J. Math. Soc. Japan
Vol. 21, No. 4, 1969

Spectral synthesis for the Kronecker sets

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(Received Feb. 28, 1969)
(Revised April 23, 1969)

Throughout this paper, let G be any locally compact abelian group and $\hat{G}$ its dual. We denote by $A(G)$ the Banach algebra consisting of the Fourier transforms of all complex-valued functions on $\hat{G}$ that are absolutely summable with respect to the Haar measure of $\hat{G}$ [2].

N. Th. Varopoulos proved in [4] that every totally disconnected Kronecker subset of G is a set of spectral synthesis (an S-set) for the algebra $A(G)$. On the other hand, every compact (Hausdorff) space is homeomorphic to a Kronecker subset of some compact abelian group (see Theorem 2). The main purpose of this paper is to show that every Kronecker set is an S-set.

DEFINITION 1. A compact subset K of the group G is called a quasi-Kronecker set, provided that: For each $\varepsilon>0$ and each real continuous function h on K ($h \in C_R(K)$), there exists a character $\gamma \in \hat{G}$ such that

$$\sup_{x \in K} |\exp[i\, h(x)]-(x, \gamma)| < \varepsilon .$$

It is then easy to see that:

(i) Every quasi-Kronecker set is independent;

(ii) A Kronecker set is a quasi-Kronecker set;

(iii) If K is a quasi-Kronecker subset of G, then we have $\|\mu\| = \|\hat{\mu}\|_\infty$ for all $\mu \in M(K)$. In particular, every quasi-Kronecker set is a Helson set.

The following theorem seems to be well-known. But the author does not know any literature about it; hence we give here a complete proof of it.

THEOREM 2. *There exists a compact abelian group which contains a quasi-Kronecker set that is not a Kronecker set. Every compact space is homeomorphic to a Kronecker subset of some compact abelian group.*

PROOF. Suppose that X is a compact space, and that a and b are two constants such that $0<a<b<1$, and take any subset F of $C_R(X)$ such that:

(2.1) We have $a \leqq f \leqq b$ for all $f \in F$;

(2.2) The functions in F separate points of X.

Let us then denote by $\mathfrak{F}$ the set of all functions in $C_R(X)$ expressible as a finite product of elements in F, and let

$$G=\prod_{g\in\mathcal{F}} T(g) \qquad (T(g)=T \text{ for all } g\in\mathcal{F}), \tag{2.3}$$

where T denotes the one-dimensional torus (the circle group). Thus every point p of G has the form

$$p=(p(g))_{g\in\mathcal{F}} \qquad (p(g)\in T(g) \text{ for all } g\in\mathcal{F}), \tag{2.4}$$

and for every $\gamma\in\hat{G}$ there exist integers $n_1, n_2, \cdots, n_k$ and functions $g_1, g_2, \cdots, g_k$ of $\mathcal{F}$ such that

$$(p, \gamma)=\prod_{j=1}^{k}\{p(g_j)\}^{n_j} \qquad (p\in G). \tag{2.5}$$

We now define a mapping t from X into G by

$$t(x)=(\exp[2\pi i\, g(x)])_{g\in\mathcal{F}} \qquad (x\in X). \tag{2.6}$$

It is then trivial that t is a homeomorphism from X onto $K=t(X)$. If $h\in C_R(K)$, then there exists $h'\in C_R(X)$ such that $2\pi\, h'(x)=h(t(x))$. If $\gamma\in\hat{G}$ has the form (2.5), we see from (2.6) that

$$\begin{aligned}&|\exp[i\,h(t(x))]-(t(x),\gamma)|\\ &\quad=|\exp[2\pi i\,h'(x)]-\exp[2\pi i\sum_{j=1}^{k}n_j g_j(x)]|\\ &\quad\leqq 2\pi|h'(x)-\sum_{j=1}^{k}n_j g_j(x)| \qquad (x\in X).\end{aligned}$$

Thus, in order to prove that K is a quasi-Kronecker set, it suffices to apply an analogous argument as in [**2**: p. 104].

Suppose now that X is homeomorphic to T, and that s is a homeomorphism of K onto T. It then follows from (2.5) and (2.6) that

$$\begin{aligned}\inf_{\gamma\in\hat{G}}\{\sup_{p\in K}|s(p)-(p,\gamma)|\} &\geqq \inf_{g\in C_R(K)}\{\sup_{p\in K}|s(p)-\exp[i\,g(p)]|\}\\ &=\inf_{h\in C_R(T)}\{\sup_{z\in T}|z-\exp[i\,h(z)]|\}>0.\end{aligned}$$

Thus K is not a Kronecker set although it is a quasi-Kronecker set, and this establishes the first statement.

Suppose again that X is any compact space, and let $\mathcal{F}$ in (2.3) be the set of all complex-valued functions $g\in C(X)$ with $|g|\equiv 1$. Defining a mapping τ from X into G by

$$\tau(x)=(g(x))_{g\in\mathcal{F}} \qquad (x\in X), \tag{2.7}$$

one can now easily show that τ is a homeomorphism from X onto $K=\tau(X)$, and that K is a Kronecker set of G.

This completes the proof.

We now introduce some notations. For any closed subset E of G, let us denote by:

$$I(E)=\{f\in A(G): f=0 \text{ on } E\};$$

$$I_0(E)=\{f\in A(G): E\cap \operatorname{supp} f=\emptyset\};$$

$$J(E)=\text{the closure of } I_0(E).$$

Thus $I(E)$ (resp. $J(E)$) is the largest (resp. the smallest) closed ideal in $A(G)$ whose zero-set is E. We also denote by $PM(E)$ the space of all pseudo-measures P on G with $\operatorname{supp} P\subset E$, and for any $P\in PM(E)$ $\hat{P}$ will be always chosen to be continuous if this is possible, where $\hat{P}$ denotes the bounded Borel function on $\hat{G}$ corresponding to P. We call E an SH-set if and only if E is both an S-set and a Helson set. It is trivial that this condition is equivalent to the one $PM(E)=M(E)$, and that such a set is a set of spectral resolution (an SR-set) [**1**].

