mu)\cup\alpha$ (disjoint sum, without loss of generality we can assume $D(\mu)$ and α is disjoint) and $\mu\oplus\alpha\restriction D(\mu)=\mu$, $(\mu\oplus\alpha)(\xi)=0$ for $\xi<\alpha$. If μ, μ' are types, $\mu\subseteqq\mu'$ means that $D(\mu)\subseteqq D(\mu')$ and $\mu'\restriction D(\mu)=\mu$. A given type μ, a system $\mathfrak{A}=\langle A, R_i\rangle_{i\in D(\mu)}$ formed by non empty set A and $\mu(i)$-ary relation R_i on A if $\mu(i)>0$, i-th element R_i of A if $\mu(i)=0$, is a relational system of type μ, or briefly μ-system whose universe $|\mathfrak{A}|=A$, i-th relation $\mathfrak{A}^{(i)}=R_i$ if $\mu(i)>0$, i-th element $\mathfrak{A}^{(i)}=R_i$ if $\mu(i)=0$ and power $\overline{\overline{\mathfrak{A}}}=\overline{\overline{A}}$. If $\mathfrak{A}$ is a μ-system and $a\in|\mathfrak{A}|^\alpha$, then $(\mathfrak{A}, a)$ is the $\mu\oplus\alpha$-system, whose ξ-th element is $a(\xi)$ for $\xi<\alpha$. $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ are used to denote μ-systems and $M(\mu)$ denote the class of all μ-systems.

For each type μ, we consider the first order predicate calculus with equality $L(\mu)$. We assume that the reader is familiar with the syntactical and semantical notions related to it.

$L(\mu)$ has the identity symbol $\doteq$, (individual) variables $\{v_n\}_{n<\omega}$, for each $i\in D(\mu)$ such that $\mu(i)>0$, $\mu(i)$-ary predicate symbol P_i, and for each $i\in D(\mu)$ such that $\mu(i)=0$, individual constant c_i. We shall use ordinary logical symbols, $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$, $\exists$, $\forall$. The set of formulas, the set of formulas having no free variables except $\{v_m\}_{m<k}$, the set of sentences will be denoted by $F(\mu)$, $F_k(\mu)$, $S(\mu)$ respectively. Let $F^*(\mu)=\bigcup_{\alpha\in\Omega}F(\mu\oplus\alpha)$.

If $\mathfrak{A}$ is a μ-system, $\theta\in F(\mu)$ and $a\in|\mathfrak{A}|^\omega$, we shall write $\models_{\mathfrak{A}}\theta[a]$ to mean that the sequence a satisfies θ in $\mathfrak{A}$. In case that $\theta\in F_k(\mu)$, we may write, instead $\models_{\mathfrak{A}}\theta[a(0),\cdots,a(k-1)]$. If $\theta\in S(\mu)$, we write $\models_{\mathfrak{A}}\theta$ to mean that θ is true in $\mathfrak{A}$. $T_n\mathfrak{A}$ is the set of all sentences of $L(\mu)$ which are true in $\mathfrak{A}$. For $\mathfrak{A},\mathfrak{B}\in M(\mu)$, $\mathfrak{A}\cong\mathfrak{B}$, $\mathfrak{A}\equiv\mathfrak{B}$ and $\mathfrak{A}<\mathfrak{B}$ mean respectively, $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$, $\mathfrak{A}$ is elementary equivalent to $\mathfrak{B}$ (i.e. $T_n\mathfrak{A}=T_n\mathfrak{B}$) and $\mathfrak{A}$ is elementary subsystem of $\mathfrak{B}$ (i.e. $|\mathfrak{A}|\subseteqq|\mathfrak{B}|$ and for any $\theta\in F(\mu)$, any $a\in|\mathfrak{A}|^\omega$, $\models_{\mathfrak{A}}\theta[a]$ is equivalent to $\models_{\mathfrak{B}}\theta[a]$). A function f from $|\mathfrak{A}|$ to $|\mathfrak{B}|$ is said to be elementary embedding if f is injection and the image $f(\mathfrak{A})$ of $\mathfrak{A}$ by f is elementary subsystem of $\mathfrak{B}$.

For $\Sigma\subseteqq F_k(\mu)$, $\mathfrak{A}\in M(\mu)$, Σ is satisfiable in $\mathfrak{A}$ if there is an $a\in|\mathfrak{A}|^k$ such that $\models_{\mathfrak{A}}\theta[a]$ for all $\theta\in\Sigma$ and Σ is said to be finitely satisfiable if every finite subset of Σ is satisfiable in $\mathfrak{A}$. $\mathfrak{A}$ is said to be κ-saturated if for any $\alpha<\kappa$,

$a \in |\mathfrak{A}|^{\alpha}$, $\Sigma \subseteq F_1(\mu \oplus \alpha)$, if Σ is finitely satisfiable in $(\mathfrak{A}, a)$, then Σ is satisfiable in $(\mathfrak{A}, a)$. $\mathfrak{A}$ is said to be *saturated* if $\mathfrak{A}$ is $\overline{\overline{\mathfrak{A}}}$-saturated. $\mathfrak{A}$ is said to be *special* if there is an elementary chain $\{\mathfrak{A}_\kappa\}_{\kappa<\overline{\overline{\mathfrak{A}}}}$ of elementary subsystems of $\mathfrak{A}$ such that $\mathfrak{A}_\kappa$ is κ^+-saturated and $\bigcup_{\kappa<\overline{\overline{\mathfrak{A}}}} \mathfrak{A}_\kappa = \mathfrak{A}$, when $\{\mathfrak{A}_\kappa\}_{\kappa<\overline{\overline{\mathfrak{A}}}}$ is said to be a *specializing chain* of $\mathfrak{A}$.

