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Transformations of pseudo-Riemannian manifolds

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An m-dimensional pseudo-Riemannian manifold (M, g) is by definition a differentiable manifold M with a definite or indefinite Riemannian metric tensor g of signature (r, s). If the signature of g is $(m, 0)$, then we say that (M, g) is a Riemannian manifold. The purpose of this note is to generalize the results on transformations of Riemannian manifolds to those of pseudo-Riemannian manifolds.

In section 1 we give the basic relations of connections or various tensors satisfied by projective or conformal transformations. In section 2 we consider affine transformations and, for example, we get

COROLLARY 2.5. *If (M, g) is a compact irreducible pseudo-Riemannian manifold of signature (r, s) satisfying $r \neq s$, then any affine transformation of M is an isometry.*

In sections 3, 4, 5 and 6 we study projective and conformal transformations leaving some tensors invariant, in a similar way as in K. Yano and T. Nagano's paper [**10**]. However, some statements of theorems in [**10**] seem to be imcomplete, and so we give here complete statements and prove them in pseudo-Riemannian manifolds. For example we have

PROPOSITION 5.1. *Let (M, g) and $(N, 'g)$ be pseudo-Riemannian manifolds of dimension $m \geqq 4$. If there is a conformal transformation φ of M to N which leaves the covariant derivatives of the Weyl conformal curvature tensors invariant and if the set of points where φ is non-affine is dense in M, then M and N are conformally flat.*

As a consequence of this proposition we have

PROPOSITION 5.3. *Let M $(m \geqq 4)$ be an irreducible locally symmetric pseudo-Riemannian manifold of signature (r, s), $r \neq s$. Then we have either*

(i) *M is of constant curvature, or*

(ii) *M does not admit any non-homothetic conformal transformation.*

In the last section we give examples which support our statements of Proposition 3.1 and Proposition 5.1.

§ 1. Preliminaries.

(i) Let M and N be differentiable manifolds with linear connections ∇ and $'\nabla$. If φ is a transformation (diffeomorphism) of M to N, then φ induces a map of geometric objects $'K$ on N to those on M denoted by ${}^{\varphi}K$. Especially for $'\nabla$ on N we have an induced connection ${}^{\varphi}\nabla$ defined by

$$(1.1) \qquad {}^{\varphi}\nabla_X Y = \varphi^{-1} \cdot {}'\nabla_{\varphi X}(\varphi Y)$$

for any vector fields X and Y on M, where φ itself denotes the differential of φ. From now on by V, X, Y and Z we denote vector fields on M. Since the difference of the connections ${}^{\varphi}\nabla$ and ∇ makes a tensor field of type (1, 2) we denote it by W, and we define W_X by

$$(1.2) \qquad {}^{\varphi}\nabla_X Y - \nabla_X Y = W(Y, X) = W_X(Y).$$

If K is a tensor field of type (1, 1), for example, then we have

$$({}^{\varphi}\nabla_X K - \nabla_X K)Y = W_X(KY) - K \cdot W_X(Y).$$

In the last equation if we replace K by ${}^{\varphi}K$, and notice the relation ${}^{\varphi}\nabla_X {}^{\varphi}K = {}^{\varphi}({}'\nabla_{\varphi X}{}'K)$, then we get

LEMMA 1.1. *Let φ be a transformation of (M, ∇) to $(N, '\nabla)$ and let $'K$ be a tensor field of type* (1, 1), *for example, on N. Then we have*

$$(1.3) \qquad ({}^{\varphi}({}'\nabla_{\varphi X}{}'K) - \nabla_X {}^{\varphi}K)Y = W_X({}^{\varphi}KY) - {}^{\varphi}K \cdot W_X(Y).$$

(ii) Suppose that the linear connections ∇ and $'\nabla$ are symmetric. A transformation φ of M to N is projective if and only if we have a 1-form p on M such that

$$(1.4) \qquad W_X(Y) = W(Y, X) = p(Y)X + p(X)Y.$$

We say that φ is non-affine at x of M if $p_x \neq 0$. The Riemannian curvature tensors R and ${}^{\varphi}R$, the Ricci curvature tensors R_1 and ${}^{\varphi}R_1$ are, as is well known (for example, see [1]), related by

$$(1.5) \qquad {}^{\varphi}R(X, Y)Z = R(X, Y)Z + (\nabla_Y p)(Z)X - (\nabla_X p)(Z)Y + p(Z)p(X)Y \\ - p(Z)p(Y)X + ((\nabla_Y p)(X) - (\nabla_X p)(Y))Z,$$

$$(1.6) \qquad {}^{\varphi}R_1(X, Y) = R_1(X, Y) + (m-1)(p(X)p(Y) - (\nabla_Y p)(X)) \\ + (\nabla_X p)(Y) - (\nabla_Y p)(X).$$