Now, for any $f\in A(G)$ let $\sigma(f, E)$ be the set of all points $x\in G$ at which f does not belong to $J(E)$ locally, and put

$$\sigma(E)=\bigcup_{f\in I(E)}\sigma(f, E).$$

It is well-known ([2], [**3**]) that $\sigma(E)$ is a union of perfect subsets of ∂E (the boundary of E), and that E is an S-set if and only if $\sigma(E)$ is empty. One can also show that $\sigma(E)$ is closed if G is metrizable.

LEMMA 3. *Suppose that E is the union of two S-sets E_1 and E_2 of G, then we have $\sigma(E)\subset\partial E_1\cap\partial E_2\cap\partial E$. In particular, it follows that E is an S-set if either $\partial E_1\cap\partial E_2\cap\partial E$ contains no perfect subset or there exists a C-set C such that $\partial E_1\cap\partial E_2\cap\partial E\subset C\subset E$.*

PROOF. It is trivial that $\sigma(E)\subset\partial E$. To show that every function of $I(E)$ belongs to $J(E)$ locally at any point in the complement of $E_1\cap E_2$, take $f\in I(E)$ and $x\in E\setminus(E_1\cap E_2)$ arbitrarily. Without loss of generality, we may assume that $x\in E_1$. Choose $u\in I_0(E_2)$ so that $u=1$ on some neighborhood of x. Since E_1 is an S-set by our assumption, it follows that there is a sequence $\{g_n\}$ in $I_0(E_1)$ such that $\lim_{n\to\infty}\|f-g_n\|=0$. Then $g_n u\in I_0(E)$ for all $n=1, 2, \cdots$, and $\lim_{n\to\infty}\|fu-g_n u\|=0$, which implies $fu\in J(E)$. Since $fu=f$ on some neighborhood of x, it follows that f belongs to $J(E)$ locally at x. Therefore we have

$$\sigma(E)\subset E_1\cap E_2\cap\partial E=\partial E_1\cap\partial E_2\cap\partial E,$$

and this establishes the first statement.

If $\partial E_1\cap\partial E_2\cap\partial E$ contains no perfect subset, then $\sigma(E)$ is empty, and hence E is an S-set. Finally, suppose that E contains a C-set C such that $C\supset\partial E_1\cap\partial E_2\cap\partial E$. Then for every $f\in I(E)$ we can find a sequence $\{v_n\}_1^\infty$ in $I_0(C)$ so that $\lim_{n\to\infty}\|f-fv_n\|=0$. Since each fu_n belongs to $J(E)$ at all points of G by what we have proved above, it follows that $fv_n\in J(E)$ for all

$n=1, 2, \cdots$, and hence we have $f \in J(E)$. Since $f \in I(E)$ was arbitrary, this gives the desired conclusion.

The proof is now complete.

THEOREM 4. *The union of an SH-set and an S-set is an S-set.*

PROOF. Suppose that H and S be an SH-set and an S-set of G, respectively. There exists then a finite positive constant C such that to every $k \in C(H)$ corresponds a $g \in A(G)$ with

$$g|_H = k, \quad \text{and} \quad \|g\| \leqq C\|k\|_\infty. \tag{4.1}$$

Let us take $f \in I(H \cup S)$ and $P \in PM(H \cup S)$ arbitrarily. Since $\sigma(H \cup S) \subset H \cap S$ by Lemma 3, it is easy to verify that $\operatorname{supp} fP \subset H \cap S$. Therefore the assumption that H is an SH-set guarantees that fP is a measure on $H \cap S$. To show that $fP=0$, let $\mathcal{V}$ be an arbitrarily fixed basis of open neighborhoods of $H \cap S$, and for each $U \in \mathcal{V}$ denote by $\mathcal{K}(U)$ the set of all $g \in A(G)$ such that

$$\operatorname{supp} g|_H \subset U, \qquad g=1 \text{ on } H \cap S, \quad \text{and} \quad \|g\| \leqq C. \tag{4.2}$$

It follows then from (4.1) that each $\mathcal{K}(U)$, $U \in \mathcal{V}$, is non-empty. Thus the sets $\mathcal{L}(U)=\{gP: g \in \mathcal{K}(U)\}$, $U \in \mathcal{V}$, have the finite intersection property, and it is trivial that they are all contained in the closed ball of $PM(G)$ with radius $C\|P\|$; hence they have a common weak-star cluster point $Q \in PM(G)$. We then claim that $\operatorname{supp} Q \subset S$ and $fQ=fP$.

To show this, let $h \in I_0(S)$ be arbitrary, and take an open neighborhood V of S on which h vanishes. If $U \in \mathcal{V}$ is such that $U \subset V$, and if $g \in \mathcal{K}(U)$, then we have $hg \in I(H \cup S)$ and so $hg \in J(H \cup S)$ by Lemma 3, since $hg=0$ on $V \supset H \cap S$. This yields that $hgP=0$ for all $g \in \mathcal{K}(U)$, and hence $hQ=0$ since Q belongs to the weak-star closure of $\mathcal{L}(U)$. But $h \in I_0(S)$ was arbitrary, and so we conclude that $\operatorname{supp} Q \subset S$. On the other hand, for any $U \in \mathcal{V}$ and $g \in \mathcal{K}(U)$, it must be $fgP=fP$ since $fP \in M(H \cap S)$ and $g=1$ on $H \cap S$ by (4.2), which yields $fQ=fP$. Finally we have $fP=fQ=0$, since $Q \in PM(S)$, $f \in I(S)$, and S is an S-set.

This completes the proof.

COROLLARY 5. *Every finite union of SH-sets is an SR-set.*

PROOF. Since every closed subset of an SH-set is also an SH-set, it suffices to show that every finite union of SH-sets is an S-set. But this follows at once from Theorem 4 by induction.

COROLLARY 6. *Every Helson set that is a finite union of S-sets is an SH-set.*

PROOF. Trivial.

We shall now prove four lemmas, the first two of which are essentially contained in [4]. To make the paper self-contained, we give their complete proofs.

LEMMA 7. *To each $\varepsilon>0$ corresponds a constant $\varepsilon>\eta(\varepsilon)>0$ with the following property: For any compact subset K of G, any complex number α with $|\alpha|=1$, and any characters $\gamma_1, \gamma_2 \in \hat{G}$ such that*

(7.1) $$\sup_{x\in K} |\alpha(x, \gamma_1)-(x, \gamma_2)| < \eta(\varepsilon),$$

we can find $h\in A(G)$ so that

(7.2) $$\|h\| < \varepsilon, \quad \textit{and} \quad h=\alpha\gamma_1-\gamma_2$$

on some neighborhood of K.

