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On the asymptotic behaviour of the Green operators for elliptic boundary problems and the pure imaginary powers of some second order operators

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§ 0. Introduction.

In this note we shall generalize the results of the author's previous papers [7] and [8] to the case of general elliptic boundary problems of even order.

Suppose X and Y are respectively smooth vector bundles over a compact oriented Riemannian manifold M and its boundary ∂M. Let A be an elliptic partial differential operator operating on smooth sections of X and let B be a boundary differential operator mapping sections of X to those of Y. We denote by A_B the closed extension of A considered under the homogeneous boundary condition $Bu=0$. Under a certain condition posed on the pair (A, B) (cf. § 3), we construct the Green operator $(A_B+z)^{-1}$ in § 4. Our expression of the operator $(A_B+z)^{-1}$ enables us to know the asymptotic behaviour of $(A_B+z)^{-1}$ when z tends to infinity along ray of minimal growth introduced in Agmon [1]. Using this, we obtain the asymptotic expansion of Trace e^{-tA} when $t\to 0$ and of Trace $(A_B+\lambda)^{-1}$ when $\lambda\to\infty$. In the latter case, we of course assume that the order of A is larger than the dimension of M.

The behaviour of the pure imaginary power $A_B^{\kappa i}$ of A_B is, in general, very delicate even in L^2-theory. The simplest case is treated in § 6. If A is a single second order principally real operator and if B is the linear combination of the Neumann and the Dirichlet condition, then we can prove that $A_B^{\kappa i}$ is a bounded operator in L^p $(1<p<\infty)$ space and its norm can be estimated using the above results. This enables us to determine the domain $D(A_B^\theta)$ of fractional power A_B^θ $(0<\theta<1)$ of A_B in L^p space. If B includes derivatives which are tangential to ∂M, $A_B^{\kappa i}$ is, in general, unbounded except for $\kappa=0$ even in L^2 space.

All these results are obtained by using a special class of pseudo-differential operators treated in [6].

Results similar to those presented in § 2, § 4 and § 5 were announced by several authors (Seeley [18], Shimakura, Asano and Arima).

§ 1. Pseudo-differential operators.

Given a C^∞ n-manifold M countable at infinity and a differentiable vector bundle X over M with fibre $\boldsymbol{C}^{l_1}$, we shall denote by $\mathcal{D}(M, X)$ the space of C^∞ sections of X with compact support and by $\mathcal{E}(M, X)$ the space of C^∞ sections of X. They are provided with usual topologies. When X is the trivial line bundle $\mathbf{1}_M$, we shall respectively use the notations $\mathcal{D}(M)$ and $\mathcal{E}(M)$ instead of $\mathcal{D}(M, \mathbf{1}_M)$ and $\mathcal{E}(M, \mathbf{1}_M)$. $\mathcal{S}(\boldsymbol{R}^m)$ and $\mathcal{S}'(\boldsymbol{R}^m)$ mean the space of rapidly decreasing C^∞ functions on $\boldsymbol{R}^n$ and its dual space. $\mathcal{O}_M(\boldsymbol{R}^m)$ stands for the space of slowly increasing C^∞ functions on $\boldsymbol{R}^m$. (cf. L. Schwartz [16]).

Let $\tilde{\omega}: M\times\boldsymbol{R}^m\to M$ be the projection. Then we denote the induced bundle of X by $\tilde{\omega}^{-1}X$. Then $\tilde{\omega}^{-1}X$ is isomorphic to the exterior tensor product $X\otimes\mathbf{1}_{R^m}$. We have $\mathcal{E}(M\times\boldsymbol{R}^m, \tilde{\omega}^{-1}X)=\mathcal{E}(M, X)\hat{\otimes}\mathcal{E}(\boldsymbol{R}^m)$, where $\mathcal{E}(M, X)\hat{\otimes}\mathcal{E}(\boldsymbol{R}^m)$ is the completion of $\mathcal{E}(M, X)\otimes\mathcal{E}(\boldsymbol{R}^m)$ by its projective topology (cf. A. Grothendieck [10]).

Now we assume that another vector bundle Y with fibre $\boldsymbol{C}^{l_2}$ is given over M.

DEFINITION 1.1. A continuous linear mapping $\mathcal{P}$ from $\mathcal{D}(M, X)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^m)$ into $\mathcal{E}(M, Y)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^m)$ is called a β-pseudo-differential operator of order z_0, if there is a sequence of complex numbers $\{z_j=s_j+it_j\}_{j=0}$ with real parts $s_j>s_{j+1}>\cdots>-\infty$ satisfying the following properties:

(i) For any $f\in\mathcal{D}(M, X)$ and compact set $\mathcal{K}$ in $\mathcal{E}(M)$ consisting of real elements g with $dg\neq 0$ on supp f, $e^{-i\lambda(g\rho+s\sigma)}\mathcal{P}(fe^{i\lambda(g\rho+s\sigma)})$ $(\lambda, \rho\in\boldsymbol{R}^1$ $s, \sigma\in\boldsymbol{R}^m)$ is the pull back of a section $p(f, g, x, \rho, \sigma, \lambda)$, $x\in M$ in $\mathcal{E}(M, Y)$.

(ii) There exist sections $p_j(f, g, x, \rho, \sigma)\in\mathcal{E}(M, Y)$ $(j=0, 1, 2, \cdots)$ such that for any integer $N>0$,

$$\lambda^{-s_N}\Big(p(f, g, x, \rho, \sigma, \lambda)-\sum_{j=0}^{N-1}p_j(f, g, x, \rho, \sigma)\lambda^{z_j}\Big)$$

is bounded in $\mathcal{E}(M, Y)\hat{\otimes}\mathcal{E}(S_1)$, when λ tends to infinity, where $S_1=\Big\{(\rho, \sigma)\in\boldsymbol{R}^{m+1};\ \frac{1}{2}\leqq\rho^2+|\sigma|^2\leqq 2\Big\}$.

As usual, here we have denoted

$$s\cdot\sigma=\sum_{j=1}^{m}s_j\sigma_j \quad\text{and}\quad |\sigma|=\Big(\sum_{j=1}^{m}\sigma_j^2\Big)^{1/2}.$$

We shall call the formal sum $\Sigma(\mathcal{P})(f, g)=\sum_{j=0}^{\infty}p_j(f, g, x, \rho, \sigma)\lambda^{z_j}$ the (f, g) symbol of $\mathcal{P}$. Clearly $p_j(f, g, x, \rho, \sigma)$ is homogeneous of degree z_j in g. $p_0(f, g, x, \rho, \sigma)$ has the following meaning. Let $dg(x)$ be the cotangent vector

1) In author's previous note [6], only the case $m=1$ was treated. However all of the discussion there are valid for $m\geqq 1$.

to M at x determined by g. Then $\rho\, dg(x)+\sigma\neq 0$ represents a cotangent vector to $M\times \boldsymbol{R}^m$ at (x, s). There is a linear map $\sigma(\mathcal{P})(x, \rho\, dg, \sigma)$ from X_x, the fibre of X over x, to Y_x such that

$$p_0(f, g, x, \rho, \sigma)=\sigma(\mathcal{P})(x, \rho\, dg, \sigma)f(x)\,.$$

We call $\sigma(\mathcal{P})$ the principal symbol of $\mathcal{P}$. Since $\sigma(\mathcal{P})$ is independent of s, we can consider $\sigma(\mathcal{P})$ as a section of the bundle $\Pi^{-1}\operatorname{Hom}(X\otimes \boldsymbol{1}_{R^m}, Y\otimes \boldsymbol{1}_{R^m})$ over $\mathfrak{S}=$ the set of non-zero elements in $T^*(M\times \boldsymbol{R}^m)$ over $M\times\{0\}$. Where Π is the projection of the cotangent bundle $T^*(M\times \boldsymbol{R}^m)$ of $M\times \boldsymbol{R}^m$.

Suppose both X and Y are trivial over a coordinate neighborhoods $U_1\subset M$. For any $\Psi\in\mathcal{D}(U_1)$ such that $\Psi\equiv 1$ in some open set $U_2\subset U_1$, and for any section v of $X_{|U}$ which is constant with respect to the trivialization of $X_{|U}$ and for the coordinate functions $(x_1, x_2, \cdots, x_n)$ valid in U_1 and $^\forall\xi\in \boldsymbol{R}^n$, $^\forall s, {}^\forall\sigma\in \boldsymbol{R}^m$, $e^{-i(x\cdot\xi+s\cdot\sigma)}P(\Psi v e^{i(x\cdot\xi+s\cdot\sigma)})$ depends linearly on v. So we write this as $e^{-i(x\cdot\xi+s\cdot\sigma)}P(\Psi v e^{(x\cdot\xi+s\cdot\sigma)})=P(\Psi\,;\,x, \xi, \sigma)v$. We call $P(\Psi\,;\,x, \xi, \sigma)$ the Fourier integral kernel of $\mathcal{P}\Psi$ with respect to the local coordinates. Clearly $P(\Psi\,;\,x, \xi, \sigma)$ has an asymptotic expansion

$$P(\Psi\,;\,x, \xi, \sigma)\sim \sum_j p_j(\Psi\,;\,x, \xi, \sigma)$$

which we shall call the symbol of $\mathcal{P}\Psi$ with respect to local coordinates. When $x\in U_2$, $p_j(\Psi, x, \xi, \sigma)$ is independent of Ψ, so we shall denote this by $p_j(x, \xi, \sigma)$, omitting the symbol Ψ.

Let $\{\Psi_j\}_{j=1,2,3,\dots}$ be a C_0^∞ partition of unity on M. Then $\mathcal{P}$ is a β-pseudo-differential operator if and only if the mappings $u\to\Psi_j\mathcal{P}\Psi_k u$, $u\in\mathcal{D}(M, X)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^m)$, are all β-pseudo-differential operators. Therefore, given a smooth section over the set $\mathfrak{S}$ of $\Pi^{-1}\operatorname{Hom}(X\otimes \boldsymbol{1}_{R^m}, Y\otimes \boldsymbol{1}_{R^m})$ homogeneous in the fibre, we can find a β-pseudo-differential operator $\mathcal{P}$ whose principal symbol coincides with it. A β-pseudo-differential operator $\mathcal{P}$ is called elliptic if $\sigma(\mathcal{P})$ is the section of $\Pi^{-1}\operatorname{Isom}(X\otimes \boldsymbol{1}_{R^m}, Y\otimes \boldsymbol{1}_{R^m})$. In addition to the results in [6], we shall use the following properties of β-pseudo-differential operators in the subsequent sections of this note.

Given a β-pseudo-differential operator $\mathcal{P}$ of order z_0 mapping $\mathcal{D}(M, X)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^m)$ to $\mathcal{E}(M, Y)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^m)$, it is clear that for any $\sigma''\in \boldsymbol{R}^{m-k}$, $0\leqq k\leqq m$, the mapping $f\to e^{-is''\sigma''}\mathcal{P}(fe^{is''\sigma''})$, is a β-pseudo-differential operator of order z_0 mapping $\mathcal{D}(M, X)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^k)$ into $\mathcal{E}(M, Y)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^k)$. We shall denote this by $\mathcal{P}_{\sigma''}$.

THEOREM 1.1. *If $\mathcal{P}$ is an elliptic β-pseudo-differential operator of order z_0 mapping $\mathcal{D}(M, X)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^m)$ to $\mathcal{E}(M, X)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^m)$ and if $|\sigma''|$ is larger than a certain constant C, then $\mathcal{P}_{\sigma''}^{-1}$ exists and is a β-pseudo-differential operator of order $-z_0$.*

Moreover if $\mathcal{P}(t)$ is a family of elliptic β-pseudo-diereffntial operators depend-

ing on a parameter t and if the asymptotic expansion (1.1) *is uniform in t, then we can so choose the above constant $C>0$ that C is independent of t and the asymptotic expansion of $\mathcal{P}_{\sigma''}(t)^{-1}$ of the form* (1.1) *is uniform in t.*

We shall give a proof of this theorem in the appendix.

Now, we shall treat behaviour at the boundary of β-pseudo-differential operators. We assume that $M=U\times I$, where U is an $n-1$ manifold and I is the interval $(-1, 1)$. We shall denote the generic point x in M as (x', x_n), $x'\in U$, $x_n\in I$.

Let $\mathcal{P}$ be a β-pseudo-differential operator of order z_0 mapping $\mathcal{D}(M, X)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^m)$ to $\mathcal{E}(M, X)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^m)$. Further we assume that when supp f is in a coordinate neighborhood Ω where coordinate functions $(x_1, \cdots, x_{n-1}, x_n)$ are valid, for fixed (x, ξ', σ) every term $p_j(x, \xi, \sigma)$ of the asymptotic expansion

$$e^{-i(x\cdot\xi+s\cdot\sigma)}\mathcal{P}(fe^{i(x\cdot\xi+s\cdot\sigma)})\sim\sum_j p_j(x, \xi, \sigma)$$

is a rational function of ξ_n in the upper half plane. We denote by δ_{x_n} the Dirac distribution in $\mathcal{D}'(I)$. Clearly for any $\phi\in\mathcal{D}(U, X)$, $\phi\otimes\delta_{x_n}$ is in $\mathcal{D}'(M, X)$.

We assert that

THEOREM 1.2. *If $\varphi\in\mathcal{D}(U, X_U)\hat{\otimes}\mathcal{X}$, where $\mathcal{X}$ is one of the spaces $\mathcal{S}'(\boldsymbol{R}^m)$, $\mathcal{O}_M(\boldsymbol{R}^m)$ and $\mathcal{S}(\boldsymbol{R}^m)$, then $\mathcal{P}(\varphi\otimes\delta_{x_n})$ is defined in $\mathcal{D}'(M, X)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^m)$ and there exists a distribution ψ in $\mathcal{E}(M, X)\hat{\otimes}\mathcal{X}$ such that*

$$\psi=\mathcal{P}(\varphi\otimes\delta_{x_n}) \quad \text{in} \quad U^+,$$

where $U^+=\{x\in M;\ x_n>0\}$.

This and the next Theorem 1.3 are analogues of Theorem 2.1.4 of Hörmander [**12**].

THEOREM 1.3. *Under the same hypothesis, the mapping*

$$\mathcal{D}(U, X)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^m)\ni\varphi\to\mathcal{P}(\varphi\otimes\delta_{x_n})|_{x_n=+0}\in\mathcal{E}(U, X)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^m)$$

is a β-pseudo-differential operator of order z_0+1. For any functions $\varphi_1, \varphi_2\in\mathcal{D}(M)$ such that coordinates $x=(x_1, \cdots, x_n)$ are valid in some neighbourhood of supp $\varphi_1\cup$ supp φ_2, *we have the asymptotic expansion*

$$e^{-i(x'\cdot\xi'+s\cdot\sigma)}D_n^\nu\varphi_1\mathcal{P}(\varphi_2\otimes\delta_{x_n}^{(\mu)}e^{-i(x'\cdot\xi'+s\cdot\sigma)})|_{x_n=+0}$$
$$\sim\sum_{j=0}^{\infty}\int_{\Gamma(x,\xi',\sigma)}(D_n+\xi_n)^\nu p_j(x', 0, \xi', \xi_n, \sigma)\xi_n^\mu d\xi_n$$

where p_j are given by the expansion

$$e^{-i(x\cdot\xi+s\cdot\sigma)}\varphi_1\mathcal{P}(\varphi_2e^{i(x\cdot\xi+s\cdot\sigma)})\sim\sum_{j=1}^{\infty}p_j(x, \xi, \sigma).$$

Proofs of Theorems 1.2, 1.3 will be given in the appendix.

§ 2. Elliptic operators on compact manifold without boundary.

We shall assume that $X=Y$, $l_1=l_2=l$, and that M is a compact oriented Riemannian manifold without boundary. We assume that X has a smooth hermitian metric. We denote this metric by $(u|v)$, $u, v \in X_x$.

In the following we consider an elliptic differential operator A of even order m, operating in $\mathcal{D}(M, X)$, satisfying the following

ASSUMPTION $(A\text{-}I)$: For any $x \in M$, $0 \neq \xi \in T_x^*(M)$ and any positive number z, $\sigma(A)(x, \xi)+zI$ is an isomorphism from X_x to X_x. Where I is the identity mapping in X_x.

REMARK 2.1. Since M is compact and $\sigma(A)(x, \xi)$ is homogeneous in ξ, if A satisfies the assumption $(A\text{-}I)$, then A satisfies the following assumption (A_Θ) with some Θ. $(\boldsymbol{A_\Theta})$: For any $x \in M$, $0 \neq \xi \in T_x^*(M)$ and for any z in the sector

$$\Sigma_\Theta = \{z \in \boldsymbol{C};\ |\arg z| < \Theta\}, \qquad 0 < \Theta < \pi,$$

$\sigma(A)(x, \xi)+zI$ is an isomorphism from X_x to X_x.

The aim of this section is to know the asymptotic behaviour of $(A+z)^{-1}$ when z tends to ∞ in the sector Σ_Θ. This was treated in [7] and [8]. (Seeley [17] and [18] treated general case in a little different manner.) In the following, we shall repeat it briefly. Let $\zeta(\tau)$ be a C^∞-function in $\boldsymbol{R}^3$ satisfying

$$\zeta(\tau) = \begin{cases} |\tau| & \text{when } |\tau| \geqq 1 \\ 0 & \text{when } |\tau| \leqq \frac{1}{2}. \end{cases}$$

We denote by $\zeta(D)$ the following operator

$$\mathcal{S}'(\boldsymbol{R}^3) \ni T \to (\zeta(\tau)\hat{T})^{\wedge -1} \in \mathcal{S}'(\boldsymbol{R}^3),$$

where $\hat{T}$ means the Fourier transform of T.

Now we assume that A satisfies A_Θ. Then with any z in a compact set in Σ_Θ,

$$\mathcal{A}^{(z)} = A + z(\zeta(D))^m \tag{2.1}$$

is an elliptic β-pseudo-differential operator of order m operating on $\mathcal{D}(M, X) \hat{\otimes} \mathcal{S}'(\boldsymbol{R}^3)$.

The principal symbol of $\mathcal{A}^{(z)}$ is, for any $(x, \xi, \sigma) \in \mathfrak{S}$,

$$\sigma(\mathcal{A}^{(z)})(x, \xi, \sigma) = \sigma(A)(x, \xi) + z|\sigma|^m I. \tag{2.2}$$

It is clear that the expansion of the form (1.1) is uniform in z. If $|\sigma_3|$ is large enough the operator $\mathcal{A}_{\sigma_3}^{(z)}$ defined in section 1 is a β-pseudo operator

$A+Z(-\Delta+\sigma_3^2)^{\frac{m}{2}}$, where Δ is the Laplacian operator on $\boldsymbol{R}^2$.

From Theorem 1.1, we have

THEOREM 2.1. *If $|\sigma_3|$ is larger than some constant >0, there is a β-pseudo-differential operator $(A+z(-\Delta+\sigma_3^2)^{\frac{m}{2}})^{-1}$ of order $-m$ operating on $\mathscr{D}(M, X)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^2)$. The asymptotic expansion of the form (1.1) for $(A+z(-\Delta+\sigma_3^2)^{\frac{m}{2}})^{-1}$ is uniform in z in some compact set in Σ_Θ.*

COROLLARY 2.2. *If τ_0 is large enough, the inverse $(A+z(D_s^2+\tau_0^2)^{\frac{m}{2}})^{-1}$ exists as a β-pseudo-differential operator of order $-m$ operating in $\mathscr{D}(M, X)\hat{\otimes}\mathcal{S}'(R^1)$. Where $D_s=-i\frac{\partial}{\partial s}$. The asymptotic expansions of the form (1.1) for $(A+z(D_s^2+\tau_0^2)^{\frac{m}{2}})^{-1}$ is uniform in z.*

COROLLARY 2.3. *If $\tau>0$ is larger than a constant, the Green operator $(A+z\tau^m)^{-1}$ exists.*

We denote the kernel function of the Green operator by $G(x, y, z\tau^m)$. Let $\mathscr{P}^{(z)}$ be any parametrix of $\mathscr{A}^{(z)}$, that is, $\mathscr{P}^{(z)}$ is a β-pseudo-differential operator operating on $\mathscr{D}(M, X)\hat{\otimes}\mathcal{S}'(R^3)$ satisfying $\Sigma(\mathscr{A}^{(z)}\mathscr{P}^{(z)})(f, g)=I$ or $\Sigma(\mathscr{P}^{(z)}\mathscr{A}^{(z)})(f, g)=I$. And let $\tilde{G}^{(z)}(x, g, \tau^m)$ be the kernel function of the mapping $\mathscr{P}_\sigma^{(z)}$ with $\sigma=(\tau, 0, 0)$. Then

THEOREM 2.4. *For any $x, y\in M$ and for any integer $N>0$, there is a constant $c>0$ such that*

$$|G(x, y, z\tau^m)-\tilde{G}^{(z)}(x, y, \tau^m)|\leqq c(|\tau|+1)^{-N}. \tag{2.3}$$

Given two points x and y on M, we choose a not necessarily connected neighbourhood U of x and y, where the bundle X is trivial. $\mathscr{A}^{(z)}$ is identified with $l\times l$ matrix-valued differential operator. Let $a(x, \xi, z^{\frac{1}{m}}\sigma)$ be a $l\times l$ matrix valued function of $x\in U$, $(\xi, \sigma)\in\boldsymbol{R}^{n+3}-\{0\}$, such that

$$a(x, \xi, z^{\frac{1}{m}}\sigma)\boldsymbol{v}=e^{-ix\cdot\xi}Ae^{ix\cdot\xi}\boldsymbol{v}+z|\sigma|^m\boldsymbol{v}$$

where $\boldsymbol{v}$ is a constant vector in $\boldsymbol{C}^l$ and $x\cdot\xi=\sum_i^n x_i\xi_i$ is a real linear function of coordinate function $x_1, \cdots, x_n$. We determine the formal sum $b(x, \xi, z^{\frac{1}{m}}\sigma)=\sum_{j=0}^\infty b_j(x, \xi, z^{\frac{1}{m}}\sigma)$ of $l\times l$ matrix valued functions $b_j(x, \xi, z^{\frac{1}{m}}\sigma)$ homogeneous of degree $-m-j$ in (ξ, σ) by Leibniz's rule

$$\sum_\alpha\frac{(i)^{|\alpha|}}{\alpha!}D_x^\alpha a(x, \xi, z^{\frac{1}{m}}\sigma)D_\xi^\alpha b(x, \xi, z^{\frac{1}{m}}\sigma)=I \tag{2.4}$$

or

$$\sum_\alpha\frac{(i)^{|\alpha|}}{\alpha!}D_\xi^\alpha a(x, \xi, z^{\frac{1}{m}}\sigma)D_x^\alpha b(x, \xi, z^{\frac{1}{m}}\sigma)=I, \tag{2.5}$$

where $\alpha=(\alpha_1, \cdots, \alpha_n)$ is a multi-index and D_x^α is the partial differential operator $\left(\frac{\partial}{i\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{i\partial x_n}\right)^{\alpha_n}$.