§ 1. Normal operations on models.

A function U from $M(\mu)$ to $M(\mu')$ is said to be an *operation on models* (o.p.m.) of type (μ, μ'). A binary function τ is said to be an *assignment transformation* (a.s.t.) associate to a given o.p.m. U of type (μ, μ') if there is a unique natural number n and for any $\mathfrak{A} \in M(\mu)$, $\tau(\mathfrak{A}, *)$ is a *bijection* from $|U(\mathfrak{A})|$ to $|\mathfrak{A}|^n$, when we shall denote $\tau(\mathfrak{A}, *)$ and n by $\tau_{\mathfrak{A}}(*)$ and $d(\tau)$.

If τ is an a.s.t. associate to an o.p.m. U, then for any ordinal number α, and any $\mathfrak{A} \in M(\mu)$, we can define a bijection $\tau_{\mathfrak{A}}^*$ from $|U(\mathfrak{A})|^{\alpha}$ to $|\mathfrak{A}|^{d(\tau)\alpha}$ by

$$(\tau_{\mathfrak{A}}^*(a))(\delta + d(\tau)m + j) = (\tau_{\mathfrak{A}}(a(\delta+m)))(j) \qquad \text{for } a \in |U(\mathfrak{A})|^{\alpha},$$

$\delta+m < \alpha$, $j < d(\tau)$, $m < \omega$, where δ is a limit ordinal. The definition of $\tau_{\mathfrak{A}}^*$ depends upon α but without any confusion we can use $\tau_{\mathfrak{A}}^*$ for any ordinal α.

A function F from $S(\mu)$ to $S(\mu')$ is said to be an *operation on sentences* (o.p.s.) of type (μ, μ'). A function F from $F(\mu)$ to $F(\mu')$ is said to be an *operation on formulas* (o.p.f.) of type (μ, μ') if there is a natural number n (the least n denoted by $d(F)$) such that for any $\theta \in F_k(\mu)$, $F(\theta) \in F_{nk}(\mu')$ for $k < \omega$. A function F from $F^*(\mu)$ to $F^*(\mu')$ is said to be a *strong operation on formulas* (s.o.f.) of type (μ, μ') if there is an n (the least n denoted by $d(F)$) such that for any $\alpha \in \Omega$, if $\theta \in F_k(\mu \oplus \alpha)$ then $F(\theta) \in F_{nk}(\mu' \oplus n\alpha)$.

Suppose F is an o.p.s. (o.p.f. or s.o.f.) of type (μ, μ'). F is said to be *commutable* with $\wedge$ ($\vee$ or $\neg$) if for any $\theta, \varphi \in S(\mu)(F(\mu)$ or $F^*(\mu))$,

$$\vdash F(\theta \wedge \varphi) \leftrightarrow (F(\theta) \wedge F(\varphi)) \ (\vdash F(\theta \vee \varphi) \leftrightarrow (F(\theta) \vee F(\varphi)) \text{ or } \vdash F(\neg\theta) \leftrightarrow \neg F(\theta)).$$

Suppose U is an o.p.m. of type (μ, μ') and τ is an a.s.t. associate to U.

U is said to be *normal with respect to an o.p.s.* F of type (μ', μ) if for any $\theta \in S(\mu')$ and $\mathfrak{A} \in M(\mu)$, $\models_{U(\mathfrak{A})} \theta$ is equivalent to $\models_{\mathfrak{A}} F(\theta)$.

(U, τ) is said to be *normal with respect to an o.p.f.* F of type (μ', μ) if $d(\tau) = d(F)$ and for any $\theta \in F_k(\mu')$, $\mathfrak{A} \in M(\mu)$, $a \in |U(\mathfrak{A})|^k$, $\models_{U(\mathfrak{A})} \theta[a]$ is equivalent to $\models_{\mathfrak{A}} F(\theta)[\tau_{\mathfrak{A}}^*(a)]$.

(U, τ) is said to be *normal with respect to a s.o.f.* F of type (μ', μ) if $d(\tau) = d(F)$ and for any $\alpha \in \Omega$, $\theta \in F_k(\mu' \oplus \alpha)$, $\mathfrak{A} \in M(\mu)$, $a \in |U(\mathfrak{A})|^k$, $b \in |U(\mathfrak{A})|^{\alpha}$, $\models_{(U(\mathfrak{A}), b)} \theta[a]$ is equivalent to $\models_{(\mathfrak{A}, \tau_{\mathfrak{A}}^*(b))} F(\theta)[\tau_a^*(a)]$.

U is said to be *weakly normal* if U is normal with respect to some o.p.s. of type (μ', μ).

(U, τ) is said to be (*strongly*) *normal* if U is normal with respect to some o.p.f. (s.o.f.) of type (μ', μ).

U is said to be (*strongly*) *normal* if there is an a.s.t. τ associate to U such that (U, τ) is (strongly) normal.

Then the following four propositions are immediate from the above definitions.

PROPOSITION 1.1.

(I) *If U is strongly normal, then U is normal.*

(II) *If U is normal, then U is weakly normal.*

PROPOSITION 1.2.

(I) *If U is normal with respect to two o.p.s.s F and G of type (μ', μ),*

$$\vdash F(\theta) \leftrightarrow G(\theta) \qquad \text{for all } \theta \in S(\mu').$$

(II) *If (U, τ) is normal with respect to two o.p.f.s (s.o.p.s) F and G of type (μ', μ),*

$$\vdash F(\theta) \leftrightarrow G(\theta) \qquad \text{for all } \theta \in F(\mu'),\ (F^*(\mu')).$$

By Proposition 1.2, weakly normal U and (strongly) normal (U, τ) have the unique corresponding o.p.s and o.p.f. (s.o.f.) respectively if we neglect the difference between logically equivalent formulas. So, we denote them by F_U and $F_{(U,\tau)}$.