Now we define a tensor P_1 of type (0, 2) by

$$(1.7) \qquad (m^2-1)P_1(X, Y) = -mR_1(X, Y) - R_1(Y, X).$$

Then the Weyl projective curvature tensor P defined by

$$(1.8)\qquad P(Z, X, Y) = R(X, Y)Z + P_1(Z, X)Y - P_1(Z, Y)X + (P_1(X, Y) - P_1(Y, X))Z$$

is invariant under any projective transformation, i.e. ${}^{\varphi}P = P$. For $m \geqq 3$ we define a tensor Q of type (0, 3) by defining $(m-2)Q(Z, X, Y)$ to be the trace of the map $V \to (\nabla_V P)(Z, X, Y)$. If the Ricci tensor is symmetric, then (1.8) is written as

$$(1.8)'\qquad P(Z, X, Y) = R(X, Y)Z - (m-1)^{-1}(R_1(X, Z)Y - R_1(Y, Z)X).$$

LEMMA 1.2. *For a projective transformation φ of a differentiable manifold $(M, \nabla)(m \geqq 3)$ with symmetric connection and symmetric Ricci tensor to another such $(N, '\nabla)$ we have*

$$(1.9)\qquad {}^{\varphi}Q(Z, X, Y) - Q(Z, X, Y) = p(P(Z, X, Y)).$$

PROOF. If we apply (1.3) to the projective curvature tensor P, then, using ${}^{\varphi}P = P$, we get

$$(m-2)({}^{\varphi}Q(Z, X, Y) - Q(Z, X, Y)) = \mathrm{trace}\,[V \longrightarrow W_V P(Z, X, Y) - P(W_V Z, X, Y) - P(Z, W_V X, Y) - P(Z, X, W_V Y)].$$

By applying (1.4), the right hand side is written as

$$\mathrm{trace}\,[V \longrightarrow p(P(Z, X, Y))V - 2p(V)P(Z, X, Y) - p(Z)P(V, X, Y) - p(X)P(Z, V, Y) - p(Y)P(Z, X, V)].$$

By (1.8)′ we see that trace $[V \to P(Z, X, V)] = 0$. Similarly we get trace $[V \to P(V, X, Y)] = 0$ and trace $[V \to P(Z, V, Y)] = 0$. Then (1.9) follows from

$$\mathrm{trace}\,[V \to p(P(Z, X, Y))V - 2p(V)P(Z, X, Y)] = (m-2)p(P(Z, X, Y)).$$

(iii) Let φ be a conformal transformation of a pseudo-Riemannian manifold (M, g) to $(N, 'g)$ such that ${}^{\varphi}g = e^{2\alpha}g$ for a function α on M. With respect to the Riemannian connections ∇ and $'\nabla$ on M and N we have

$$(1.10)\qquad W_X Y = (X\alpha)Y + (Y\alpha)X - g(X, Y)\,\mathrm{grad}\,\alpha,$$

where grad α is a vector field associated with $d\alpha$ defined by the metric tensor g. We say that φ is non-homothetic at x of M if $(d\alpha)_x \neq 0$. The relation between the Riemannian curvature tensors is

$$(1.11)\qquad {}^{\varphi}R(X, Y)Z = R(X, Y)Z + F(Z, Y)X - F(Z, X)Y + g(Z, Y)F(X) - g(Z, X)F(Y),$$

where

$$(1.12)\qquad F(Z, Y) = (\nabla_Z d\alpha)(Y) - (Z\alpha)(Y\alpha) + 2^{-1}g(\mathrm{grad}\,\alpha, \mathrm{grad}\,\alpha)g(Z, Y)$$

