J. Math. Soc. Japan
Vol. 21, No. 2, 1969

On Mordell's conjecture for the curve over function field with arbitrary constant field

By Megumu MIWA

(Received Feb. 12, 1968)
(Revised Oct. 11, 1968)

§ 1. Introduction.

The purpose of this paper is to improve and correct[1] the results of author's paper [3]. For the convenience to the readers we restate here the results of H. Grauert and J. P. Samuel in a form which fits in our discussion. In the following C_K means the set of all rational points, of an algebraic curve, over a field K.

THEOREM OF MANIN-GRAUERT ([1]). *Let k be an algebraically closed field of characteristic 0, K a function field over k and C a complete non-singular algebraic curve, of genus $\geqq 2$, defined over K. Then the set of all rational points C_K, of C, over K is infinite if and only if there exist an algebraic curve C' defined over k and a birational isomorphism $u: C \to C'$ defined over K. In this case, $C_K - u^{-1}(C'_k)$ is a finite set.*

THEOREM OF SAMUEL ([5]). *Let k be an algebraically closed field of characteristic $p=0$, K a function field over k and C be a complete non-singular curve, of genus $\geqq 2$, defined over K.* i) *If C is not isomorphic to any algebraic curve defined over a finite field, then C_K is infinite if and only if there exist an algebraic curve C' defined over k and a birational isomorphism $u: C \to C'$ defined over K. In this case, $C_K - u^{-1}(C'_k)$ is a finite set.* ii) *If C is isomorphic to an algebraic curve C' defined over a finite field F_q with q elements and all the elements of* Aut (C') *are defined over F_q, then C_K is infinite if and only if there exist a finite Galois extension K'/K, a birational isomorphism $u: C \to C'$ defined over K', an injective homomorphism $j: G(K'/K) \to \mathrm{Aut}(C')$ such that $j(s) = u^s \cdot u^{-1}$ for all s in $G(K'/K)$ and either* (1) *there exists an element z in $C'_{K'} - C'_k$ such that $j(s)z = z^s$ for all elements s of $G(K'/K)$ or* (2) *(only when $K'=K$) C'_k is infinite. In this case there exists a finite set $(x_i)_{i \in I}$ of points in $C'_{K'}$, with $j(s)x_i = x_i^s$ for all s of $G(K'/K)$, such that every point of C_K can be written either in the form $u^{-1}(f^n(x_i))$ or (only when $K'=K$) $u^{-1}(x)$ with $x \in C'_k$, where f is the*

1) Proposition [3] is not correct. The statement of Theorem 2 of [3] is true only in the cases of a) and b) i) of Theorem in this paper.

Frobenius morphism: $x \to x^q$ *of* C'.

We shall prove in this paper the following

THEOREM. *Let* k *be an arbitrary field,* K *a function field over* K *(i.e. a finite type regular extension of* k*) and* C *a complete non-singular algebraic curve, of genus* $\geqq 2$*, defined over* K.

a) *Let* k *be of characteristic* 0. *Then the set* C_K *is infinite if and only if there exist an algebraic curve defined over* k *and a birational isomorphism* $u: C \to C'$ *defined over* K *and the set* C'_k *is infinite. In this case,* $C_K - u^{-1}(C'_k)$ *is a finite set.*

b) *Let* k *be of characteristic* $p \neq 0$. *Then there are two cases.*

i) *Assume that* C *is not isomorphic to any algebraic curve defined over a finite field. Then the set* C_K *is infinite if and only if there exist an algebraic curve* C' *defined over* k *and a birational isomorphism* $u: C \to C'$ *defined over* K *and the set* C'_k *is infinite. In this case* $C_K - u^{-1}(C'_k)$ *is a finite set.*