THEOREM 2.5. *Then the kernel function $G(x, y, z\sigma^m)$ has the estimate*

$$\left|G(x, y, z\sigma^m)-\frac{1}{(2\pi)^n}\sum_{j=0}^{N-1}\int_{R^n} b_j(x, \xi, z^{\frac{1}{m}}\sigma)e^{i\langle x-y\rangle\cdot\xi}d\xi\right|\leqq c(1+|\sigma|)^{s_N}$$

where the constant $c>0$ is independent of x, y, z.

Setting $x=y$, we have

THEOREM 2.6. *If $m>n$, we have the asymptotic expansions*

$$\text{trace}\, G(x, x, z\sigma^m)\sim\frac{1}{(2\pi)^n}\sum_{j=0}^{\infty}\sigma^{n-m-j}\times\int_{R^n}\text{trace}\, b_j(x, \xi, z^{\frac{1}{m}})d\xi \tag{2.7}$$

and

$$\text{trace}\,(A+z\sigma^m)\sim\frac{1}{(2\pi)^n}\sum_{j=0}^{\infty}\sigma^{n-m-j}\int_M b_j(x)d\mu(x)\,, \tag{2.8}$$

where $b_j=\int_{R^n}\text{trace}\, b_j(x, \xi, z^{\frac{1}{m}})d\xi$, *and $d\mu(x)$ is the Riemannian volume element.*

Proofs of Theorems 2.4, 2.5 and 2.6 are omitted because we can find them in [8].

Finally we must note that $b_j(x, \xi, z^{\frac{1}{m}})$ are holomorphic in $z\in\Sigma_\Theta$ because of Theorem 1.1.

§ 3. Formulation of boundary value problems.

We assume that M has the boundary ∂M and X also has the boundary ∂X.

We denote by $\hat{M}$ and $\hat{X}$ the copies of M and X respectively. Gluing X with $\hat{X}$ and M with $\hat{M}$ along their respective boundary, we have another vector bundle $\tilde{X}$ which we may assume endowed with hermitian metric (1). The bundle $\tilde{X}$ which we shall call the double of X satisfies the following properties:

(i) There is an isometric map $\Phi_1: X\to\tilde{X}$ which is linear in each fibre.

(ii) There is a diffeomorphism $\Phi_2: \tilde{X}\to\tilde{X}$ which is linear in each fibre.

(iii) $\Phi_1=\Phi_2\circ\Psi$ on ∂X, where Ψ is the natural isomorphism from X to $\hat{X}$.

The existence of $\tilde{X}$ is assured by the following fact: there is a neighbourhood V of ∂M such that V is diffeomorphic to $[0, 1)\times\partial M$ and the part of X over V is induced from $X|_{\partial M}$, the part of X over ∂M, by the retraction: $V\to\partial M$.

DEFINITION 3.1. A smooth linear elliptic partial differential operator A given on M is called uniformly elliptic if with some constant $c>0$ we have the estimate

$$c^{-1}|\xi|^m\|u\|_x\leqq\|\sigma(A)(x, \xi)u\|_x\leqq c|\xi|^m\|u\|_x \tag{3.1}$$

for any $u\in X_x$ and for any non-zero $\xi\in T_x^*(M)$. Where $\|u\|_x$ stands for the norm of u in the fibre X_x.

A Theorem of H. Whitney [19] proves the following.

PROPOSITION 3.2. *Given a uniformly elliptic partial differential operator A on $\mathcal{E}(M, X)$, we can find an elliptic partial differential operator on $\mathcal{E}(\tilde{M}, \tilde{X})$ which is an extension of A.*

Further, if A satisfies the condition (A_Θ), the extension also satisfies this.

In the following, we shall denote this extension by the same symbol A, which will not cause any confusion. The Theorem of H. Whitney or Calderón [3] also proves

PROPOSITION 3.3. *There is an extension mapping*

$$\Lambda: \mathcal{E}(M, X) \to \mathcal{E}(\tilde{M}, \tilde{X})$$

which is linear and continuous.

Recall that the principal symbol $\sigma(\mathcal{A}^{(z)})(x, \xi, \sigma)$, $(x, \xi, \sigma) \in \mathfrak{S}$, is an isomorphism in X_x. We shall denote its determinant by $\det \sigma(\mathcal{A}^{(z)})(x, \xi, \sigma)$. In addition to the condition $(A\text{-}I)$ we shall assume the following condition called the root condition. (cf. S. Agmon, A. Douglis and L. Nirenberg [2]).

Root condition (R). For any linearly independent $(\eta, 0)$ and $(\xi, \sigma) \in \mathfrak{S}$ $\det \sigma(\mathcal{A})(x, \xi+\tau\eta, \sigma)$ is a polynomial of τ of even degree 2ρ. The equation $\det \sigma(\mathcal{A}^{(z)})(x, \xi+\tau\eta, \sigma)=0$ has just ρ roots with positive imaginary parts. In the following we use a fixed z. So we omit z and write $\mathcal{A}^{(z)}$ as $\mathcal{A}$.

We shall respectively denote by $\tau_1^+(x, \xi, \sigma), \cdots, \tau_\rho^+(x, \xi, \sigma)$ and by $\tau_1^-(x, \xi, \sigma), \cdots, \tau_\rho^-(x, \xi, \sigma)$ the roots with positive imaginary part and with negative imaginary part.

We shall denote by $\mathfrak{N}$ the conormal bundle of ∂M with the orientation compatible with the unit inner conormal vector ν_x at $x \in \partial M$. Then we can define the following polynomials in τ,

$$L^+(x, \xi, \sigma, \tau) = \prod_{j=1}^{\rho} (\tau - \tau_j^+(x, \xi, \sigma)) \tag{3.4}$$

$$L^-(x, \xi, \sigma, \tau) = \prod_{j=1}^{\rho} (\tau - \tau_j^-(x, \xi, \sigma)). \tag{3.5}$$

Setting

$$L^+(x, \xi, \sigma, \tau) = \tau^\rho + a_1^+ \tau^{\rho-1} + \cdots + a_\rho^+, \tag{3.6}$$

and

$$L^-(x, \xi, \sigma, \tau) = \tau^\rho + a_1^- \tau^{\rho-1} + \cdots + a_\rho^-, \tag{3.7}$$

we obtain functions $a_j^+(x, \xi, \sigma)$ and $a_j^-(x, \xi, \sigma)$ which are homogeneous in (ξ, σ) of degree j. Further a_j^+ and a_j^- are C^∞ in (x, ξ, σ) if $(\xi, \sigma) \neq 0$. This is because $\sigma(\mathcal{A})(x, \xi+\tau\nu, \sigma)$ has no real roots.

Using a_j^+ and a_j^-, we can define for any k, $0 \leqq k \leqq \rho-1$, (3.8) $$L_k^+(x, \xi, \sigma, \tau)=\tau^k+a_1^+\tau^{k-1}+\cdots+a_k^+ ,$$

and

(3.9) $$L_k^-(x, \xi, \sigma, \tau)=\tau^k+a_1^-\tau^{k-1}+\cdots+a_k^- .$$

$L_k^+(x, \xi, \sigma, \tau)$ and $L_k^-(x, \xi, \sigma, \tau)$ are homogeneous in (ξ, σ, τ) of degree k.

Now we can formulate our boundary conditions. Let us denote by Y_j $(j=1, 2, \cdots, k)$ vector bundle over ∂M and by B_j $(j=1, 2, \cdots, k)$ boundary differential operators of order μ_j mapping $\mathcal{D}(\hat{M}, \hat{X})$ to $\mathcal{D}(\partial M, Y_j)$. We define $\mu=\max_j \mu_j$. If $\mu_j<\mu$, we consider $(1-\Delta)^{\frac{\mu-\mu_j}{2}} B_j$ instead of B_j. Where Δ is the Laplacian operator associated with the Riemannian structure of $\partial M\times \boldsymbol{R}^3$. So we may assume all B_j's are of order μ. Set $Y=Y_1\oplus\cdots\oplus Y_k$ and $\mathcal{B}=B_1\oplus\cdots\oplus B_k$, then we assume

(B-1) $$\dim Y=\rho .$$

Now introduce the cofactor matrix $\tilde{A}(x, \xi, \sigma, \tau)$ of $\sigma(\tilde{\mathcal{A}})(x, \xi+\tau\nu, \sigma)$ for any (x, ξ, σ) in $\mathfrak{X}=T^*(\partial M)\oplus \boldsymbol{R}^3-\{0\}$. Considering $\sigma(\mathcal{B})(x, \xi+\tau\nu)\tilde{A}(x, \xi, \sigma, \tau)$, $x\in\partial M$, as a $\mathrm{Hom}(X_x, Y_x)$ valued polynomial in τ, we can find a polynomial H such that

(3.10) $$\sigma(B)(x, \xi+\tau\nu)\tilde{A}(x, \xi, \sigma, \tau)=H(x, \xi, \sigma, \tau)$$

modulo $L^+(x, \xi, \sigma, \tau)$. It is clear that $H(x, \xi, \sigma, \tau)$ is homogeneous in (ξ, σ, τ) of degree $\mu+2\rho-m$ and for fixed (ξ, σ) in $|\xi|^2+|\sigma|^2=1$, $H(x, \xi, \sigma, \tau)$ is rational in τ in the upper complex half-plane. We regard $H(x, \xi, \sigma, \tau)$ as an element in $\sum_{j=0}^{\rho-1}\mathfrak{N}_x^j\otimes \mathrm{Hom}(X_x, Y_x)$ where $\mathfrak{N}_x$ is the j-tensor product $\overbrace{\mathfrak{N}_x\otimes\cdots\otimes\mathfrak{N}_x}^{j}$ of $\mathfrak{N}_x$. Therefore $H(x, \xi, \sigma, \tau)$ defines a linear map from $\sum_{j=0}^{\rho-1}\mathfrak{N}_x^{-j}\otimes X_x$ to Y_x, where $\mathfrak{N}_x^{-j}$ is the j-tensor product $\mathfrak{N}_x^{-1}\otimes\cdots\otimes\mathfrak{N}_x^{-1}$ of normal vector space $\mathfrak{N}_x^{-1}$. Especially, setting $\tau=1$, we have for ${}^\forall(x, \xi, \sigma)\in\mathfrak{X}$

$$H(x, \xi, \sigma, 1): \sum_{j=0}^{\rho-1}\mathfrak{N}_x^{-j}\otimes X_x\rightarrow Y_x .$$

Our assumption on the boundary operator $\mathcal{B}$ is the condition (B-1) and (B-2). The mapping

$$H(x, \xi, \sigma, 1): \sum_{j=0}^{\rho-1}\mathfrak{N}_x^{-j}\otimes X_x\rightarrow Y_x$$

is onto when $(x, \xi, \sigma)\in\mathfrak{X}_1$.

This was called complementing condition in S. Agmon, A. Douglis and L. Nirenberg [2].

Now we shall explain the condition (B-2) in the rest of this section. First we define $H_j(x, \xi, \sigma)$ by

$$H(x, \xi, \sigma, \tau)=\sum_{j=0}^{\rho-1} H_j(x, \xi, \sigma)\tau^{\rho-1-j}. \tag{3.11}$$

$H_j(x, \xi, \sigma)$ is a $\mathrm{Hom}(X_x, Y_x)$ valued function in $(x, \xi, \sigma)\in\mathfrak{T}$ which is homogeneous in (ξ, σ) of degree $\mu+\rho-m+j+1$. The mapping $H(x, \xi, \sigma, 1)$ maps an element

$$u=u_{\rho-1}\oplus(u_{\rho-2}\otimes\nu^*)+\cdots+(u_0\otimes\nu^{*\rho-1})$$

into

$$Hu=H_{\rho-1}(u_{\rho-1})+H_{\rho-2}(u_{\rho-2})+\cdots+H_0(u_0). \tag{3.12}$$

where ν^* stands for the unit outer normal to ∂M at x.

This mapping is given also in the following manner.

$$Hu=\frac{1}{2\pi i}\int_\gamma \frac{H(x, \xi, \sigma, \tau)}{L^+(x, \xi, \sigma, \tau)}\Big(\sum_{j=0}^{\rho-1} L_j^+(x, \xi, \sigma, \tau)u_j\Big)d\tau \tag{3.13}$$

where γ is a complex contour enclosing τ_j^+, $(1\leqq j\leqq \rho)$. In fact, the right hand side of (3.13) is equal to

$$\frac{1}{2\pi i}\sum_{j,k=0}^{\rho-1}\int_\gamma \frac{H_k(x, \xi, \sigma)}{L^+(x, \xi, \sigma, \tau)}\tau^{\rho-1-k}L_j^+(x, \xi, \sigma, \tau)u_j d\tau=\sum_j H_j(x, \xi, \sigma)u_j. \tag{3.14}$$

Because

$$\frac{1}{2\pi i}\int_\gamma \tau^{\rho-1-k}\frac{L_j^+(x, \xi, \sigma, \tau)}{L^+(x, \xi, \sigma, \tau)}d\tau=\delta_j^k, \tag{3.15}$$

where δ_j^k is the Kronecker's symbol.

Let $Z(x, \xi, \sigma)$ be the kernel of $H(x, \xi, \sigma, 1)$, that is we have the exact sequence:

$$0\longrightarrow Z(x, \xi, \sigma)\longrightarrow\sum_{j=0}^{\rho-1}\mathfrak{N}_x^{-j}\otimes X_x \overset{H(x,\xi,\sigma,1)}{\longrightarrow} Y_x\longrightarrow 0.$$

The complementing condition assures that $\mathfrak{Z}=\bigcup_{(x,\xi,\sigma)} Z(x, \xi, \sigma)$ is a vector subbundle of $\Pi^{-1}\Big(\sum_{j=0}^{\rho-1}\mathfrak{N}^{-j}\otimes X\Big)$ over $\mathfrak{T}$. Since $\mathfrak{N}^{-j}\otimes X$ is endowed with the metric, we have a splitting

$$\Pi^{-1}\Big(\sum_{j=0}^{\rho-1}\mathfrak{N}^{-j}\otimes X\Big)=Z\oplus W \tag{3.16}$$

over $\mathfrak{T}$ and the lift up

$$K(x, \xi, \sigma):\ \Pi^{-1}Y\to\Pi^{-1}\Big(\sum_{j=0}^{\rho-1}\mathfrak{N}^{-j}\otimes X\Big).$$

(3.17)

$$H(x, \xi, \sigma, 1)K(x, \xi, \sigma)=I.$$

Decompose $K(x, \xi, \sigma)$ into components and we obtain for any $u\in Y_x$

$$K(x, \xi, \sigma)u=(K_{\rho-1}(x, \xi, \sigma)u, \cdots, K_0(x, \xi, \sigma)\otimes\nu^{*\rho-1}).$$

$K_j(x, \xi, \sigma)$ maps Y_x to X_x. And

$$\sum_{j=0}^{\rho-1} H_j(x, \xi, \sigma, 1)K_j(x, \xi, \sigma)=I \tag{3.18}$$

$K_j(x, \xi, \sigma)$ is a homogeneous function in (ξ, σ) of degree $-(\mu+\rho-m+j+1)$.

For any $x\in\partial M$ let us consider the following ordinary differential equations.

$$\sigma(\mathcal{A})(x, \xi+D_t, \sigma)u(t)=0\,, \qquad t>0 \tag{3.19}$$

$$\sigma(B)(x, \xi+D_t, \nu)u|_{t=+0}=v\,, \qquad v\in Y_x \tag{3.20}$$

where $D_t=\frac{1}{i}\frac{d}{dt}$

PROPOSITION 3.4. *The complementing condition* (B-I) *is equivalent to the fact that the exponentially decaying solution of* (3.19) (3.20) *exists uniquely for every* $v\in Y_x$.

Proof is omitted here. See S. Agmon, A. Douglis and L. Nirenberg [2]. However it is important to note that the solution of (3.19) and (3.20) is given by

$$u(t)=\frac{1}{2\pi i}\int_\gamma \frac{\tilde{A}(x, \xi, \sigma, \tau)}{L^+(x, \xi, \sigma, \tau)} \times\Big(\sum_{j=0}^{\rho-1} L_j^+(x, \xi, \sigma, \tau)K_j(x, \xi, \sigma)v\Big)e^{it\tau}d\tau\,, \tag{3.21}$$

where γ is a complex contour enclosing the roots of $L^+(x, \xi, \sigma, \tau)$. We shall verify this.

$$\begin{aligned}&\sigma(\mathcal{A})(x, \xi+D_t\nu, \sigma)u(t)\\ &=\frac{1}{2\pi i}\int_\gamma \frac{\det\sigma(\mathcal{A})(x, \xi+\tau\nu, \sigma)}{L^+(x, \xi, \sigma, \tau)}\Big(\sum_{j=0}^{\rho-1} L_j(x, \xi, \sigma, \tau)K_j(x, \xi, \sigma)v\Big)e^{it\tau}d\tau\\ &=0\end{aligned}$$

and we have

$$\begin{aligned}&\sigma(B)(x, \xi+D_+\nu)u|_{t=+0}\\ &=\frac{1}{2\pi i}\int_\gamma \frac{B(x, \xi+\tau\nu)\tilde{A}(x, \xi, \sigma, \tau)}{L^+(x, \xi, \sigma, \tau)}\Big(\sum_{j=0}^{\rho-1} L_j^+(x, \xi, \sigma, \tau)K_j(x, \xi, \sigma)v\Big)d\tau\\ &=\frac{1}{2\pi i}\int_\gamma \frac{H(x, \xi, \sigma, \tau)}{L^+(x, \xi, \sigma, \tau)}\Big(\sum_{j=0}^{\rho-1} L_j^+(x, \xi, \sigma, \tau)K_j(x, \xi, \sigma)v\Big)d\tau\\ &=v\,.\end{aligned} \tag{3.22}$$

Here we have used (3.10), (3.13) and (3.18).

§ 4. The Poisson kernels and the Green kernels.

In § 3 we considered vector bundles $\mathfrak{N}$, Y, etc. and homomorphisms over ∂M. We shall extend these to some neighbourhood of ∂M in $\tilde{M}$.

We choose such a neighbourhood U of ∂M in $\tilde{M}$ that U is diffeomorphic to $\partial M\times I$, $I=(-1,1)$ and that $U\cap M$ and $U\cap\hat{M}$ correspond respectively to $\partial M\times[0,1)$ and $\partial M\times(-1,0]$ by this diffeomorphism. We shall denote the generic point x in U as (x', x_n), $x'\in\partial M$ and $x_n\in(-1,1)$. Taking U sufficiently small we may assume further that the segments $x'=$ const. are orthogonal to ∂M and the vector bundles over U, $X|_U$ and $T^*(U)$ are induced by the retraction $U\to\partial M$ from $X|_{\partial M}$ and $T^*(M)|_{\partial M}$. Hence $X_U=X|_{\partial M}\otimes 1_I$. Since $T^*(M)|_{\partial M}=T^*(\partial M)\oplus\mathfrak{N}$ (Whitney sum), $T^*(U)$ also splits. Using the same symbols, we shall denote this as

$$T^*(U)=T^*(\partial M)\oplus\mathfrak{N}\,. \tag{4.1}$$

Since (4.1) holds over U, $\sigma(\mathcal{A})(x,\xi+\tau\nu,\sigma)$, $\tilde{A}(x,\xi,\sigma,\tau)$, $L^{\pm}(x,\xi,\sigma,\tau)$ and $L_k^{\pm}(x,\xi,\sigma,\tau)$, $k=0,\cdots,\rho-1$, are defined in $\mathfrak{S}_1=$ the part of $\mathfrak{S}$ over U. $a_j^{\pm}(x,\xi,\sigma)$ are defined in $\mathfrak{T}=T^*(\partial M)\oplus\boldsymbol{R}^3-\{0\}$. We shall define the induced bundle from Y by the retraction $U\to\partial M$ and denote it by the same symbol Y. The boundary operator $\mathcal{B}$ is also extended over U. Therefore $H(x,\xi,\sigma,\tau)$ is defined over $\mathfrak{S}_1$. $\sum_j\mathfrak{N}^j\otimes\mathrm{Hom}(X,Y)$ and $\sum_j\mathfrak{N}^{-j}\otimes\mathrm{Hom}(X,Y)$ are defined over U. Taking U small, we may assume that the complementing condition (B-II) holds over U. $H_j(x,\xi,\sigma)$ $j=0,1,\cdots,\rho-1$ are defined over $\mathfrak{T}$. The vector bundle Z is defined over $\mathfrak{T}$ and is a subbundle of $\Pi^{-1}(\sum_j\mathfrak{N}^{-j}\otimes X)$. The splitting (3.16) holds over $\mathfrak{T}$ and defines a bundle W over $\mathfrak{T}$ and a lift up

$$K(x,\xi,\gamma):\ \Pi^{-1}Y\to\Pi^{-1}\Big(\sum_{j=0}^{\rho-1}\mathfrak{N}^{-j}\otimes X\Big)$$

satisfying (3.17) over $\mathfrak{T}$. $K_j(x,\xi,\sigma)$ are defined and satisfy (3.18) over $\mathfrak{T}$.

