Parabolic and pseudo-parabolic partial differential equations*

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§ 1. Introduction.

Let G be a bounded domain in an Euclidean n-space and $\bar{G}$ its closure. Let $C^k(G)$, $0 \leq k < \infty$, be the class of real-valued functions defined and k-times continuously differentiable in G and $C_0^k(G)$ the subset of $C^k(G)$ consisting of those functions with compact support in G. As usual [1-4], we denote by $\hat{H}^k(G)$ the pre-Hilbert space consisting of functions in $C^k(G)$ with finite k-fold Dirichlet norm, $\| \; \|_k$, and denote by $H^k(G)$ the Hilbert space being the completion of $\hat{H}^k(G)$ under the norm $\| \; \|_k$. In a completely similar way one defines the pre-Hilbert space $\hat{H}_0^k(G)$ and the Hilbert space $H_0^k(G)$. In the following discussions the domain G will be fixed. We shall for simplicity write H_0^k for $H_0^k(G)$ etc.. It may be noted that H_0^0 is the space $L_2(G)$.

Consider the two self-adjoint elliptic partial differential operators,

$$(1.1) \qquad L \equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(l_{ij}(x) \frac{\partial}{\partial x_j} \right) - l(x) ,$$

$$M \equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(m_{ij}(x) \frac{\partial}{\partial x_j} \right) - m(x) .$$

It will be assumed that the given real-valued functions $l_{ij}(x)$, $l(x)$, $m_{ij}(x)$ and $m(x)$ are bounded measurable in G and that $l(x) \geq 0$, $m(x) \geq 0$ almost everywhere in G. Further we shall restrict L and M to be elliptic in the sense that there are constants k_L, K_L, k_M and K_M such that almost everywhere in G

$$(1.2) \qquad k_L \sum_{i=1}^{n} (\xi_i)^2 \leq \sum_{i,j=1}^{n} l_{ij}(x)\xi_i\xi_j \leq K_L \sum_{i=1}^{n} (\xi_i)^2 ,$$

$$k_M \sum_{i=1}^{n} (\xi_i)^2 \leq \sum_{i,j=1}^{n} m_{ij}(x)\xi_i\xi_j \leq K_M \sum_{i,j=1}^{n} (\xi_i)^2 ,$$

for all real vectors ξ.

We shall be concerned with the solutions of the following initial value problems. Let $u_0(x)$ be a given function in $H_0^1 \cap H^2$. Find the function $u_\lambda(t, x)$, $\lambda > 0$, such that $u_\lambda(t, x)$ together with its time derivative $u_\lambda'(t, x)$, defined as the limit (in H_0^1) of $[u_\lambda(t+\tau, x) - u_\lambda(t, x)]/\tau$ as $\tau \to 0$, belongs to $H_0^1 \cap H^2$ for all $-\infty < t < \infty$ and that it satisfies the equations

$$(1.3) \qquad u_\lambda' - \frac{1}{\lambda} M u_\lambda' = L u_\lambda , \qquad -\infty < t < \infty ,$$

$$u_\lambda(0, x) = u_0(x) ,$$

in the $\| \ \|_1$-norm. We shall call the differential equation in (1.3) to be pseudo-parabolic. It differs from the parabolic one by the additional higher order terms,

$$\frac{1}{\lambda} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left[m_{ij}(x) \frac{\partial}{\partial x_j} \frac{\partial}{\partial t} u_\lambda \right] ,$$

where the parameter λ is restricted to be positive. Although no such type of equations has been proposed in the theory of heat conductions at the present time, it appears in theory of non-Newtonian fluids as well as soil mechanics, [6-9].

It is known [10] that a unique solution of the problem (1.3) exists and that the solution is just as regular as the coefficients and the initial conditions permit it to be. Indeed, for all positive numbers λ the operator $\lambda - M$ has a bounded inverse $(\lambda - M)^{-1}$, [4], the operator $\lambda(\lambda - M)^{-1} L$ as a mapping of H_0^1 onto H_0^1 is bounded with respect to the norm $\| \ \|_1$ and the unique solution $u_\lambda(t, x)$ is given by the formula [10],

$$(1.4) \qquad u_\lambda(t, x) \equiv E_\lambda(t) u_0(x) , \qquad -\infty < t < \infty ,$$

where λ is a positive number and where the group of operators $E_\lambda(t)$ which map H_0^1 onto H_0^1 is defined by

$$(1.5) \qquad E_\lambda(t) \equiv \exp [t\lambda(\lambda - M)^{-1} L] , \qquad \lambda > 0 , \qquad -\infty < t < \infty .$$

The objective of this note is to study the limiting behavior of the solutions $u_\lambda(t, x)$ as $\lambda \to \infty$. To this end, we consider the following parabolic initial-value problem [4]. Let $u_0(x)$ be a given function in H_0^0. Find the function $u(t, x)$ such that $u(t, x)$ together with its time derivative $u_t(t, x)$, defined as the limit (in H_0^0) of $[u(t+\tau, x) - u(t, x)]/\tau$ as $\tau \to 0$ if $t > 0$ or as $\tau \downarrow 0$ if $t = 0$, belongs to H_0^0 for all $t \geq 0$ and that it satisfies the equations,

$$(1.6) \qquad u_t = L u , \qquad t \geq 0 ,$$

$$u(0, x) = u_0(x) ,$$

in the $\| \ \|_0$-norm. It is known [4] that a unique solution to problem (1.6) exists. Indeed, we can define the semi-group of operators on H_0^0,

(1.7) $E_\lambda(t) \equiv \exp\left[t\lambda(\lambda-L)^{-1}L\right]$, $t \geqq 0$,

and the solution $u(t, x)$ of (1.6) is given by

(1.8) $u(t, x) \equiv E(t)u_0(x)$, $t \geqq 0$,

where $E(t)$ is the strong $\| \ \|_0$-limit of $E_\lambda(t)$ as $\lambda \to \infty$, [1, 4].