PROOF. We shall here regard T as the multiplicative group of the complex numbers z with $|z|=1$. Consider the function $f\in A(T)$ defined by $f(z)=1-z$, and let $\varepsilon>0$ be given. Since $f(1)=0$, there exist a function $f_\varepsilon\in A(T)$ and a constant $\varepsilon>\eta(\varepsilon)>0$ such that

(7.3) $$f_\varepsilon(z)=\sum_{n=-\infty}^{\infty} a_n z^n, \qquad \|f_\varepsilon\|=\sum_{n=-\infty}^{\infty}|a_n|<\varepsilon,$$

and such that

(7.4) $$z\in T, \qquad |1-z|<\eta(\varepsilon) \Rightarrow f_\varepsilon(z)=1-z.$$

Suppose now that K, α, γ_1 and γ_2 satisfy the condition (7.1), and define a function g on G by

$$g(x)=\alpha(x, \gamma_1)f_\varepsilon(\bar{\alpha}(x, \gamma_2-\gamma_1)).$$

It is then easy to see from (7.3) and (7.4) that g is the Fourier-Stieltjes transform of a measure on G with norm $<\varepsilon$, and that $g=\alpha\gamma_1-\gamma_2$ on some open set containing K. To complete the proof, take $\delta>0$ and $k=k_\delta\in A(G)$ so that $\|k\|<1+\delta$ and $k=1$ on some neighborhood of K. Setting $h_\delta=gk$, we see that for a sufficiently small $\delta>0$, $h=h_\delta\in A(G)$ satisfies (7.2).

This establishes the Lemma.

LEMMA 8. *Let K be a quasi-Kronecker subset of G, let $\{Q_j\}_1^n$ be n pseudo-measures in $PM(K)$ such that*

(8.1) $$\operatorname{supp} Q_i \cap \operatorname{supp} Q_j = \emptyset \qquad (1\leqq i<j\leqq n),$$

and put $Q=\sum_{j=1}^{n} Q_j$. Then we have

(8.2) $$\sup_{\gamma\in\hat{G}} |\hat{Q}(\gamma_1+\gamma)-\sum_{j=1}^{n}\alpha_j\hat{Q}_j(\gamma)| \leqq \varepsilon\|Q\|$$

for any $\varepsilon>0$, any $\gamma_1\in\hat{G}$, and any choice $\{\alpha_j\}_1^n$ of complex numbers with $|\alpha_j|=1$ $(1\leqq j\leqq n)$ such that

(8.3) $$|(x, \gamma_1)-\alpha_j|<\eta(\varepsilon) \qquad (x\in \operatorname{supp} Q_j,\ 1\leqq j\leqq n),$$

where $\eta(\varepsilon)$ is a constant as in Lemma 7. In particular, we have

$$(8.4)\qquad \sup_{\gamma\in\hat{G}}\sum_{j=1}^{n}|\hat{Q}_j(\gamma)|\leqq\|Q\|.$$

PROOF. Let ε, γ_1, and $\{\alpha_j\}_1^n$ be as in (8.3), and take $\delta>0$ so that the inequality in (8.3) remains valid even if the right term is replaced by $\eta(\varepsilon)-\delta$. Since K is a quasi-Kronecker set, and since $\operatorname{supp} Q=\bigcup_{j=1}^{n}\operatorname{supp} Q_j\subset K$, we can find $\gamma'\in\hat{G}$ so that

$$|\alpha_j-(x,\gamma')|<\eta(\delta)\qquad(x\in\operatorname{supp} Q_j,\ 1\leqq j\leqq n).$$

It follows then that for all $x\in\operatorname{supp} Q$ we have

$$|(x,\gamma_1)-(x,\gamma')|\leqq\min_{1\leqq j\leqq n}\{|(x,\gamma_1)-\alpha_j|+|\alpha_j-(x,\gamma')|\}$$
$$<\{\eta(\varepsilon)-\delta\}+\eta(\delta)<\eta(\varepsilon).$$

Therefore we see from Lemma 7 that for all $\gamma\in\hat{G}$

$$|\hat{Q}(\gamma_1+\gamma)-\sum_{j=1}^{n}\alpha_j\hat{Q}_j(\gamma)|$$
$$\leqq|\hat{Q}(\gamma_1+\gamma)-\hat{Q}(\gamma'+\gamma)|+\sum_{j=1}^{n}|\hat{Q}_j(\gamma'+\gamma)-\alpha_j\hat{Q}_j(\gamma)|$$
$$\leqq\varepsilon\|Q\|+\delta\sum_{j=1}^{n}\|Q_j\|.$$

Since $\delta>0$ can be taken as small as one pleases, we obtain (8.2).

To complete the proof, let $\gamma\in\hat{G}$ be given, and take α_j so that $|\alpha_j|=1$ and $\alpha_j\hat{Q}_j(\gamma)=|\hat{Q}_j(\gamma)|$ for all $j=1,2,\cdots,n$. Then for any $\varepsilon>0$, there exists $\gamma_1\in\hat{G}$ which satisfies (8.3). This fact, combined with (8.2), yields (8.4).

The proof is now established.

LEMMA 9. *Suppose that K is a quasi-Kronecker subset of G, that $P\in PM(K)$, and that $\{E_k\}_1^n$ are n closed, pairwise disjoint, subsets of K. Then there exist n pseudo-measures $\{P_k\}_1^n$ such that:*

(9.1) *For all $k=1,2,\cdots,n$, we have*

$$P_k\in PM(E_k),\qquad\|\sum_{k=1}^{n}P_k\|\leqq\|P\|,$$
$$\|P-P_k\|\leqq\|P\|,\quad\text{and}\quad P-P_k\in PM(\overline{K\backslash E_k});$$

(9.2) *For all $k=1,2,\cdots,n$ and any neighborhood $\hat{U}$ of $\hat{0}$ of $\hat{G}$,*

$$\sup_{\gamma-\gamma'\in\hat{U}}|\hat{P}_k(\gamma)-\hat{P}_k(\gamma')|\leqq\sup_{\gamma-\gamma'\in\hat{U}}|\hat{P}(\gamma)-\hat{P}(\gamma')|,$$

and

$$\sup_{\gamma-\gamma'\in\hat{U}}|(P-P_k)\hat{}(\gamma)-(P-P_k)\hat{}(\gamma)|\leqq\sup_{\gamma-\gamma'\in\hat{U}}|\hat{P}(\gamma)-\hat{P}(\gamma')|.$$

PROOF. Fix any $\chi\in A(G)$ so that $\chi=1$ on some neighborhood of K.