Now we can construct the Poisson kernel for $A+Z|\sigma|^m$ and B.

Since $a_j^{\pm}(x,\xi,\sigma)$, $j=1,\cdots,\rho$, are defined over $\mathfrak{T}$ and homogeneous of degree j, we can define β-pseudo-differential operators $\mathcal{A}_j^{\pm}$ with the principal symbol $\sigma(\mathcal{A}_j^{\pm})=a_j^{\pm}(x,\xi,\sigma)$ operating on $\mathcal{D}(\partial M)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^3)$. Using these, we can define operators

$$\tilde{\mathcal{L}}^{+}=I\otimes D_n^{\rho}+\mathcal{A}_1^{+}\otimes D_n^{\rho-1}+\cdots+\mathcal{A}_\rho^{+}\otimes I \tag{4.2}$$

$$\tilde{\mathcal{L}}^{-}=I\otimes D_n^{\rho}+\mathcal{A}_1^{-}\otimes D_n^{\rho-1}+\cdots+\mathcal{A}_\rho^{-}\otimes I \tag{4.3}$$

$$\tilde{\mathcal{L}}_k^{+}=I\otimes D_n^{k}+\mathcal{A}_1^{+}\otimes D_n^{k-1}+\cdots+\mathcal{A}_k^{+}\otimes I \tag{4.4}$$

and

$$\tilde{\mathcal{L}}_k^{-}=I\otimes D_n^{k}+\mathcal{A}_1^{-}\otimes D_n^{k-1}+\cdots+\mathcal{A}_k^{-}\otimes I \tag{4.5}$$

where we denoted by D_n the partial differential operator $-i\frac{\partial}{\partial x_n}$. All of them map $\mathcal{D}(U, X)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^3)$ to $\mathcal{E}(U, X)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^3)$.

Let $\varphi(t)$ be a C^∞ function with carrier in $(-1, 1)$ and $\varphi(t)=1$ in $\left(-\frac{1}{2}, \frac{1}{2}\right)$. We define

$$\mathcal{L}^{\pm}=\varphi(x_n)\tilde{\mathcal{L}}^{\pm}\varphi(x_n)\,, \qquad \mathcal{L}_k^{\pm}=\varphi(x_n)\tilde{\mathcal{L}}_k^{\pm}\varphi(x_n)\,.$$

From Theorem 2.1 we have the β-pseudo-differential operator $(A+z(-\Delta+\sigma_3^2)^{\frac{m}{2}})^{-1}$ on $\mathcal{D}(\tilde{M}, \tilde{X})\otimes\mathcal{S}'(\boldsymbol{R}^2)$ if $|\sigma_3|$ is large. We denote this by $\mathcal{F}_{\sigma_3}$. We denote by $\mathcal{L}_{\sigma_3}^-$ the operator $\mathcal{L}_{(0,0,\sigma_3)}^-$ defined in § 1.

THEOREM 4.1. *$\mathcal{F}_{\sigma_3}\mathcal{L}_{\sigma_3}^-$ is an operator mapping $\mathcal{D}(\tilde{M}, \tilde{X})\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^2)$ to $\mathcal{D}(\tilde{M}, \tilde{X})\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^2)$. $\mathcal{F}_{\sigma_3}\mathcal{L}_{\sigma_3}^-$ has the following properties:*

(i) *For any $\phi\in\mathcal{D}(\partial M, \tilde{X}|_{\partial M})\otimes\mathcal{S}'(\boldsymbol{R}^2)$ and integer $k\geqq 0$,*

$$\operatorname{supp}(A+z(-\Delta+\sigma_3^2)^{\frac{m}{2}})\mathcal{F}_{\sigma_3}\cdot\mathcal{L}_{\sigma_3}^-(\phi\otimes\delta_{x_n}^{(k)})\subset\partial M\times\boldsymbol{R}^2\,.$$

(ii) *For any $\phi\in\mathcal{D}(\partial M, \tilde{X}|_{\partial M})\hat{\otimes}\mathcal{X}$, there is a $\psi\in\mathcal{D}(\tilde{M}, \tilde{X})\hat{\otimes}\mathcal{X}$ such that $\mathcal{F}_{\sigma_3}\cdot\mathcal{L}_{\sigma_3}^-(\phi\otimes\delta_{\partial M}^{(k)})=\psi$ on M, where $\mathcal{X}$ is the one of the spaces $\mathcal{S}'(\boldsymbol{R}^2)$, $\mathcal{O}_M(\boldsymbol{R}^2)$ and $\mathcal{S}'(\boldsymbol{R}^2)$.*

(iii) *the mapping*

$$\phi\to\mathcal{F}_{\sigma_3}\cdot\mathcal{L}_{\sigma_3}^-(\phi\otimes\delta_{x_n}^{(k)})|_{x_n=+0}$$

is a β-pseudo-differential operator mapping

$$\mathcal{D}(\partial M, X|_{\partial M})\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^2) \quad \text{to} \quad \mathcal{D}(\partial M, X|_{\partial M})\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^2)\,.$$

PROOF. We shall prove (i)

$$\begin{aligned}&(A+z(\Delta+\sigma_3^2)^{\frac{m}{2}})\mathcal{F}_{\sigma_3}\mathcal{L}_{\sigma_3}^-(\phi\otimes\delta_{x_n}^{(k)})\\ &=\mathcal{L}_3^-(\phi\otimes\delta_{x_n}^{(k)})\\ &=\sum_{j=0}^{\rho}\mathcal{A}_j^-(\phi)\otimes\varphi(x_n)\cdot D_n^{\rho-j}(\varphi(x_n)\cdot\delta_{x_n}^{(k)})+\phi D_n^{\rho}(\varphi(x_n)\delta_{x_n}^{(k)})\,.\end{aligned}$$

$$\operatorname{supp}(A+z(\Delta+\sigma_3^2)^{\frac{m}{2}})\mathcal{F}_{\sigma_3}\cdot\mathcal{L}_{\sigma_3}^-(\phi\otimes\delta_{x_n}^{(k)})\subset\partial M\,.$$

Our fundamental results are the following.

THEOREM 4.2. *If $\sigma''=(\sigma_2, \sigma_3)$ has large $|\sigma''|=(\sigma_2^2+\sigma_3^2)^{\frac{1}{2}}$, there are β-pseudo-differential operators $\mathcal{K}_j$ $j=0, 1, \cdots, \rho-1$ of order $-(\mu+\rho-m+j+1)$ mapping*

$$\mathcal{D}(\partial M, Y)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^1) \quad \text{to} \quad \mathcal{D}(\partial M, X|_{\partial M})\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^1)$$

such that

$$\mathcal{B}_{\sigma''}\mathcal{F}_{\sigma''}\mathcal{L}_{\sigma''}^-\Big(\sum_{j=0}^{\rho-1}\mathcal{L}_{j,\sigma''}^+(\mathcal{K}_j\otimes\delta_{x_n})\Big)\Big|_{x_n=+0}=I$$

and

$$(A+z(D_{s_1}^2+|\sigma''|^2)^{\frac{m}{2}})\mathcal{F}_{\sigma''}\mathcal{L}_{\sigma''}^{-}\sum_j \mathcal{L}_{j,\sigma''}^{+}(\mathcal{K}_j\otimes\delta_{x_n})=0 \qquad on \quad M$$

where $\mathcal{F}_{\sigma''}$ *and* $\mathcal{L}_{j,\sigma''}^{+}$ *respectively denote* $\mathcal{F}_{(0,\sigma'')}$, $\mathcal{L}_{j,(0,\sigma'')}^{+}$ *and* $D_{s_1}=\frac{1}{i}\frac{\partial}{\partial s_1}$.

PROOF. Let us recall that we can find a β-pseudo-differential operators $\tilde{\mathcal{K}}_j$, $j=0, 1, \cdots, \rho-1$ with the principal symbol $\sigma(\tilde{\mathcal{K}}_j)(x, \xi, \sigma)=\mathcal{K}_j(x, \xi, \sigma)$ operating

$$\mathcal{D}(\partial M, Y)\hat{\otimes}\mathcal{S}'(\mathbf{R}^3) \text{ into } \mathcal{D}(\partial M, X)\hat{\otimes}\mathcal{S}'(\mathbf{R}^3).$$

Then we have for ${}^\forall v\in Y_x$ and $\sigma'=(\sigma_1, \sigma_2)$

$$\begin{aligned}(4.7)\qquad &\sigma(\mathcal{B}_{\sigma_3}\mathcal{F}_{\sigma_3}\mathcal{L}_{\sigma_3}^{-}(\sum_j \mathcal{L}_{j,\sigma_3}^{+}(\mathcal{K}_{j,\sigma_3}\otimes\delta_{x_n})))(x, \xi, \sigma')v\\ &=\frac{1}{2\pi i}\int_\gamma \sigma(\mathcal{B}\ \mathcal{F}_{\sigma_3})(x, \xi, \sigma, \tau)\sum_j L_j^{+}(x, \xi, \sigma, \tau)K_j(x, \xi, \sigma)v\,d\tau\\ &=\frac{1}{2\pi i}\int_\gamma \frac{H(x, \xi, \sigma, \tau)}{L^{+}(x, \xi, \sigma, \tau)}\sum_j L_j^{+}(x, \xi, \sigma, \tau)K_j(x, \xi, \sigma)v\,d\tau\\ &=v.\end{aligned}$$

It follows that there is a β-pseudo-differential operator $\mathcal{K}'$ of degree <0 from

$$\mathcal{D}(\partial M, Y)\hat{\otimes}\mathcal{S}'(\mathbf{R}^2) \text{ to } \mathcal{D}(\partial M, X)\hat{\otimes}\mathcal{S}'(\mathbf{R}^2)$$

such that

$$\mathcal{B}_{\sigma_3}\mathcal{L}_{\sigma_3}^{-}(\sum_j \mathcal{L}_{j,\sigma_3}^{+}(\tilde{\mathcal{K}}_j\otimes\delta_{x_n}))=I+\mathcal{K}'.$$

Hence, it follows from Theorem 1.1 that if σ_2 is large enough, there is β-pseudo-differential operator $(I+\mathcal{K}'_{\sigma_2})^{-1}$ of order 0. If we define $\mathcal{K}_j$ as $\tilde{\mathcal{K}}_{j,\sigma''}(I+\mathcal{K}'_{\sigma_2})^{-1}$, then $\mathcal{K}_j$ satisfies the property required.

As a corollary to Theorem 4.1 and Theorem 1.2 we have

THEOREM 4.3. *Let* $\mathcal{X}$ *be one of the spaces,* $\mathcal{S}'(\mathbf{R}^1)$, $\mathcal{O}_M(\mathbf{R}^1)$ *and* $\mathcal{S}(\mathbf{R}^1)$ *and let* φ *be in* $\mathcal{D}(\partial M, X)\hat{\otimes}\mathcal{X}$. *Then there is a* $u\in\mathcal{D}(\tilde{M}, \tilde{X})\hat{\otimes}\mathcal{X}$ *such that*

$$u=\mathcal{F}_{\sigma''}\mathcal{L}_{\sigma''}^{-}\Big(\sum_{j=0}^{\rho-1}\mathcal{L}_{j,\sigma''}^{+}(\mathcal{K}_j\varphi\otimes\delta_{x_n})\Big)$$

on M.

DEFINITION 4.4. We shall call the map

$$\varphi\to\mathcal{F}_{\sigma''}\mathcal{L}_{\sigma''}^{-}\Big(\sum_{j=0}^{\rho-1}\mathcal{L}_{j,\sigma''}^{+}((\mathcal{K}_j\varphi)\otimes\delta_{x_n})\Big)$$

the Poisson operator and its kernel the Poisson kernel.

REMARK 4.5. Theorem 4.2 and Theorem 4.3 are very general. However it is not pleasant that we have used the splitting (3.16) which is not unique. If the vector bundle X is a line bundle, we can avoid this. In fact, in this case,

$$\dim \sum_{j=0}^{\rho-1} \mathcal{N}_x^j \otimes X_x = \rho = \dim Y_x .$$

The complementing condition (B-2) means that $H(x, \xi, \sigma, \tau)$ is an isomorphism. Therefore we must define K as $H(x, \xi, \sigma, \tau)^{-1}$.

Now we wish to solve the boundary value problems:

$$(4.8) \qquad \begin{cases} (A+z(D_{s_1}^2+|\sigma''|^2)^{\frac{m}{2}})u=f & \text{on} \quad M\times \boldsymbol{R}^1 \\ Bu|_{\partial M\times R^1}=\phi \end{cases}$$

$$f\in \mathcal{E}(\tilde{M}, X)\hat{\otimes} \mathcal{S}'(\boldsymbol{R}^1), \qquad \phi\in \mathcal{D}(\partial M, Y)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^1)$$

and

$$(4.9) \qquad \begin{cases} (A+z|\sigma|^m)v=g & \text{on} \quad M \\ Bv|_{\partial M}=\varphi & \text{on} \quad M \end{cases}$$

$$g\in \mathcal{E}(\bar{M}, X), \qquad \phi\in \mathcal{D}(\partial M, Y).$$

The following theorems follow from Theorems 4.2 and 4.3.

THEOREM 4.6. *Let Λ be the extension mapping from $\mathcal{E}(\bar{M}, X)$ to $\mathcal{D}(\tilde{M}, \tilde{X})$. Then*

$$u=\mathcal{F}_{\sigma''}(\Lambda\otimes I)f+\mathcal{F}_{\sigma''}\mathcal{L}_{\sigma''}^{-}\Big(\sum_{j=0}^{\rho-1}\mathcal{L}_{j,\sigma''}^{+}(\mathcal{K}_j(\phi-\mathcal{B}_{\sigma''}\mathcal{F}_{\sigma''}(\Lambda\otimes I)f|_{\partial M})\otimes\delta_{x_n})\Big)$$

gives the unique solution of (4.8).

COROLLARY 4.7.

$$u=\mathcal{F}_{\sigma''}(\Lambda\otimes I)f-\mathcal{F}_{\sigma''}\mathcal{L}_{\sigma''}^{-}\Big(\sum_{j=0}^{\rho-1}\mathcal{L}_{j,\sigma''}^{+}(\mathcal{K}_j(\mathcal{B}_{\sigma''}\mathcal{F}_{\sigma''}(\Lambda\otimes I)f|_{\partial M})\otimes\delta_{x_n})\Big)$$

gives the unique solution of the problem (4.8) *with* $\phi=0$.

THEOREM 4.8.

$$v=\mathcal{F}_{\sigma}(\Lambda g)+\mathcal{F}_{\sigma}\mathcal{L}_{\sigma}^{-}\Big(\sum_{j=0}^{\rho-1}\mathcal{L}_{j,\sigma}^{+}(\mathcal{K}_{j,\sigma_1}(\varphi-\mathcal{B}_{\sigma}\mathcal{F}_{\sigma}(\Lambda g))\otimes\delta_{x_n})\Big)$$

gives the unique solution of (4.9).

COROLLARY 4.9.

$$v=\mathcal{F}_{\sigma}(\Lambda g)-\mathcal{F}_{\sigma}\mathcal{L}_{\sigma}^{-}\Big(\sum_{j=0}^{\rho-1}\mathcal{L}_{j,\sigma}^{+}(\mathcal{K}_{j,\sigma_1}\mathcal{B}_{\sigma}\mathcal{F}_{\sigma}(\Lambda g)\otimes\delta_{x_n})\Big)$$

gives the unique solution of (4.9) *with* $\varphi=0$.

As to the proof of Theorem 4.6 and 4.8, the only point that we must prove is the uniqueness. However, this was done in S. Agmon [**1**]. So we omit it here.

DEFINITION 4.10. Let us denote by g the following mapping from $\mathcal{D}(\tilde{M}, \tilde{X})\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^1)$ to $\mathcal{D}(\tilde{M}, \tilde{X})\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^1)$:

$$u\rightarrow \mathcal{F}_{\sigma''}u-\mathcal{F}_{\sigma''}\mathcal{L}_{\sigma''}^{-}\Big(\sum_{j=0}^{\rho-1}\mathcal{L}_{j,\sigma''}^{+}((\mathcal{K}_j\mathcal{B}_{\sigma''}\mathcal{F}_{\sigma''}u)\otimes\delta_{x_n})\Big).$$

DEFINITION 4.11. We call the mapping defined in Corollary 4.9 the Green operator G_σ corresponding to the system

$$\{(A+z|\sigma|^m), B\}\,.$$

Or we call it simply the Green operator, if there is no fear of confusion.

In the next section we shall consider the kernel function of the Green operators, that is, the Green function.

REMARK 4.12. All the argument above is valid uniformly in $z\in\Sigma_\Theta$ because of Theorem 1.1.

§ 5. The asymptotic behaviour of Trace $(Z+A)^{-1}$ and Trace e^{-tA}.

We shall denote the Green function of $\{(A+z|\sigma|^m), B\}$ by $G(x, y, z|\sigma|^m)$, that is, for any $u, v\in\mathcal{E}(M, X)$

$$\langle G_\sigma u, v\rangle=\int_{M\times M}(G(x, y, z|\sigma|^m)u(y)|v(x))d\mu(x)d\mu(g)\,,$$

where $d\mu(x)$ is the measure on M associated with the Riemannian structure of M.

We shall divide G_σ into two parts.

$$G_\sigma=G_\sigma^{(1)}-G_\sigma^{(2)}\,, \tag{5.1}$$

$$G_\sigma^{(1)}u=\mathcal{F}_\sigma\Lambda u\,, \tag{5.2}$$

$$G_\sigma^{(2)}u=\mathcal{F}_\sigma\mathcal{L}_\sigma^-\Big(\sum_{j=0}^{\rho-1}\mathcal{L}_{j,\sigma}^+\mathcal{K}_{j,\sigma}(\mathcal{B}_\sigma\mathcal{F}_\sigma\Lambda)u\otimes\delta_{x_n}\Big)\,. \tag{5.3}$$

And we shall denote corresponding kernel functions by $G^{(1)}(x, y, z|\sigma|^m)$ and $G^{(2)}(x, y, z|\sigma|^m)$ respectively. Since $G_\sigma^{(1)}$ was considered in § 2, here we shall treat only $G_\sigma^{(2)}$.

Setting $\mathcal{P}_{\sigma''}=\mathcal{B}_{\sigma''}\mathcal{F}_{\sigma''}$,

$$q_{\sigma''}=\mathcal{L}_{\sigma''}^-\Big(\sum_{j=0}^{\rho-1}\mathcal{L}_{j,\sigma''}^+\mathcal{K}_j\Big)=\sum_i q_{\sigma'',(j)}D_n^{2\rho-1-j}$$

$$\mathcal{P}_\sigma=(\mathcal{P}_{\sigma''})_{(\sigma_1,0,0)}\quad\text{and}\quad Q_\sigma=(Q_{\sigma''})_{(\sigma_1,0,0)}\,,$$

we have, for any $u, v\in\mathcal{E}(M, X)$,

$$\begin{aligned}\langle G_\sigma^{(2)}u, v\rangle&=\langle\mathcal{F}_\sigma q_\sigma(\mathcal{P}_\sigma\Lambda u)\delta_{\partial M}, v\rangle\\&=\sum_j\int_{\partial M}((\mathcal{P}_\sigma)\Lambda u\,|\,q_\sigma\mathcal{F}_\sigma^*v)d\mu'(x')\end{aligned}$$

where $d\mu'(x)$ is the volume element of ∂M.

Let $\sum_{j=1}^{J}\Psi_j=1$ be the smooth partition of unity on M such that for any six functions of them, there is a coordinate neighbourhood (not necessarily connected) which contains the union of support of them.