To establish the limiting behavior of the solutions, $u_\lambda(t, x)$ as $\lambda \to \infty$, we add the following additional restrictions on the coefficients of the operators L and M and on the boundary, ∂G, of G. Some of these restrictions may not be really needed for our results to be true, but to overcome certain technical difficulties we have made use of the a priori L_2 estimates for the solutions of the elliptic partial differential equations, [2, 16-18]. Our assumptions on L, M and the domain G are as follows:

(1.9) The functions $l_{ij}(x)$ and $m_{ij}(x)$ are in $C^3(\bar{G})$, $l(x)$ and $m(x)$ are in $C^2(\bar{G})$ and $m_{ij}(x)$ has a modulus of continuity for $i, j = 1, 2, \cdots, n$;

(1.10) ∂G can be covered by a finite number of n-dimensional neighborhoods N_j such that each $\bar{N}_j \cap \bar{G}$ can be mapped in a 1-1 way onto the closure of an n-dimensional hemisphere Σ, with $\bar{N}_j \cap \partial G$ mapped onto the flat face of Σ, by a mapping T_j which together with its inverse has continuous and bounded first two derivatives, [16, pp. 704-706].

Having described the above two types of problems and the additional restrictions on L, M and G, we now state our results as a

THEOREM. *If the problems* (1.3) *and* (1.6) *are posed for the same domain* G *and with the same initial condition* $u_0(x)$ *in* $H_0^1 \cap H^2$ *and if the assumptions in* (1.9) *and* (1.10) *are satisfied by* L, M *and* G, *then for all* $t \geqq 0$ *the solution* $u(t, x)$ *of the problem* (1.6) *is the* $\| \ \|_0$-*limit of the solutions* $u_\lambda(t, x)$ *of the problems* (1.3), *i. e.,*

$$\lim_{\lambda \to \infty} \|u_\lambda(t, x) - u(t, x)\|_0 = 0, \qquad t \geqq 0.$$

§2. Proof of the Theorem.

We shall follow Yosida's idea in his proof of Hille-Yosida Theorem, [5, 12]. An additional case should be taken for the fact that the problems (1.3) and (1.6) involve two different norms. Also, instead of the semi-group of operators in (1.7) we now have the group of operators in (1.5) which differ from (1.7) in replacing $(\lambda-L)^{-1}$ by $(\lambda-M)^{-1}$ with M being an arbitrary elliptic differential operator. To overcome this difficulty we introduce on H_0^1 one-parameter family of norms induced by the elliptic differential operator M, namely,

(2.1) $p\{u ; \lambda-M\} \equiv ((\lambda-M)u, u)^{1/2}$, $u \in H_0^1$,

where

$$((\lambda-M)u,\ u) \equiv \int_G \Big\{ \sum_{i,j=1}^{n} m_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + (m(x)+\lambda)u^2(x) \Big\} dx .$$

It is clear that for all positive finite λ, $\frac{1}{\sqrt{\lambda}} p\{\ ;\lambda-M\}$ and $\|\ \|_1$ are equivalent norms on the space H_0^1. On the other hand for fixed u in H_0^1 we have

$$\lim_{\lambda\to\infty} \frac{1}{\sqrt{\lambda}} p\{u\,;\lambda-M\} = \|u\|_0 .$$

To prove the theorem we first show that for all $t \geq 0$ and for all u in $H_0^3 \cap H^4$

(2.2) $$\|E_\lambda(t)u - E_\mu(t)u\|_0 \to 0 \quad \text{as} \quad \lambda, \mu \to \infty,$$

so that we can define an operator $E(t)$ on the subspace $H_0^3 \cap H^4$ of H_0^0 as the strong $\|\ \|_0$-limit of $E_\lambda(t)$ as $\lambda \to \infty$. We then proceed to verify that the operator $E(t)$ so defined is bounded with respect to the norm $\|\ \|_0$. Since $H_0^3 \cap H^4$ is $\|\ \|_0$-dense in H_0^0, we can extend $E(t)$ to the whole space H_0^0 by continuity. Our proof is then completed by verifying that for all u_0 in $H_0^1 \cap H^2$ the function defined by $u(t,\ x) \equiv E(t)u_0(x)$, $t \geq 0$, solves the parabolic initial-value problem (1.6).