and $F(X)$ is defined by $g(F(X), Y)=F(X, Y)$. We have also the relations between the Ricci curvature tensors, and scalar curvatures S and ${}^{\varphi}S$. The Weyl conformal curvature tensor C defined for $m\geqq 3$ by

$$\begin{aligned}(1.13)\qquad C(Z, X, Y)=R(X, Y)Z-(m-2)^{-1}(R_1(Z, X)Y-R_1(Z, Y)X\\ +g(Z, X)R^1(Y)-g(Z, Y)R^1(X))\\ +(m-1)^{-1}(m-2)^{-1}S(g(Z, X)Y-g(Z, Y)X)\end{aligned}$$

is invariant under any conformal transformation, where $R^1(X)$ is defined by $g(R^1(X), Y)=R_1(X, Y)$. If $m=3$, then we have $C=0$. For $m\geqq 4$, we define $(m-3)E(Z, X, Y)$ to be the trace of the map $V\rightarrow(\nabla_V C)(Z, X, Y)$. Then E is a tensor field of type $(0, 3)$. Similarly to Lemma 1.2, we have

LEMMA 1.3. *For a conformal transformation φ of a pseudo-Riemannian manifold $(M, g)(m\geqq 4)$ to another $(N, 'g)$ we have*

$$(1.14)\qquad {}^{\varphi}E(Z, X, Y)-E(Z, X, Y)=d\alpha(C(Z, X, Y)).$$

PROOF. If we apply (1.3) to the conformal curvature tensor C, then, using ${}^{\varphi}C=C$, we get

$$\begin{aligned}(m-3)({}^{\varphi}E(Z, X, Y)-E(Z, X, Y))=\operatorname{trace}[V\longrightarrow W_V C(Z, X, Y)\\ -C(W_V Z, X, Y)-C(Z, W_V X, Y)-C(Z, X, W_V Y)].\end{aligned}$$

By (1.10) the right hand side is written as

$$\begin{aligned}\operatorname{trace}[V\longrightarrow d\alpha(C(Z, X, Y))V-2(V\alpha)(C(Z, X, Y)\\ -g(V, C(Z, X, Y))\operatorname{grad}\alpha-(Z\alpha)C(V, X, Y)\\ -(X\alpha)C(Z, V, Y)-(Y\alpha)C(Z, X, V)\\ +g(V, Z)C(\operatorname{grad}\alpha, X, Y)+g(V, X)C(Z, \operatorname{grad}\alpha, Y)\\ +g(V, Y)C(Z, X, \operatorname{grad}\alpha)].\end{aligned}$$

First we have

$$\operatorname{trace}[V\longrightarrow d\alpha(C(Z, X, Y))V-2(V\alpha)C(Z, X, Y)]=(m-2)d\alpha(C(Z, X, Y)),$$

$$\begin{aligned}\operatorname{trace}[V\longrightarrow -g(V, C(Z, X, Y))\operatorname{grad}\alpha]&=-g(\operatorname{grad}\alpha, C(Z, X, Y))\\ &=-d\alpha(C(Z, X, Y)).\end{aligned}$$

Next by (1.13) we get $\operatorname{trace}[V\rightarrow C(V, X, Y)]=0$, $\operatorname{trace}[V\rightarrow C(Z, V, Y)]=0$ and $\operatorname{trace}[V\rightarrow C(Z, X, V)]=0$. If we write $C(Z, X, Y)=C(X, Y)Z$, then it is known that C satisfies the same algebraic equations as those satisfied by the Riemannian curvature tensor R, and so we have

$$\operatorname{trace}[V \longrightarrow g(V, Z)C(\operatorname{grad}\alpha, X, Y)] = g(Z, C(\operatorname{grad}\alpha, X, Y))$$
$$= -g(\operatorname{grad}\alpha, C(Z, X, Y))$$
$$= -d\alpha(C(Z, X, Y)),$$

$$\operatorname{trace}[V \longrightarrow g(V, X)C(Z, \operatorname{grad}\alpha, Y) + g(V, Y)C(Z, X, \operatorname{grad}\alpha)]$$
$$= g(X, C(Z, \operatorname{grad}\alpha, Y)) + g(Y, C(Z, X, \operatorname{grad}\alpha))$$
$$= -g(Z, C(X, \operatorname{grad}\alpha, Y) + C(Y, X, \operatorname{grad}\alpha))$$
$$= g(Z, C(\operatorname{grad}\alpha, Y, X))$$
$$= g(\operatorname{grad}\alpha, C(Z, X, Y))$$
$$= d\alpha(C(Z, X, Y)).$$

Therefore, adding these results together we have (1.14).