PROPOSITION 1.3.

(I) *If U is weakly normal, then F_U is commutable with $\wedge$, $\vee$ and $\neg$.*

(II) *If (U, τ) is (strongly) normal, $F_{(U,\tau)}$ is commutable with $\wedge$, $\vee$ and $\neg$.*

PROPOSITION 1.4.

If U is normal, then $\overline{\overline{U(\mathfrak{A})}} = \overline{\overline{\mathfrak{A}}}$ for any $\mathfrak{A} \in M(\mu)$ such that $\overline{\overline{\mathfrak{A}}} \geqq \omega$ and $\overline{\overline{U(\mathfrak{A})}} < \omega$ for any $\mathfrak{A} \in M(\mu)$ such that $\overline{\overline{\mathfrak{A}}} < \omega$.

LEMMA 1.5. *Suppose (U, τ) is normal.*

(I) *If $\mathfrak{A} < \mathfrak{B}$, then $U(\mathfrak{A})$ is elementary embeddable in $U(\mathfrak{B})$ by $\tau_{\mathfrak{B}}^{-1} \circ \tau_{\mathfrak{A}} = f_{\mathfrak{A}\mathfrak{B}}$.*

(II) *If $\mathfrak{A} < \mathfrak{B}$, $\mathfrak{B} < \mathfrak{C}$, then $\mathfrak{A}' < \mathfrak{B}' < U(\mathfrak{C})$, where*

$$\mathfrak{A}' = f_{\mathfrak{A}\mathfrak{C}}(U(\mathfrak{A})) \qquad \text{and} \qquad \mathfrak{B}' = f_{\mathfrak{B}\mathfrak{C}}(U(\mathfrak{B})).$$

PROOF. (I) Let $\theta \in F_k(\mu')$, $\langle a_0, \cdots, a_{k-1} \rangle \in |U(\mathfrak{A})|^k$, $k < \omega$. Suppose $\models_{U(\mathfrak{A})} \theta[a_0, \cdots, a_{k-1}]$. Then $\models_{\mathfrak{A}} F(\theta)[\tau_{\mathfrak{A}}(a_0), \cdots, \tau_{\mathfrak{A}}(a_{k-1})]$. Since $\mathfrak{A} < \mathfrak{B}$, $\models_{\mathfrak{B}} F(\theta)[\tau_{\mathfrak{A}}(a_0), \cdots, \tau_{\mathfrak{A}}(a_{k-1})]$. By the normality of (U, τ), $\models_{U(\mathfrak{B})} \theta[\tau_{\mathfrak{B}}^{-1}(\tau_{\mathfrak{A}}(a_1)), \cdots, \tau_{\mathfrak{B}}^{-1}(\tau_{\mathfrak{A}}(a_{k-1}))]$. So we get $\models_{U(\mathfrak{B})} \theta[f_{\mathfrak{A}\mathfrak{B}}(a_1), \cdots, f_{\mathfrak{A}\mathfrak{B}}(a_{k-1})]$. Hence $f_{\mathfrak{A}\mathfrak{B}}$ is an elementary embedding of $U(\mathfrak{A})$ to $U(\mathfrak{B})$.

(II) By (I) $\mathfrak{A}' < U(\mathfrak{C})$ and $\mathfrak{B}' < U(\mathfrak{C})$. So it is sufficient to prove that $|\mathfrak{A}'| \subseteqq |\mathfrak{B}'|$. But this is obvious from the definitions. q. e. d.

LEMMA 1.6. *Suppose* (U, τ) *is strongly normal of type* (μ, μ') *and* $d(\tau)=n$ *and* κ *is a cardinal number. If* $\mathfrak{A}$ *is* $n\kappa$*-saturated, then* $U(\mathfrak{A})$ *is* κ*-saturated.*

PROOF. Let $F=F_{(U,\tau)}$. Assume that $\mathfrak{A}$ is $n\kappa$-saturated. Let $\alpha<\kappa$, $a\in|U(\mathfrak{A})|^{\alpha}$, $\Sigma\subseteq F_1(\mu'\oplus\alpha)$. Suppose that Σ is finitely satisfiable in $(U(\mathfrak{A}), a)$. Let $\Sigma'=\{F(\theta)\,;\,\theta\in\Sigma\}$. Then $\Sigma'\subseteq F_n(\mu\oplus n\alpha)$. Let $\{\theta_0, \cdots, \theta_m\}$ is an arbitrary finite subset of Σ. Since Σ is finitely satisfiable in $(U(\mathfrak{A}), a)$ there is a $b\in|U(\mathfrak{A})|$ such that $\models_{(U(\mathfrak{A}),a)} \theta_j[b]$ for all $j\leqq m$. By the strong normality of (U, τ), $\models_{(\mathfrak{A},\tau^*_{\mathfrak{A}}(a))} F(\theta_j)[\tau_{\mathfrak{A}}(b)]$ for all $j\leqq m$. Therefore Σ' is finitely satisfiable in $(\mathfrak{A}, \tau^*_{\mathfrak{A}}(a))$. Since $\mathfrak{A}$ is $n\kappa$-saturated, $\Sigma'\subseteq F_n(\mu\oplus n\alpha)$ and $n\alpha<n\kappa$, Σ' is satisfiable in $(\mathfrak{A}, \tau^*_{\mathfrak{A}}(a))$ (see [11] Lemma 2). So there is an $e\in|\mathfrak{A}|^n$ such that $\models_{(\mathfrak{A},\tau^*_{\mathfrak{A}}(a))} F(\theta)[e]$ for all $\theta\in\Sigma$. Hence $\models_{(U(\mathfrak{A}),a)} \theta[\tau^{-1}_{\mathfrak{A}}(e)]$ for all $\theta\in\Sigma$. This shows that Σ is satisfiable in $(U(\mathfrak{A}), a)$. Therefore $U(\mathfrak{A})$ is κ-saturated. q. e. d.