To establish the limit relations in (2.2) we shall always, without loss of generality, assume $\mu > \lambda$ and proceed to show that for every u in $H_0^3 \cap H^4$ and for all $t \geq 0$,

(2.3) $$\frac{1}{\sqrt{\lambda}} p\{E_\lambda(t)u - E_\mu(t)u\,;\lambda-M\} \to 0 \quad \text{as} \quad \lambda \to \infty .$$

Indeed, for every positive λ and μ and non-negative t

(2.4) $$\frac{1}{\sqrt{\lambda}} p\{E_\lambda(t)u - E_\mu(t)u\,;\lambda-M\} \geq \|E_\lambda(t)u - E_\mu(t)u\|_0 .$$

Hence (2.2) is a direct consequence of (2.3).

To establish the limit relations in (2.3) we note that for every positive number λ the generator,

(2.5) $$A_\lambda \equiv \lambda(\lambda-M)^{-1}L ,$$

is bounded as a one-one mapping of H_0^1 onto H_0^1. Hence its exponential,

(2.6) $$E_\lambda(t) \equiv \exp[tA_\lambda], \qquad t \geq 0, \quad \lambda > 0 ,$$

is also a bounded operator with respect to the norm $\|\ \|_1$ and that $E_\lambda(0) = I$, [10].

Basing on Poincaré's inequality and the restrictions in (1.2) for L and M, we can choose, with L and M being fixed, a constant c so large that for all u in H_0^1

(2.7) $\left(-\dfrac{1}{c}Lu, u\right) \le (-Mu, u)$.

Upon writing $L/c \equiv N$, we have the following decomposition for A_λ on the space H_0^1:

(2.8) $A_\lambda = -c\lambda + c\lambda(\lambda - M)^{-1}(\lambda - M + N)$, $\lambda > 0$.

It is easy to verify that the operator $(\lambda - M)^{-1}(\lambda - M + N)$ is also bounded with respect to the norm $\| \ \|_1$ as a mapping of H_0^1 onto H_0^1. Consequently, we can extract a scalar factor from the operator $E_\lambda(t)$ as follows:

(2.9) $E_\lambda(t) = \exp[-ct\lambda] \cdot \exp[ct\lambda(\lambda - M)^{-1}(\lambda - M + N)]$, $\lambda > 0$, $t \ge 0$.

For simplicity we shall write

(2.10) $B_\lambda \equiv (\lambda - M)^{-1}(\lambda - M + N)$,

and proceed to show that for every u in H_0^1

(2.11) $p\{B_\lambda u ; \lambda - M\} \le \left(1 - \dfrac{k}{\lambda}\right)p\{u ; \lambda - M\}$,

where k is a positive constant depending only on L and M and the domain G. That is, the operator B_λ in (2.10) maps H_0^1 into H_0^1 and it is bounded by $1 - k/\lambda$ with respect to the norm $p\{ \ ; \lambda - M\}$. To establish the essential inequality (2.11) we first appeal to direct computation to obtain the identity,

(2.12) $[p\{B_\lambda u ; \lambda - M\}]^2 = ((\lambda - M + N)u, u + (\lambda - M)^{-1}Nu)$

                        $= [p\{u ; \lambda - M\}]^2 + 2(Nu, u) + (Nu, (\lambda - M)^{-1}Nu)$.

Next we give an estimate for the last term on the right-hand side of (2.12) so as to transform it into an inequality. Indeed,

$(Nu, (\lambda - M)^{-1}Nu) = (-Nu, (\lambda - M)^{-1}(-Nu))$

$\le (-Nu, u)^{1/2}(-N(\lambda - M)^{-1}(-Nu), (\lambda - M)^{-1}(-Nu))^{1/2}$

$\le (-Nu, u)^{1/2}((\lambda - M)(\lambda - M)^{-1}(-Nu), (\lambda - M)^{-1}(-Nu))^{1/2}$

$\le (-Nu, u)^{1/2}(-Nu, (\lambda - M)^{-1}(-Nu))^{1/2}$,

where the first inequality is that of Schwarz and the second inequality follows from the choice of the constant c in (2.7). After cancelling the common factor from the both sides of the above inequality, we obtain the desired estimate,

(2.13) $(Nu, (\lambda - M)^{-1}Nu) \le (-Nu, u)$.

Upon applying the inequality (2.13) to the last term on the right-hand side of the identity (2.12), it yields

(2.14) $[p\{B_\lambda u ; \lambda-M\}]^2 \leqq [p\{u ; \lambda-M\}]^2 - [p\{u ; -N\}]^2$

$$= \Big\{1 - \Big[\frac{p\{u ; -N\}}{p\{u ; \lambda-M\}} \Big]^2 \Big\}[p\{u ; \lambda-M\}]^2,$$

where $p\{u ; -N\} \equiv (-Nu, u)^{1/2}$. According to Poincaré there is a constant k_1 independent of u such that

(2.15) $k_1(-Nu, u) \geqq (u, u)$ for all u in H_0^1.

Since $N \equiv L/c$ with c being chosen in (2.7), it is clear that we can also choose a constant c_1 so large that

(2.16) $c_1(-Nu, u) \geqq (-Mu, u)$ for all u in H_0^1.