By $A(M)$, $H(M)$ and $I(M)$ we denote the group of affine ($W=0$), homothetic ($d\alpha=0$) and isometric ($\alpha=0$) transformations of M, respectively.

If a transformation φ of (M, g) to $(N, 'g)$ satisfies ${}^{\varphi}g = -e^{2\alpha}g$ we say that φ is an anti-conformal transformation, an anti-homothety, or an anti-isometry.

§ 2. Affine transformations.

In the previous paper [6] generalizing [2] we obtained the following

PROPOSITION 2.1. *Let (M, g) and $(N, 'g)$ be irreducible pseudo-Riemannian manifolds and assume that the signature (r, s) of g satisfies $r \neq s$. If there is an affine transformation φ of M to N, then the signature of $'g$ is (r, s) or (s, r) and φ is a homothety or an anti-homothety, respectively.*

REMARK 2.2. Any 2-dimensional orientable pseudo-Riemannian manifold (M, g) of signature $(1, 1)$ is reducible. In fact for any point x of M each 1-dimensional subspace of the tangent space M_x at x defined by null vectors is invariant by the restricted homogeneous holonomy group.

REMARK 2.3. Since the distinction between g and $-g$ in a pseudo-Riemannian manifold M is not essential, in many cases we may assume that the signature (r, s) of g satisfies $r \geqq s$.

PROPOSITION 2.4. *Any homothety (or anti-homothety) of a compact pseudo-Riemannian manifold (M, g) is an isometry (or anti-isometry).*

PROOF. We assume that M is orientable. Then we have the volume element $(\varepsilon \det g)^{1/2} dx^1 \wedge \cdots \wedge dx^m$ defined by the determinant of g in each coordinate neighborhood (the order $(x^1, \cdots, x^m)$ being compatible with the orientation and ε being the sign of $\det g$). Then the proof for a homothety is the same as in the Riemannian case (cf. [5]). An anti-homothety can exist only when the signature of g is (r, r). And for an anti-homothety φ of (M, g), we consider a homothety φ of (M, g) to $(M, -g)$.

By Propositions 2.1. and 2.4 we get

COROLLARY 2.5. *If (M, g) is a compact irreducible pseudo-Riemannian manifold of signature (r, s) satisfying $r \neq s$, then any affine transformation of M is an isometry.*

Similarly to [2], we have

COROLLARY 2.6. *Let M be an irreducible pseudo-Riemannian manifold of signature (r, s) satisfying $r \neq s$. Then we have:*

(i) *Any compact subgroup of $A(M)$ is a subgroup of $I(M)$.*

(ii) *The commutator subgroup $[A(M), A(M)]$ is a subgroup of $I(M)$.*

REMARK 2.7. Let M and N be pseudo-Riemannian manifolds. If an affine transformation of M to N is isometric at some point of M, then it is an isometry (see [9], p. 57).

§ 3. ∇P-preserving projective transformations.

PROPOSITION 3.1. *Let $M(m \geqq 3)$ and N be differentiable manifolds with symmetric connections ∇ and $'\nabla$, and symmetric Ricci tensors R_1 and $'R_1$. If there is a projective transformation φ of M to N which leaves the covariant derivatives of the Weyl projective curvature tensors invariant and if the set of points where φ is non-affine is dense in M, then M and N are projectively flat.*

PROOF. By ${}^{\varphi}('\nabla' P) = \nabla P$, we have ${}^{\varphi}Q = Q$. Next by Lemma 1.2 we have $p(P(Z, X, Y)) = 0$. If we apply (1.3) to P, using ${}^{\varphi}('\nabla' P) = \nabla P$ and ${}^{\varphi}P = P$, we get

$$0 = W_V(P(Z, X, Y)) - P(W_V Z, X, Y) - P(Z, W_V X, Y) - P(Z, X, W_V Y).$$

By (1.4), using $p(P(Z, X, Y)) = 0$, we get

$$0 = 2p(V)P(Z, X, Y) + p(Z)P(V, X, Y) + p(X)P(Z, V, Y) + p(Y)P(Z, X, V). \tag{3.1}$$

Take a point x of M such that $p_x \neq 0$. Then we have a basis $(e_1, \cdots, e_m)$ of M_x and the dual basis $(w^1, \cdots, w^m)$ such that $w^1 = p_x$.