Note then that $\chi P=P$, and that $\|\hat{l}\chi P\|\leqq\|l\|\|P\|$ for all $l\in M(\hat{G})$, since K is a compact set containing the support of P.

Let $\mathcal{U}$ be the set of all tuples $u=(\varepsilon; U_1, U_2, \cdots, U_n)$ of $0<\varepsilon<1$ and open neighbourhoods U_k of E_k such that the sets $\bar{U}_k$, $1\leqq k\leqq n$, are pairwise disjoint. If we introduce an order "$\prec$" in $\mathcal{U}$ by

$$(9.3)\qquad (\varepsilon_1; U_{11}, U_{21}, \cdots, U_{n1})\prec(\varepsilon_2; U_{12}, U_{22}, \cdots, U_{n2}) \Leftrightarrow \varepsilon_1>\varepsilon_2, \text{ and } U_{k1}\supset U_{k2} \text{ for all } k\ (1\leqq k\leqq n),$$

then $\mathcal{U}$ is clearly a directed set. Fixing $u=(\varepsilon; U_1, U_2, \cdots, U_n)$ in $\mathcal{U}$, we shall now define two pseudo-measures Q_u and R_u of $PM(K)$ as follows. Take $h_u \in C_R(K)$ so that

$$(9.4)\qquad 0\leqq h_u\leqq\pi, \quad h_u=0 \text{ on } \bigcup_{k=1}^{n} E_k, \quad\text{and}\quad h_u=\pi \text{ on } \bigcap_{k=1}^{n} K\backslash U_k.$$

Since K is a quasi-Kronecker set, there exists $\gamma_u\in\hat{G}$ such that

$$(9.5)\qquad |\exp[i\,h_u(x)]-(x,\gamma_u)|<\eta(\varepsilon)/2 \qquad (x\in K),$$

where $\eta(\varepsilon)$ is as in Lemma 7. We then define

$$(9.6)\qquad Q_u=(1+\gamma_u)\chi P/2, \quad\text{and}\quad R_u=(1-\gamma_u)\chi P/2.$$

It is trivial that

$$(9.7)\qquad P=Q_u+R_u, \quad\text{and}\quad \|Q_u\|,\ \|R_u\|\leqq\|P\| \qquad (u\in\mathcal{U}).$$

This assures that a subnet of the net $\{Q_u\}_u$ (resp. $\{R_u\}_u$) converges to some Q (resp. R) of $PM(K)$ in the weak-star topology of $PM(G)$ such that

$$(9.8)\qquad P=Q+R, \quad\text{and}\quad \|Q\|, \|R\|\leqq\|P\|.$$

We claim then that

$$(9.9)\qquad \operatorname{supp} Q\subset\bigcup_{k=1}^{n} E_k, \quad\text{and}\quad \operatorname{supp} R\subset F,$$

where F denotes the closure of $\bigcap_{k=1}^{n} K\backslash E_k$. To show this, take $f\in I_0\left(\bigcup_{k=1}^{n} E_k\right)$ arbitrarily. Then for some open set U containing $\bigcup_{k=1}^{n} E_k$ we have $\operatorname{supp} fP \subset K\backslash U$. On the other hand, for all $u=(\varepsilon; U_1, U_2, \cdots, U_n)\in\mathcal{U}$ with $\bigcup_{k=1}^{n} U_k\subset U$, we have by (9.4) and (9.5)

$$|1+\gamma_u|<\eta(\varepsilon)/2 \quad\text{on}\quad K\backslash U,$$

and so that

$$\|fQ_u\|=\|(1+\gamma_u)fP\|/2\leqq\varepsilon\|f\|\|P\|.$$

Since Q is a cluster point of the net $\{Q_u\}_u$, this implies $fQ=0$; since f was an arbitrary function of $I_0\left(\bigcup_{k=1}^{n} E_k\right)$, it follows that $\operatorname{supp} Q\subset\bigcup_{k=1}^{n} E_k$. Similarly we have $\operatorname{supp} R \subset F$, and obtain (9.9).

We now decompose Q into the sum of n pseudo-measures $\{P_k\}_1^n$ such that

$$Q=\sum_{k=1}^{n} P_k\,, \quad \text{and} \quad \operatorname{supp} P_k \subset E_k \qquad (1 \leqq k \leqq n)\,, \tag{9.10}$$

and show that these $\{P_k\}_1^n$ satisfy the conditions (9.1) and (9.2).

The first two of (9.1) immediately follow from (9.8) and (9.10). To prove the remainder parts, let $\{Q_{u(\alpha)}\}_\alpha$ be any subnet of the net $\{Q_u\}_u$ that converges to Q. Fixing $u=(\varepsilon;\, U_1, U_2, \cdots, U_n) \in \mathcal{U}$, we see from (9.4) that the function on K defined by

$$h'_u=\begin{cases} h_u & \text{on} \quad K \cap U_1 \\ \pi & \text{on} \quad K \backslash U_1 \end{cases} \tag{9.4$'$}$$

is continuous; it follows that there exists $\gamma'_u \in \hat{G}$ with

$$|\exp[i\,h'_u(x)]-(x, \gamma'_u)| < \eta(\varepsilon)/2 \qquad (x \in K)\,. \tag{9.5$'$}$$