Accordingly, we have, for all u in H_0^1,

(2.17) $$\frac{(-Nu, u)}{\lambda(u, u) + (-Mu, u)} \geqq \frac{1}{\lambda k_1} \Big[\frac{\lambda(u, u)}{\lambda(u, u) + (-Mu, u)} \Big]$$

$$= \frac{1}{\lambda k_1} \Big[1 - \frac{(-Mu, u)}{\lambda(u, u) + (-Mu, u)} \Big]$$

$$\geqq \frac{1}{\lambda k_1} \Big[1 - \frac{c_1(-Nu, u)}{\lambda(u, u) + (-Mu, u)} \Big],$$

where the first and the second inequalities follow from the choices of the constants k_1 and c_1 in (2.15) and (2.16) respectively. Upon solving (2.17) and restricting $\lambda \geqq 1$, we find

(2.18) $$\Big[\frac{p\{u ; -N\}}{p\{u ; \lambda-M\}} \Big]^2 \geqq \frac{1}{1 + c_1/\lambda k_1} \cdot \frac{1}{\lambda k_1} \geqq \frac{1}{\lambda} \frac{1}{c_1 + k_1}$$

$$\geqq k_2/\lambda, \qquad k_2 \equiv 1/(c_1 + k_1).$$

Thus, (2.11) follows from (2.14) and (2.18) by setting $k \equiv k_2/2$. The above proof shows that k depends only on L, M and G. For the following application, we emphasize that k is independent of the function u as well as the parameter $\lambda \geqq 1$.

We proceed to combine the identity (2.9) and the inequality (2.11) to derive an estimate for $p\{E_\lambda(t)u ; \lambda-M\}$ for all u in H_0^1. Indeed, for all u in H_0^1 one has

(2.19) $p\{E_\lambda(t)u ; \lambda-M\} = \exp[-ct\lambda] \cdot p\{\exp[ct\lambda B_\lambda] \cdot u ; \lambda-M\}$

$$\leqq \exp[-ct\lambda] \cdot \exp[ct\lambda p\{B_\lambda ; \lambda-M\}] \cdot p\{u ; \lambda-M\}$$

$$\leqq \exp[-ct\lambda] \cdot \exp\Big[ct\lambda\Big(1 - \frac{k}{\lambda}\Big)\Big] \cdot p\{u ; \lambda-M\}$$

$$\leqq \exp[-ctk] \cdot p\{u ; \lambda-M\},$$

with $c, k > 0$ and $t \geqq 0$. This proves that for all $t \geqq 0$ the one-parameter family

of operators $E_\lambda(t)$, $\lambda > 0$, are *uniformly* bounded with respect to the corresponding one-parameter family of norms $p\{\ ;\ \lambda-M\}$. More precisely, as a mapping of H_0^1 onto H_0^1

$$(2.20) \qquad p\{E_\lambda(t)\ ;\ \lambda-M\} \leq \exp[-ckt], \qquad c, k > 0, \quad t \geq 0,$$

where the constants c and k depend only on L and M and the domain G. This accomplishes the first step of the proof.

Our next step is to derive certain estimates for $p\{(A_\lambda-A_\mu)u\ ;\ 1-M/\lambda\}$. First, we shall derive the following identity which holds for all numbers λ and μ with $\mu > \lambda > 0$ and for all u in $H_0^1 \cap H^2$,

$$(2.21) \qquad \left[p\left\{(A_\lambda-A_\mu)u\ ;\ 1-\frac{M}{\lambda}\right\}\right]^2 = \left(\frac{1}{\lambda}-\frac{1}{\mu}\right)^2\left(\left(1-\frac{M}{\lambda}\right)^{-1}MA_\mu u,\ MA_\mu u\right).$$

Indeed, direct computation gives that for all u in $H_0^1 \cap H^2$ and for all numbers λ and μ with $\mu > \lambda > 0$

$$(2.22) \qquad \left(1-\frac{M}{\lambda}\right)(A_\lambda-A_\mu)u = \left(\frac{1}{\lambda}-\frac{1}{\mu}\right)MA_\mu u.$$

By applying the operator $(1-M/\lambda)^{-1}$ on the left to the both sides of the above identity, we find that

$$(2.23) \qquad (A_\lambda-A_\mu)u = \left(\frac{1}{\lambda}-\frac{1}{\mu}\right)\left(1-\frac{M}{\lambda}\right)^{-1}MA_\mu u.$$

Upon taking the inner product of the corresponding sides of (2.22) and (2.23) we obtain the desired identity (2.21). The usefulness of such an identity lies in the fact that it expresses the norm of $(A_\lambda-A_\mu)u$ in terms of a functional on $A_\mu u$ alone which is easier to be estimated.

We proceed now to transform the identity in (2.21) into an inequality. To this end, we consider two functions v, w in $H_0^1 \cap H^2$ such that

$$(2.24) \qquad\qquad Mw = Lv, \quad \text{i.e.,} \quad w = M^{-1}Lv.$$

For all v, w in $H_0^1 \cap H^2$ and related by (2.24) we have

$$(2.25) \qquad (-Mw,\ w)^{1/2}(-Mv,\ v)^{1/2} \geq (-Mw,\ v)$$

$$= (-Lv,\ v)$$

$$\geq \frac{1}{k'}(-Mv,\ v),$$

where the positive constant k' depends only on L, M and G and is independent of v and w. On the other hand, by the same argument as for deriving the estimate in (2.13) we also have that for all v in $H_0^1 \cap H^2$ and for all $\lambda > 0$,

$$(2.26) \qquad \left(\left(1-\frac{M}{\lambda}\right)^{-1}Mv,\ Mv\right) \leq \lambda(-Mv,\ v).$$

Thus, we have from (2.25) and (2.26) that for all v, w in $H_0^1 \cap H^2$ and for all $\lambda > 0$,

$$(2.27) \qquad \left(\left(1 - \frac{M}{\lambda} \right)^{-1} Mv, Mv \right) \leqq \lambda (k')^2 (-Mw, w),$$

where $w \equiv M^{-1} Lv$.