If we put $V = e_1$, $Z = e_l$, $X = e_j$, $Y = e_k$ in (3.1), then we get $P(e_l, e_j, e_k) = 0$ for $j, k, l \neq 1$.

If we put $V = Z = e_1$, $X = e_j$, $Y = e_k$ in (3.1), then we have $P(e_1, e_j, e_k) = 0$ for $j, k \neq 1$.

If we put $V = X = e_1$, $Z = e_l$, $Y = e_k$ in (3.1), then we get $P(e_l, e_1, e_k) = 0$ for $k, l \neq 1$.

Finally if we put $V = Z = X = e_1$, $Y = e_k$ in (3.1), then we have $P(e_1, e_1, e_k) = 0$ for $k \neq 1$.

Therefore we have $P=0$ at x. Since the set of points x such that $p_x \neq 0$ is dense in M, we have $P=0$ on M.

REMARK 3.2. In section 7, we give an example showing necessity of the assumption that "the set of points where φ is non-affine is *dense* in M" in the above Proposition.

A pseudo-Riemannian manifold M is said to be of constant curvature at x, if the Riemannian curvature tensor satisfies

$$R(X, Y)Z = k_x(g(Z, Y)X - g(Z, X)Y)$$

at x for some real number k_x. If k_x is constant on M, M is said to be of constant curvature. It is known that any projectively flat pseudo-Riemannian manifold is of constant curvature. Thus we get

PROPOSITION 3.3. *Let (M, g) and $(N, 'g)$ be pseudo-Riemannian manifolds $(m \geqq 3)$. If there is a projective transformation φ of M to N which leaves the covariant derivatives of the Weyl projective curvature tensors invariant and if the set of points where φ is non-affine is dense in M, then M and N are of constant curvature.*

COROLLARY 3.4. *Suppose that a pseudo-Riemannian manifold $M(m \geqq 3)$ is not of constant curvature on any open set in M. Then any projective transformation of M to another N which leaves the covariant derivatives of the Weyl projective curvature tensors invariant is affine.*

PROPOSITION 3.5. *Let $(M, g)(m \geqq 3)$ be a locally symmetric pseudo-Riemannian manifold. Then either*

(i) *M is of constant curvature, or*

(ii) *M does not admit any non-affine projective transformation.*

PROOF. Since M is locally symmetric we have $\nabla R = 0$ and hence $\nabla P = 0$. Suppose that M is not of constant curvature. Then P does not vanish at some point of M. Since P is a parallel tensor field, it does not vanish anywhere. Thus any projective transformation of M is necessarily affine.

REMARK 3.6. When the metric is positive definite, Proposition 3.3 and Corollary 3.4 for non-affine infinitesimal projective transformation were stated by K. Yano and T. Nagano in [**10**] without the condition that the set of points where φ is non-affine is dense in M.

Proposition 3.5 is a generalization of a result due to T. Sumitomo [**5**] on Riemannian manifolds.

§ 4. Ricci-curvature-tensor-preserving projective transformations.

First we remark that a projective transformation leaves the Ricci curvature tensor invariant if and only if it leaves the Riemannian curvature tensor invariant. In fact, each condition is equivalent to $(\nabla_X p)(Y) = p(X)p(Y)$ in

(1.5) and (1.6).

PROPOSITION 4.1. *Let M and N be irreducible pseudo-Riemannian manifolds and assume that the signature (r, s) of g satisfies $r \neq s$. Then any projective transformation of M to N which leaves the Ricci curvature tensors invariant is a homothety, or anti-homothety.*

Especially, further, if both Ricci curvature tensors of M and N vanish, then any projective transformation is a homothety, or anti-homothety.

PROOF. Since the restricted holonomy group of M is irreducible, it has no invariant covector. By S. Ishihara's result ([**1**], p. 209) any Ricci-curvature-tensor-preserving projective transformation is affine. So if we apply Proposition 2.1, then the proof is completed.

§ 5. ∇C-preserving conformal transformations.