Take now any $g_1 \in A(G)$ so that $g_1=1$ on a neighborhood V_1 of E_1 and $g_1=0$ on a neighborhood W_1 of $\bigcup_{k=2}^{n} E_k$. Then for all $u=(\varepsilon;\, U_1, U_2, \cdots, U_n) \in \mathcal{U}$ with $U_1 \subset V_1$ and $\bigcup_{k=2}^{n} U_k \subset W_1$, we see from (9.4), (9.4) , (9.5) and (9.5)$'$ that

$$|\gamma'_u-\gamma_u| < \eta(\varepsilon) \quad \text{on} \quad K \backslash W_1\,,$$

and

$$|1+\gamma'_u| < \eta(\varepsilon) \quad \text{on} \quad K \backslash V_1\,.$$

Therefore, taking into account the fact that $\operatorname{supp} g_1 P \subset K \backslash W_1$, we have for such $u \in \mathcal{U}$

$$\begin{aligned} &\|(1+\gamma'_u)\chi P/2-g_1 Q_u\| \\ &\quad \leqq 2^{-1}\|(1+\gamma'_u)g_1\chi P-(1+\gamma_u)g_1\chi P\|+2^{-1}\|(1+\gamma'_u)(1-g_1)\chi P\| \\ &\quad \leqq \|(\gamma'_1-\gamma_u)g_1 P\|+\varepsilon\|(1-g_1)\chi P\| \\ &\quad \leqq \varepsilon(\|g_1 P\|+\|1-g_1\|\cdot\|P\|)\,, \end{aligned}$$

from which it follows at once that

$$\begin{aligned} P_1 &= g_1 Q \\ &= \lim_\alpha g_1 Q_{u(\alpha)} \\ &= \lim_\alpha [(1+\gamma'_{u(\alpha)})\chi P/2+\{g_1 Q_{u(\alpha)}-(1+\gamma'_{u(\alpha)})\chi P/2\}] \\ &= \lim_\alpha (1+\gamma'_{u(\alpha)})\chi P/2\,, \end{aligned} \tag{9.11}$$

and so that

$$P-P_1=\lim_\alpha (1-\gamma'_{u(\alpha)})\chi P/2\,. \tag{9.12}$$

In particular, we have $\| P-P_1 \| \leqq \| P \|$, and also it follows from (9.8), (9.9) and (9.10) that

$$\operatorname{supp}(P-P_1)=\operatorname{supp}\Big(R+\sum_{k=2}^{n} P_k\Big)\subset\Big[\Big(K\setminus\bigcup_{k=1}^{n} E_k\Big)^{-}\Big]\cup\Big[\bigcup_{k=2}^{n} E_k\Big]$$
$$\subset (K\setminus E_1)^{-}.$$

Suppose now that γ, $\gamma'\in\hat{G}$ are arbitrary, then we see from (9.11) that

$$\hat{P}_1(\gamma)-\hat{P}_1(\gamma')=2^{-1}\lim_{\alpha}\,[\{\hat{P}(\gamma)+\hat{P}(\gamma+\gamma'_{u(\alpha)})\}-\{\hat{P}(\gamma')+\hat{P}(\gamma'+\gamma'_{u(\alpha)})\}]$$
$$=2^{-1}\lim_{\alpha}\,[\{\hat{P}(\gamma)-\hat{P}(\gamma')\}+\{\hat{P}(\gamma+\gamma'_{u(\alpha)})-\hat{P}(\gamma'+\gamma'_{u(\alpha)})\}],$$

which yields

$$\sup_{\gamma-\gamma'\in\hat{U}}|\hat{P}_1(\gamma)-\hat{P}_1(\gamma')|\leqq\sup_{\gamma-\gamma'\in\hat{U}}|\hat{P}(\gamma)-\hat{P}(\gamma')|$$

for all neighborhoods $\hat{U}$ of $\hat{O}\in\hat{G}$. Similarly it follows from (9.12) that this last inequality holds with P_1 replaced by $P-P_1$.

Applying the same arguments for all k $(1\leqq k\leqq n)$, we see that the $\{P_k\}_1^n$ have all the required properties, and this completes the proof.

LEMMA 10. *Suppose that K is a compact subset of G, then for each neighborhood U of $O\in G$, there exists a natural number $N=N(U)$ with the following property:*

For any natural number n, we can find $N\times n$ compact subsets $\{E_{jk}\}$, $1\leqq j\leqq N$, $1\leqq k\leqq n$, of K such that;

(a) *The sets $\{E_{jk}\}_{k=1}^n$ are pairwise disjoint for each $j=1, 2, \cdots, N$.*

(b) *To any choice $\{k(j)\}_{j=1}^n$ of natural numbers $k(j)$ with $1\leqq k(j)\leqq n$ $(1\leqq j \leqq N)$, there correspond finitely many, pairwise disjoint, closed subsets $\{K_l\}_l$ of K such that*

$$\bigcap_{j=1}^{N} K\setminus E_{jk(j)}\subset\bigcup_{l} K_l,\quad \textit{and}\quad \bigcup_{l}(K_l-K_l)\subset U.$$

PROOF. We shall first show this lemma in case that G has the form

(10.1) $$G=\prod_{\alpha\in A} T(\alpha)\qquad (T(\alpha)=T \text{ for all } \alpha\in A)$$

as a topological group. We then denote by $S(\alpha)$ a copy of S for any subset S of T and $\alpha\in A$. Suppose now that U is any fixed neighborhood of $O\in G$. It follows then from the definition of the product topology that we can find a neighborhood W of $O\in T$ and a finite subset A_1 of A so that

(10.2) $$(W;A_1)=\prod_{\alpha\in A_1} W(\alpha)\times\prod_{\alpha\in A\setminus A_1} T(\alpha)\subset U.$$

We then define $N=N(U)$ to be the number of the elements of A_1.

Suppose that n be an arbitrary natural number. Let us then take n closed, pairwise disjoint, subsets $\{F_k\}_1^n$ of T so that: For each k $(1\leqq k\leqq n)$, the closure of $T\backslash F_k$ consists of finitely many connected components (i. e., closed arcs) $\{C_{km}\}_m$ such that $\bigcup_m (C_{km}-C_{km})\subset W$. Denoting by $\{\alpha_1, \alpha_2, \cdots, \alpha_N\}$ the elements of A_1, we define the sets E_{jk} by

$$E_{jk}=K\cap[F_k(\alpha_j)\times \prod_{\alpha\neq\alpha_j} T(\alpha)] \qquad (1\leqq j\leqq N,\ 1\leqq k\leqq n). \tag{10.3}$$

It is then easy to verify that so defined $\{E_{jk}\}$ satisfy both the required conditions (a) and (b).