We wish to compare the value of the expression on the left of (2.27) with $((1-M/\mu)^{-1} Lv, Lv)$, $\mu > \lambda$. To this end, we note that the fractional power, $(-M)^{1/2}$, of the self-adjoint elliptic partial differential operator M can be defined [1, 13-15]. Furthermore, it is known [14, II] that $(-M)^{1/2}$ is also self-adjoint and it commutes with $(1-M/\mu)^{-1}$ on the domain of $(-M)^{1/2}$ and that for all u, v in H_0^1

$$(2.28) \qquad (-Mu, v) = ((-M)^{1/2} u, (-M)^{1/2} v) = (u, -Mv).$$

Consequently, for all v in $H_0^1 \cap H^2$ and for all $\mu > \lambda > 0$,

$$
\begin{aligned}
(2.29) \qquad & ((1-M/\mu)^{-1} Lv, Lv) \\
& = ((1-M/\mu)^{-1} M M^{-1} Lv, M M^{-1} Lv) \\
& = ((1-M/\mu)^{-1} Mw, Mw) \\
& = \mu \{ (-Mw, w) - \mu ((\mu - M)^{-1} w, -Mw) \} \\
& = \mu \{ (-Mw, w) - \mu ((\mu - M)^{-1} (-M)^{1/2} w, (-M)^{1/2} w) \},
\end{aligned}
$$

where the second equality sign follows from the defining relation in (2.24) and the last one follows from the non-trivial identities in (2.28) and from the commutativity of $(-M)^{1/2}$ and $(1-M/\mu)^{-1}$ on H_0^1. Now

$$
\begin{aligned}
(2.30) \qquad & \mu ((\mu - M)^{-1} (-M)^{1/2} w, (-M)^{1/2} w) \\
& \leqq \mu \| (\mu - M)^{-1} (-M)^{1/2} w \|_0 \, \| (-M)^{1/2} w \|_0 \\
& \leqq \mu \| (\mu - M)^{-1} \|_0 \, \| (-M)^{1/2} w \|_0^2 \\
& \leqq \frac{\mu}{\mu + k''} ((-M)^{1/2} w, (-M)^{1/2} w) \\
& = \frac{\mu}{\mu + k''} (-Mw, w),
\end{aligned}
$$

where the positive constant k'' depends only on M and G and is independent of λ, μ and w, and the last equality sign again follows from the identities in (2.28). By combining (2.29) and (2.30) it follows that for all v in $H_0^1 \cap H^2$ and for all numbers λ and μ with $\mu > \lambda > 0$,

$$
\begin{aligned}
(2.31) \qquad & \left(\left(1 - \frac{M}{\mu} \right)^{-1} Lv, Lv \right) \geqq \mu \left\{ \left(1 - \frac{\mu}{\mu + k''} \right) (-Mw, w) \right\} \\
& = \frac{k'' \mu}{\mu + k''} (-Mw, w),
\end{aligned}
$$

where $w \equiv M^{-1}Lv$.

Thus, we conclude from the inequalities in (2.27) and (2.31) that for all v in $H_0^1 \cap H^2$ and for all $\mu > \lambda > 0$,

(2.32) $\left(\left(1-\dfrac{M}{\lambda}\right)^{-1}Mv, Mv\right) \leqq \dfrac{\lambda(k')^2(\mu+k'')}{k''\mu}\left(\left(1-\dfrac{M}{\mu}\right)^{-1}Lv, Lv\right)$,

where k' depends only on L, M and G and k'' depends only on M and G.

For every given $\lambda > 0$, A_λ is bounded as a one-one mapping of H_0^1 onto H_0^1. So is $A_\lambda E_\lambda(t-s)$ for all $0 < t-s < \infty$. Consequently, for all $\mu > \lambda > 0$,

(2.33) $\left(\left(1-\dfrac{M}{\lambda}\right)(A_\lambda - A_\mu)E_\mu(t-s)u, (A_\lambda - A_\mu)E_\mu(t-s)u\right)$

$= \left(\dfrac{1}{\lambda}-\dfrac{1}{\mu}\right)^2\left(\left(1-\dfrac{M}{\lambda}\right)^{-1}MA_\mu E_\mu(t-s)u, MA_\mu E_\mu(t-s)u\right)$

$\leqq \left(1-\dfrac{\lambda}{\mu}\right)^2\dfrac{\mu+k''}{\mu k''}\dfrac{(k')^2}{\lambda}\left(\left(1-\dfrac{M}{\mu}\right)^{-1}LA_\mu E_\mu(t-s)u, LA_\mu E_\mu(t-s)u\right)$

$= \left(1-\dfrac{\lambda}{\mu}\right)^2\dfrac{\mu+k''}{\mu k''}\dfrac{(k')^2}{\lambda}\left(A_\mu^2 E_\mu(t-s)u, \left(1-\dfrac{M}{\mu}\right)A_\mu^2 E_\mu(t-s)u\right)$

$= \left(1-\dfrac{\lambda}{\mu}\right)^2\dfrac{\mu+k''}{\mu k''}\dfrac{(k')^2}{\lambda}\left(\left(1-\dfrac{M}{\mu}\right)E_\mu(t-s)A_\mu^2 u, E_\mu(t-s)A_\mu^2 u\right)$,

where the first equality sign follows from the identity in (2.21), the inequality sign follows from (2.32), the second equality sign follows from the definition of A_μ in (2.5) and the last equality sign follows from the commutativity of A_μ and E_μ. Indeed, it is to make use of the commutativity of A_μ and E_μ that we derived the estimate in (2.32) to replace the operator M by L and to replace $(1-M/\lambda)^{-1}$ by $(1-M/\mu)^{-1}$.