An analogous proposition to Proposition 3.3 is as follows.

PROPOSITION 5.1. *Let (M, g) and $(N, 'g)$ be pseudo-Riemannian manifolds $(m \geqq 4)$. If there is a conformal transformation φ of M to N which leaves the covariant derivatives of the Weyl conformal curvature tensors invariant and if the set of points where φ is non-homothetic is dense in M, then we have $C=0$ and, M and N are conformally flat.*

PROOF. By ${}^{\varphi}C=C$ and ${}^{\varphi}('\nabla'C)=\nabla C$, we have ${}^{\varphi}E=E$. Then by Lemma 1.3 we get $d\alpha(C(Z, X, Y))=0$, and this also implies $C(\operatorname{grad}\alpha, X, Y)=0$. If we apply (1.3) to C, then we get

$$0=W_V(C(Z, X, Y))-C(W_VZ, X, Y)-C(Z, W_VX, Y)-C(Z, X, W_VY).$$

Using (1.10) and above relations, we get

$$\begin{aligned}0=&\,2(V\alpha)C(Z, X, Y)+g(V, C(Z, X, Y))\operatorname{grad}\alpha\\&+(Z\alpha)C(V, X, Y)+(X\alpha)C(Z, V, Y)+(Y\alpha)C(Z, X, V).\end{aligned}\tag{5.1}$$

Taking the inner product with U we get

$$\begin{aligned}0=&\,2(V\alpha)g(U, C(Z, X, Y))+(U\alpha)g(V, C(Z, X, Y))\\&+(Z\alpha)g(U, C(V, X, Y))+(X\alpha)g(U, C(Z, V, Y))\\&+(Y\alpha)g(U, C(Z, X, V)).\end{aligned}\tag{5.2}$$

If $d\alpha \neq 0$ at x of M, then we can take a basis $(e_1, \cdots, e_m)$ of M_x and the dual basis $(w^1, \cdots, w^m)$ at x such that $w^1=d\alpha$. In the following calculation we read $(V\alpha)=d\alpha(V)$, etc.

If we put $V=e_1$, $X=e_j$, $Y=e_k$, $Z=e_l$, $U=e_i$ in (5.2), then we have $g(e_i, C(e_l, e_j, e_k))=0$ for $i, j, k, l \neq 1$.

If we put $V=U=e_1$, $X=e_j$, $Y=e_k$, $Z=e_l$ in (5.2), we get $g(e_1, C(e_l, e_j, e_k))$

$=0$ for $j, k, l \neq 1$.

If we put $V=U=X=e_1$, $Y=e_k$, $Z=e_l$ in (5.2), then we have $g(e_1, C(e_l, e_1, e_k))=0$ for $k, l \neq 1$.

Thus we have $g(U, C(Z, X, Y))=0$ at x for any U, Z, X, and Y, and we get $C=0$ at x. Since the set of points x such that $(d\alpha)_x \neq 0$ is dense in M, we have $C=0$ on M.

COROLLARY 5.2. *Suppose that a pseudo-Riemannian manifold $M(m \geqq 4)$ is conformally non-flat on any open set in M. Then any conformal transformation of M to another N which preserves the covariant derivatives of the Weyl conformal curvature tensors is a homothety.*

PROPOSITION 5.3. *Let $M(m \geqq 4)$ be an irreducible locally symmetric pseudo-Riemannian manifold of signature (r, s), $r \neq s$. Then we have either*

(i) *M is of constant curvature, or*

(ii) *M does not admit any non-homothetic conformal transformation.*

PROOF. By local symmetry of M we have $\nabla R=0$ and $\nabla C=0$. If C is not trivial at some point, then it is not trivial anywhere. So we have (ii). Otherwise we have $C=0$ on M, and so if we show the next Lemma, we get (i).

LEMMA 5.4. *If an irreducible pseudo-Riemannian manifold (M, g) has signature (r, s) satisfying $r \neq s$ and has parallel Ricci curvature tensor, then it is an Einstein space.*

In fact, if we define a (1, 1)-tensor A by $R_1(X, Y)=g(X, AY)$, then by the same argument as in [**6**] we have $A=aI$, where a is constant since g and R_1 are parallel. Therefore M is an Einstein space.