Returning to the general case, suppose that G is any locally compact abelian group, and that K is any compact subset of it. We can find then a cardinal number Ω and a compact subset $\tilde{K}$ of the product group T^{Ω} for which there exists a homeomorphism s from $\tilde{K}$ onto K (cf. the proof of Theorem 2). Fixing any neighborhood U of $0\in G$, take a neighborhood $\tilde{U}$ of $0\in T^{\Omega}$ so that

$$\tilde{x}, \tilde{y}\in \tilde{K}, \quad \text{and} \quad \tilde{x}-\tilde{y}\in \tilde{U}\Rightarrow s(\tilde{x})-s(\tilde{y})\in U. \tag{10.4}$$

For T^{Ω} and this $\tilde{U}$, choose a natural number N as before. Then for any natural number n, we can find $N\times n$ compact subsets $\{\tilde{E}_{jk}\}$ of $\tilde{K}$ that satisfy (a) and (b) with K and $\{E_{jk}\}$ replaced by $\tilde{K}$ and $\{\tilde{E}_{jk}\}$. If we define E_{jk} to be $s(\tilde{E}_{jk})$ for $1\leqq j\leqq N$ and $1\leqq k\leqq n$, it is easy to see from (10.4) that these sets $\{E_{jk}\}$ have the required properties.

This completes the proof.

THEOREM 11. *Every quasi-Kronecker subset K of G is an SH-set.*

PROOF. We must prove that $P\in PM(K)$ implies $P\in M(K)$.

Fix any $P\in PM(K)$; we shall first show that for any compact subset $\hat{C}$ of $\hat{G}$ and $\varepsilon>0$ there exists a measure $\mu=\mu(\hat{C}, \varepsilon)\in M(K)$ such that

$$\|\mu\|\leqq\|P\|, \quad \text{and} \quad |\hat{\mu}(\gamma)-\hat{P}(\gamma)|\leqq\varepsilon(\|P\|+1) \qquad (\gamma\in\hat{C}). \tag{11.1}$$

To do this, take $\varepsilon>0$ and a compact subset $\hat{C}$ of $\hat{G}$, and put

$$U=U(\hat{C}, \varepsilon)=\{x\in G: \sup_{\gamma\in\hat{C}}|1-(x, \gamma)|<\eta(\varepsilon)\}, \tag{11.2}$$

which is a neighborhood of $0\in G$. Let $N=N(U)$ be a natural number as in Lemma 10. Since P has compact support, $\hat{P}$ is a uniformly continuous function on $\hat{G}$; it follows that there exists a neighborhood $\hat{V}$ of $\hat{0}\in\hat{G}$ such that

$$\sup_{\gamma-\gamma'\in\hat{V}}|\hat{P}(\gamma)-\hat{P}(\gamma')|<\varepsilon/2N. \tag{11.3}$$

Since $\hat{C}$ is compact, we can find finitely many elements of $\hat{C}$, say $\gamma_1, \gamma_2, \cdots, \gamma_r$ so that

$$\hat{C}\subset\bigcup_{i=1}^{r}(\gamma_i+\hat{V}). \tag{11.4}$$

Let us now take a positive integer M with $\|P\|<M\varepsilon/2N$, and put $n=rM$.

There exist $N\times n$ compact subsets $\{E_{jk}\}$ $(1\leqq j\leqq N,\ 1\leqq k\leqq n)$ of K satisfying the conditions (a) and (b) in Lemma 10. Since the sets $\{E_{1k}\}_1^n$ are pairwise disjoint, Lemma 9 applies, and we can find n pseudo-measures $\{P_k\}_1^n$ so that:

$$(11.5)\qquad \begin{cases} P_k\in PM(E_{1k}), \qquad P-P_k\in PM(\overline{K\setminus E_{1k}}), \\ \left\|\sum_{k=1}^{n} P_k\right\|\leqq \|P\|, \ \text{ and } \ \|P-P_k\|\leqq\|P\| \end{cases} \qquad (1\leqq k\leqq n);$$

(11.6) For any neighborhood $\hat{U}$ of $\hat{O}\in\hat{G}$, we have

$$\sup_{\gamma-\gamma'\in\hat{U}}|\hat{P}_k(\gamma)-\hat{P}_k(\gamma')|\leqq \sup_{\gamma-\gamma'\in\hat{U}}|\hat{P}(\gamma)-\hat{P}(\gamma')|,$$

and

$$\sup_{\gamma-\gamma'\in\hat{U}}|(P-P_k)\hat{}(\gamma)-(P-P_k)\hat{}(\gamma')|\leqq \sup_{\gamma-\gamma'\in\hat{U}}|\hat{P}(\gamma)-\hat{P}(\gamma')|$$

for all $k=1, 2, \cdots, n$.

We then claim that $\sup_{\gamma\in\hat{C}}|\hat{P}_k(\gamma)|<\varepsilon/N$ for at least one k $(1\leqq k\leqq n)$. Otherwise, there exist n elements $\{\gamma_k'\in\hat{C}\}_1^n$ with $|\hat{P}_k(\gamma_k')|\geqq\varepsilon/N$ for all k $(1\leqq k\leqq n)$. It follows from (11.4) that some $\gamma_i+\hat{V}$, say $\gamma_1+\hat{V}$, contains M elements of the set $\{\gamma_k'\}_1^n$, say $\gamma_1', \gamma_2', \cdots, \gamma_M'$ (note that $n=rM$). Therefore we have by (11.3) and (11.6)

$$\begin{aligned}|\hat{P}_k(\gamma_1)| &\geqq |\hat{P}_k(\gamma_k')|-|\hat{P}_k(\gamma_k')-\hat{P}_k(\gamma_1)| \\ &\geqq \varepsilon/N-\sup_{\gamma-\gamma'\in\hat{V}}|\hat{P}_k(\gamma)-\hat{P}_k(\gamma')| \\ &\geqq \varepsilon/2N \qquad (1\leqq k\leqq M).\end{aligned}$$