We have established the uniform boundedness of the operators $E_\lambda(t)$ in (2.20) and the essential inequality (2.33). To establish (2.3) we still need the following fact, namely, for all $\lambda > 0$ and for all t

$$\lim_{\tau \to 0} p\left\{\dfrac{1}{\tau}[E_\lambda(t+\tau)-E_\lambda(t)]u - A_\lambda E_\lambda(t)u ; 1-M/\lambda\right\} = 0$$

for all u in $H_0^1 \cap H^2$. That is,

(2.34) $\dfrac{d}{dt}E_\lambda(t) = A_\lambda E_\lambda(t)$, $\lambda > 0$, $-\infty < t < \infty$,

in the strong sense with respect to the norm, $p\{\ ; 1-M/\lambda\}$. To establish the formulas in (2.34) we note the group property of the operators $E_\lambda(t)$,

$$E_\lambda(t_1+t_2) = E_\lambda(t_1)E_\lambda(t_2), \qquad -\infty < t_1, t_2 < \infty,$$

with respect to the operator topology induced by the norm, $p\{\ ; 1-M/\lambda\}$. Thus,

(2.35) $\lim\limits_{\tau \to 0} p\left\{\dfrac{1}{\tau}\left[E_\lambda(t+\tau)-E_\lambda(t)\right]u-A_\lambda E_\lambda(t)u \; ; \; 1-M/\lambda\right\}$

$$= \lim\limits_{\tau \to 0} p\left\{\dfrac{1}{\tau}\left[E_\lambda(\tau)-1-\tau A_\lambda\right]E_\lambda(t)u \; ; \; 1-M/\lambda\right\}$$

$$\leqq p\left\{E_\lambda(t) \; ; \; 1-\dfrac{M}{\lambda}\right\} \cdot \lim\limits_{\tau \to 0} p\left\{\dfrac{1}{\tau}\left[E_\lambda(\tau)-1-\tau A_\lambda\right]u \; ; \; 1-M/\lambda\right\}$$

$$\leqq e^{-ckt}p\{u \; ; \; 1-M/\lambda\} \cdot \lim\limits_{\tau \to 0} p\left\{\dfrac{1}{\tau}\left[E_\lambda(\tau)-1-\tau A_\lambda\right] \; ; \; 1-M/\lambda\right\}$$

$$\leqq e^{-ckt} \cdot p\{u \; ; \; 1-M/\lambda\} \lim\limits_{\tau \to 0} \sum\limits_{n=2}^{\infty} \tau^{n-1}\left[p\left\{A_\lambda \; ; \; 1-\dfrac{M}{\lambda}\right\}\right]^n /n! \, ,$$

where we have used the estimates in (2.11) and (2.20). Since for all u in $H_0^1 \cap H^2$,

$$\left[p\left\{A_\lambda u \; ; \; 1-\dfrac{M}{\lambda}\right\}\right]^2 = \lambda((\lambda-M)^{-1}Lu, Lu) \, , \qquad \lambda > 0 \, ,$$

and that analogous to (2.13)

$$\lambda((\lambda-M)^{-1}Lu, Lu) \leqq c\lambda(-Lu, u) \leqq k_3^2((1-M/\lambda)u, u)$$

with k_3 being a positive constant depending only on λ, L, M and the domain G. Thus,

(2.36) $p\{A_\lambda \; ; \; 1-M/\lambda\} \leqq k_3 \, .$

Hence our formulas in (2.34) follow from (2.35) and (2.36). In an analogous way, we can also establish the formula,

(2.37) $\dfrac{d}{dt}E_\lambda(t)E_\mu(t) = E_\lambda(t)(A_\lambda+A_\mu)E_\mu(t) \, ,$

which holds in the strong sense with respect to the norm, $p\{ \; ; \; (1-M/\lambda)\}$.