ii) *Assume that* C *is isomorphic to an algebraic curve* C' *defined over a finite field* F_q *with* q *elements contained in* k *over which all the elements of* Aut(C') *are defined. Then there exist a Galois extension* K'/K*, a birational isomorphism* $u: C \to C'$ *defined over* K'*, an injective homomorphism* $j: G(K'/K) \to \mathrm{Aut}(C')$. *The set* C_K *is infinite if and only if either* (1) *there exists a point* $z \in C'_{K'} - C'_{k'}$ *such that* $j(s)\, z = z^s$ *for all* $s \in G(K'/K)$*, where* $k' = \bar{k} \cap K'$*, with the algebraic closure* $\bar{k}$ *of* k*, or* (2) *(only when* $K' = K \cdot k'$*) the set* $\{x \in C'_{k'} \mid j(s)x = x^s$ *for all* $s \in G(K'/K)\}$ *is infinite. At any rate, in this case there exists a finite set* $(x_i)_{i \in I}$ *of points of* $C'_{K'}$ *such that every point of* C_K *can be written either in the form* $u^{-1}(f^n(x_i))$ *or (only when* $K' = K \cdot k'$*)* $u^{-1}(x)$ *with* $x \in C'_{k'}$*, where* f *is the Frobenius morphism:* $x \to x^q$ *of* C'.

Here we notice that in this paper "genus" and "non-singular" are used all in the absolute sense.

The author wishes to express his hearty thanks to Professor Y. Kawada for his valuable advice and encouragement and to Professor S. Koizumi who kindly informed the author of the useful Lemma 1 by which the proof became very clear and short.

§ 2. Several Lemmas.

LEMMA 1 (Koizumi). *Let* C *and* C' *be complete non-singular algebraic curves, of same genus* $g \geqq 2$*, defined over a field* k *and* σ *be a birational isomorphism from* C *to* C'. *Then* σ *is defined over a finite separably algebraic extension of* k.

PROOF. Let (J, φ) and (J', φ') be the Jacobian varieties of C and C' respectively, where J and J' are defined over k and φ and φ' are defined over a finite separably algebraic extension of k. Then there exist a birational isomorphism $h: J \to J'$ and a point a of J' such that $h \cdot \varphi = \varphi' \cdot \sigma + a$. By Chow's Theorem (p. 26, [2]), h is defined over a finite separably algebraic extension of k. We have $h(\varphi(C)) = \varphi'(C') + a$. For the Θ-divisor $\Theta' = \varphi'(C') + \cdots + \varphi'(C')$ on J', we have $\Theta'_a = \Theta' + a = \varphi'(C') + \cdots + \varphi'(C') + h(\varphi(C))$. Since $h(\varphi(C))$ and $\varphi'(C')$ are defined over a finite separably algebraic extension ($= k'$) of k, the divisor $\Theta'_a - \Theta'$ is rational over k'. Hence by Corollary 2 of Theorem 32 [7], a is rational over k'. Therefore, $\sigma = (\varphi')^{-1} \cdot (h \cdot \varphi - a)$ is defined over a finite separably algebraic extension of k. Q. E. D.

LEMMA 2. *Let k be a field, k_1 a finite separably algebraic extension of k and K be an algebraic function field over k (i. e. a finite type regular extension of k). Let C_1 and C_2 be complete non-singular algebraic curves, of genera $\geqq 2$, defined over k_1 and K respectively and f be a birational isomorphism from C_1 to C_2 defined over $k_1 \cdot K$. In this case, there exists a complete non-singular algebraic curve C_0 defined over k, which is birationally isomorphic to C_1 (resp. C_2) over k_1 (resp. K) compatibly with f.*

PROOF. Let (σ, τ) be a pair of isomorphisms of k_1 over k. Then (σ, τ) can be considered as a pair of isomorphisms of $k_1 \cdot K$ over K. The birational isomorphism $f_{\tau,\sigma} = (f^\tau)^{-1} \cdot f^\sigma : C_1^\sigma \to C_1^\tau$ is defined over a finite separably algebraic extension of k by Lemma 1. Clearly we have 1) $f_{\tau,\sigma} \cdot f_{\sigma,\rho} = f_{\tau,\rho}$ for a triple (σ, τ, ρ) of isomorphisms of k_1 over k and 2) $f_{\tau\omega,\sigma\omega} = (f_{\tau,\sigma})^\omega$ for every automorphism ω of the separably algebraic closure of k. Therefore, by the Theorem of Weil (p. 12 [2]), there exist a complete non-singular algebraic curve C_0 defined over k and a birational isomorphism $f_1: C_0 \to C_1$ defined over k_1 such that $f_{\tau,\sigma} = f_1^\tau \cdot (f_1^\sigma)^{-1}$. Since we have $(f^\tau)^{-1} \cdot f^\sigma = f_1^\tau \cdot (f_1^\sigma)^{-1}$, we get $(f \cdot f_1)^\sigma = (f \cdot f_1)^\tau$. Hence the birational isomorphism $f \cdot f_1 : C_0 \to C_2$ is defined over K. Thus our Lemma is proved. Q. E. D.