For any double index $\alpha=(\alpha_1, \alpha_2)$ we shall denote the operator $\Psi_{\alpha_1}\mathcal{P}_\sigma\Psi_{\alpha_2}$ by ${}_\alpha\mathcal{P}_\sigma$. Let α, β and γ be arbitrary three double indices. We can choose coordinate functions $(x_1, \cdots, x_n)$ valid in some neighbourhood U of supports of six functions $\Psi_{\alpha_1}, \Psi_{\alpha_2}$, etc.. We denote by $\sqrt{g(x)}dx$ the volume element of M. Then we shall denote the Fourier integral kernels of ${}_\alpha\mathcal{P}_{\sigma''}\frac{1}{g}$, ${}_\gamma\mathcal{F}^*_{\sigma''}$ and ${}_\beta q_{\sigma''}$ with respect to local coordinates by

$$(5.7) \qquad {}_\alpha\mathcal{P}_{\sigma''}(x, \xi, \sigma_1) \sim \sum_{j=m-\mu}^{\infty} {}_\alpha\mathcal{P}_{\sigma'', -j}(x, \xi, \sigma_1)$$

$$(5.8) \qquad {}_\gamma f^*_{\sigma''}(x, \xi, \sigma_1) \sim \sum_{j=-m} {}_\gamma f^*_{\sigma'', -j}(x, \xi, \sigma_1) \qquad (|\xi|+|\sigma|\to\infty)$$

and

$$(5.9) \qquad {}_\beta q_{\sigma''}(x, \xi, \sigma_1) \sim \sum_{j=\mu-m+1}^{\infty} {}_\beta q_{\sigma'', -j}(x, \xi, \sigma_1) \qquad (|\xi'|+|\sigma_1|\to\infty)\,.$$

Set ${}_{\beta\gamma}\mathcal{R}_{\sigma''}={}_\beta q_{\sigma''\gamma}\mathcal{F}^*_{\sigma''}$. Then ${}_{\beta\gamma}\mathcal{R}_{\sigma''}$ also has Fourier integral kernel $r(x, \xi, \sigma_1)$ with respect to the local coordinates. And if $|\sigma_1|>1$, we have the asymptotic expansion;

$$r(x, \xi, \sigma_1) \sim \sum_{j,k,\alpha} \frac{(iD_\xi)^\alpha}{\alpha!} {}_\beta q_{\sigma'', -j}(x, \xi, \sigma_1)(D_x)^\alpha {}_\gamma f^*_{\sigma'', -k}(x, \xi, \sigma_1)$$
$$= \sum_{j=\mu+1}^{\infty} r_{-j}(x, \xi, \sigma_1)\,.$$

From these, we obtain

$$\int_{\partial M} (({}_\alpha\mathcal{P}_\sigma \Lambda u)(x', 0) \,|\, {}_{\beta\gamma}\mathcal{R}_{\sigma''}(x', 0)v d\mu'(x'))$$
$$=(2\pi)^{-2n}\int_{\partial M}\Big(\int_{R^n} {}_\alpha p_{\sigma''}(w', 0, \xi, \sigma_1)e^{iw'\cdot\xi'}\widehat{\Lambda u}(\xi)d\xi \,|$$
$$\int_{R^n} {}_{\beta\gamma} r_{\sigma''}(w', 0, \eta, \sigma_1)e^{iw'\cdot\eta}\hat{v}(\eta)d\eta\Big)d\mu'(w')\,.$$

Therefore

$$(5.11) \qquad \text{trace}\; {}_{\beta\gamma}\mathcal{R}_{\sigma''}\,{}_\alpha\mathcal{P}_\sigma$$
$$=(2\pi)^{-2n}\int_{R^n_+}dy\int_{\partial M}\sqrt{g(w')}dw'\int_{R^{n+n}}\text{trace}\, r^*_{\sigma''}(w', 0, \eta, \sigma_1)$$
$$\times {}_\alpha p_{\sigma''}(w', 0, \xi, \sigma_1)e^{i(w'-y)\xi}e^{-i(w'-y)\eta}d\eta\, d\xi$$
$$=(2\pi)^{-2n}\int_{R^n_+}dy\int_{\partial M}\sqrt{g(w')}dw'\int_{R^{2n}}\text{trace}\, r^*_{\sigma''}(w', 0, \eta, \sigma_1)$$
$$\times {}_\alpha p_{\sigma''}(w', 0, \xi, \sigma_1)e^{-iy\cdot\xi}e^{iy\cdot\eta}d\eta\, d\xi$$
$$=(2\pi)^{-n}\int_{\partial M}\sqrt{g(w')}dw'\int_0^\infty dy_n\int_{R^2}e^{iy_n(\eta_n-\xi_n)}d\xi_n d\eta_n$$
$$\int_{R^{n-1}}\text{trace}\, r^*_{\sigma''}(w', 0, \xi', \xi_n, \sigma_1)\,{}_\alpha p_{\sigma''}(w', 0, \xi', \eta_n, \sigma_1)d\xi'\,.$$

The last equality follows from Parseval's equality. Replacing (5.7) and (5.10) into (5.11) we have the asymptotic expansion

$$(5.12)\qquad \operatorname{trace}{}_{\beta r}\mathcal{R}_{\sigma''\,\alpha}p_\sigma = (2\pi)^{-n}\sum_{r=m+1}^{\infty}\sigma_1^{-r+n}\sum_{j+k=r}^{\infty}\int_{\partial M}\sqrt{g(w')}dw' \int_{R^n}\operatorname{trace} r_{-j}^*(w',0,\xi,1)\,{}_\alpha p_{\sigma'',-k}(w',0,\xi,1)d\xi\,.$$

On the other hand for any fixed $x\in M$ we can choose such partition of unity $\{\Psi_j\}_{j\in J}$ that $\Psi_1(x)\equiv 1$ in some neighbourhood of x. Then for any $\alpha=(\alpha_1,\alpha_2)$ which is not equal to $(1,1)$ we have ${}_\alpha p_{\sigma'',-j}(x,\xi,\sigma_1)=0$ and ${}_\alpha f_{\sigma'',-j}(x,\xi,\sigma_1)=0$ and ${}_\alpha q_{\sigma'',-j}(x,\xi,\sigma_1)=0$. Therefore we have proved

THEOREM 5.2. *When $m>n$ we have the following asymptotic expansion*

$$\operatorname{trace}(A+z(|\sigma''|^2+\sigma_1^2)^{\frac{m}{2}})^{-1}\sim\sum_{r=m}^{\infty}a_r\sigma_1^{n-r}-\sum_{r=m+1}^{\infty}b_r\sigma_1^{n-r}$$

a_j and b_j are calculated in the following manner. We can trivialize X and Y near an arbitrary point $x\in M$ and identify operators above with matrix valued β-pseudo-differential operators. We choose $\Psi(x)$ in $\mathcal{D}(M)$ which is identically 1 near x. We consider the asymptotic expansions:

$$(5.13)\qquad e^{-i(x\xi+s_1\sigma_1)}(A+z(|\sigma''|^2+D_{s_1}^2)^{\frac{m}{2}})^{-1}\Big(\frac{\Psi}{\sqrt{g}}e^{i(x\xi+s_1\sigma_1)}\Big)\sim\sum_{j=m}^{\infty}f_{\sigma'',-j}(x,\xi,\sigma_1)\,;$$

$$(5.14)\qquad e^{-i(x\xi+s_1\sigma_1)}\mathcal{B}_{\sigma''}\mathcal{F}_{\sigma''}\Big(\frac{\Psi(x)}{\sqrt{g(x)}}e^{i(x\xi+s_1\sigma_1)}\Big)\sim\sum_{j=m-\mu}^{\infty}p_{\sigma'',-j}(x,\xi,\sigma_1)\,;$$

$$(5.15)\qquad e^{-i(x\xi+s_1\sigma_1)}\Big(\sum_k\mathcal{F}_{\sigma''}\mathcal{L}_{\sigma''}^{-}\mathcal{L}_{k,\sigma''}^{+}\mathcal{K}_k\Big)(\Psi(x)e^{i(x\xi+s_1\sigma_1)})\sim\sum_{j=\mu+1}^{\infty}r_{\sigma'',-j}(x,\xi,\sigma_1)\qquad |\sigma_1|>1\,,\quad |\xi'|>D\,,$$

where $f_{\sigma'',-j}$, $p_{\sigma'',-j}$ and $q_{\sigma'',-j}$ are homogeneous in (ξ,σ_1) of degree $-j$. Then, we have

$$(5.16)\qquad a_r=\frac{1}{(2\pi)^n}\int_M\sqrt{g(x)}dx\int_{R^n}\operatorname{trace} f_r(x,\xi,1)d\xi$$

and

$$(5.17)\qquad b_r=\frac{1}{(2\pi)^n}\sum_{j+k=r}\int_{\partial M}\sqrt{g(w')}dw' \times\int_{R^n}\operatorname{trace} r_{\sigma'',-j}(w',0,\xi,1)p_{\sigma'',-k}(w',0,\xi,1)d\xi\,.$$

Now we assume $\Theta<\frac{\pi}{2}$. In this case the operator $-A$ considered under boundary condition B generates a strong continuous semigroup e^{-tA} of bounded operators in $L^2(M, X)$ which is holomorphic in $|\arg t| < \Theta$. (cf. S. Agmon [**1**], K. Yosida [**20**].) We shall treat the asymptotic behaviour of trace e^{-tA} when $t \to +0$. In the following, we omit the assumption $m > n$.

To do this, we must note that the operators G_σ, $\mathcal{F}_\sigma$, $\mathcal{P}_\sigma$, q_σ, ${}_\alpha\mathcal{P}_{\sigma''}$ and ${}_\beta q_{\sigma''}$ depend holomorphically on z in Σ_Θ, and that the asymptotic expansion of them of the type (1.1) are uniform in z when z remains in a compact set in Σ_Θ. To express the dependence of them on z, we denote respectively $G_{\zeta\sigma}$, $\mathcal{F}_{\zeta\sigma}$, $\mathcal{P}_{\zeta\sigma}$, $q_{\zeta\sigma}$, ${}_\alpha\mathcal{P}_{\zeta\sigma''}$, ${}_\beta q_{\zeta\sigma''}$, ${}_\alpha f_{\zeta\sigma'',-j}(x, \xi, \zeta\sigma_1)$, ${}_\alpha p_{\zeta\sigma'',-j}(x, \xi, \zeta\sigma_1)$ instead of G_σ, $\mathcal{F}_\sigma$, $\mathcal{P}_\sigma$, q_σ, ${}_\alpha\mathcal{P}_{\sigma''}$, ${}_\alpha q_{\sigma''}$, ${}_\alpha f_{\sigma'',-j}(x, \xi, \sigma)$, ${}_\alpha p_{\sigma'',-j}(x, \xi, \sigma_1)$ and ${}_\alpha q_{\sigma'',-j}(x, \xi, \sigma_1)$ with $\zeta = z^{\frac{1}{m}}$.

The operator e^{-tA} is given for any $u, v \in \mathcal{D}(M, X)$ by

$$\langle e^{-tA}u, v\rangle = \frac{1}{2\pi i}\int_\gamma e^{tz|\sigma|^m}\langle G_{\zeta\sigma}u, v\rangle\, d(z|\sigma|^m) \tag{5.18}$$

where γ is the complex contour $\{z|\sigma|^m;\ |\arg z|\sigma|^m - \tau_0| = \theta\}$ with $\frac{\pi}{2} < \theta < \pi - \Theta$ and sufficient large τ_0.

For any double index $\alpha = (\alpha_1, \alpha_2)$, we denote the operator $\Psi_{\alpha_1} G_{\zeta\sigma} \Psi_{\alpha_2}$ by ${}_\alpha G_{\zeta\sigma}$ and define functions ${}_\alpha g_{\sigma'',-j}(x, \xi, \zeta\sigma_1)$ by

$$\begin{aligned} (5.19)\qquad & e^{-i(x\xi + s_1\sigma_1)}\Psi_{\alpha_1}(A + z(|\sigma''|^2 + D_{s_1}^2)^{\frac{m}{2}})^{-1}\Big(\frac{\Psi}{\sqrt{g}}e^{i(x\xi + s_1\sigma_1)}\Big) \\ & = {}_\alpha g_{\zeta\sigma''}(x, \xi, \zeta\sigma) \sim \sum_{j=m}^{\infty} g_{\zeta\sigma'',-j}(x, \xi, \zeta\sigma_1)\,. \end{aligned}$$

Then we have

$$\begin{aligned} (5.20)\qquad & \operatorname{Trace}\frac{1}{2\pi i}\int_\gamma e^{tz|\sigma|^m}{}_\alpha G_{\zeta\sigma}^{(1)}d(z|\sigma|^m) \\ & = \frac{1}{(2\pi)^n}\int_M \sqrt{g}\, dx \\ & \quad \times \int_{R^n}\operatorname{trace}\Big[\frac{1}{(2\pi i)}\int_\gamma e^{tz|\sigma|^m}{}_\alpha g_{\zeta\sigma''}(x, \xi, \zeta\sigma)d(z|\sigma|^m)\Big]d\xi\,. \end{aligned}$$

Therefore we have the asymptotic expansion

$$\begin{aligned} (5.21)\qquad & \operatorname{Trace}\frac{1}{2\pi i}\int_\gamma e^{tz|\sigma|^m}{}_\alpha G_{\zeta\sigma}^{(1)}d(z|\sigma|^m) \sim \frac{1}{(2\pi)^n}\sum_{j=m}^{\infty}\int_M \sqrt{g(x)}\, dx \\ & \quad \times \int_{R^n}\operatorname{trace}\Big[\frac{1}{2\pi i}\int_\gamma e^{tz|\sigma|^m}g_{\zeta\sigma'',-j}(x, \xi, \zeta\sigma_1)d(z|\sigma|^m)\Big]d\xi \\ & = \frac{1}{(2\pi)^n}\sum_{j=n}^{\infty} t^{\frac{-n+j-m}{m}}\int_M \sqrt{g(x)}\, dx \\ & \quad \times \int_{R^n}\operatorname{trace}\Big[\frac{1}{2\pi i}\int_\gamma e^{z|\sigma|^m}g_{\zeta\sigma'',-j}(x, \xi, \zeta\sigma_1)d(z|\sigma|^m)\Big]d\xi \end{aligned}$$

where we have used that $g_{\zeta\sigma'',-j}(x, \xi, \zeta\sigma_1)$ is a rational function of $\zeta(\sigma)$. Similarly for any two double indices α, β, we have

$$\text{Trace}\,\frac{1}{2\pi i}\int_{\gamma} e^{tz|\sigma|^m}{}_\beta r^*_{\zeta\sigma}\,{}_\alpha p_{\zeta\sigma} d(z|\sigma|^m)$$

$$=(2\pi)^{-n}\int_{\partial M}\sqrt{g(w')}\,dw'$$

$$\times\int_{R^n}\text{trace}\Big[\frac{1}{2\pi i}\int_{\gamma} e^{tz|\sigma|^m} r_{\zeta\sigma''}(w',0,\xi,\zeta\sigma_1)_\alpha p_{\zeta\sigma''}(w',0,\xi,\zeta\sigma_1)d(z|\sigma|^m)\Big]\,d\xi$$

and we have the following asymptotic expansion

$$\text{Trace}\,\frac{1}{2\pi i}\int_{\gamma} e^{tz|\sigma|^m}{}_\beta r^*_{\zeta\sigma}\cdot{}_\alpha p_{\zeta\sigma}\, d(z|\sigma|^m)\sim(2\pi)^{-n}\sum_{r=m+1}^{\infty}\sum_{j+k=r}\int_{\partial M}\sqrt{g(w')}\,dw'$$

$$\times\int_{R^n}d\xi\,\text{trace}\Big[\frac{1}{2\pi i}\int_{\gamma} e^{tz|\sigma|^m}{}_\beta r^*_{\zeta\sigma'',-j}(w',0,\xi,\zeta\sigma_1)_\alpha p_{\zeta\sigma'',-k}(w',0,\xi,\zeta\sigma_1)d(z|\sigma|^m)\Big]$$

$$=(2\pi)^{-n}\sum_{r=m+1}^{\infty}t^{\frac{r-m-n}{m}}\sum_{j+k=r}\int_{\partial M}\sqrt{g(w')}\,dw'$$

$$\times\int_{R^n}d\xi\,\text{trace}\Big[\frac{1}{2\pi i}\int_{\gamma} e^{z|\sigma|^m}{}_\beta r^*_{\zeta\sigma'',-j}(w',0,\xi,\zeta\sigma_1)_\alpha p_{\zeta\sigma'',-k}(w',0,\xi,\zeta\sigma_1)d(z|\sigma|^m)\Big].$$

Therefore adding these terms, we have proved

THEOREM 5.3 *We have the following asymptotic expansion*

$$\text{Trace}\,e^{-tA}\sim\sum_{r=m}^{\infty}a_r t^{\frac{r-n-m}{m}}+\sum_{r=m+1}^{\infty}b_r t^{\frac{r-m-n}{m}},\qquad t\to 0$$

where

$$a_r=(2\pi)^{-n}\int_M\sqrt{g(x)}\,dx$$

$$\times\int_{R^n}\text{trace}\Big[\frac{1}{2\pi i}\int_{\gamma} e^{z|\sigma|^m} g_{\zeta\sigma'',-r}(x,\xi,\zeta\sigma_1)d(z|\sigma|^m)\Big]\,d\xi$$

$$b_r=(2\pi)^{-n}\sum_{j+k=r}\int_{\partial M}\sqrt{g(w')}\,dw'$$

$$\times\int_{R^n}d\xi\,\text{trace}\Big[\frac{1}{2\pi i}\int_{\gamma} e^{z|\sigma|^m} r^*_{\zeta\sigma'',-j}(w',0,\xi,\zeta\sigma_1)p_{\zeta\sigma'',-k}(w',0,\xi,\zeta\sigma_1)d(z|\sigma|^m)\Big].$$

§ 6. The pure imaginary power of some elliptic operators of the second order.

In this section we assume that X is a trivial line bundle and $\tilde{A}$ is an elliptic differential operator of the second order operating $\mathcal{E}(\tilde{M},\tilde{X})$. We assume that with respect to some coordinate system, $\tilde{A}$ is represented by

$$A=\sum_{ij}a_{ij}(x)D_iD_j+\sum_i a_i(x)D_i+a(x)\,,$$

where a_{ij} are real and $\sum a_{ij}\xi_i\xi_j$ is a positive definite quadratic form. Further we assume that the boundary operator is represented by either

$$Bu = D_n u(x) + b(x)u(x)$$

or

$$Bu = u(x) .$$

Let $L^p(M, X)$ $(1 < p < \infty)$ be the Banach space of L^p sections of X. It is well known that the operator A considered under the boundary condition $Bu|_{\partial M} = 0$ has minimal closed extension A_B in the space $L^p(M, X)$. Adding, if necessary, some large positive number to A_B, we may assume that the sector $|\arg(\tau+1)| \leqq \frac{3}{4}\pi$ belongs to the resolvent set of $-A_B$ in $L^p(M, X)$. For any u in the domain of A_B, we can define $(A_B+\tau_0)^\theta u$ $(-1 < \mathrm{Re}\,\theta < 0)$ by

$$(A_B+\tau_0)^\theta u = \frac{1}{2\pi i}\int_\gamma (-\lambda)^\theta(\lambda+\tau_0+A_B)^{-1} u\, d\lambda \tag{6.1}$$

where γ is a complex contour lying in the resolvent of A_B and enclosing $\left(-\infty, \frac{-\tau_0}{2}\right)$ and the branch of z^θ is so taken that $1^\theta = 1$. The aim of this section is to prove

THEOREM 6.1. *$(A_B+\tau_0)^{\kappa i}$, $\kappa \in \boldsymbol{R}^1$ is a bounded operator in $L^p(M, X)$. For any $\varepsilon > 0$, there is a constant $C > 0$ such that*

$$\|(A_B+\tau_0)^{\kappa i}\| < Ce^{\varepsilon\kappa} . \tag{6.2}$$

First we shall prepare a lemma.

LEMMA 6.2. *Let $\mathcal{P}$ be a pseudo-differential operator of order $i\kappa$, $\kappa \in \boldsymbol{R}$, operating $\mathcal{E}(\tilde{M}, \tilde{X})$. Then there exists a unique continuous extension of $\mathcal{P}$ operating in $L^p(\tilde{M}, \tilde{X})$ $(1 < p < \infty)$.*

PROOF. Let $p(x, \xi)$ be a function in $\mathcal{B}(\boldsymbol{R}^n)\hat{\otimes}\mathcal{O}_M(\boldsymbol{R}^n)$ and let $p_0(x, \xi)$ be a function in $\mathcal{B}(\boldsymbol{R}^n)\hat{\otimes}\mathcal{E}(\boldsymbol{R}^n-\{0\})$ homogeneous of degree $i\kappa$ in ξ. Further we assume that for any multi-index α; $|\alpha| \leqq n+1$, and $|\xi| > 1$, we have the estimate

$$|D_\xi^\alpha(p(x, \xi)-p_0(x, \xi))| \leqq M_1|\xi|^{-\gamma-|\alpha|} ,$$

with some $\gamma > 0$. Then the mapping T defined by

$$Tf(x) = (2\pi)^{-n}\int_{\boldsymbol{R}^n} p(x, \xi)\hat{f}(\xi)e^{ix\cdot\xi}d\xi \tag{6.3}$$

is a bounded map in $L^p(\boldsymbol{R}^n)$ and we have the estimate

$$\|Tf\| \leqq C_{n,p}(M_1+M_2)\|f\| \tag{6.4}$$

where

$$M_2 = \sup_{\substack{x\in \boldsymbol{R}^n \\ |\xi|=1,\, |\alpha|<2n}} |D_\xi^\alpha p_0(x, \xi)| . \tag{6.5}$$

Clearly lemma 6.2 follows from (6.4).

Now we shall prove (6.4).

Let $\varphi(t)$ be a C^∞-function on $t \geqq 0$, such that $\varphi(t)=0$ for $0 \leqq {}^\forall t \leqq \frac{1}{2}$, and $\varphi(t)=1$ on $t \geqq 1$. We can write

$$(6.6) \qquad Tf(x)=(2\pi)^{-n}\int_{R^n} p_0(x,\xi)\varphi(|\xi|)\hat{f}(\xi)e^{ix\cdot\xi}d\xi$$
$$+(2\pi)^{-n}\int_{R^n}(p(x,\xi)-p_0(x,\xi)\varphi(|\xi|))\hat{f}(\xi)e^{ix\cdot\xi}d\xi\,.$$

Let $Y_{nm}(\xi)$ be the spherical harmonics of degree r. Then we can expand $p_0(x,\xi)\varphi(|\xi|)$ as

$$p_0(x,\xi)\varphi(|\xi|)=\sum_{nm} a_{nm}(x)|\xi|^{i\kappa}\varphi(|\xi|)Y_{nm}(\xi)\,.$$

Using Mikhilin's theorem on the Fourier multiplier (cf. Mikhilin [**14**]), we can estimate the first term of (6.6).