THEOREM 1.7. *Suppose that* U *is strongly normal of type* (μ, μ').

(I) *If* $\mathfrak{A}\in M(\mu)$ *is saturated, then* $U(\mathfrak{A})$ *is saturated.*

(II) *If* $\mathfrak{A}\in M(\mu)$ *is special, then* $U(\mathfrak{A})$ *is special.*

PROOF. (I) is obvious from Proposition 1.4 and Lemma 1.6. (II) Let τ be an a. s. t. associate to U such that (U, τ) is strongly normal. Let $d(\tau)=n$, $F_{(U,\tau)}=F$. By Proposition 1.4, it is enough to consider the case that $\overline{\overline{\mathfrak{A}}}\geqq\omega$ (see [3] Corollary 6.2.2). Let $\{\mathfrak{A}_\kappa\}_{\kappa<\overline{\overline{\mathfrak{A}}}}$ be a specializing chain of $\mathfrak{A}$. Define $\mathfrak{A}'_\kappa$ for $\kappa<\overline{\overline{U(\mathfrak{A})}}=\overline{\overline{\mathfrak{A}}}$ by $\mathfrak{A}'_\kappa=f_{\mathfrak{A}_\kappa\mathfrak{A}}(U(\mathfrak{A}_\kappa))$ if $\kappa\geqq\omega$ and $\mathfrak{A}'_\kappa=f_{\mathfrak{A}_{n(\kappa+1)}\mathfrak{A}}(U(\mathfrak{A}_{n(\kappa+1)}))$ if $\kappa<\omega$, where $f_{\mathfrak{A}_\kappa\mathfrak{A}}$ is $\tau^{-1}_{\mathfrak{A}}\circ\tau_{\mathfrak{A}_\kappa}$. Then by Lemma 1.5 and Lemma 1.6, $\{\mathfrak{A}'_\kappa\}_{\kappa<\overline{\overline{U(\mathfrak{A})}}}$ is an elementary chain of elementary subsystems of $U(\mathfrak{A})$ such that $\mathfrak{A}'_\kappa$ is κ^+-saturated for $\kappa<\overline{\overline{U(\mathfrak{A})}}$. In order to show that $\{\mathfrak{A}'_\kappa\}_{\kappa<\overline{\overline{U(\mathfrak{A})}}}$ is a specializing chain of $U(\mathfrak{A})$, we only need to prove that $U(\mathfrak{A})=\bigcup_{\kappa<\overline{\overline{U(\mathfrak{A})}}}\mathfrak{A}'_\kappa$. By Lemma 1.5, $U(\mathfrak{A})>\bigcup_{\kappa<\overline{\overline{U(\mathfrak{A})}}}\mathfrak{A}'_\kappa$ (see [17] Theorem 1.9). Let a be an arbitrary element of $U(\mathfrak{A})$. Then $\tau_{\mathfrak{A}}(a)\in|\mathfrak{A}|^n$. But $|\mathfrak{A}|=\bigcup_{\kappa<\overline{\overline{\mathfrak{A}}}}|\mathfrak{A}_\kappa|$, so there is a $\kappa<\overline{\overline{\mathfrak{A}}}$ such that $\tau_{\mathfrak{A}}(a)\in|\mathfrak{A}_\kappa|^n$. In case $\kappa<\omega$, $\tau_{\mathfrak{A}}(a)\in|\mathfrak{A}_{n(\kappa+1)}|^n$ because $|\mathfrak{A}_\kappa|\subseteq|\mathfrak{A}_{n(\kappa+1)}|$. Let $\kappa'=\kappa$ for $\kappa\geqq\omega$ and $\kappa'=n(\kappa+1)$ for $\kappa<\omega$. Then $\tau_{\mathfrak{A}}(a)\in|\mathfrak{A}_{\kappa'}|^n$. Hence $\tau^{-1}_{\mathfrak{A}_{\kappa'}}\circ\tau_{\mathfrak{A}}(a)\in|U(\mathfrak{A}_{\kappa'})|$. But $a=\tau^{-1}_{\mathfrak{A}}(\tau_{\mathfrak{A}_{\kappa'}}(\tau^{-1}_{\mathfrak{A}_{\kappa'}}, (\tau_{\mathfrak{A}}(a))))=f_{\mathfrak{A}_{\kappa'}\mathfrak{A}}(\tau^{-1}_{\mathfrak{A}_{\kappa'}}\circ\tau_{\mathfrak{A}}(a))$ and $f_{\mathfrak{A}_{\kappa'}\mathfrak{A}}(\tau^{-1}_{\mathfrak{A}_{\kappa'}}\circ\tau_{\mathfrak{A}}(a))\in|\mathfrak{A}'_\kappa|$. Hence $U(\mathfrak{A})=\bigcup_{\kappa<\overline{\overline{U(\mathfrak{A})}}}\mathfrak{A}'_\kappa$. Therefore $U(\mathfrak{A})$ is special. q. e. d.

§ 2. Examples of normal operations on models.

In this section, we shall find some normal operations on models. Especially example 3 (n-direct powers and n-conjunctive formulas) will illustrate the ideas of this paper.