This, combined with Lemma 8 and (11.5), shows

$$\|P\|\geqq\left\|\sum_{k=1}^{n}P_k\right\|\geqq\sum_{k=1}^{n}|\hat{P}_k(\gamma_1)|\geqq\sum_{k=1}^{M}|\hat{P}_k(\gamma_1)|\geqq M\varepsilon/2N,$$

which contradicts our choice of M. Thus there exists an integer $k(1)$ $(1\leqq k(1)\leqq n)$ with $\sup_{\gamma\in\hat{C}}|\hat{P}_{k(1)}(\gamma)|<\varepsilon/N$. Putting $P_1'=P_{k(1)}$, we have a decomposition of P such that:

$$(11.7)\qquad \begin{cases} P=(P-P_1')+P_1', \qquad \|P-P_1'\|\leqq\|P\|, \\ \sup_{\gamma\in\hat{C}}|\hat{P}_1'(\gamma)|<\varepsilon/N, \qquad P-P_1'\in PM(\overline{K\setminus E_{1k(1)}}), \\ \sup_{\gamma-\gamma'\in\hat{V}}|(P-P_1')\hat{}(\gamma)-(P-P_1')\hat{}(\gamma')|<\varepsilon/2N. \end{cases}$$

Repeating the same arguments for $P-P_1'\in PM(\overline{K\setminus E_{1k(1)}})$ and the compact subsets $\{E_{2k}\cap(\overline{K\setminus E_{1k(1)}})\}_{k=1}^n$ of $\overline{K\setminus E_{1k(1)}}$, and so on, we can find N integers $\{k(j)\}_{j=1}^N$ with $1\leqq k(j)\leqq N$ and N pseudo-measures $\{P_j'\}_1^N$ so that:

(11.8) $$P=Q+\sum_{j=1}^{N} P'_j, \qquad \sup_{1\leqq j\leqq N}\{\sup_{\gamma\in\hat{C}} |\hat{P}'_j(\gamma)|\}<\varepsilon/N,$$

and

(11.9) $$\|Q\|\leqq\|P\|, \qquad \operatorname{supp} Q\subset \text{the closure of } \bigcap_{j=1}^{N} K\backslash E_{jk(j)}.$$

It then follows from (b) of Lemma 10 that there exist finitely many, pairwise disjoint, closed subsets $\{K_l\}_l$ of K such that

(11.10) $$\bigcap_{j=1}^{N} K\backslash E_{jk(j)}\subset\bigcup_l K_l, \quad \text{and} \quad \bigcup_l (K_l-K_l)\subset U.$$

Therefore we have a decomposition of Q of the form

(11.11) $$Q=\sum_l Q_l, \qquad Q_l\in PM(K_l) \qquad \text{for all } l.$$

Letting $\{x_l\in K_l\}_l$ be any choice of points, we now define

(11.12) $$\mu\in M(K) \quad \text{by} \quad \mu=\sum_l \hat{Q}_l(0)\delta(x_l),$$

where in general $\delta(x)$ denotes the unit mass at the point x. Observe then that $\|\mu\|\leqq\sum_l|\hat{Q}_l(0)|$, which together with (11.9), (11.11) and Lemma 8 gives $\|\mu\|\leqq\|Q\|\leqq\|P\|$. We have also by (11.8) and (11.12)

$$|\hat{\mu}(\gamma)-\hat{P}(\gamma)|\leqq|\sum_l(x_l,\gamma)\hat{Q}_l(0)-\hat{Q}(\gamma)|+\sum_{j=1}^{N}|\hat{P}'_j(\gamma)|$$
$$\leqq|\hat{Q}(\gamma)-\sum_l(x_l,\gamma)\hat{Q}_l(0)|+\varepsilon \qquad (\gamma\in\hat{C}).$$

This, combined with Lemma 8, (11.2), (11.10) and (11.11) shows

$$|\hat{\mu}(\gamma)-\hat{P}(\gamma)|\leqq\varepsilon\|Q\|+\varepsilon\leqq\varepsilon(\|P\|+1) \qquad (\gamma\in\hat{C}),$$

and we have proved the existence of a measure $\mu\in M(K)$ satisfying (11.1).

But it is clear that (11.1) implies that P is the Fourier-Stieltjes transform of a measure of $M(K)$, which follows at once from the fact that every closed (bounded) ball of $M(K)$ is weak-star compact.

This establishes the Theorem.

COROLLARY 12. *Every finite union of quasi-Kronecker sets is an SR-set.*

PROOF. This is evident from Theorem 11 and Corollary 5.

THEOREM 13 (cf. [5]). *Suppose that $\{K_j\}_0^n$ are $n+1$, pairwise disjoint, compact subsets of G such that:*

(13.1) *The set $\bigcup_{j=0}^{n} K_j$ is a quasi-Kronecker set;*

(13.2) *Any K_j contains no perfect subset $(1\leqq j\leqq n)$.*

Then the set $K_0+K_1+\cdots+K_n$ is an SR-set.

PROOF. We prove this by induction on n. When $n=0$, the statement is nothing but Theorem 11. Suppose that the conclusion of the Theorem holds with n replaced by $n-1$ for some natural number n, and that the sets $\{K_j\}_0^n$ satisfy the above conditions. Put then

$$L=K_0+K_1+\cdots+K_{n-1}, \quad \text{and} \quad D=K_n,$$

and let $W=\{1, 2, \cdots, \alpha, \alpha+1, \cdots\}$ be any well-ordered set having cardinal number larger than that of D. For any compact subset E of D, we shall define a family $\{E(\alpha);\ \alpha\in W\}$ of subsets of E as follows. Let $E(1)$ be the set of all accumulation points of E, and suppose that $E(\alpha)$ has already defined for every $\alpha\in W$ with $\alpha<\alpha_0$. We then define the set $E(\alpha_0)$ to be the set $\bigcap_{\alpha<\alpha_0} E(\alpha)$ if α_0-1 does not exist, and to be the set of all accumulation points of $E(\alpha_0-1)$ if α_0-1 exists. By transfinite induction, we obtain the family $\{E(\alpha);\ \alpha\in W\}$.

Suppose now that E is a closed subset of D. If $E(1)=\emptyset$, then E is finite, and so $L+E$ is a finite disjoint union of translates of L by (13.1). Since L is an SR-set by the hypothesis of the induction, it is easy to see that $L+E$ is an SR-set. We shall now fix $\alpha_0>1$ $(\alpha_0\in W)$ and assume that $L+E$ is an SR-set for every compact subset E of D with $E(\alpha)=\emptyset$ for some $\alpha<\alpha_0$.