Set

$$p'(x,z)=(2\pi)^{-n}\int_{R^n}(p(x,\xi)-p_0(x,\xi)\varphi(|\xi|))e^{iz\cdot\xi}d\xi$$

then

$$|p(x,z)| \leqq C_{n,p}M_1(1+|z|)^{-n-1}$$

and we can estimate the second term of (6.6).

Note that both the mappings $v \to \mathcal{F}_{\zeta\sigma}v$ and

$$v \to \mathcal{F}_{\zeta\sigma}\mathcal{L}^-_{\zeta\sigma}\Big(\sum_{j=0}^{\rho-1}\mathcal{L}^+_{j\cdot\zeta\sigma}\mathcal{K}_{j\sigma_1}(\mathcal{B}_\sigma\mathcal{F}_{\zeta\sigma}v)\otimes\delta_{x_n}\Big)$$

can be extended continuously from $L^p(M,X)$ to $L^p(M,X)$ and $G_{\zeta\sigma}$ is independent of the mapping Λ. Hence as far as we consider $G_{\zeta\sigma}$ in $L^p(M,X)$, we can omit Λ in (5.2) and (5.3).

Now we consider $A_B^{\kappa i}$. For any $u \in D(A_B)$

$$A_B^{\kappa i}u=\frac{1}{2\pi i}\int_{\gamma_2}(-\lambda)^{\kappa i}(\lambda+A_B)^{-1}u\,d\lambda$$
$$=\frac{1}{2\pi i}\int_{\gamma_1}(-z\sigma^2)^{\kappa i}\mathcal{F}_{\zeta\sigma}u\;d(z\sigma^2)$$
$$+\frac{1}{2\pi i}\int_{\gamma_1}(-z\sigma^2)^{\kappa i}G^{(2)}_{\zeta\sigma}u\;d(z\sigma^2)$$

where γ_1 is the complex contour $\arg(z\sigma^2+1)=\frac{3}{4}\pi$ and $\zeta=z^{\frac{1}{2}}$. Consider the operator

$$\frac{1}{2\pi i}\int_{\gamma_1}(-z\sigma^2)^{\kappa i}\mathcal{F}_{\zeta\sigma}u\;d(z\sigma^2)\,.$$

It is clear this coincides with the pure imaginary power $A^{\kappa i}$ of A considered on $\tilde{M}$.

PROPOSITION 6.3. *$A^{\kappa i}$ is a pseudo-differential operator of order κi. The principal symbol of $A^{\kappa i}$ is given by*

$$\sigma(A^{\kappa i})=\sigma(A)^{\kappa i}.$$

(This was first proved in [17].)

PROOF. Let $\{\Psi_k\}$ be the partition of unity used in §5. For any double index $\alpha=(\alpha_1, \alpha_2)$ and coordinate functions $(x_1, \cdots, x_n)$ valid in $\operatorname{supp}\Psi_{\alpha_1} \cup \operatorname{supp}\Psi_{\alpha_2}$ and for any $(\xi_1, \xi_2, \cdots, \xi_n)$ in $\boldsymbol{R}^n$, we have

$$\begin{aligned}(6.8)\qquad & e^{-ix\cdot\xi}\Psi_{\alpha_2}(A^{\kappa i})\Psi_{\alpha_1}e^{ix\cdot\xi} \\ &=\frac{1}{2\pi i}\int_{\gamma_2}(-z\sigma^2)^{\kappa i}e^{-ix\cdot\xi}\Psi_{\alpha_2}\mathcal{F}_{\zeta\sigma}\Psi_{\alpha_1}e^{ix\cdot\xi}d(z\sigma^m) \\ &\sim\sum_{j=-2m}^{-\infty}\int_{\gamma_2}(-z\sigma^2)^{\kappa i}f_{\alpha,j}(x, \xi, \zeta\sigma)\,d(z\sigma^m)\end{aligned}$$

where $\sum_{j=-2}^{\infty} f_{\alpha,j}(x, \xi, \zeta\sigma)$ is the asymptotic expansion of $e^{-ix\cdot\xi}\Psi_{\alpha_2}\mathcal{F}_{\zeta\sigma}\Psi_{\alpha_1}e^{ix\cdot\xi}$ and γ_2 is the complex contour $|\arg(z\sigma^2+1)|=\frac{3}{4}\pi$.

To treat the second term of (6.7) we recall that

$$G_{\zeta\sigma}^{(2)}u=\mathcal{F}_{\zeta\sigma}\mathcal{Q}_{\zeta\sigma}((\mathcal{P}_{\zeta\sigma}u)|_{\partial M}\otimes\delta_{x_n}).$$

Let $\{\Psi_j\}_{j\in J}$ be the partition of unity treated in §5. As in §5, for any double index $\alpha=(\alpha_1, \alpha_2)\in J\times J$, we denote by ${}_\alpha\mathcal{P}_{\zeta\sigma}$ the operator $\Psi_{\alpha_1}\mathcal{P}_{\zeta\sigma}\Psi_{\alpha}$. We shall respectively denote the Fourier integral kernels of ${}_\alpha\mathcal{P}_{\zeta\sigma''}$, ${}_\beta q_{\zeta\sigma''}$ and ${}_\gamma\mathcal{F}_{\zeta\sigma''}$, with respect to local coordinates by

$$(6.13)\qquad {}_\alpha p_{\zeta\sigma''}(x, \xi, \zeta\sigma_1)\sim\sum_{j=2-\mu}^{\infty}{}_\alpha p_{\zeta\sigma'',-j}(x, \xi, \zeta\sigma_1)$$

$$(6.14)\qquad {}_\gamma f_{\zeta\sigma''}(x, \xi, \zeta\sigma_1)\sim\sum_{j=2}^{\infty}{}_\gamma f_{\zeta\sigma'',-j}(x, \xi, \zeta\sigma_1) \quad \text{if}\quad (\xi, \sigma_1)\neq 0$$

$$(6.15)\qquad {}_\beta q_{\zeta\sigma''}(x, \xi, \zeta\sigma_1)\sim\sum_{j=\mu-1}^{\infty}{}_\beta q_{\zeta\sigma'',-j}(x, \xi, \zeta\sigma_1), \quad \text{if}\quad (\xi', \sigma_1)\neq 0.$$

And we define the operator M by

$$\begin{aligned}Mu=&(2\pi)^{-n}(2\pi i)^{-1}\int_{\gamma_1}(-z\sigma^2)^{-\kappa i}d(z\sigma^2)\int_{R^{n-1}}e^{ix'\cdot\xi'}d\xi'\int_{\Gamma}e^{ix_n\cdot\eta}d\eta \\ &\times\int_{R^1}{}_\gamma f_{\zeta\sigma'',-2}(x', 0, \xi', \eta, \sigma_1)\,{}_\beta q_{\zeta\sigma'',1-\mu}(x', 0, \xi, \sigma_1) \\ &\times{}_\alpha p_{\zeta\sigma'',\mu-2}(x', 0, \xi, \sigma_1)\hat{u}(\xi)d\xi_n\end{aligned}$$

where Γ is a complex contour enclosing $\tau_1^+(x', 0, \xi', \sigma_1)$. Then

$$v(x)=\frac{1}{2\pi i}\int_{\gamma_1} r\mathscr{F}_{\zeta\sigma''}\circ{}_\beta q_{\zeta\sigma'}\circ{}_\alpha p_{\zeta\sigma'',\mu-2}(u)(-z\sigma^2)^{-\kappa i}d(z\sigma^2)-Mu$$
$$=\frac{1}{2\pi i}\int_{\gamma_1}(-z\sigma^2)^{-\kappa i}d(z\sigma^2)\int_{R^n}(C_1(x',0,\xi,\sigma_1\zeta)+C_2(x,\xi,\sigma_1\zeta))u(\xi)e^{ix\cdot\xi}d\xi$$

where

$$(1+|\xi|+|\zeta\sigma_1|)^m(1+|\xi'|+|\zeta\sigma_1|)C_1(x',0,\xi,\sigma_1\zeta)$$

is bounded in $\mathscr{D}(U)$ and

$$(1+|\xi|+|\zeta\sigma_1|)^m(1+|\xi'|+|\zeta\sigma_1|)^2C_2(x,\xi,\sigma_1\zeta)$$

is bounded in $\mathscr{D}(U)$, (cf. Appendix). Using these and the generalized Mikhilin's theorem, we can prove that $\|v\|_{L^p(M)}\leqq C\|u\|_{L^p(M)}$ where C is independent of κ and of u.

Near the boundary ∂M we choose a coordinate system such that ∂M is represented by $x_n=0$, and $\sigma(A)(x,\xi)=\sum_{i,j=1}^{n-1}a_{ij}(x)\xi_i\xi_j+\xi_n^2$ at $x\in\partial M$. In the following we shall denote $\xi'\cdot\xi'=\sum_{i,j=1}^{n-1}a_{ij}(x)\xi_i\xi_j$. Then

$$\sigma(A)(x,\xi)=\xi'\cdot\xi'+\xi_n^2\,. \tag{6.18}$$

For any $\xi'\in T_x^*(\partial M)$

$$\sigma(\mathscr{A})(x,\xi'+\tau\nu,\rho\zeta)=\tau^2+\xi'\cdot\xi'+\rho^2z \qquad \zeta=z^{\frac{1}{2}}\,. \tag{6.19}$$

And $\sigma(\mathscr{A})(x,\xi'+\tau\nu,\rho\zeta)=0$ has roots

$$\tau^+(x,\xi',\rho\zeta)=i(\xi'\cdot\xi'+\rho^2\zeta^2)^{\frac{1}{2}}\,, \tag{6.20}$$

$$\tau^-(x,\xi',\rho\zeta)=-i(\xi'\cdot\xi'+\rho^2\zeta^2)^{\frac{1}{2}}\,, \tag{6.21}$$

where the branch of $(\xi'\cdot\xi'+\rho^2\zeta^2)^{\frac{1}{2}}$ is so taken that $1^{\frac{1}{2}}=1$. Thus, $\operatorname{Im}\tau^+>0$, $\operatorname{Im}\tau^-<0$. Therefore, we have

$$L^+(x,\xi,\rho,\tau)=\tau-i(\xi'\cdot\xi'+\rho^2\zeta^2)^{\frac{1}{2}} \tag{6.22}$$

$$L^-(x,\xi,\rho,\tau)=\tau+i(\xi'\cdot\xi'+\rho^2\zeta^2)^{\frac{1}{2}} \tag{6.23}$$

$$L_0^+=1\,. \tag{6.24}$$

In the case of $\mathscr{B}=1$ (Dirichlet boundary condition). In this case $\mathscr{K}_{0,\rho\zeta}=1$. Therefore we have only to estimate the function

$$v(x)=(2\pi)^{-n}\int_{R^{n-1}}e^{ix'\cdot\xi'}d\xi'\int_{\gamma_1}(-\rho^2\zeta^2)^{\kappa i}d(\rho^2\zeta^2) \times\int_\Gamma\frac{e^{ix_n\tau}d\tau}{(\tau-i(\rho^2\zeta^2+\xi'\cdot\xi')^{1/2})}\int_{R^1}\frac{\hat{u}(\xi',\eta)d\eta}{\xi'\cdot\xi'+\eta^2+\rho^2\zeta^2} \tag{6.25}$$

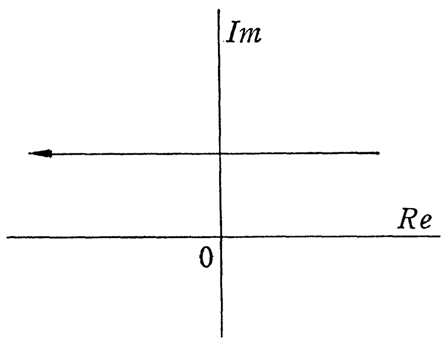
where γ_1 is the complex contour $|\arg(\zeta^2\rho^2+1)|=\frac{3}{4}\pi$ and Γ is the complex contour enclosing $i(\rho^2\zeta^2+\xi'\cdot\xi')^{1/2}$. Therefore integrating first by τ, we have

$$(6.26)\qquad v(x)=(2\pi)^{-n}\int_{R^{n-1}}e^{ix'\cdot\xi'}\int_{\gamma_1}(-\rho^2\zeta^2)^{\kappa i}e^{-x_n(\rho^2\zeta^2+\xi'\cdot\xi')^{1/2}}d(\rho^2\zeta^2)$$
$$\int_{R^1}\frac{\hat{u}(\xi',\eta)}{\xi'\cdot\xi'+\eta^2+\rho^2\zeta^2}d\eta \qquad \text{for}\quad x_n>0\,.$$

Now we replace $i(\rho^2\zeta^2+\xi'\cdot\xi')^{1/2}$ with ξ_n. Then we have

$$(6.27)\qquad v(x)=-2(2\pi)^{-n}\int_{R^{n-1}}e^{ix'\cdot\xi'}\int_{\gamma_2}(\xi_n^2+\xi'\cdot\xi')^{\kappa i}e^{ix\cdot\xi}\xi_n d\xi_n$$
$$\times\int_{R^1}\frac{\hat{u}(\xi',\eta)d\eta}{\eta^2-\xi_n^2}\,.$$

where γ_2 is the complex contour represented in the following figure:



Since support of u is contained in $x_n>0$, $\hat{u}(\xi',\eta)$ is holomorphic in $\operatorname{Im}\eta<0$, and continuous and uniformly bounded on $\operatorname{Im}\eta\leqq 0$. And we have

$$(6.28)\qquad \int_{R^1}\frac{\hat{u}(\xi^1,\eta)d\eta}{\eta^2-\xi_n^2}=\frac{-\hat{u}(\xi',-\xi_n)}{2\xi_n}\,,\qquad \text{for}\quad \forall\xi_n\in\gamma_2\,.$$

Substituting this for (6.27), we have

$$(6.29)\qquad v(x)=(2\pi)^{-n}\int_{R^{n-1}}\int_{R^1}(\xi_n^2+\xi'\cdot\xi')^{\kappa i}\hat{u}(\xi',-\xi_n)e^{ix\cdot\xi}d\xi'd\xi_n\,.$$

Therefore by Lemma 6.2 we obtain

$$\|v\|_{L^p(M,X)}\leqq c\|u\|_{L^p(M,X)}$$

where c is independent of κ.

Now we shall treat the case

$$\sigma(\mathscr{B})(x,\xi'+\tau\nu)=\tau\,.$$

In this case

$$(6.30)\qquad \sigma(\mathscr{B})(x,\xi'+\tau\nu)\equiv i(\xi'\cdot\xi'+\rho^2\zeta^2)^{1/2}\,,\qquad \operatorname{mod} L^+\,,$$

$$K_{0,\rho\zeta}(x, \xi', \rho\zeta) = \frac{1}{i(\xi'\cdot\xi'+\rho^2\zeta^2)^{1/2}}. \tag{6.31}$$

Therefore, we have only to estimate the function

$$v(x) = \frac{1}{i(2\pi)^{n+2}} \int_{R^{n-1}} e^{ix'\cdot\xi'} d\xi' \int_{\gamma_1} \frac{(-\rho^2\zeta^2)^{\kappa i} d(\rho^2\zeta^2)}{(\xi'\cdot\xi'+\rho^2\zeta^2)^{1/2}} \int_{\Gamma} \frac{e^{ix_n\tau} d\tau}{(\tau - i(\xi'\cdot\xi'+\rho^2\zeta^2))} \times \int_{R^1} \frac{\eta u(\xi', \eta)}{(\xi'\cdot\xi'+\eta^2+\rho^2\zeta^2)} d\eta . \tag{6.32}$$

Integrating by τ and substituting $i(\xi'\cdot\xi'+\rho^2\zeta^2) = \xi_n$, we have

$$v(x) = \frac{-2}{(2\pi)^{n+1}} \int_{R^{n-1}} e^{ix'\cdot\xi'} \int_{\gamma_2} \frac{(\xi_n^2+\xi'\cdot\xi')^{\kappa i} \xi_n e^{ix\cdot\xi_n} d\xi_n}{\xi_n} \times \int_{R^1} \frac{\eta\, \hat{u}(\xi', \eta)}{\eta^2-\xi_n^2} d\eta . \tag{6.33}$$

When $u \in C_0^\infty(M)$ and support u is contained in $x_n > \delta > 0$, then $\hat{u}(\xi', \eta)$ is holomorphic in $\operatorname{Im} \eta < \delta$ and $\hat{u}(\xi', \eta)$ decreases exponentially as $\operatorname{Im} \eta$ tends to $-\infty$. Hence we have

$$\int_{R^1} \frac{\eta\, \hat{u}(\xi', \eta)}{\eta^2-\xi_n^2} d\eta = \frac{(2\pi i)\hat{u}(\xi'-\xi_n)}{2}. \tag{6.34}$$

Substituting this into (6.33) we have, for $x_n > 0$,

$$v(x) = \frac{-i}{(2\pi)^n} \int_{R^n} \hat{u}(\xi', -\xi_n)(\xi_n^2 - \xi'\cdot\xi')^{\kappa i} e^{ix\cdot\xi} d\xi .$$

Therefore from Lemma 6.2 we obtain

$$\|v\|_{L^p(M, X)} \leqq c\|u\|_{L^p(M, X)} . \tag{6.35}$$

The estimate (6.35) was proved only for sections $C_0^\infty(M, X)$ whose support is contained in $x_n > \delta$ with some $\delta > 0$. Since these are dense in $L^p(M, X)$, (6.35) holds for any $u \in L^p(M, X)$. This and (6.29) complete the proof of Theorem 6.1.

REMARK. If B includes tangential derivatives, then $A_B^{\kappa i}$ is not bounded for $\kappa \neq 0$. This can be proved by the same method as in the proof of Theorem 6.1.

Now we shall describe some consequences of Theorem 6.1.

THEOREM 6.5. *The family of bounded operators* $\{(A_B+\tau_0)^\mu\}_{\operatorname{Re}\mu\leqq 0}$ *forms a holomorphic semigroup in* $\operatorname{Re}\mu \lneqq 0$ *and continuous in* $\operatorname{Re}\mu \leqq 0$ (cf. Yosida [20]). *Its generator is* $\log(A_B+\tau_0)$.

PROOF. We have only to prove that the generator of $\{(A_B+\tau_0)^\mu\}_{\operatorname{Re}\mu\leqq 0}$ is $\log(A_B+\tau_0)$. We may assume $\operatorname{Re}\tau \geqq 0$ is contained in the resolvent set of $-A_B$ and assume $\tau_0 = 0$. Recall that the operator $\log A_B$ is the minimal closed extension of the operator

(6.36) $$(\log A_B)u=\frac{1}{2\pi i}\int_\Gamma \frac{\log\lambda}{\lambda}A_B(\lambda-A_B)^{-1}ud\lambda$$

defined for those x that satisfy

(6.37) $$\int_\Gamma \frac{\log\lambda}{\lambda}\|A_B(\lambda-A_B)^{-1}u\|_{L^p(M,X)}d\lambda<\infty .$$

We denote by D_L the set of u satisfying (6.37) and by $\log A_B$ the infinitesimal generator of $\{A_B^\mu\}_{\mathrm{Re}\,\mu\leqq c}$. Let D^∞ stand for those elements in $L^p(M, X)$ defined by

$$\int_0^\infty \chi(\mu)A_B^{-\mu}u\,d\mu$$

with some $\chi\in C_0^\infty(0,\infty)$ and some $u\in L^p(M, X)$.

To prove Theorem 6.5, it is sufficient that we show

(6.38) $$D_L\supset D^\infty ,$$

(6.39) $$D_L\subset \text{the domain of } \mathrm{Log}\,A_B ,$$

and

(6.40) $$\log A_B=\mathrm{Log}\,A_B \quad \text{on} \quad D_L .$$

Take any $u\in L^p(M, X)$ and $\chi\in C_0^\infty(0,\infty)$, let δ be the lower bound of support χ. We define

$$v=\int_\delta^\infty \chi(\mu)A_B^{-\mu}u\,d\mu .$$

For any $\mu>0$

$$\|A_B^{1-\mu}(\lambda-A_B)^{-1}u\|\leqq C\|A_B(\lambda-A_B)^{-1}u\|^{1-\mu}\|(\lambda-A_B)^{-1}u\|^{\mu}$$
$$\leqq C_u(1+|\lambda|)^{-\mu} .$$

Then

$$\int_\Gamma\left|\frac{\log\lambda}{\lambda}\right|\|A_B(\lambda-A_B)^{-1}u\|_{L^p(M,X)}d\lambda$$
$$\leqq\int_\Gamma\left|\frac{\log\lambda}{\lambda}\right|d\lambda\int_\delta^\infty|\chi(\mu)|\,\|A_B^{1-\mu}(\lambda-A_B)^{-1}u\|_{L^p(M,X)}d\mu$$
$$\leqq C\int_0^\infty|\chi(\mu)|\,d\mu\int_\Gamma|\log\lambda|\,|\lambda|^{-\mu-1}d\lambda .$$

(6.38) has been proved.

Let $u\in D_L$, then

$$\frac{1}{h}(A_B^{-h}-I)u=\frac{1}{h}\int_\Gamma(\lambda^{-h-1}-\lambda^{-1})A(\lambda-A)^{-1}ud\lambda$$
$$=\int_\Gamma\frac{1}{h}\int_0^h\lambda^{-1-s}\log\lambda\,ds\,A(\lambda-A)^{-1}ud\lambda$$
$$=\frac{1}{h}\int_0^h ds\int_\Gamma\lambda^{-1-s}\log\lambda\,A(\lambda-A)^{-1}ud\lambda .$$

Thus

$$s-\lim_{h\to 0}(A_B^{-h}-I)u=(\log A_B)u\,.$$

We have thus proved (6.39) and (6.40) and this completes the proof of Theorem 6.5.