EXAMPLE 1 (Reduct). Let μ, μ' be two types such that $\mu' \subseteqq \mu$. Then the operation U defined by $U(\mathfrak{A}) = \mathfrak{A} \upharpoonright \mu'$ for $\mathfrak{A} \in M(\mu)$, is a strongly normal operation. For a. s. t. τ and $F_{(U,\tau)}$, we can take identity map on $|\mathfrak{A}|$ and $F^*(\mu)$.

EXAMPLE 2 (Relativization). Let μ, μ' be two types such that $\mu' \subseteqq \mu$ and $D(\mu) = \alpha+1$, $R(\mu) \not\ni 0$ $\mu(\alpha) = 1$ where $D(\mu') = \alpha$. Then the operation U defined by $U(\mathfrak{A}) = \mathfrak{A}_\alpha$, where $\mathfrak{A}_\alpha$ is the μ'-system such that $\mathfrak{A}_\alpha = \langle B, B_\xi \rangle_{\xi<\alpha}$, $B = R_\alpha$, $B_\xi = R_\xi \cap (R_\alpha)^{\mu(\xi)}$ for $\xi < \alpha$ and $\mathfrak{A} = \langle A, R_\xi \rangle_{\xi<\alpha+1}$. For F_U, we use $F_U(\theta) = (\exists v_0)P(v_0) \to \theta^{(P\alpha)}$, where $\theta^{(P\alpha)}$ is the relativization of θ by P_α (see [**16**]). Then U is weakly normal.

REMARK. In Example 2 we must define $T_n\mathfrak{A} = S(\mu')$ for μ'-system whose universe is empty set.

It is well known that relativization preserves saturated models and special models (see [**14**] Theorem 3.7 and [**18**] Theorem 4.2). But in our definition of a. s. t. τ, we require that $\tau_{\mathfrak{A}}$ is bijection for $\mathfrak{A} \in M(\mu)$, and this is not true if we take $\tau_{\mathfrak{A}}$ as inclusion map from R_α to A. So, in order to extend Theorem 1.7 to include relativization we must modify the definition of a. s. t. and need more complicated treatments.

To describe the next two examples we require some preliminaries.

Let $n < \omega$, $j < n$. For $\theta \in F^*(\mu)$, define $\theta^{(n,j)}$ by the result of proper simultaneously substitution of v_m by v_{nm+j} and c_ξ by $c_{n\xi+j}$ for $m < \omega$, $\xi \in \Omega$. Then obviously if $\theta \in F_k(\mu \oplus \alpha)$, then $\theta^{(n,j)} \in F_{nk}(\mu \oplus n\alpha)$.

EXAMPLE 3 (n-direct power and n-conjunctive formulas). Fix a natural number n. Let $U(\mathfrak{A}) = \mathfrak{A}^n$ for $\mathfrak{A} \in M(\mu)$ and $\tau_{\mathfrak{A}}(a) = a$ for $a \in |U(\mathfrak{A})|$. Define $F(\theta) = \bigwedge_{j<n} \theta^{(n,j)}$ if θ is atomic, and $F(\theta_1 \wedge \theta_2) = F(\theta_1) \wedge F(\theta_2)$, $F(\theta_1 \vee \theta_2) = F(\theta_1) \vee F(\theta_2)$, $F(\urcorner\theta) = \urcorner F(\theta)$, $F((\forall v_m)\theta(v_m)) = (\forall v_{nm}) \cdots (\forall v_{nm+n-1})F(\theta(v_m))$, $F((\exists v_m)\theta(v_m)) = (\exists v_{nm}) \cdots (\exists v_{nm+n-1})F(\theta(v_m))$, then (U, τ) is strongly normal with respect to F (see [**15**], this information is due to C. C. Chang and H. J. Keisler).

Hence U is strongly normal. Any $\theta \in F(\mu)$ and $\varphi \in S(\mu)$ such that $\theta \in R(F)$ and $\varphi \in R(F)$ are called *n-conjunctive formulas* and *n-conjunctive sentences.*

EXAMPLE 4 (n-direct sum and n-disjunctive formulas). Suppose $n < \omega$ and μ is a type such that $\mu(i) > 0$ for all $i \in D(\mu)$ (for convenience's sake). For $\mathfrak{A} = \langle A, R_i \rangle_{i \in D(\mu)}$, define $n\mathfrak{A} = \langle A^n, nR_i \rangle_{i \in D(\mu)}$ where $\langle a_0, \cdots, a_{\mu(i)-1} \rangle \in nR_i$ if and only if for some $j < n$, $\langle a_0(j), \cdots, a_{\mu(i)-1}(j) \rangle \in R_i$ for $\langle a_0, \cdots, a_{\mu(i)-1} \rangle \in (A^n)^{\mu(i)}$, $i \in D(\mu)$. $n\mathfrak{A}$ is called the n-direct sum of $\mathfrak{A}$. Define U by $U(\mathfrak{A}) = n\mathfrak{A}$, and τ by $\tau_{\mathfrak{A}}(a) = a$ for $a \in |U(\mathfrak{A})|$, for $\mathfrak{A} \in M(\mu)$. Define F by $F(\theta) = \bigvee_{j<n} \theta^{(n,j)}$ if θ is atomic and other cases are the same as Example 3. Then (U, τ) is strongly normal with respect to F, so U is strongly normal. Any element in $R(F) \cap F(\mu)(R(F) \cap S(\mu))$ is called as *n-disjunctive formula* (*sentence*).

REMARK. We can treat Example 3 and 4 more generally as a generalized power defined on n (see [**4**]).

§ 3. Preservation theorems.

We shall consider in this section a generalized form of preservation theorems. At first, we consider functions whose domain is $M(\mu)$ and whose range is the collection of subclasses of $M(\mu)$ $(P(M(\mu)))$. Weglorz called this type of function an operation, but we use this term by another meaning. So we call them operators (see [**12**]).