REMARK 5.5. Proposition 5.1 (as well as Corollary 5.2) for a Riemannian manifold was first stated by K. Yano and T. Nagano [**10**] for a non-homothetic infinitesimal conformal transformation without specifying that the set of points where φ is non-homothetic is *dense* in M. We give an example in the last section which shows that this condition is necessary.

REMARK 5.6. Proposition 5.3 is a generalization of T. Sumitomo's result [**5**] on Riemannian manifolds.

§ 6. Ricci-curvature-tensor-preserving conformal transformations.

As in the case of a projective transformation, a conformal transformation leaves the Ricci curvature tensor invariant if and only if it leaves the Riemannian curvature tensor invariant.

Now we prove

PROPOSITION 6.1. *Let $(M, g)(m \geqq 3)$ be a pseudo-Riemannian manifold such that the Riemannian connection is complete. Then any Ricci-curvature-tensor-preserving conformal transformation of (M, g) to another $(N, 'g)$ is a homothety.*

PROOF. By ${}^{\varphi}R_1 = R_1$, we have ${}^{\varphi}R = R$ and $F = 0$:

$$\nabla d\alpha - d\alpha \otimes d\alpha + 2^{-1} g(\operatorname{grad} \alpha, \operatorname{grad} \alpha) g = 0 . \tag{6.1}$$

If $\operatorname{grad} \alpha$ is a null vector field everywhere, then we have $\nabla d\alpha = d\alpha \otimes d\alpha$. Since ∇ is complete, we have $d\alpha = 0$ by S. Ishihara's Lemma ([1], p. 210). If $\operatorname{grad} \alpha$ is not a null vector at some point x of M, then we apply the argument of S. Ishihara's Lemma ([1], p. 216). Transvecting (6.1) with $\operatorname{grad} \alpha$, we have

$$2\nabla_{\operatorname{grad} \alpha} \operatorname{grad} \alpha = g(\operatorname{grad} \alpha, \operatorname{grad} \alpha) \operatorname{grad} \alpha . \tag{6.2}$$

This implies that each trajectory of $\operatorname{grad} \alpha$ is a geodesic. So we take a trajectory $x(t)$ of $\operatorname{grad} \alpha$ passing through x. Since $\operatorname{grad} \alpha$ is not null at x, we can assume that the parameter t is the arc-length parameter. Consider a function λ defined by

$$\lambda^2 = \varepsilon g(\operatorname{grad} \alpha, \operatorname{grad} \alpha) = |\operatorname{grad} \alpha|^2$$

on $x(t)$ such that $\lambda > 0$ at x, ε being the sign of $g(\operatorname{grad} \alpha, \operatorname{grad} \alpha)$. Let $X = (\operatorname{grad} \alpha)/\lambda$ in the domain where $\lambda > 0$. Then we have

$$\begin{aligned} 2\lambda d\lambda/dt &= \varepsilon \nabla_X (g(\operatorname{grad} \alpha, \operatorname{grad} \alpha)) \\ &= \varepsilon 2(1/\lambda) g(\operatorname{grad} \alpha, \nabla_{\operatorname{grad} \alpha} \operatorname{grad} \alpha) \\ &= \varepsilon (1/\lambda)(g(\operatorname{grad} \alpha, \operatorname{grad} \alpha))^2 \qquad \text{by (6.2)} . \end{aligned}$$

Thus we have $2d\lambda/dt = \varepsilon\lambda^2$ and $\lambda = -2\varepsilon/(t-c)$ for some constant c. Now notice that the arc-length parameter t for a non-light-like geodesic is also an affine parameter (in our case we have $\nabla_{\operatorname{grad} \alpha}((\operatorname{grad} \alpha)/\lambda) = 0$). By completeness of the Riemannian connection, λ^2 must be defined for $t = c$. But this is impossible, namely, we have $\lambda = 0$ everywhere, and α must be constant on M.

§ 7. Examples.

EXAMPLE 7.1. *There exist projectively non-flat differentiable manifolds* (M, ∇) *and* $(N, '\nabla)$ *with symmetric connections and symmetric Ricci tensors, such that they admit a non-affine projective transformation which maps* ∇P *into* $'\nabla' P$.