We denote by $D(A_B^\theta)$ the domain of A_B^θ. Then another direct consequence of Theorem 6.1 is the following

THEOREM 6.6.

$$D(A_B^\theta)=[L^p(M,X),\,D(A_B)]_\theta \qquad 0\leqq\theta\leqq 1\,. \tag{6.41}$$

The right hand side of (6.41) *is the complex interpolation space between* $L^p(M, X)$ *and* $D(A_B)$. (cf. Calderón [4].)

PROOF. This is a consequence of general theory for complex interpolation spaces (see, for example, [13]). So we omit the proof here.

The identification of $[L^p(M, X), D(A_B)]_\theta$ with some function spaces was done in the author's previous note [9]. Using these, we have proved.

THEOREM 6.7.

1° *if* $\sigma(B)(x,\xi'+\tau\nu)=\tau$, *for* $x\in\partial M$,

$$D(A_B^\theta)=\{u\in H^{2\theta,p}(M,X)\,|\ Bu|_{\partial M}=0\} \qquad \textit{when}\ \ 1+\frac{1}{p}<2\theta\leqq 2\,,$$

$$D(A_B)=H^{2\theta,p}(M,X) \qquad \textit{when}\ \ 0\leqq 2\theta<1+\frac{1}{p}\,.$$

2° *if* $\sigma(B)(x,\xi'+\tau\nu)=1$ *for* $x\in\partial M$,

$$D(A_B^\theta)=\{u\in H^{2\theta,p}(M,X)\,|\ Bu|_{\partial M}=0\} \qquad \textit{when}\ \ \frac{1}{p}<2\theta\leqq 2\,,$$

$$D(A_B^\theta)=H^{2\theta,p}(M,X) \qquad \textit{when}\ \ 0\leqq 2\theta<\frac{1}{p}$$

where $H^{s,p}(M, X)$ *is the space of sections of* X *which can locally be identified with functions in* $H^{s,p}(R^n)$.[1]

Appendix I

First we assume $\partial M=\phi$ and we shall prove Theorem 1.1. Since there is a β-pseudo-differential operator $\mathcal{P}'$ such that $\mathcal{P}\circ\mathcal{P}'-I$ and $\mathcal{P}'\circ\mathcal{P}-I$ are β-pseudo-differential operators of order $-\infty$, we may assume from the first that $\mathcal{P}=I+q$ where q is a β-pseudo-differential operator of order $-\infty$. Theorem 31 in [6] asserts that for any $N>0$, there is a constant C and we have the estimate, for any $u\in L^2(M, X)$ and $\sigma''\in \boldsymbol{R}^{m-k}$

1) $H^{s,p}(\boldsymbol{R}^n)$ is the image of the convolution mapping $L^p(\boldsymbol{R}^n)\ni f\to J^{-s}*f$. Where $J^{-s}(x)=\int_{\boldsymbol{R}^n}e^{+ix\cdot\xi}(1+|\xi|^2)^{-\frac{s}{2}}d\xi$.

(A-1) $$\| q_{\sigma''}(u\otimes 1)\|_{L^2(M,X)} \leqq C(1+|\sigma''|)^{-N}\|u\|_{L^2(M,X)}.$$

When $|\sigma''|$ is larger than a constant, $\mathcal{P}_{\sigma''}^{-1}$ exists as a bounded operator operating in $L^2(M, X)$ and is given by

$$\mathcal{P}_{\sigma''}^{-1}=\sum_{j=0}^{\infty}(-Q_{\sigma''})^j.$$

We must prove that $\mathcal{P}_{\sigma''}^{-1}$ is a β-pseudo-differential operator of order 0. For any $\sigma', s'\in \boldsymbol{R}^k$, $N>0$ and $u\in \mathcal{D}(M, X)$, we have from Theorem 31 in [**6**]

(A-2) $$\|e^{-is'\cdot\sigma'}Q_{\sigma''}(e^{is'\cdot\sigma'}u)\|_{H^a(M,X)} \leqq C_a(1+|\sigma'|+|\sigma''|)^{-N}\|u\|_{L^2(M,X)}$$

where $H^a(M, X)$ is the space of Sobolev sections of X of order a and of exponent 2 and C_a is a constant depending on a but independent of σ' and σ''. Hence, for any $j\geqq 1$,

$$\|e^{-is'\cdot\sigma'}(Q_{\sigma''})^j(e^{is'\cdot\sigma'}u)\|_{H^a(M,X)} \leqq C_0^{j-1}C_a(1+|\sigma''|+|\sigma'|)^{-jN}\|u\|_{L^2(M,X)}.$$

Thus $e^{-is'\cdot\sigma'}(\mathcal{P}_{\sigma''})^{-1}(e^{is'\cdot\sigma'}u)$ is the pull back of a section of X and if we denote this again by $e^{-is'\cdot\sigma'}(\mathcal{P}_{\sigma''}^{-1})(e^{is'\cdot\sigma'}u)$,

(A-3) $$\|e^{-is'\cdot\sigma'}(\mathcal{P}_{\sigma''}^{-1})(e^{is'\cdot\sigma'}u)\|_{H^a(M,X)}$$
$$\leqq\left\{\left(\frac{C_a(1+|\sigma''|+|\sigma'|)^{-N}}{1-C_0(1+|\sigma''|+|\sigma''|)^{-N}}\right)+1\right\}\|u\|_{L^2(M,X)}$$

(A-3) gives that the map $u\to\mathcal{P}_{\sigma}^{-1}(1\otimes u)$ is continuous from $\mathcal{D}(M, X)$ to $\mathcal{D}(M, X)$. $\mathcal{P}_{\sigma''}^{-1}$ satisfies the equation

(A-4) $$\mathcal{P}_{\sigma''}^{-1}=1+Q_{\sigma''}^{-1}\mathcal{P}_{\sigma''}^{-1}=1+\mathcal{P}_{\sigma''}^{-1}Q_{\sigma''}^{-1}$$

and for any $1\leqq j\leqq k$

(A-5) $$\frac{\partial}{\partial\sigma_j}(e^{-is'\cdot\sigma'}\mathcal{P}_{\sigma''}^{-1}e^{is'\cdot\sigma'}u)$$
$$=-\frac{\partial}{\partial\sigma_j}(e^{-is'\cdot\sigma'}Q_{\sigma''}e^{is'\cdot\sigma'}e^{-is'\cdot\sigma'}\mathcal{P}_{\sigma''}^{-1}\mathcal{P}_{\sigma''}^{-1}e^{is'\cdot\sigma'}u).$$

This together with (A-3) proves that

$$e^{-is'\cdot\sigma'}\mathcal{P}_{\sigma''}^{-1}e^{is'\cdot\sigma'}u\in\mathcal{D}(M, X)\widehat{\otimes}\mathcal{O}_M(\boldsymbol{R}^k).$$

We define for any $u\otimes T\in\mathcal{D}(M, X)\otimes\mathcal{S}'(\boldsymbol{R}^k)$,

$$\mathcal{R}_{\sigma''}(u\otimes T)=(2\pi)^{-k}\int_{\boldsymbol{R}^k}(e^{-is'\cdot\sigma'}\mathcal{P}_{\sigma''}^{-1}(e^{is'\cdot\sigma'}u)\cdot\hat{T})e^{is'\cdot\sigma'}d\sigma$$

where the integral of the right hand side represents symbolically partial Fourier inverse transform of $e^{-is'\cdot\sigma'}\mathcal{P}_{\sigma''}^{-1}(e^{is'\cdot\sigma'}u)\hat{T}$ over $\boldsymbol{R}^k$ space. $\mathcal{R}_{\sigma''}$ coincides with $\mathcal{P}_{\sigma''}^{-1}$ on $\mathcal{D}(M, X)\otimes\mathcal{S}(\boldsymbol{R}^k)$ and is continuous linear mapping from $\mathcal{D}(M, X)\widehat{\otimes}\mathcal{S}'(\boldsymbol{R}^k)$ to $\mathcal{D}(M, X)\widehat{\otimes}\mathcal{S}'(\boldsymbol{R}^k)$. Thus defining $\mathcal{P}_{\sigma''}^{-1}(u\otimes T)=\mathcal{R}_{\sigma''}(u\otimes T)$, we obtain a continuous linear map from $\mathcal{D}(M, X)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^k)$ to $\mathcal{D}(M, X)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^k)$.

Now let Ψ_j $(1\leqq j\leqq 4)$ be functions in $\mathcal{D}(M)$. We assume that $(x_1, x_2, \cdots, x_n)$ be coordinate functions valid in some neighbourhood U of $\bigcup_{j=1}^{4}\operatorname{supp}\Psi_j$. Then, by assumption, for any $N>0$ and $\lambda>1$

$$\lambda^N e^{-i\lambda(x\cdot\xi+s'\cdot\sigma')}\Psi_2 Q_{\sigma''}\Psi_1(e^{i\lambda(x\cdot\xi+s'\cdot\sigma')}u) \tag{A-6}$$

remains bounded in $\mathcal{E}(U\times S_1)$.

Therefore we have only to prove the mapping

$$u\to e^{i\lambda(x\cdot\xi+s'\cdot\sigma')}\Psi_4\mathcal{P}_{\sigma''}^{-1}\Psi_3 e^{i\lambda(x\cdot\xi+s'\cdot\sigma')}u$$

is equicontinuous in $\mathcal{E}(U\times S_1)$, when λ tends to infinity. By Theorem 29 of [7], for any $N>0$ and $a, b\in\boldsymbol{R}$

$$\begin{aligned}\|e^{-i\lambda(x\cdot\xi+s\cdot\sigma)}\Psi_4 q_{\sigma''}\Psi_3(e^{i\lambda(x\cdot\xi+s'\cdot\sigma')}u)\|_{H^{a+b}(M,X)}\\ \leqq C(1+|\lambda|)^{-N}\|u\|_{H^a(M,X)}\,.\end{aligned} \tag{A-7}$$

Hence

$$\begin{aligned}&\|e^{-i\lambda(x\cdot\xi+s'\cdot\sigma')}\Psi_4\mathcal{P}_{\sigma''}^{-1}\Psi_3(e^{i\lambda(x\cdot\xi+s'\cdot\sigma')}u)\|_{H^a(M,X)}\\ &\quad\leqq\|\Psi_4\Psi_3 u\|_{H^a(M,X)}\\ &\qquad+\|e^{-i\lambda(x\cdot\xi+s'\cdot\sigma')}\Psi_4 Q_{\sigma''}\Psi_j e^{i\lambda(x\cdot\xi+s'\cdot\sigma')}e^{-i\lambda(x\cdot\xi+s'\cdot\sigma')}\Psi_k\mathcal{P}_{\sigma''}^{-1}\Psi_3 e^{i\lambda(x\cdot\xi+s'\cdot\sigma')}u\|_{H^a(M,X)}\\ &\quad\leqq\|u\|_{H^a(M,X)}+\|e^{-i\lambda(x\cdot\xi+s'\cdot\sigma')}\Psi_k\mathcal{P}_{\sigma''}^{-1}\Psi_3 e^{i(x\cdot\xi+s'\cdot\sigma')}u\|_{L^2(M,X)}\\ &\quad\leqq C(\|u\|_{H^a(M,X)}+\|u\|_{L^2(M,X)})\,.\end{aligned}$$

Thus the mappings $u\to e^{i\lambda(x\cdot\xi+s'\cdot\sigma')}\Psi_4\mathcal{P}_{\sigma''}^{-1}\Psi_3 e^{i\lambda(x\cdot\xi+s'\cdot\sigma')}u$ are equicontinuous in $\mathcal{E}(M, X)\hat{\otimes}\mathcal{C}(S_1)$. Equicontinuity in $\mathcal{E}(M, X)\hat{\otimes}\mathcal{E}(S_1)$ follows from (A-5) and

$$\begin{aligned}&\frac{\partial}{\partial\xi_j}e^{-i\lambda(x\cdot\xi+s'\cdot\sigma')}\Psi_4\mathcal{P}_{\sigma''}^{-1}\Psi_3 e^{i\lambda(x\cdot\xi+s'\cdot\sigma')}u\\ &\qquad=-\lambda x_j e^{-i\lambda(x\cdot\xi+s'\cdot\sigma')}\Psi_4\mathcal{P}_{\sigma''}^{-1}\Psi_3 e^{i\lambda(x\cdot\xi+s'\cdot\sigma')}u\\ &\qquad\quad+\lambda e^{-i\lambda(x\cdot\xi+s'\cdot\sigma')}\Psi_4\mathcal{P}_{\sigma''}^{-1}\Psi_3 e^{i\lambda(x\cdot\xi+s'\cdot\sigma')}x_j u\,.\end{aligned} \tag{A-8}$$

The above argument holds uniformly in t, if $\mathcal{P}$ depends on a parameter t and satisfies the condition stated in Theorem 1.1. Therefore we have proved Theorem 1.1.

Let U be an open set in $\boldsymbol{R}^n$ and I be the unit interval $I=(-1, 1)$. We shall denote by $x=(x', x_n)$ the generic point of $U\times I$. $(\xi, \sigma)=(\xi', \xi_n, \sigma)$ stands for the point in $\boldsymbol{R}^{n+m}$. We are given functions $K(x, \xi, \sigma)$ in $\mathcal{E}(U\times I)\hat{\otimes}\mathcal{O}_M(\boldsymbol{R}^{n+m})$ and $K_j(x, \xi, \sigma)$, $j=1, 2, 3, \cdots$ which are in $C^\infty(U\times I\times(\boldsymbol{R}^{n+m}-\{0\}))$ and homogeneous in (ξ, σ) of degree $z_j=s_j+it_j$ with real parts s_j decreasing to $-\infty$. And

$$(|\xi|+|\sigma|)^{-s_N+|\alpha_1|+|\alpha_2|}D_\xi^{\alpha_1}D_\sigma^{\alpha_2}(K(x, \xi, \sigma)-\sum_0^{N-1}K_j(x, \xi, \sigma))$$

are bounded in $\mathcal{E}(U\times I)$, when $|\xi|+|\sigma|\rightarrow\infty$. We assume that $K_j(x, \xi, \sigma)$ is holomorphic in ξ_n for $\operatorname{Im}\xi_n>-\tau_0(\xi', \sigma)$, where $\tau_0(\xi', \sigma)$ is a positive homogeneous function of $(\xi', \sigma)\in \boldsymbol{R}^{n+m-1}-\{0\}$ of degree 1.

PROPOSITION A-1. *Define the mapping*

$$\mathcal{K}:\ \mathcal{D}(U\times I)\otimes\mathcal{S}(\boldsymbol{R}^m)\longrightarrow\mathcal{E}(U\times I)\otimes\mathcal{S}(\boldsymbol{R}^m)$$

$$\text{(A-9)}\qquad \mathcal{K}(\varphi\otimes\phi\otimes T)=(2\pi)^{-n-m}\int_{\boldsymbol{R}^{n+m}}K(x, \xi, \sigma)\hat{\varphi}(\xi')\hat{\phi}(\xi_n)\hat{T}(\sigma)e^{i(x\cdot\xi+s\cdot\sigma)}d\xi\cdot d\sigma\,.$$

Then we can uniquely extend K as a continuous linear mapping from $\mathcal{E}'(U\times I)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^m)$ to $\mathcal{D}'(U\times I\times\boldsymbol{R}^m)$. If $s_0<-1$, then we have for any $\varphi\in\mathcal{D}(U)$, $T\in\mathcal{S}'(\boldsymbol{R}^m)$

$$\mathcal{K}(\varphi\otimes\delta_{x_n}\otimes T)$$
$$=(2\pi)^{-n-m}\int_{\boldsymbol{R}^{n+m-1}}\hat{\varphi}(\xi')\hat{T}(\sigma)e^{i(x'\cdot\xi'+s'\cdot\sigma')}d\xi'd\sigma\int_{\boldsymbol{R}^1}K(x, \xi, \sigma)e^{ix_n\xi_n}d\xi_n$$

and $\mathcal{K}(\varphi\otimes\delta_{x_n}\otimes T)$ belongs to $\mathcal{E}(U)\hat{\otimes}\mathcal{C}(I)\hat{\otimes}\mathcal{X}$, where $\mathcal{X}$ is one of the spaces $\mathcal{S}(\boldsymbol{R}^m)$, $\mathcal{O}_M(\boldsymbol{R}^m)$ and $\mathcal{S}'(\boldsymbol{R}^m)$.

PROOF. A β-pseudo-differential operator maps continuously $\mathcal{D}(U\times I\times\boldsymbol{R}^m)$ into $\mathcal{E}(U\times I)\hat{\otimes}\mathcal{S}(\boldsymbol{R}^m)$. Theorem 19 of [6] implies that $\mathcal{K}$ maps continuously $(\mathcal{E}(U\times I)\hat{\otimes}\mathcal{S}(\boldsymbol{R}^m))'=\mathcal{E}'(U\times I)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^m)$ into $\mathcal{D}'(U\times I\times\boldsymbol{R}^m)$. To complete the proof it is sufficient to prove that $\mathcal{K}(\varphi\otimes\delta_{x_n}\otimes T)\in\mathcal{E}(U)\hat{\otimes}\mathcal{C}(I)\otimes\mathcal{S}'(\boldsymbol{R}^m)$ if $s_0<-1$. It is clear that

$$\int_{\boldsymbol{R}^1}K(x, \xi, \sigma)e^{ix_n\xi_n}d\xi_n\in\mathcal{E}(U)\hat{\otimes}\mathcal{C}(I)\hat{\otimes}\mathcal{O}_M(\boldsymbol{R}^{n+m-1})\,.$$

Therefore, we have $\mathcal{K}(\varphi\otimes\delta_{x_n}\otimes T)\in\mathcal{E}(U)\hat{\otimes}\mathcal{C}(I)\hat{\otimes}\mathcal{X}$.

COROLLARY A-2. *If* $s_0<-m-1$,

$$\mathcal{K}(\varphi\otimes\delta_{x_n}\otimes T)\in\mathcal{E}(U)\hat{\otimes}\mathcal{C}^m(I)\hat{\otimes}\mathcal{X}\,,\qquad for\quad \varphi\in\mathcal{D}(U)\,,\quad T\in\mathcal{X}\,,$$

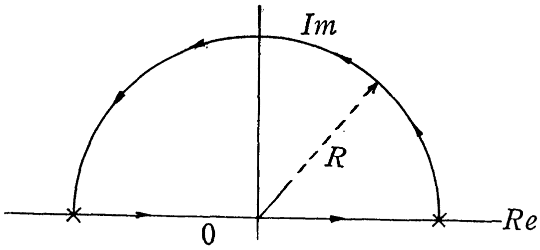
where $\mathcal{X}$ *is one of the spaces* $\mathcal{S}(\boldsymbol{R}^m)$, $\mathcal{O}_M(\boldsymbol{R}^m)$ *and* $\mathcal{S}'(\boldsymbol{R}^m)$.

PROPOSITION A-3. *Let* $K_j(x, \xi, \sigma)$ *be the function defined above, and* $\Psi(\xi', \xi_n, \sigma)$ *be a* C^∞ *function on* $\boldsymbol{R}^{n-1}\times\boldsymbol{C}\times\boldsymbol{R}^m$ *which is identically* 0 *near the origin and* 1 *if* $|\sigma|^2+|\xi'|^2+|\xi_n|^2\geqq 1$.

Set $K=H_j=K_j\Psi$ in (A-9), then we obtain the mapping $\mathcal{H}_j$: $\mathcal{E}'(U\times I)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^m)$ to $\mathcal{D}'(U\times I\times\boldsymbol{R}^m)$. For any $\varphi\in\mathcal{D}(U)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^m)$, $\mathcal{H}_j(\varphi\otimes\delta_{x_n})$ coincides in $x_n>0$ with

$$\text{(A-10)}\qquad \tilde{\mathcal{H}}_j(\varphi)=(2\pi)^{-n-m}\int_{\boldsymbol{R}^{n-1+m}}\hat{\varphi}(\xi', \sigma)e^{i(x'\cdot\xi'+s\cdot\sigma)}d\xi'd\sigma$$
$$\times\int_{\Gamma(\xi',\sigma')}H_j(x, \xi', \xi_n, \sigma)e^{ix_n\xi_n}d\xi_n$$

where $\Gamma(\xi', \sigma')$ is the complex contour enclosing the pole of $K_j(x, \xi', \xi_n, \sigma)$ and represented in the following figure:



PROOF. We have only to prove $\tilde{\mathscr{H}}_j(\psi\otimes\phi)$ coincides with $\mathscr{H}_j(\psi\otimes\delta_{x_n}\otimes\phi)$ in $x_n>0$ when $\psi\in\mathscr{D}(U)$, $\phi\in\mathcal{S}(\boldsymbol{R}^m)$. However this is clear.

COROLLARY A-4. *If $\varphi\in\mathscr{D}(U)\hat{\otimes}\mathscr{X}$, then $\tilde{\mathscr{H}}_j(\varphi)$ belongs to $\mathscr{E}(U\times I)\hat{\otimes}\mathscr{X}$, where $\mathscr{X}$ is one of the spaces $\mathcal{S}(\boldsymbol{R}^m)$, $\mathcal{O}_M(\boldsymbol{R}^m)$ and $\mathcal{S}'(\boldsymbol{R}^m)$.*

PROOF. It is clear that

$$\int_{\Gamma(\xi',\sigma')} H_j(x,\xi',\xi_n,\sigma)e^{ix_n\cdot\xi_n}d\xi_n\in\mathscr{E}(U\times I)\hat{\otimes}\mathcal{O}_M(\boldsymbol{R}^{n+m-1})\,.$$

The corollary immediately follows from this.