LEMMA 3. *Let k be a field, k_1 a purely inseparable extension of k and K be an algebraic function field over k. Let C_1 and C_2 be complete non-singular algebraic curves defined over k_1 and K respectively and f be a birational isomorphism from C_1 to C_2 defined over $k_1 \cdot K$. In this case there exists a complete non-singular algebraic curve C_0 defined over k, which is birationally isomorphic to C_1 (resp. C_2) over k_1 (resp. K) compatibly with f.*

PROOF. Let T be a model of the function field K/k and t, t', t'' be the independent generic points of T over k such that $k(t) = K$. We extend the generic specialization $t \overset{k_1}{\leftrightarrow} t'$ to the generic specialization $(t, C_2 = C_t, f = f_t, C_1) \overset{k_1}{\leftrightarrow} (t', C_{t'}, f_{t'}, C_1)$. Then $f_{t'}$ is a birational isomorphism from C_1 to $C_{t'}$ and $f_{t',t} = f_{t'} \cdot f_t^{-1} : C_t \to C_{t'}$ is a birational isomorphism defined over $k_1(t, t')$ and over $k(t, t')$ by Lemma 1. Clearly we have $f_{t'',t'} \cdot f_{t',t} = f_{t'',t}$. Therefore, by Weil's Theorem (p. 12, [2]), there exist a complete non-singular curve C_0 defined over k and a birational isomorphism $g_t: C_0 \to C_t = C_2$ defined over $k(t) = K$ such that $f_{t',t} = f_{t'} \cdot f_t^{-1} = g_{t'} \cdot g_t^{-1}$. On the other hand the birational isomorphism $f_t^{-1} \cdot g_t$ is defined over $k_1 \cdot K$. Hence, by Lemma 1, $f^{-1} \cdot g_t : C_0 \to C_1$ is defined over k_1. Thus Lemma is proved. Q. E. D.

Unifying the Lemma 2 and Lemma 3, we get

LEMMA 4. *Let k be a field, k_1 a finite algebraic extension of k and K a function field over k. Let C_1 and C_2 be the complete non-singular algebraic curves, of genera $\geqq 2$, defined over k_1 and K respectively and f be a birational isomorphism from C_1 to C_2 defined over $k_1 \cdot K$. In this case, there exists a complete non-singular algebraic curve C_0 defined over k, which is birationally isomorphic to C_1 (resp. C_2) over k_1 (resp. K) compatibly with f.*

§ 3. The proof of Theorem.

Let us prove the Theorem written in the introduction.

a) and b) i). We prove the cases a) and b) i) at the same time. In these cases we have only to prove the necessity. Let $\bar{k}$ be the algebraic closure of k. Since $C_{\bar{k}\cdot K}$ $(\supset C_K)$ is infinite set, by Theorem of Manin-Grauert for the case a) and Theorem of Samuel i) for the case b) i), there exist a complete non-singular algebraic curve C_1 defined over $\bar{k}$ and a birational isomorphism $u: C \to C_1$ defined over $\bar{k} \cdot K$. Since C_1 and u_1 are defined over finitely generated field over the prime field, we may replace $\bar{k}$ by a finite algebraic extension k_1 of k. Then, by Lemma 4, there exist a complete non-singular algebraic curve C' defined over k and a birational isomorphism $u: C \to C'$ defined over K. In this case $C'_{\bar{k}\cdot K} - C'_{\bar{k}}$ is a finite set and $C'_K - C'_k$ is a subset of $C'_{\bar{k}\cdot K} - C'_{\bar{k}}$. Thus we can conclude that $C'_K - C'_k$ is a finite set. Q. E. D.