We are now ready to prove the limit relation in (2.3). For all u in $H_0^3 \cap H^4$ and for all $\mu > \lambda \geqq 1$, we have

(2.38) $p\{E_\lambda(t)u-E_\mu(t)u \; ; \; 1-M/\lambda\}$

$$= p\left\{\int_0^t \dfrac{d}{ds}\left[E_\lambda(s)E_\mu(t-s)\right] \cdot u\,ds \; ; \; 1-M/\lambda\right\}$$

$$\leqq \left(\dfrac{1}{\lambda}\right)^{1/2} \int_0^t p\{E_\lambda(s)(A_\lambda-A_\mu)E_\mu(t-s)u \; ; \; \lambda-M\}\,ds$$

$$\leqq \left(\dfrac{1}{\lambda}\right)^{1/2} \int_0^t e^{-cks}p\{(A_\lambda-A_\mu)E_\mu(t-s)u \; ; \; \lambda-M\}\,ds$$

$$= \int_0^t e^{-cks}p\{(A_\lambda-A_\mu)E_\mu(t-s)u \; ; \; 1-M/\lambda\}\,ds$$

$$\leqq \left(1-\dfrac{\lambda}{\mu}\right)\left[\dfrac{\mu+k''}{\mu k''}\dfrac{1}{\lambda}\right]^{1/2} k' \int_0^t e^{-cks}p\left\{E_\mu(t-s)A_\mu^2 u \; ; \; 1-\dfrac{M}{\mu}\right\}\,ds$$

$$\leqq \left(1-\frac{\lambda}{\mu}\right)\left[\frac{\mu+k''}{\mu k''}\frac{1}{\lambda}\right]^{1/2}k'\int_0^t e^{-ckt}p\left\{A_\mu^2 u\,;\,1-\frac{M}{\mu}\right\}ds$$

$$=\left(1-\frac{\lambda}{\mu}\right)\left[\frac{\mu+k''}{\mu k''}\frac{1}{\lambda}\right]^{1/2}k'te^{-ckt}p\left\{A_\mu^2 u\,;\,1-\frac{M}{\mu}\right\},$$

where the first equality sign follows from (2.37), the second and the last inequality follow from (2.20) and the third inequality follows from (2.33). Thus, to complete our proof for (2.3) it suffices to show that for all u in $H_0^3 \cap H^4$

$$(2.39) \qquad\qquad p\left\{A_\mu^2 u\,;\,1-\frac{M}{\mu}\right\} < \text{const} \cdot K$$

uniformly in μ.

To establish (2.39) we first note the inequality

$$(2.40) \qquad\qquad \left[p\left\{A_\mu^2 u\,;\,1-\frac{M}{\mu}\right\}\right]^2 \leqq \frac{\mu}{\mu+k''}\left\|L\left(1-\frac{M}{\mu}\right)^{-1}Lu\right\|_0^2.$$

This relation indicates that the boundedness relation (2.39) seems to be a trivial one. But to prove it we shall apply the results from the a priori L_2 estimates for the solutions of elliptic partial differential equations. First, we note that M and $(1-M/\mu)^{-1}$ commute on $H^2 \cap H_0^1$. Hence, if u belongs to $H_0^3 \cap H^4$, then

$$(2.41) \qquad\qquad \left\|M\left(1-\frac{M}{\mu}\right)^{-1}Lu\right\|_0 = \left\|\left(1-\frac{M}{\mu}\right)^{-1}MLu\right\|_0$$

$$\leqq \frac{\mu}{\mu+k''}\|MLu\|_0.$$

The right-hand side of this inequality is clearly bounded uniformly in μ if u is in $H_0^3 \cap H^4$ and if the conditions in (1.9) are satisfied by the operators L and M. We wish to show that for all u in $H_0^3 \cap H^4$ and for all λ and μ with $\mu > \lambda \geqq 1$,

$$(2.42) \qquad\qquad \left\|L\left(1-\frac{M}{\mu}\right)^{-1}Lu\right\|_0 \leqq k_4\left\{\left\|M\left(1-\frac{M}{\mu}\right)^{-1}Lu\right\|_0 + \|Lu\|_0\right\}$$

with k_4 being a constant depending only on L, M and G and independent of λ and μ.

To establish (2.42) we note from (1.9) that for all w in H^2

$$(2.43) \qquad\qquad \|Lw\|_0 \leqq \text{const.}\,\|w\|_2,$$

where the constant depends only on L and G. On the other hand for a given w in H^2, it may be regarded as the solution of the following Dirichlet problem, namely,

$$Mw = Mw \quad \text{in} \quad G, \qquad w = w \quad \text{on} \quad \partial G,$$

where Mw and w on the right-hand sides of the above equations are regarded to be given function while w on the left is regarded as the solution function.

Since the operator M and the boundary ∂G satisfy the restrictions in (1.9) and (1.10) respectively, it is a well-known a priori L_2 estimate, [14, pp. 704-706], that

$$(2.44) \qquad \|w\|_2 \leqq \text{const.} \{\|Mw\|_0 + \|w\|_{2-1/2} + \|w\|_0\}$$

$$\leqq \text{const.} \{\|Mw\|_0 + \|w\|_{2-1/2}\} \,,$$

where the constant depends only on M and G and where

$$(2.45) \qquad \|w\|_{2-1/2} \equiv \text{g. l. b.} \|v\|_2 \,, \qquad v \in H^2 \,, \qquad v = w \quad \text{on} \quad \partial G \,.$$

By combining (2.43) and (2.44) we conclude that

$$(2.45') \qquad \|Lw\|_0 \leqq k\{\|Mw\|_0 + \|w\|_{2-1/2}\} \,, \qquad w \in H^2 \,,$$

with k being a constant depending only on L, M and G. To see (2.42) is, indeed, valid we recall that the operator $(1 - M/\mu)^{-1}$ was so uniquely defined that $(1 - M/\mu)^{-1} Lu$ is the unique solution of the Dirichlet problem:

$$\left(1 - \frac{M}{\mu}\right) v = Lu \,, \qquad v \in H_0^1 \cap H^2 \,.$$

It follows that

$$(2.45'') \qquad \left(1 - \frac{M}{\mu}\right)^{-1} Lu \in H_0^1 \cap H^2 \,, \qquad 0 < \mu < \infty \,.$$

Furthermore, it is easy to see that

$$(2.45''') \qquad \lim_{\mu \to \infty} \|(1 - M/\mu)^{-1} Lu - Lu\|_0 = 0 \,.$$

Thus, our estimate (2.42) follows from (2.45'-2.45''').