Let M be a sphere with the natural metric g^*. The Riemannian connection ∇^* is symmetric and the Ricci tensor R_1^* is also symmetric. Since g^* is of constant curvature, (M, ∇^*) is projectively flat. Take a small open set U^* in M and define a non-constant positive C^∞-function f^* on M such that f^* takes value 1 outside U^*. Let $'\nabla$ be the Riemannian connection defined by $f^* g^*$. Then we have $'\nabla = \nabla^*$ outside U^* and there is a point x in U^* where $'\nabla$ is not projectively flat (because, as is known, any projectively flat Riemannian manifold is of constant curvature, but $f^* g^*$ is not of constant curvature in U^*). Notice that $'R_1$ is symmetric. Take an open set U outside U^* and take a non-trivial C^∞-function f on M which vanishes outside U. Then we have a 1-form p defined by $p=df$ on M vanishing outside U. Now define a connection ∇ by

$$\nabla_X Y = '\nabla_X Y + p(X)Y + p(Y)X\,.$$

Then ∇ is symmetric and the Ricci curvature tensor R_1 is also symmetric by (1.6), since $'R_1$ is symmetric and p is a derived form. By the way the identity transformation $\varphi: (M, \nabla) \to (N=M, '\nabla)$ is projective on M and affine on U^*. Therefore on U^* we have ${}^{\varphi}('\nabla P) = \nabla P$. Outside U^* we have $'P = P = 0$ and hence ${}^{\varphi}('\nabla 'P) = \nabla P$. Since f and p are not trivial, φ is not affine at some point. Moreover, we have $'P = P \neq 0$ at x.

EXAMPLE 7.2. *There exists a Riemannian manifold which is not conformally flat and which admits a non-homothetic (infinitesimal) conformal transformation which leaves the covariant derivative of the Weyl conformal curvature tensor invariant.*

A simple example is constructed on an odd dimensional sphere $M = S^{2n+1}$. Since M admits a Sasakian structure, namely, a normal contact metric structure, we denote the structure tensors by (ϕ, ξ, η, g) where ξ is a unit Killing vector field with respect to the metric g induced from that in E^{2n+2} (cf. [4]). Let x be an arbitrary point of M and take two small neighborhoods U and V such that the closure of U is contained in V. Since ξ generates a 1-parameter group of isometries $\exp t\xi$, we have a great circle $(\exp t\xi \cdot x\,;\, 0 \leqq t < 2\pi)$ and its tubular neighborhoods $*U = (\exp t\xi \cdot U\,;\, 0 \leqq t < 2\pi)$ and $*V = (\exp t\xi \cdot V\,;\, 0 \leqq t < 2\pi)$. We define a non-negative C^∞-function f on M such that

(i) f is invariant by $\exp t\xi$,
(ii) $f = 1$ on $*U$,
(iii) $f = 0$ outside $*V$.

Now we define a new metric $*g$ on M for a constant $\alpha > 1$ by

$$*g = g + (\alpha - 1) f (g + \alpha \eta \otimes \eta)\,. \tag{7.1}$$

Then $*g$ on $*U$ is $\alpha g + (\alpha^2 - \alpha)\eta \otimes \eta$, and this is an associated Riemannian metric with respect to another Sasakian structure on $*U$. But $*g$ on $*U$ is not of constant curvature (cf. [8]). On the other hand, if the associated Riemannian metric of a Sasakian structure is conformally flat, then it is of constant curvature ([3], [7]). Therefore $*g$ is not conformally flat on $*U$. Since ξ leaves η invariant too, by (7.1) ξ is a Killing vector field also with respect to $*g$. Next we take a small open set W outside $*V$ and define a positive C^∞-function h such that

(iv) there is a point y in W where $h(y) \neq 1$, and
(v) for any z outside W we have $h(z) = 1$.

Then the metric G defined by $G=h^*g$ is the one required. Namely, we have

(vi) G is not conformally flat (on $^*U$).

(vii) M admits an infinitesimal conformal transformation ξ which is a Killing vector field with respect to G outside W and which is non-homothetic on some open set in W, since $L_\xi G=(L_\xi h)(1/h)G$.

(viii) The covariant derivative ∇C of the Weyl conformal curvature tensor may be non-vanishing only in $^*V$. Since ξ is a Killing vector field on $^*V$ we have $L_\xi \nabla C=\nabla L_\xi C=0$ on $^*V$ and hence on M.

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