A function O from $M(\mu)$ to $P(M(\mu))$ is said to be an *operator on models* of type μ (o. r. m.). For o. r. m. O of type μ, let $\Delta(O)$ be the set of sentences θ such that if $\underset{\mathfrak{A}}{\models}\theta$, then $\underset{\mathfrak{B}}{\models}\theta$ for all $\mathfrak{B}\in O(\mathfrak{A})$. This definition of $\Delta(O)$ is due to Weglorz (see [**12**]).

Weglorz also defined *perfect* o. r. m. O by the following:

O is perfect if and only if $T_n\mathfrak{A}\cap\Delta(O)\subseteqq T_n\mathfrak{A}$ implies that there are $\mathfrak{A}_1>\mathfrak{A}$, $\mathfrak{B}_1>\mathfrak{B}$ such that $\mathfrak{B}_1\in O(\mathfrak{A}_1)$ for any $\mathfrak{A},\mathfrak{B}\in M(\mu)$. This definition is very natural if we consider classical proofs of some preservation theorems (see [**10**], [**5**]). But we shall use the following stronger definition of perfect o. r. m. to represent the key method of special models and saturated models in preservation theorems (see [**6**], [**7**], [**8**]).

DEFINITION 3.1. Suppose O is an o. r. m. of type μ. O is *perfect* if $T_n\mathfrak{A}\cap\Delta(O)\subseteqq T_n\mathfrak{B}$ implies $\mathfrak{B}\in O(\mathfrak{A})$ for any special models $\mathfrak{A}$ and $\mathfrak{B}$ such that $\overline{\overline{\mathfrak{A}}}=\overline{\overline{\mathfrak{B}}}$ or $\overline{\overline{\mathfrak{A}}}<\omega$ or $\overline{\overline{\mathfrak{B}}}<\omega$.

From now on, we assume that U is an o. p. m. of type (μ,μ') and O is an o. r. m. of type μ'. Then $U^{-1}\circ O\circ U$ is defined by $\mathfrak{B}\in(U^{-1}\circ O\circ U)(\mathfrak{A})$ if and only if $U(\mathfrak{B})\in O(U(\mathfrak{A}))$. Hence $U^{-1}\circ O\circ U$ is an o. r. m. of type μ. For $\Delta\subseteqq S(\mu)$, $F(\Delta)=\{F(\theta);\theta\in\Delta\}$.

THEOREM 3.2. *Suppose that U is strongly normal, O is perfect, $\mathfrak{A}$ and $\mathfrak{B}$ are special such that $\overline{\overline{\mathfrak{A}}}=\overline{\overline{\mathfrak{B}}}$ or $\overline{\overline{\mathfrak{A}}}<\omega$ or $\overline{\overline{\mathfrak{B}}}<\omega$. $T_n\mathfrak{A}\cap F_U(\Delta(O))\subseteqq T_n\mathfrak{B}$ is equivalent to $U(\mathfrak{B})\in O(U(\mathfrak{A}))$.*

PROOF. It is obvious that $U(\mathfrak{B})\in O(U(\mathfrak{A}))$ implies $T_n\mathfrak{A}\cap F_U(\Delta(O))\subseteqq T_n\mathfrak{B}$. Suppose $T_n\mathfrak{A}\cap F_U(\Delta(O))\subseteqq T_n\mathfrak{B}$. Then clearly $T_nU(\mathfrak{A})\cap\Delta(O)\subseteqq T_nU(\mathfrak{B})$. Since U is strongly normal, by Proposition 1.4 and Theorem 1.7, $U(\mathfrak{A})$ and $U(\mathfrak{B})$ are special and $\overline{\overline{U(\mathfrak{A})}}=\overline{\overline{U(\mathfrak{B})}}$ or $\overline{\overline{U(\mathfrak{A})}}<\omega$ or $\overline{\overline{U(\mathfrak{B})}}<\omega$. Since O is perfect, $U(\mathfrak{B})\in O(U(\mathfrak{A}))$. q. e. d.

For $\Delta\subseteqq S(\mu)$, let Δ' be the logical closure of Δ.

THEOREM 3.3. *Suppose that U is strongly normal and O is perfect. Then $\Delta(U^{-1}\circ O\circ U)=(F_U(\Delta(O)))'$.*

PROOF. It is obvious from the definition and assumption that $(F_U(\Delta(O)))'\subseteqq\Delta(U^{-1}\circ O\circ U)$. Let θ be an arbitrary element of $\Delta(U^{-1}\circ O\circ U)$ and $\Gamma=\{\phi;\phi\in F_U(\Delta(O)),\vdash\theta\rightarrow\phi\}$. Suppose that $\Gamma\cup\{\neg\theta\}$ is consistent. Let $\mathfrak{B}$ be one of its models and $\Gamma' = \{\neg\psi\,;\ \psi \in F_U(\Delta(O)),\ \models_{\mathfrak{B}} \neg\psi\}$. Since $\Delta(O)$ is closed under disjunction and F_U is commutable with disjunction, $\Gamma' \cup \{\theta\}$ is consistent. Let $\mathfrak{A}$ be a model of $\Gamma' \cup \{\theta\}$. Then $\models_{\mathfrak{A}} \theta$, $\models_{\mathfrak{B}} \neg\theta$ and $T_n\mathfrak{A} \cap F_U(\Delta(O)) \subseteqq T_n\mathfrak{B}$. Let $\mathfrak{A}_1$, $\mathfrak{B}_1$ be two special models such that $\mathfrak{A}_1 \equiv \mathfrak{A}$, $\mathfrak{B}_1 \equiv \mathfrak{B}$ and $\bar{\bar{\mathfrak{A}}}_1 = \bar{\bar{\mathfrak{B}}}_1$ or $\bar{\bar{\mathfrak{A}}}_1 < \omega$ or $\bar{\bar{\mathfrak{B}}}_1 < \omega$ (see [14]). Then $\models_{\mathfrak{A}_1} \theta$, $\models_{\mathfrak{B}_1} \neg\theta$ and $T_n\mathfrak{A}_1 \cap F_U(\Delta(O)) \subseteqq T_n\mathfrak{B}_1$. By Theorem 3.2, $U(\mathfrak{B}_1) \in O(U(\mathfrak{A}_1))$. Since $\models_{\mathfrak{A}_1} \theta$ and $\theta \in \Delta(U^{-1} \circ O \circ U)$, $\models_{\mathfrak{B}} \theta$. This is a contradiction. Hence $\Gamma \cup \{\neg\theta\}$ is inconsistent. By the compactness theorem and the fact that Γ is closed under conjunction, $\theta \in (F_U(\Delta(O)))'$. This means that $\Delta(U^{-1} \circ O \circ U) = (F_U(\Delta(O)))'$. q. e. d.