b) ii). By Lemma 1, the birational isomorphism u, we write, from C to C' is defined over a finite Galois extension K' of K, If we put $j(s) = u^s \cdot u^{-1}$ for the element s of the Galois group $G(K'/K)$ of the extension K'/K, then j defines a homomorphism $j: G(K'/K) \to \mathrm{Aut}(C')$. If j is not injective, we can replace K' by the elementwise fixed subfield of K' under the kernel of j, and then j will be injective. Then we have $C_K = \{u^{-1}(x) \mid x \in C'_{K'},\ j(s)x = x^s \text{ for all } s \in G(K'/K)\}$. In fact $(u^{-1}(x))^s = (u^s)^{-1}(x^s) = (u^s)^{-1}(j(s)x) = (u^s)^{-1}(u^s \cdot u^{-1}(x)) = u^{-1}(x)$ for $x \in C'_{K'}$ with $j(s)x = x^s$ and for $y = u^{-1}(x) \in C_K$, $j(s)x = u^s \cdot u^{-1}(x) = u^s(y) = u^s(y^s)$ $= (u(y))^s = x^s$. When we have $K' \neq k' \cdot K$ with $k' = \bar{k} \cap K'$, the set $\{x \in C'_{k'} \mid j(s)x = x^s = x \text{ for all } s \in G(K'/K)\}$ is a finite set, because the set of fixed points of non-trivial automorphism of C' is finite. Therefore, if C_K is infinite there exists a point $z \in C'_{K'} - C'_{k'}$ such that $j(s)z = z^s$ for all $s \in G(K'/K)$. Conversely, for such a point z, all $f^n(z)$ $(n > 0)$ are distinct and satisfy the condition $j(s)(f^n(z)) = (f^n(z))^s$ for all $s \in G(K'/K)$. Therefore, the existence of such a point z implies the infiniteness of C_K. (b) ii) (1)). The assertion b) ii) (2) is trivial by the above discussions. Now we show the last assertion. By the Theorem of Severi (p. 73 [**6**]) and the finiteness of $\mathrm{Aut}(C')$, there are only finitely many points $(x_i)_{i\in I}$ in $C'_{K'}$ with $j(s)x_i=x_i^s$ (for all $s\in G(K'/K)$) such that $k(x_i)\not\subset K'^q$. If a point $z\in C'_{K'}-C'_{k'}$ satisfies $j(s)z=z^s$ (for all $s\in G(K'/K)$), then we have $j(s)(z^{q^{-n}})=(z^{q^{-n}})^s$ and $k\subset k(z^{q^{-n}})\subset K'$, $k(z^{q^{-n}})\not\subset K'^q$ for some integer n. Hence we have $z^{q^{-n}}=x_i$ for some $i\in I$, i.e. $z=x_i^{q^n}=f^n(x_i)$. If we recall the finiteness of $C_K\cap u^{-1}(C'_{k'})$ in the case $K'\neq k'\cdot K$, we can conclude our last assertion. Q.E.D.

University of Tokyo

References

[1] H. Grauert, Mordell's Vermutung über rationale Punkte auf algebraishe Kurven und Funktionenkörper, Publ. Math. I.H.E.S. N°29, 1965.
[2] S. Lang, Abelian varieties, Interscience Publ., New York-London, 1959.
[3] M. Miwa, On Mordell's conjecture of the curve over function field, J. Math. Soc. Japan, **18** (1966), 182-188.
[4] J.L. Mordell, On the rational solutions of the indeterminate equations of third and fourth degree, Proc. Cambrige Philos. Soc., **21** (1922), 179-192.
[5] J.P. Samuel, Compléments á un articles de Hans Grauert sur la conjecture de Mordell, Publ. Math. I.H.E.S. N°30, 1966, 311-318.
[6] J.P. Samuel, Lecture note at Tata institute (India), 1967.
[7] A. Weil, Variétés abéliennes et courbes algébriques, Hermann, Paris, 1948.
[8] A. Weil, The field of definition of a variety, Amer. J. Math., **78** (1956), 509-524.