Let us then take any closed subset E of D with $E(\alpha_0)=\emptyset$. In case that α_0-1 does not exist, then $E(\alpha_0)=\bigcap_{\alpha<\alpha_0} E(\alpha)$ by the definition of $E(\alpha_0)$; it follows that $E(\alpha)=\emptyset$ for some $\alpha<\alpha_0$, since each $E(\alpha)$, $\alpha\in W$, is compact, and since we have $E(\alpha)\supset E(\alpha')$ for all $\alpha, \alpha'\in W$ with $\alpha<\alpha'$. Thus $L+E$ is an SR-set by our hypothesis of the transfinite induction. If $\alpha_1=\alpha_0-1$ exists, then $E(\alpha_1)$ must be finite. Taking any closed subset F of $L+E$, $f\in I(F)$, and $P\in PM(F)$, we want to show that $fP=0$.

First of all we have

$$\text{supp}\, fP\subset F\cap(L+E(\alpha_1)). \tag{13.3}$$

In fact, let $u\in I_0(F\cap(L+E(\alpha_1)))$ be arbitrary; there exists an open set U such that $U\supset E(\alpha_1)$ and $(\text{supp}\, u)\cap(F\cap(L+U))=\emptyset$; we have then

$$\text{supp}\, uP\subset(\text{supp}\, u)\cap F\subset F\backslash(L+U)\subset L+(E\backslash U).$$

But $(E\backslash U)(\alpha_1)\subset E(\alpha_1)\backslash U=\emptyset$; it follows from our assumption that $L+(E\backslash U)$ is an SR-set, and so that we have $ufP=0$. Since $u\in I_0(F\cap(L+E(\alpha_1)))$ was arbitrary, this establishes (13.3). Note also that $L+E(\alpha_1)$ is an SR-set since $E(\alpha_1)$ is a finite subset of D.

Let $\varepsilon>0$ be arbitrary; there exists $f_\varepsilon\in A(G)$ with

$$\text{supp}\, f_\varepsilon\cap(F\cap(L+E(\alpha_1)))=\emptyset, \quad \text{and} \quad \|f-f_\varepsilon\|<\varepsilon. \tag{13.4}$$

Since $D=K_n$ contains no perfect subset by (13.2), D is totally disconnected;

thus E, as a compact subset of D, is 0-dimensional. Therefore we can find an open set U so that:

$$\begin{cases} U \supset E(\alpha_1), \quad \text{and} \quad E \cap U \text{ is compact;} \\ (\operatorname{supp} f_\varepsilon) \cap F \cap (L+U) = \emptyset . \end{cases} \tag{13.5}$$

For each $\eta > 0$, there exists $\gamma \in \hat{G}$ such that;

$$|\gamma-1| < \eta \quad \text{on} \quad (E \cap U) \cup \bigcup_{j=0}^{n-1} K_j ;$$
$$|\gamma+1| < \eta \quad \text{on} \quad E \backslash U ;$$

because of (13.1). Consequently we can find $\gamma_\varepsilon \in \hat{G}$ so that:

$$|\gamma_\varepsilon-1| < \eta(\varepsilon) \quad \text{on} \quad L+(E \cap U);$$
$$|\gamma_\varepsilon+1| < \eta(\varepsilon) \quad \text{on} \quad L+(E \backslash U).$$

This, together with (13.3), (13.4) and (13.5) gives

$$\begin{aligned} &\| fP-(\gamma_\varepsilon+1)(f-f_\varepsilon)P/2 \| \\ &\quad \leqq 2^{-1}\{\|(1-\gamma_\varepsilon)fP\|+\|(\gamma_\varepsilon+1)f_\varepsilon P\|\} \\ &\quad \leqq 2^{-1}\varepsilon(\|fP\|+\|f_\varepsilon P\|) \leqq \varepsilon(\|fP\|+\varepsilon\|P\|), \end{aligned}$$

and hence

$$\begin{aligned} \|fP\| &\leqq \varepsilon(\|fP\|+\varepsilon\|P\|)+\|(\gamma_\varepsilon+1)(f-f_\varepsilon)P/2\| \\ &\leqq \varepsilon(\|fP\|+\varepsilon\|P\|)+\varepsilon\|P\| . \end{aligned}$$

Letting $\varepsilon \to 0$, we have $fP=0$. Thus F is an S-set, and we have proved that $L+E$ is an SR-set for every compact subset E of D with $E(\alpha_0)=\emptyset$.

By transfinite induction, we see that $L+E$ is an SR-set for every compact subset E of D such that $E(\alpha)=\emptyset$ for some $\alpha \in W$. But it is easy to see that $D(\alpha)=\emptyset$ for some $\alpha \in W$, since D contains no perfect subset and since the cardinal number of W is larger than that of D. Thus the set $L+D=K_0+K_1+\cdots+K_n$ is an SR-set.

This completes the induction, and so establishes the Theorem.

We finish up this paper with:

THEOREM 14. *For n compact spaces $\mathcal{K}=\{K_j\}_1^n$, let $V=V(\mathcal{K})$ be the tensor algebra over the spaces $\mathcal{K}=\{K_j\}_1^n$ (for the definition, see* [**6**; p. 59]). *Then, if at least $n-1$ spaces K_j do not contain any perfect subsets, spectral synthesis holds in the algebra V.*

PROOF. Without loss of generality, we can and will assume that $\{K_j\}_1^n$ are pairwise disjoint compact subsets of some compact abelian group G such that their union is a Kronecker set (see the proof of Theorem 2). Then we can identify isometrically and algebraically V to the quotient algebra $A(\tilde{K})=$

$A(G)/I(\tilde{K})$, where $\tilde{K}=K_1+K_2+\cdots+K_n$ [**6**; p. 73]. Thus our statement follows at once from Theorem 13 (cf. [**6**; § 4]).

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References

[1] P. Malliavin, Ensembles de résolution spectrale, Proc. I.C.M. Stockholm, 1962, 368–378.
[2] W. Rudin, Fourier analysis on groups, New York, Interscience, 1962.
[3] S. Saeki, An elementary proof of a theorem of Henry Helson, Tôhoku Math. J., **20** (1968), 244–247.
[4] N. Th. Varopoulos, Sur les ensembles parfaits et les séries trigonométriques, C.R. Acad. Sci. Paris, **260** (1955), 3831–3834.
[5] N. Th. Varopoulos, Sur les ensembles parfaits et les séries trigonométriques, C. R. Acad. Sci. Paris, **260** (1955), 4668–4670.
[6] N. Th. Varopoulos, Tensor algebras and harmonic analysis, Acta Math., **119** (1967), 51–112.