COROLLARY 3.4. (Keisler [4], Theorem 4.1 and Corollary 4.2).

(I) *If $\mathfrak{A}$ and $\mathfrak{B}$ are special of same power, then the following two conditions are equivalent:*

(i) $\mathfrak{A}^n \cong \mathfrak{B}^n$

(ii) *Every n-conjunctive sentence true in $\mathfrak{A}$ is also true in $\mathfrak{B}$.*

(II) *The following two conditions are equivalent:*

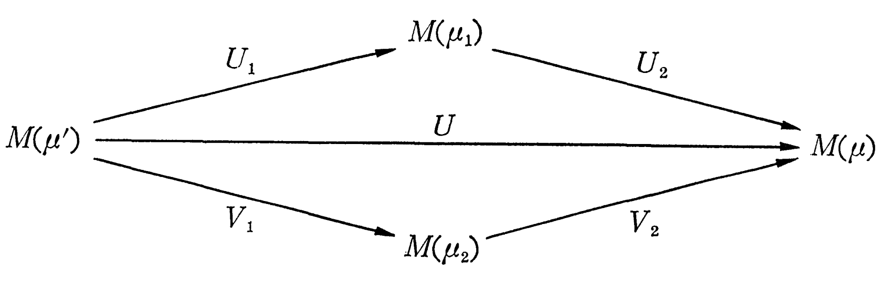
(i) *If $\models_{\mathfrak{A}} \theta$ and $\mathfrak{A}^n \cong \mathfrak{B}^n$, then $\models_{\mathfrak{B}} \theta$.*

(ii) *θ is equivalent to some n-conjunctive sentence.*

PROOF. In Theorem 3.3, take O as isomorphism and U as n-direct power. q. e. d.

§ 4. Interpolation theorem.

In this section, we shall get a generalized form of the Craig-Lyndon interpolation theorem using normal operations and perfect operators on models. Throughout this section we fix U_1, U_2, V_1, V_2, U as strong normal operations on models of type (μ', μ_1), (μ_1, μ), (μ', μ_2), (μ_2, μ), (μ', μ), respectively such that $U = U_2 \circ U_1 = V_2 \circ V_1$ (i. e. the following diagram commutes)

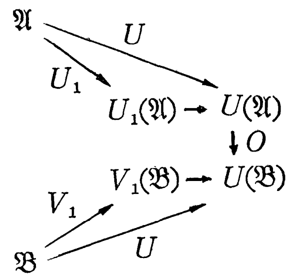


and $F_U = F_{U_2} \circ F_{U_1} = F_{V_2} \circ F_{V_1}$.

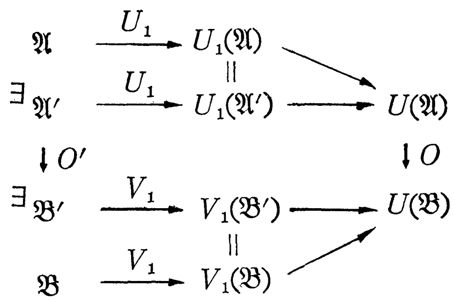
Moreover O, O' are operators on models of type μ, μ'. $O \subseteqq O'$ means that for any $\mathfrak{A}, \mathfrak{B} \in M(\mu')$, if $U(\mathfrak{B}) \in O(U(\mathfrak{A}))$, then there are $\mathfrak{A}', \mathfrak{B}' \in M(\mu')$ such that $U_1(\mathfrak{A}') = U_1(\mathfrak{A})$, $V_1(\mathfrak{B}') = V_1(\mathfrak{B})$ and $\mathfrak{B}' \in O'(\mathfrak{A}')$.

(i. e.

If



then



THEOREM 4.1 (interpolation theorem). *Suppose that O is perfect and $O \subseteqq O'$, $\theta \in R(F_{U_1})$, $\varphi \in R(F_{V_1})$ and $\theta \in \Delta(O')$ or $\varphi \in \Delta(O')$.*

If $\vdash \theta \to \varphi$, then there is a $\psi \in F_U(\Delta(O))$ such that $\vdash \theta \to \psi$ and $\vdash \psi \to \varphi$.