This Corollary A-4 together with Corollary A-2 proves Theorem 1.2.

THEOREM A-5. *For any $\varphi\in\mathscr{D}(U)$, $\xi'\in\boldsymbol{R}^{n-1}$, $s,\sigma\in\boldsymbol{R}^m$ and $\forall N>0$*

$$\text{(A-11)}\qquad \lambda^{-s_N}\Big[e^{-i\lambda(x'\cdot\xi'+s\cdot\sigma)}\mathscr{K}(\varphi\otimes\delta_{x_n}e^{i\lambda(x'\cdot\xi'+s\cdot\sigma)})\Big(x',\frac{x_n}{\lambda}\Big)$$
$$-\sum_j\sum_{|\alpha|<N}\sum_{k<l}\frac{D^{\alpha'}\varphi(x')}{\alpha!}\frac{1}{k!}\Big(\frac{ix_n}{\lambda}\Big)^k$$
$$\int_{\Gamma(\lambda\xi',\lambda\sigma)}D^{\alpha'}_{\lambda\xi'}D^k_{x_n}K_j(x',0,\lambda\xi',\xi_n,\lambda\sigma)e^{ix_n\cdot\xi_n}d\xi_n\Big]$$

is bounded in $\mathscr{E}(U\times I_1\times S)$, where $I_1=0\leqq x_n\leqq 1$

$$S=\{(\xi',\sigma)\in\boldsymbol{R}^{n+m-1},\ 2^{-1}\leqq|\xi'|^2+|\sigma|^2\leqq 2\}\,.$$

PROOF. By definition we have

$$\text{(A-12)}\qquad e^{-i\lambda(x'\cdot\xi'+s\cdot\sigma)}\mathscr{K}(\varphi\otimes\delta_{x_n}e^{i\lambda(x'\cdot\xi'+s\cdot\sigma)})$$
$$=e^{-i\lambda x'\cdot\xi'}\int_{\boldsymbol{R}^n}\Big(K(x,\eta',\eta_n,\lambda\sigma)-\sum_{j=0}^{N-1}H_j(x,\eta',\eta_n,\lambda\sigma)\Big)e^{ix\cdot\eta}\hat{\varphi}(\eta'-\lambda\xi')d\eta$$
$$+\sum_j e^{-i\lambda x'\cdot\xi'}\int_{\boldsymbol{R}^{n-1}}e^{ix'\cdot\eta'}\hat{\varphi}(\eta'-\lambda\xi')d\eta'\int_{\Gamma(\eta',\lambda\sigma)}H_j(x,\eta',\eta_n,\lambda\sigma)e^{ix_n\cdot\eta_n}d\eta_n\,.$$

Taking $\eta'-\lambda\xi'$ as new variables, we have

$$\text{(A-13)}\qquad e^{-i\lambda(x'\cdot\xi'+s\cdot\sigma)}\mathscr{K}(\varphi\otimes\delta_{x_n}e^{i\lambda(x'\cdot\xi'+s\cdot\sigma)})$$
$$=\int_{\boldsymbol{R}^n}\Big(K(x,\eta'+\lambda\xi',\eta_n,\lambda\sigma)-\sum_{j=0}^{N-1}H_j(x,\eta'+\lambda\xi',\eta_n,\lambda\sigma)\Big)\hat{\varphi}(\eta')e^{ix\cdot\eta}d\eta$$

$$+\sum_j \int_{R^{n-1}} e^{ix'\cdot\eta'}\hat{\varphi}(\eta')d\eta' \int_{\Gamma(\eta'+\lambda\xi',\lambda\sigma)} H_j(x,\eta'+\lambda\xi',\eta_n,\lambda\sigma)e^{ix_n\cdot\eta_n}d\eta_n\,.$$

Let $D_{x'}^{\alpha'}$, $\alpha'=(\alpha_1,\cdots,\alpha_{n-1})$, be the differential operator $D_{x_1}^{\alpha_1},\cdots,D_{x_{n-1}}^{\alpha_{n-1}}$, then

$$|(K(x,\eta'+\lambda\xi',\eta_n,\lambda\sigma)-\sum_j H_j(x,\eta'+\lambda\xi',\eta_n,\lambda\sigma))|$$
$$\leqq C(1+|\eta'+\lambda\xi'|+|\eta_n|+|\lambda\sigma|)^{s_N}.$$

Therefore we have

$$\text{(A-14)}\qquad \int_{R^n}\Big(K(x,\eta+\lambda\xi,\eta_n,\lambda\sigma)-\sum_j^{N-1} H(x,\eta'+\lambda\xi',\eta_n,\lambda\sigma)\Big)\hat{\varphi}(\eta')e^{ix\cdot\eta}d\eta$$
$$\leqq C\int_{R^n}(1+|\eta'+\lambda\xi'|+|\eta_n|+|\lambda\sigma|)^{s_N}(1+|\eta'|)^{-|s_N|-n-1}d\eta$$
$$\leqq C\int_{R^{n-1}}(1+|\eta'|)^{|s_N|+1}(1+|\lambda\xi'|+|\lambda\sigma|)^{s_N+1}(1+|\eta'|)^{-|s_N|-n-1}d\eta'$$
$$\leqq C(1+\lambda|\xi'|+\lambda|\sigma|)^{s_N}\int_{R^{n-1}}(1+|\eta'|)^{-n}d\eta'\,.$$

Setting

$$\text{(A-15)}\qquad H_j(x,\eta'+\lambda\xi',\eta_n,\lambda\sigma)-\sum_{k<l}\sum_{|\alpha|<M}\frac{(ix_n)^k}{k!}\frac{(i\eta')^\alpha}{\alpha!}D_{x_n}^k D_{\lambda\xi'}^\alpha H_j(x',0,\lambda\xi',\eta_n,\lambda\sigma)$$
$$=x_n^l R(x,\eta,\lambda\xi',\lambda\sigma)\,,$$

we have

$$\text{(A-16)}\qquad |R(x,\eta,\lambda\xi',\lambda\sigma)|\leqq C|\eta'|^M(1+|\lambda\xi'|+|\eta_n|+|\lambda\sigma|)^{s_j-M}$$

if $|\eta'|<\frac{1}{2}(|\lambda\xi'|+|\lambda\sigma|)$ and otherwise we have

$$|R(x,\eta,\lambda\xi',\lambda\sigma)|\leqq C|\eta'|^M(1+|\lambda\sigma|+|\eta_n|)^{s_j-M}.$$

Hence for $x_n\geqq 0$,

$$\text{(A-18)}\qquad \Big|\int_{R^{n-1}}e^{ix'\cdot\eta'}\hat{\varphi}(\eta')d\eta'\int_{\Gamma(\eta'+\lambda\xi',\lambda\sigma)}H_j(x,\eta'+\lambda\xi',\eta_n,\lambda\sigma)e^{ix_n\cdot\eta_n}d\eta_n$$
$$-\sum_{k<l}\sum_{|\alpha|<M}\frac{(ix_n)^k}{k!}\frac{(iD_{x'})^\alpha\varphi(x')}{\alpha!}$$
$$\int_{\Gamma(\lambda\xi',\lambda\sigma)}D_{x_n}^k D_{\lambda\xi'}^\alpha H_j(x',0,\lambda\xi',\eta_n,\lambda\sigma)e^{ix_n\cdot\eta_n}d\eta_n\Big|$$
$$=x_n^l\Big|\int_{R^n}R(x,\eta,\lambda\xi',\lambda\sigma)\hat{\varphi}(\eta')e^{ix\cdot\eta}d\eta\Big|$$
$$\leqq Cx_n^l\Big[\int_{|\eta'|<\frac{1}{2}(\lambda|\xi'|+|\lambda\sigma|)}|\eta'|^M(1+|\lambda\xi'|+|\eta_n|+\lambda|\sigma|)^{s_j-M}(1+|\eta'|)^{-M-n-1}d\eta$$

$$+\int_{|\eta'|>\frac{1}{2}(\lambda|\xi'|+\lambda|\sigma|)} |\eta'|^M(1+|\lambda\sigma|+|\eta_n|)^{s_j-M}(1+|\eta'|)^{-M-n-s_j-1}d\eta\Big]$$

$$\leqq Cx_n^l\Big[(1+|\lambda\xi'|+|\lambda\sigma|)^{s_j-M+1}\int_{R^n}|\eta'|^M(1+|\eta'|)^{-M-n-1}d\eta'$$

$$+(1+|\lambda\sigma|)^{s_j-M+1}\int_{|\eta'|>\frac{1}{2}(\lambda|\xi'|+\lambda|\sigma|)}|\eta'|^M(1+|\eta'|)^{-2M-n-1}d\eta\Big]$$

$$\leqq Cx_n^l\Big[(1+\lambda|\xi'|+\lambda|\sigma|)^{s_j-M+1}+(1+\lambda|\sigma|)^{s_j-M+1}\Big(1+\frac{1}{2}\lambda|\xi'|\Big)^{-M+s_j+1}\Big]$$

$$\leqq Cx_n^l(1+\lambda|\xi'|+\lambda|\sigma|)^{s_j-M+1}.$$

Thus we have proved

$$\text{(A-19)}\qquad \lambda^{-s_N}\Big[e^{-\lambda(x'\cdot\xi'+s\cdot\sigma)}\mathcal{K}(\varphi\otimes\delta_{x_n}e^{i\lambda(x'\cdot\xi'+s\cdot\sigma)})\Big(x',\frac{x_n}{\lambda}\Big)$$

$$-\sum_j\sum_{|\alpha'|<N}\sum_{k<l}\frac{D_{x'}^{\alpha'}\varphi(x')}{\alpha!}\frac{1}{k!}\Big(\frac{ix_n}{\lambda}\Big)^k$$

$$\int_{\Gamma(\lambda\xi',\lambda\sigma)}D_{x_n}^kD_{\lambda\xi'}^{\alpha'}H_j(x',0,\lambda\xi',\eta_n,\lambda\sigma)e^{ix_n\cdot\eta_n}d\eta_n\Big]$$

$$\leqq C((1+|\lambda|))^{-M-l+1},$$

when $(x,\xi,\sigma)\in U\times I_1\times S$. The left hand side of (A-19) is bounded in $\mathcal{C}(U\times I_1\times S)$. That is

$$\text{(A-20)}\qquad e^{-i\lambda(x'\cdot\xi'+s\cdot\sigma)}\mathcal{K}(\varphi\otimes\delta_{x_n}e^{i\lambda(x'\cdot\xi'+s\cdot\sigma)})\Big(x',\frac{x_n}{\lambda}\Big)$$

has an asymptotic expansion in $\mathcal{C}(U\times I_1\times S)$.

If $1\leqq j\leqq n-1$

$$\text{(A-21)}\qquad D_{x_j}e^{-i\lambda(x'\cdot\xi'+s\cdot\sigma)}\mathcal{K}(\varphi\otimes\delta_{x_n}e^{i\lambda(x'\cdot\xi'+s\cdot\sigma)})$$

$$=e^{-i\lambda(x'\cdot\xi'+s\cdot\sigma)}(D_{x_j}\mathcal{K})(\varphi\otimes\delta_{x_n}e^{i\lambda(x'\cdot\xi'+s\cdot\sigma)})$$

$$+e^{-i\lambda(x'\cdot\xi'+s\cdot\sigma)}\mathcal{K}(D_{x_j}\varphi\otimes\delta_{x_n}e^{i\lambda(x'\cdot\xi'+s\cdot\sigma)})$$

where $(D_{x_j}\mathcal{K})$ is the operator with the kernel $D_{x_j}K$ instead of K. Thus

$$D_{x_j}e^{-i\lambda(x'\cdot\xi'+s\cdot\sigma)}\mathcal{K}(\varphi\otimes\delta_{x_n}e^{i\lambda(x'\cdot\xi'+s\cdot\sigma)})\Big(x',\frac{x_n}{\lambda}\Big)$$

has an asymptotic expansion in $\mathcal{C}(U\times I_1\times S)$.

$$\text{(A-22)}\qquad D_{x_n}e^{-i\lambda(x'\cdot\xi'+s\cdot\sigma)}\mathcal{K}(\varphi\otimes\delta_{x_n}e^{i\lambda(x'\cdot\xi'+s\cdot\sigma)})$$

$$=e^{-i\lambda(x'\cdot\xi'+s\cdot\sigma)}(D_{x_n}\mathcal{K})(\varphi\otimes\delta_{x_n}e^{i\lambda(x'\cdot\xi'\cdot+s\cdot\sigma)})$$

$$+e^{-i\lambda(x'\cdot\xi'+s\cdot\sigma)}(\mathcal{K}\xi_n)(\varphi\otimes\delta_{x_n}e^{i\lambda(x'\cdot\xi'+s\cdot\sigma)})$$

where $\mathcal{K}\xi_n$ is the operator of the above type with kernel $K(x,\xi,\sigma)\xi_n$. Hence $D_{x_n}e^{-i\lambda(x'\cdot\xi'+s\cdot\sigma)}\mathcal{K}(\varphi\otimes\delta_{x_n}e^{i\lambda(x'\cdot\xi'+s\cdot\sigma)})$ has an asymptotic expansion in $\mathcal{C}(U\times I_1\times S)$.

(A-23)
$$D_{\xi'_j} e^{-i\lambda(x'\cdot\xi'+s\cdot\sigma)}\mathcal{K}(\varphi\otimes\delta_{x_n}e^{i\lambda(x'\cdot\xi'+s\cdot\sigma)})$$
$$=\lambda x'_j e^{-i\lambda(x'\cdot\xi'+s\cdot\sigma)}\mathcal{K}(\varphi\otimes\delta_{x_n}e^{i\lambda(x'\cdot\xi'+s\cdot\sigma)})$$
$$-\lambda e^{-i\lambda(x'\cdot\xi'+s\cdot\sigma)}\mathcal{K}(x_j\varphi\otimes\delta_{x_n}e^{i\lambda(x'\cdot\xi'+s\cdot\sigma)})$$

and

(A-24)
$$D_{\sigma_j}e^{-i\lambda(x'\cdot\xi'+s\cdot\sigma)}\mathcal{K}(\varphi\otimes\delta_{x_n}e^{i\lambda(x'\cdot\xi'+s\cdot\sigma)})$$
$$=\lambda e^{-i\lambda(x'\cdot\xi'+s\cdot\sigma)}(D_\sigma\mathcal{K})(\varphi\otimes\delta_{x_n}e^{i\lambda(x'\cdot\xi'+s\cdot\sigma)})$$

where $D_{\sigma_j}\mathcal{K}$ is the operator with the kernel $D_\sigma K(x, \xi, \sigma)$. Therefore both $D_{\xi'_j} e^{-i\lambda(x'\cdot\xi'+s\cdot\sigma)}\mathcal{K}(\varphi\otimes\delta_{x_n}e^{i\lambda(x'\cdot\xi'+s\cdot\sigma)})$ and $D_{\sigma_j}e^{-i\lambda(x'\cdot\xi'+s\cdot\sigma)}\mathcal{K}(\varphi\otimes\delta_{x_n}e^{i\lambda(x'\cdot\xi'+s\cdot\sigma)})$ has an asymptotic expansion in $C^1(U\times\bar{I}_+\times S)$. Repeating these process, we can prove that $e^{-i\lambda(x'\cdot\xi'+s\cdot\sigma)}\mathcal{K}(\varphi\otimes\delta_{x_n}e^{i\lambda(x'\cdot\xi'+s\cdot\sigma)})$ has an asymptotic expansion in $\mathcal{E}(U\times\bar{I}_+\times S)$.

We shall denote by I_+ the interval $(0, 1)$ and by $\bar{I}_+$ $[0, 1)$.

DEFINITION A-6. A continuous linear operator

$$\mathcal{K}:\ \mathcal{D}(U)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^m)\rightarrow\mathcal{E}(U\times\bar{I}_+)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^m)$$

is called a β-pseudo Poisson kernel if the following conditions are satisfied:

(i) $e^{-i\lambda(x'\cdot\xi'+s\cdot\sigma)}\mathcal{K}(\varphi e^{i(x'\cdot\xi'+s\cdot\sigma)})$ is a function independent of s, which we shall denote by $k(x, \lambda\xi', \lambda\sigma)$.

(ii) $k\left(x', \frac{x_n}{\lambda}, \lambda\xi', \lambda\sigma\right)$ has an asymptotic expansion in $\mathcal{E}(U\times\bar{I}_+\times S)$

$$k\left(x', \frac{x_n}{\lambda}, \lambda\xi', \lambda\sigma\right)\sim\sum_j k_j(x', x_n, \xi', \sigma)\lambda^{-z_j}$$

where $\{z_j\}$ is a sequence of complex numbers whose real parts decreases monotonically to $-\infty$.

(iii) $\mathcal{K}$ has a unique continuous extension mapping

$$\mathcal{E}'(U)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^m)\quad\text{to}\quad\mathcal{E}(U\times I_+)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^m).$$

THEOREM A-7. *Let $\mathcal{P}$ be a β-pseudo-differential operator in $U\times I$, $I=(-1, 1)$, and let $p(x, \xi, \sigma)\sim\sum_{j=0}^{\infty}p_j(x, \xi, \sigma)$ be its Fourier integral kernel and its asymptotic expansion. Then for any $\varphi\in\mathcal{D}(U\times I)$ and for any f in $\mathcal{D}(U)$, we have the asymptotic expansion in $\mathcal{E}(U\times\bar{I}_+\times S_2)$, where $S_2=\left\{(\xi, \sigma):\ \frac{1}{2}\leqq|\xi|^2+|\sigma|^2\leqq 2,\ \frac{1}{2}\leqq|\xi'|^2+|\sigma|^2\leqq 2\right\}$*

$$e^{-i\lambda(x'\cdot\xi'+s\cdot\sigma)}\mathcal{K}(f\cdot\mathcal{P}(\varphi e^{i\lambda(x'\cdot\xi'+s\cdot\sigma)})|_{x_n=0}\otimes\delta_{x_n})$$

$$\sim\sum_{k,\beta}\sum_{j,\alpha}\sum_{l}\frac{1}{\alpha!}\frac{1}{l!}\left(\frac{ix_n}{\lambda}\right)^l\int_{\Gamma(\lambda\xi',\lambda\sigma)}D^\alpha_{\lambda\xi'}D^l_{x_n}K_j(x', 0, \lambda\xi', \eta_n, \lambda\sigma)e^{ix_n\cdot\eta_n}d\eta_n$$

$$\times D_{x'}^{\alpha}\Big(\frac{1}{\beta\,!} f D_x^{\beta}\varphi(x', 0) D_{\lambda\xi}^{\beta} p_k(x', 0, \lambda\xi, \lambda\sigma)\Big).$$

PROOF.

$$e^{-i\lambda(x\cdot\xi+s\cdot\sigma)}\mathcal{P}(\varphi e^{i\lambda(x\cdot\xi+s\cdot\sigma)})$$
$$\sim\sum_{k,\beta}\frac{1}{\beta\,!}D^k\varphi(x)D_{\lambda\xi}^{l}p_k(x, \lambda\xi, \lambda\sigma) \qquad \text{in } \mathcal{E}(U\times I\times S).$$

Therefore

$$\text{(A-16)} \qquad e^{-i\lambda(x\cdot\xi+s\cdot\sigma)}\mathcal{P}(\varphi e^{i\lambda(x\cdot\xi+s\cdot\sigma)})|_{x_n=0}$$
$$\sim\sum_{k,\beta}\frac{1}{\beta\,!}D_x^{\beta}\varphi(x', 0)D_{\lambda\xi}^{\beta}p_k(x', 0, \lambda\xi, \lambda\sigma) \qquad \text{in } \mathcal{E}(U\times S).$$

Hence

$$e^{-i\lambda(x'\cdot\xi'+s\cdot\sigma)}\mathcal{K}(f\mathcal{P}(\varphi e^{i\lambda(x\cdot\xi+s\cdot\sigma)})|_{x_n=0}\otimes\delta_{x_n})\Big(x', \frac{x_n}{\lambda}\Big)$$
$$=e^{-i\lambda(x'\cdot\xi'+s\cdot\sigma)}\mathcal{K}(fe^{i\lambda(x'\cdot\xi'+s\cdot\sigma)}(e^{-i\lambda(x\cdot\xi+s\cdot\sigma)}\mathcal{P}(\varphi e^{i\lambda(x\cdot\xi+s\cdot\sigma)})|_{x_n=0})\otimes\delta_{x_n})\Big(x', \frac{x_n}{\lambda}\Big)$$
$$\sim\sum_{k,\beta}e^{-i\lambda(x'\cdot\xi'+s\cdot\sigma)}\mathcal{K}\Big(fe^{i\lambda(x'\cdot\xi'+s\cdot\sigma)}\frac{1}{\beta\,!}D_x^{\beta}\varphi(x', 0)D_{\lambda\xi}^{\beta}p_k(x', 0, \lambda\xi, \lambda\sigma)\otimes\delta_{x_n}\Big)\Big(x', \frac{x_n}{\lambda}\Big)$$
$$\sim\sum_{k,\beta}\sum_{j,\alpha}\sum_{l}\frac{1}{\alpha\,!}\frac{1}{l\,!}\Big(\frac{ix_n}{\lambda}\Big)^l\int_{\Gamma(\lambda\xi',\,\lambda\sigma)}D_{\lambda\xi'}^{\alpha'}D_{x_n}^{l}K_j(x', 0, \lambda\xi', \eta_n, \lambda\sigma)e^{ix_n\cdot\eta_n}d\eta_n$$
$$\times D_{x'}^{\alpha'}\Big(\frac{1}{\beta\,!}fD_x^{\beta}\varphi(x', 0)D_{\lambda\xi}^{\beta}p_k(x', 0, \lambda\xi, \lambda\sigma)\Big).$$

Especially if $\varphi\equiv 1$ on $U'\times I'$, $f\equiv 1$ on U', where $U'\subset U$ and $I'\subset I$, then for $x\in U'\times I'$,

$$e^{-i\lambda(x'\cdot\xi'+s\cdot\sigma)}\mathcal{K}(f\mathcal{P}(\varphi e^{i\lambda(x\cdot\xi+s\cdot\sigma)})|_{x_n=0}\otimes\delta_{x_n})\Big(x', \frac{x_n}{\lambda}\Big)$$
$$\sim\sum_{\alpha', j, l}\frac{1}{\alpha\,!}\frac{1}{l\,!}\Big(\frac{ix_n}{\lambda}\Big)^l D_{x'}^{\alpha'}p_k(x', 0, \lambda\xi, \lambda\sigma)$$
$$\int_{\Gamma(\lambda\xi',\,\lambda\sigma)}D_{\lambda\xi'}^{\alpha'}D_{x_n}^{l}K_j(x', 0, \lambda\xi', \eta_n, \lambda\sigma)e^{ix_n\cdot\eta_n}d\eta_n\,.$$

THEOREM A-8. *Let $_q$ be a β-pseudo-differential operator mapping $\mathcal{D}(U)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^m)$ into $\mathcal{E}(U)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^m)$ which is defined by the Fourier integral:*

$$_qT=(2\pi)^{-n-m+1}\int_{\boldsymbol{R}^{n-1+m}}q(x', \xi', \sigma)\hat{T}(\xi', \sigma)e^{ix'\cdot\xi'+s\cdot\sigma}d\xi\,d\sigma$$

where $q(x', \xi', \sigma)$ admits the asymptotic expansion

$$q(x', \xi', \sigma)\sim\sum_j q_j(x', \xi', \sigma).$$

Let K be the β-pseudo-Poisson operator K treated in Theorem A-5. *Then for any $f\in\mathcal{D}(U)$, the mapping*

$$\begin{array}{ccc} \mathcal{D}(U)\widehat{\otimes}\mathcal{S}'(\boldsymbol{R}^m) & \longrightarrow & \mathcal{E}(U\times I)\widehat{\otimes}\mathcal{S}'(\boldsymbol{R}^m) \\ \downarrow & & \downarrow \\ \varphi & \longrightarrow & \mathcal{K}f_q\varphi \end{array}$$

is a β-pseudo-Poisson operator.