Now, we have from (2.40), (2.41) and (2.42) that for all u in $H_0^3 \cap H^4$ and for all $\mu > \lambda \geqq 1$,

$$(2.46) \qquad p\left\{A_\mu^2 u; 1 - \frac{M}{\mu}\right\} \leqq k_4\left\{\left\|M\left(1 - \frac{M}{\mu}\right)^{-1} Lu\right\|_0 + \|Lu\|_2\right\}\left[\frac{\mu}{\mu + k''}\right]^{1/2}$$

$$\leqq k_4\left\{\frac{\mu}{\mu + k''}\|MLu\|_0 + \|Lu\|_2\right\}\left[\frac{\mu}{\mu + k''}\right]^{1/2} \,.$$

By combining the inequalities (2.38) and (2.46), it follows that for all u in $H_0^3 \cap H^4$ and for all $\mu > \lambda \geqq 1$,

$$(2.47) \qquad p\{E_\lambda(t)u - E_\mu(t)u; 1 - M/\lambda\}$$

$$\leqq \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{1}{k''\lambda}\right)^{1/2} k' k_4 e^{-ckt}\left[\frac{\mu}{\mu + k''}\|MLu\|_0 + \|Lu\|_2\right] \,.$$

Since the restrictions in (1.9) ensure that $\|MLu\|_0$ and $\|Lu\|_2$ are finite for all u in $H_0^3 \cap H^4$, our assertion (2.3) follows immediately from (2.47).

To complete the proof of the theorem we can now define the operator $E(t)$ on the subspace $H_0^3 \cap H^4$ of H_0^0 by the formula,

(2.48) $E(t)u = \lim_{\lambda \to \infty} E_\lambda(t)u$

in the $\| \ \|_0$-norm. It follows that for all u in $H_0^3 \cap H^4$

$$\|E(t)u\|_0 = \lim_{\lambda \to \infty} \|E_\lambda(t)u\|_0$$

$$\leq \lim_{\lambda \to \infty} p\{E_\lambda(t)u \; ; \; 1-M/\lambda\}$$

$$\leq e^{-ckt} \lim_{\lambda \to \infty} p\{u \; ; \; 1-M/\lambda\}$$

$$\leq e^{-ckt}\left\{\|u\|_0^2 + \lim_{\lambda \to \infty} \frac{1}{\lambda}(-Mu, u)\right\}^{1/2}$$

$$\leq e^{-ckt}\|u\|_0 .$$

This proves that $E(t)$ is bounded on the subspace $H_0^3 \cap H^4$. Since the later set is $\| \ \|_0$-dense in H_0^0, we may extend $E(t)$ to the whole space H_0^0 by continuity.

As has been shown that for all $\lambda > 0$, $-\infty < t_1, t_2 < \infty$,

$$p\{(E_\lambda(t_1+t_2)-E_\lambda(t_1)E_\lambda(t_2))u \; ; \; 1-M/\lambda\} = 0, \qquad u \in H_0^3 \cap H^4 .$$

This implies that for all $\lambda > 0$, $-\infty < t_1, t_2 < \infty$

$$\|E_\lambda(t_1+t_2)u - E_\lambda(t_1)E_\lambda(t_2)u\|_0 = 0, \qquad u \in H_0^3 \cap H^4 .$$

Hence by the usual argument we conclude that for all $t_1, t_2 \geq 0$

$$\|E(t_1+t_2)u - E(t_1)E(t_2)u\|_0 = 0$$

for all u in H_0^0. Similarly, we conclude from the formula in (2.34) and the fact that $\|A_\lambda u - Lu\|_0 \to 0$ as $\lambda \to \infty$ that

$$\frac{d}{dt}E(t) = LE(t), \qquad t \geq 0,$$

in the strong $\| \ \|_0$-sense. Clearly, from the analogous property of $E_\lambda(t)$ we have that $E(0) = I$.

All what has been shown says that for all $u_0(x)$ in H_0^0, the function $u(t, x) \equiv E(t)u_0(x)$, with $E(t)$ defined in (2.48) is a solution of the parabolic initial-value problem (1.6). By the uniqueness theorem the solution so constructed is identical with one defined in (1.8). The proof of the theorem is now complete.

Finally, we add that if the coefficients, $l_{ij}(x)$, $l(x)$, $m_{ij}(x)$ and $m(x)$ as well as the initial data $u_0(x)$ are all sufficiently smooth, then the solution $u_\lambda(t, x)$ of the problem (1.3) is a smooth function in t and x, [10]. In this case, the solution of the problem (1.6) is also smooth in t and x, [11]. Thus, under these circumstances the convergence $u_\lambda(t, x) \to u(t, x)$ is point-wise.

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