PROOF. Suppose $\vdash \theta \to \varphi$. Let Δ be the set of sentence $\psi \in F_U(\Delta(O))$ such that $\vdash \theta \to \psi$. Suppose that $\Delta \cup \{\neg \varphi\}$ is consistent. Then by the same method in the proof of Theorem 3.3, we can get two special models $\mathfrak{A}_2$ and $\mathfrak{B}_2$ in $M(\mu')$ such that $\bar{\bar{\mathfrak{A}}}_2 = \bar{\bar{\mathfrak{B}}}_2$ or $\bar{\bar{\mathfrak{A}}}_2 < \omega$ or $\bar{\bar{\mathfrak{B}}}_2 < \omega$ and $\underset{\mathfrak{A}_2}{\models} \theta$, $\underset{\mathfrak{B}_2}{\models} \neg \varphi$, $T_n \mathfrak{A}_2 \cap F_U(\Delta(O)) \subseteqq T_n \mathfrak{B}_2$. Since O is perfect, by Theorem 3.2, $U(\mathfrak{B}_2) \in O(U(\mathfrak{A}_2))$. Since $O \subseteqq O'$, there are $\mathfrak{A}_1, \mathfrak{B}_1 \in M(\mu')$ such that $U_1(\mathfrak{A}_1) = U_1(\mathfrak{A}_2)$, $V_1(\mathfrak{B}_1) = V_1(\mathfrak{B}_2)$ and $\mathfrak{B}_1 \in O'(\mathfrak{A}_1)$. Since $\theta \in R(F_{U_1})$, $\underset{\mathfrak{A}_2}{\models} \theta$ and $U_1(\mathfrak{A}_1) = U_1(\mathfrak{A}_2)$, we get $\underset{\mathfrak{A}_1}{\models} \theta$. Since $\varphi \in R(F_{V_1})$, $\underset{\mathfrak{B}_2}{\models} \neg \varphi$ and $V_1(\mathfrak{B}_1) = V_1(\mathfrak{B}_2)$, we get $\underset{\mathfrak{B}_1}{\models} \neg \varphi$. If $\theta \in \Delta(O')$, then $\underset{\mathfrak{B}_1}{\models} \theta$ by $\underset{\mathfrak{A}_1}{\models} \theta$ and $\mathfrak{B}_1 \in O'(\mathfrak{A})$. Since $\vdash \theta \to \varphi$, $\underset{\mathfrak{B}_1}{\models} \varphi$. This contradicts $\underset{\mathfrak{B}_1}{\models} \neg \varphi$. If $\varphi \in \Delta(O')$, since $\vdash \theta \to \varphi$ and $\underset{\mathfrak{A}_1}{\models} \theta$, $\underset{\mathfrak{A}_1}{\models} \varphi$. Hence $\underset{\mathfrak{B}_1}{\models} \varphi$ by $\mathfrak{B}_1 \in O'(\mathfrak{A}_1)$. This contradicts $\underset{\mathfrak{B}_1}{\models} \neg \varphi$. Therefore $\Delta \cup \{\neg \varphi\}$ is inconsistent. By the compactness theorem and the fact that Δ is closed under conjunction, there is a $\psi \in \Delta$ such that $\vdash \psi \to \varphi$. Hence $\vdash \theta \to \psi$, $\vdash \psi \to \varphi$, $\psi \in F_U(\Delta(O))$. q. e. d.

REMARK. Let U, U_1, U_2, V_1, V_2 be reducts with some natural conditions. If we take O and O' as isomorphisms, then we get Craig's interpolation theorem. If we take O and O' as Γ-homomorphism and Γ'-homomorphism for appropriate generalized atomic formulas Γ, Γ' (see [**5**], [**9**], [**10**]), then we get Lyndon's interpolation theorem.

University of Tokyo

References

[1] C.C. Chang, Some new results in the theory of definition, Bull. Amer. Math. Soc., **70** (1964), 808-813.

[2] C.C. Chang, A note on two cardinal problems, Proc. Amer. Math. Soc., **16** (1965), 1148-1155.

[3] C.C. Chang and H.J. Keisler, Continuous model theory, Princeton U.P., 1966.

[4] S. Feferman and R. Vaught, The first order properties of products of algebraic systems, Fund. Math., **47** (1959), 57-103.

[5] H.J. Keisler, The theory of models with generalized atomic formulas, J. Symbolic Logic, **25** (1960), 1–26.

[6] H.J. Keisler, Union of relational systems, Proc. Amer. Math. Soc., **15** (1964), 540–545.

[7] H.J. Keisler, Reduced products and Horn classes, Trans. Amer. Math. Soc., **117** (1965), 307–328.

[8] H. J. Keisler, Some application of infinitely long formulas, J. Symbolic Logic, **30** (1965), 339–349.

[9] R. Lyndon, An interpolation theorem in the predicate calculus, Pacific J. Math., **9** (1959), 129–142.

[10] R. Lyndon, Properties preserved under homomorphism, Pacific J. Math., **9** (1959), 143–154.

[11] M. Makkai, A generalization of a theorem of Beth, Acta Math. Acad. Sci. Hungar., **15** (1964), 227–236.

[12] B. Weglorz, Some preservation theorems, Colloq. Math., **17** (1967), 269–276.

[13] M. Morley, Categoricity in powers, Trans. Amer. Math. Soc., **114** (1965), 514–538.

[14] M. Morley and R. Vaught, Homogeneous universal models, Math. Scand., **11** (1962), 37–57.

[15] A. Oberschelp, Über die Axiome produkt-abgeschlossener arithmetischer Klassen, Arch. Math. Logic Grundlagenforsch., **4** (1958), 95–123.

[16] A. Robinson, Introduction to model theory, North Holland, 1963.

[17] A. Tarski and R. Vaught, Arithmetical extensions of relational systems, Compositio Math., **13** (1957), 81–102.

[18] R. Vaught, Denumerable models of complete theories, Infinistic methods, Warsaw (1961), 303–321.