Moreover, if $\varphi\equiv f\equiv 1$ near x,

$$e^{-i\lambda(x'\cdot\xi'+s\cdot\sigma)}\mathcal{K}fQ(\varphi e^{i\lambda(x'\cdot\xi'+s\cdot\sigma)})$$
$$\sim\sum\frac{1}{\alpha!}D^{\alpha}_{x'}\Big(D^{\beta}_{x'}\varphi(x')\frac{D^{\beta}_{\xi'}}{\beta!}q_j(x',\xi',\sigma)\Big)\frac{1}{k!}\Big(\frac{ix_n}{\lambda}\Big)^k D^{\alpha}_{\lambda\xi'}$$
$$\int_{\Gamma(\lambda\xi',\lambda\sigma)}D^k_n k_j(x',0,\lambda\xi',\xi_n,\lambda\sigma)e^{ix_n\cdot\xi_n}d\xi_n\,.$$

This is a direct consequence of Theorem A-5.

THEOREM A-9. *In addition to the assumptions of Theorem* A-8 *we assume that* $g\in\mathcal{D}(U)$, *which is identically* 1 *near* x, *and that* $\mathcal{P}$ *is a β-pseudo-differential operator mapping* $\mathcal{D}(U\times I)\widehat{\otimes}\mathcal{S}'(\boldsymbol{R}^m)$ *into* $\mathcal{D}(U\times I)\widehat{\otimes}\mathcal{S}'(\boldsymbol{R}^m)$, *defined by the Fourier integral operator*

$$\mathcal{P}T=(2\pi)^{-n-m}\int_{\boldsymbol{R}^{n+m}}p(x,\xi,\sigma)\hat{T}(\xi,\sigma)e^{i(x\cdot\xi+s\cdot\sigma)}d\xi\,d\sigma\,,$$

where $p(x,\xi,\sigma)$ *admits the asymptotic expansion:*

$$p(x,\xi,\sigma)\sim\sum_j p_j(x,\xi,\sigma)\,.$$

Then we have the asymptotic expansion:

$$e^{-i\lambda(x'\cdot\xi'+s\cdot\sigma)}\mathcal{K}f_q\,g(\mathcal{P}(\varphi e^{i\lambda(x\cdot\xi+s\cdot\sigma)})|_{x_n=0}\otimes\delta_{\partial M})$$
$$\sim\sum\frac{1}{\alpha!}D^{\alpha}_{x'}\Big(\frac{1}{\beta!}D^{\beta}_{\xi'}fq_k(x',\lambda\xi',\lambda\sigma)D^{\beta}_{x'}gp_j(x',0,\lambda\xi,\lambda\sigma)\Big)\frac{1}{k!}\Big(\frac{ix_n}{\lambda}\Big)^k$$
$$\int_{\Gamma(\lambda\xi',\lambda\sigma)}D^k_nD^{\alpha}_{\lambda\xi}k_j(x',0,\lambda\xi',\eta_n,\lambda\sigma)e^{ix_n\cdot\eta}d\eta$$

uniform in $\mathcal{E}(\tilde{S})$.

$$\tilde{S}=\Big\{(x,\xi,\sigma):\ x\in U\times I_1,\ \frac{1}{2}<|\xi'|+|\sigma|<2,\ |\xi_n|<1\Big\}.$$

This is a consequence of Theorem A-7 and Theorem A-8.

Appendix II

Here we collect some properties of tangential β-operators. Proofs are left to the readers.

We shall denote the partial Fourier transform of $\varphi\in\mathcal{S}(\boldsymbol{R}^n)$ by

$$\tilde{\varphi}(\xi', x_n)=\int_{R^{n-1}}\varphi(x', x_n)e^{-ix'\cdot\xi'}dx'.$$

Similarly, for any $u\in\mathcal{S}(\boldsymbol{R}^{n+m})$, we shall denote

$$\tilde{u}(\xi', x_n, \sigma)=\int_{R^{n+m-1}}u(x', x_n, s)e^{-i(x'\cdot\xi'+s\cdot\sigma)}dx'ds$$

DEFINITION (A-II-1). A continuous linear mapping $\mathcal{P}$ from $\mathcal{D}(\boldsymbol{R}^n)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^m)$ to $\mathcal{E}(\boldsymbol{R}^n)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^m)$ defined by

$$\mathcal{P}(T)(x, s)=(2\pi)^{-m-n+1}\int_{R^{m+n-1}}p(x, \xi', \sigma)\tilde{T}(\xi', x_n, \sigma)e^{i(x'\cdot\xi'+s\cdot\sigma)}d\xi'd\sigma \tag{1}$$

is called a tangential β-operator of order z_0, if $p(x, \xi', \sigma)$ satisfies the following properties:

(i) $p\in\mathcal{E}(\boldsymbol{R}^n)\hat{\otimes}\mathcal{O}_{\mathcal{M}}(\boldsymbol{R}^{n+m-1})$.

(ii) There is a sequence of functions $p_j(x, \xi', \sigma)$, $j=0, 1, 2, \cdots$. The function $p_j(x, \xi', \sigma)$ is homogeneous in (ξ', σ) of degree $z_j=s_j+it_j\in\boldsymbol{C}$. It is C^∞ in x and in $(\xi', \sigma)\neq 0$. The sequence $\{s_j\}$ decreases monotonically to $-\infty$.

(iii) For any multi-indices $\alpha'=(\alpha_1, \alpha_2, \cdots, \alpha_{n-1})$ and $\beta=(\beta_1, \beta_2, \cdots, \beta_m)$

$$(|\xi'|+|\sigma|)^{-s_N+|\alpha'|+|\beta|}D_{\xi'}^{\alpha'}D_\sigma^\beta\Big(p(x, \xi', \sigma)-\sum_{j=0}^{N-1}p_j(x, \xi', \sigma)\Big) \tag{2}$$

is bounded in $\mathcal{E}(\boldsymbol{R}^n)$ when (ξ', σ) runs in $|\xi'|+|\sigma|\geqq 1$.

THEOREM (A-II-2). *A continuous linear mapping $\mathcal{P}$ from $\mathcal{D}(\boldsymbol{R}^n)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^m)$ to $\mathcal{E}(\boldsymbol{R}^n)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^m)$ is a tangential β-operator of order z_0 if and only if the following properties hold:*

(i) *There is a sequence $z_j=s_j+it_j$, $j=0, 1, 2, \cdots$, of complex numbers whose real parts s_j decrease monotonically to $-\infty$.*

(ii) *For any $f\in\mathcal{D}(\boldsymbol{R}^n)$ and for any real function g in a compact set $\mathcal{K}$ in $\mathcal{E}(\boldsymbol{R}^{n-1})$ whose gradient dg does not vanish on the support of f, $e^{-i\lambda(\rho g+s\cdot\sigma)}\mathcal{P}(fe^{i\lambda(\rho g+s\cdot\sigma)})$ is independent of s and with some functions $p_j(f, \rho g, x, \sigma)$, $j=0, 1, 2, \cdots,$*

$$\lambda^{-s_N}\Big(e^{-i\lambda(\rho g+\sigma\cdot s)}\mathcal{P}(fe^{i\lambda(\rho g+s\cdot\sigma)})-\sum_{j=0}^{N-1}p_j(f, \rho g, x, \sigma)\lambda^{z_j}\Big) \tag{3}$$

remains bounded in $\mathcal{E}(\boldsymbol{R}^n\times S_1)$ when $\lambda\to\infty$ and g runs in $\mathcal{K}$, where we denoted by S_1 the set $S_1=\Big\{(\rho, \sigma)\in\boldsymbol{R}^{m+1}: \frac{1}{2}\leqq\rho^2+|\sigma|^2\leqq 2\Big\}$.

(iii) *For any function $f\in\mathcal{D}(\boldsymbol{R}^n)$ and $\varphi\in\mathcal{D}(\boldsymbol{R})$*

$$\mathcal{P}(\varphi(x_n)f)=\varphi(x_n)\mathcal{P}(f). \tag{4}$$

COROLLARY (A-II-3). *If $\mathcal{P}$ is defined by (1), we have the asymptotic expansion*

$$e^{-i\lambda(\rho g+s\cdot\sigma)}\mathscr{P}(fe^{i\lambda(\rho g+s\cdot\sigma)})$$
$$\sim\sum_{j,\alpha'}\frac{1}{\alpha'!}\lambda^{z_j-|\alpha'|}D^{\alpha'}_{\rho\xi'}p_j(x,\rho\xi'_x,\sigma)(iD_{x'})^{\alpha'}(fe^{i\lambda\rho h_{x'}}),$$

where

$$\xi'_{x'}=\operatorname{grad}g(x'),\quad h_{x'}(y')=g(y')-g(x')-\langle y'-x',\xi'_{x'}\rangle,\quad x=(x',x_n).$$

DEFINITION (A-II-4). We call the formal sum $\sigma_{\mathscr{P}}(f-g)=\sum_j p_j(f,\rho g,x,\sigma)\lambda^{z_j}$ the symbol of $\mathscr{P}$.

REMARK (A-II-5). Theorem (A-II-2) enables us to define tangential β-operators operating in $\mathscr{D}(U)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^3)$ in § 4.

THEOREM (A-II-6). *If $\mathscr{P}$ is an operator defined by* (1), *then*

(i) *$\mathscr{P}(T)$ has its support in $\boldsymbol{R}^n_-\times\boldsymbol{R}^m$ for any T in $\mathscr{D}(\boldsymbol{R}^n_-)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^m)$.*

(ii) *$\mathscr{P}$ induces a mapping from $\mathscr{D}(\overline{\boldsymbol{R}^n_+})\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^m)$ to $\mathcal{E}(\overline{\boldsymbol{R}^n_+})\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^m)$.*

We shall denote the induced mapping by the same symbol $\mathscr{P}$.

DEFINITION (A-II-7). The space $H^{(p,q)}(\boldsymbol{R}^{n+m})$ is the completion of $\mathcal{S}(\boldsymbol{R}^{n+m})$ by the norm

$$\|u\|_{(p,q)}=\Big[\int_{\boldsymbol{R}^{n+m}}|\hat{u}(\xi,\sigma)|^2(1+|\xi|^2+|\sigma|^2)^p(1+|\xi'|^2+|\sigma|^2)^q\,d\xi\,d\sigma\Big]^{\frac{1}{2}}.$$

The space $H^{(p,q)}(\boldsymbol{R}^n)$ is the completion of $\mathcal{S}(\boldsymbol{R}^n)$ by the norm

$$\|\varphi\|_{(p,q)}=\Big[\int_{\boldsymbol{R}^n}|\varphi(\xi)|^2(1+|\xi|^2)^p(1+|\xi'|^2)^q\,d\xi\Big]^{\frac{1}{2}}.$$

THEOREM (A-II-8). *If $\mathscr{P}$ is a tangential operator of order $z_0=s_0+it_0$ and if $\varphi\in\mathscr{D}(\boldsymbol{R}^n)$, then*

(i) *For any $(p,q)\in\boldsymbol{R}^2$, there is a constant $C>0$ such that for any $(\xi',\sigma)\in\boldsymbol{R}^{n+m-1}$, and for any $u\in\mathcal{S}(\boldsymbol{R}^{n+m})$ we have*

$$\|e^{-i(x'\cdot\xi'+s\cdot\sigma)}\varphi\mathscr{P}(ue^{i(x'\cdot\xi'+s\cdot\sigma)})\|_{(p,q)}\leqq C(1+|\xi'|+|\sigma|)^{|s_0|}\|u\|_{(p,q+s_0)}.\tag{7}$$

(ii) *For any $(p,q)\in\boldsymbol{R}^2$ there is a constant $C>0$ such that for any $(\xi',\sigma)\in\boldsymbol{R}^{n+m-1}$ and for any $f\in\mathcal{S}(\boldsymbol{R}^n)$ we have*

$$\|e^{-i(x'\cdot\xi'+s\cdot\sigma)}\varphi\mathscr{P}(fe^{i(x'\cdot\xi'+s\cdot\sigma)})\|_{(p,q)}\leqq C(1+|\xi'|+|\sigma|)^{|s_0|}\|f\|_{(p,q+s_0)}.\tag{8}$$

THEOREM (A-II-9). *In addition to the assumption of Theorem* (A-II-8), *assume that s_0 is smaller than 0, then*

(i) *For any $(p,q)\in\boldsymbol{R}^2$ and $b\in[s_0,-s_0]$, there is a constant $C>0$ such that for any $(\xi',\sigma)\in\boldsymbol{R}^{n+m-1}$ and for any $u\in\mathcal{S}(\boldsymbol{R}^{n+m})$ we have*

$$\|e^{-i(x'\cdot\xi'+s\cdot\sigma)}\varphi\mathscr{P}(ue^{i(x'\cdot\xi'+s\cdot\sigma)})\|\leqq C(1+|\xi'|+|\sigma|)^{-b}\|u\|_{(p,q+b)}.\tag{9}$$

(ii) *For any $(p,q)\in\boldsymbol{R}^2$ and for any b in $[s_0,-s_0]$, there is a constant $C>0$ such that for any $(\xi',\sigma)\in\boldsymbol{R}^{n+m-1}$ and for any f in $\mathcal{S}(\boldsymbol{R}^n)$,*

(10) $$\|e^{-i(x'\cdot\xi'+s\cdot\sigma)}\varphi\mathcal{P}(fe^{i(x'\cdot\xi'+s\cdot\sigma)})\|_{(p,q)}\leqq C(1+|\xi'|+|\sigma|)^{-b}\|f\|_{(p,q+b)}\,.$$

(iii) *For any* $(p,q)\in\boldsymbol{R}^2$ *and for any* $0\leqq b\leqq -s_0$, *there is a constant* $C>0$ *such that for any* $\sigma\in\boldsymbol{R}^m$, f *in* $\mathcal{D}(\boldsymbol{R}^n)$ *we have the following estimate*

(11) $$\|e^{-is\cdot\sigma}\varphi\mathcal{P}(fe^{is\cdot\sigma})\|_{(p,q)}\leqq C(1+|\sigma|)^{s_0-b}\|f\|_{(p,q+b)}\,.$$

THEOREM (A-II-10). *To every tangential* β*-operator there is one and only one tangential* β*-operator* ${}^t\mathcal{P}$, *called the formal adjoint of* $\mathcal{P}$, *such that for any* u, v *in* $\mathcal{S}(\boldsymbol{R}^n)\hat{\otimes}\mathcal{S}(\boldsymbol{R}^m)$, $\langle\mathcal{P}u, v\rangle=\langle u, {}^t\mathcal{P}v\rangle$. *If* $\mathcal{P}$ *is the operator defined by* (1), *we have the asymptotic expansion in* $\mathcal{E}(U)$

$$e^{-i(x'\cdot\xi'+s\cdot\sigma)}\,{}^t\mathcal{P}(\varphi e^{i(x'\cdot\xi'+s\cdot\sigma)})\sim\sum_{\alpha',j}\frac{1}{\alpha'!}(-D_{x'})^{\alpha'}p_j^{(\alpha')}(x,-\xi',-\sigma)$$

where φ *is a function in* $\mathcal{D}(\boldsymbol{R}^n)$ *which is identically* 1 *in some neighbourhood of the closure of an open set* U *in* $\boldsymbol{R}^n$.

THEOREM (A-II-11). *Any tangential* β*-operator can be extended to a continuous linear mapping from* $\mathcal{E}'(\boldsymbol{R}^n)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^m)$ *to* $\mathcal{D}'(\boldsymbol{R}^{n+m})$.

THEOREM (A-II-12). *Let* $\mathcal{P}$ *be a tangential* β*-operator defined by definition* (A-II-1). *Then for any* $T\in\mathcal{D}(\boldsymbol{R}^{n-1})\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^m)$,

$$D_{x_n}^l\mathcal{P}(T\otimes D_{x_n}^l\delta(x_n))=(2\pi)^{-m-n+1}\int_{\boldsymbol{R}^{m+n-1}}D_{x_n}^{l+k}p(x', x_n, \xi', \sigma)\hat{T}(\xi',\sigma)d\xi'd\sigma\,,$$

where $\delta(x_n)$ *is the Dirac's* δ*-distribution in* x_n*-space.*

THEOREM (A-II-13). *Let* S *be the set* $S=\left\{(\xi',\sigma)\in\boldsymbol{R}^{n+m-1}:\ \frac{1}{2}\leqq|\xi'|^2+|\sigma|^2\leqq 2\right\}$ *and let* $\mathcal{P}$ *be a tangential* β*-operator defined in Definition* (A-II-1). *Then for any* φ *in* $\mathcal{D}(\boldsymbol{R}^n)$ *and for any* (ξ,σ) *in* S, *we have the asymptotic expansion in* $\mathcal{E}(\boldsymbol{R}^n\times S)$

$$e^{-i\lambda(x'\cdot\xi'+s\cdot\sigma)}\mathcal{P}(\varphi e^{i\lambda(x'\cdot\xi'+s\cdot\sigma)})\Big(x', \frac{x_n}{\lambda}\Big)$$
$$\sim\sum_{j,\beta_1,\beta_2,\alpha'}\frac{1}{\beta_1!\,\beta_2!\,\alpha'!}\lambda^{z_j-\beta_1-\beta_2-|\alpha'|}x_n^{\beta_1+\beta_2}D_{\xi'}^{\alpha'}D_{x_n}^{\beta_1}p(x',0,\xi',\sigma)$$
$$\times(iD_{x'})^{\alpha'}D_{x_n}^{\beta_2}\varphi(x',0)\,.$$

THEOREM (A-II-14). *Let* $\mathcal{K}$ *be a* β*-pseudo Poisson kernel defined by Fourier integral kernel* $k(x,\xi',\sigma)$ *with the expansion*

$$k\Big(x', \frac{x_n}{\lambda}, \lambda\xi', \lambda\sigma\Big)\sim\sum_{j=0}^{\infty}k_j(x', x_n, \xi', \sigma)\lambda^{z'j}\,.$$

(See Definition A-6). *Then for any* f *in* $\mathcal{D}(\boldsymbol{R}^n)$, *the mapping* $\varphi\to\mathcal{P}f\mathcal{K}(\varphi)$ *is a* β*-pseudo Poisson kernel. Let* U *be an open set in* $\overline{\boldsymbol{R}_+^n}$. *If* f *and* φ *are identically* 1 *in some neighbourhood of* $\bar{U}$, *then we have the asymptotic expansion* *in* $\mathcal{E}(U)$

$$e^{-i\lambda(x'\cdot\xi'+s\cdot\sigma)}\mathcal{P}f\mathcal{K}(\varphi e^{i\lambda(x'\cdot\xi'+s\cdot\sigma)})\left(x', \frac{x_n}{\lambda}\right)$$

$$\sim \sum_{\alpha',\beta_1,\beta_2} \frac{1}{\alpha'!\,\beta_1!\,\beta_2!} \lambda^{z_j+z'_e-|\alpha'|-\beta_1-\beta_2} x_n^{\beta_1+\beta_2} D_{\xi'}^{\alpha'} D_{x_n}^{\beta_1} p_j(x', 0, \xi', \sigma)$$

$$\times (iD_{x'})^{\alpha'} D_{x_n}^{\beta_2} k_l(x', 0, \xi', \sigma).$$

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