J. Math. Soc. Japan
Vol. 21, No. 2, 1969

Some perturbation theorems for nonnegative contraction semigroups*

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(Received Oct. 29, 1968)

§ 1. Introduction.

Let $G^+(M, \alpha)$ be the set of infinitesimal generators of strongly continuous semigroups T_t, $t \geqq 0$, of nonnegative linear operators on a Banach lattice $\mathfrak{B}$ such that $\|T_t\| \leqq Me^{\alpha t}$. We consider additive and multiplicative perturbation of operators $A \in G^+(1, \alpha)$, and prove Theorem 2.1 for additive perturbation $A+B$ and Theorems 3.1-3.3 for multiplicative perturbation B_2AB_1. A key role is played by the condition of α-dispersiveness defined below, and the main requirement for the perturbed operator to belong to $G^+(1, \alpha')$ is that it is α'-dispersive.

Define $\tau(f, g) = \lim_{\varepsilon \to 0+} \varepsilon^{-1}(\|f+\varepsilon g\| - \|f\|)$ for any f and g in $\mathfrak{B}$, and $\sigma(f, g) = \inf \tau(f, (g+h) \vee (-bf))$ for $f \geqq 0$ and any g, where the infimum is taken over all h and b satisfying $f \wedge |h| = 0$ and b a nonnegative real number. Let α be a real number: we call an operator A *α-dispersive in the strict sense* or *α-dispersive* (*s*) if $\sigma(f^+, Af) \leqq \alpha \|f^+\|$ for all $f \in \mathfrak{D}(A)$, *α-dispersive in the wide sense* or *α-dispersive* (*w*) if $\sigma(f^+, -Af) \geqq -\alpha \|f^+\|$ for all $f \in \mathfrak{D}(A)$. 0-dispersive (s) and (w) are the same as dispersive (s) and (w) defined in [**6**], where the functional σ was introduced and shown to possess the following properties. Let $f \geqq 0$:

(1.1) $$-\|g^-\| \leqq \sigma(f, g) \leqq \|g^+\|;$$

(1.2) $$\sigma(f, ag) = a\sigma(f, g), \qquad a \geqq 0;$$

(1.3) $$\sigma(f, af+g) = a\|f\| + \sigma(f, g), \qquad \text{any } a;$$

(1.4) $$\sigma(f, g+h) \leqq \sigma(f, g) + \sigma(f, h);$$

(1.5) $$g \leqq h \Rightarrow \sigma(f, g) \leqq \sigma(f, h);$$

(1.6) $$f \wedge |h| = 0 \Rightarrow \sigma(f, g) = \sigma(f, g+h).$$

* Partially supported by the National Science Foundation while the first named author was at the University of Minnesota and the second named author was visiting there.

As a consequence, we have $-\sigma(f, -g) \leqq \sigma(f, g)$, so that α-dispersive (s) implies α-dispersive (w).

Operators in $G^+(1, \alpha)$ are characterized as follows.

THEOREM 1.1. *Let A be a densely defined linear operator with $\mathfrak{R}(\lambda I - A) = \mathfrak{B}$ for some $\lambda > \alpha$. Then the following three properties are equivalent: $A \in G^+(1, \alpha)$; A is α-dispersive (s); and A is α-dispersive (w).*

In fact, the above theorem was proved in [**6**, Theorems 1 and 2] if $\alpha = 0$, and the general case reduces to the case $\alpha = 0$ by the following easily proved lemmas.

LEMMA 1.1. *$A \in G^+(M, \alpha)$ if and only if $A + \beta I \in G^+(M, \alpha + \beta)$.*

LEMMA 1.2. *A is α-dispersive (s) if and only if $A + \beta I$ is $(\alpha + \beta)$-dispersive (s). The same is true with (s) replaced by (w).*

REMARK 1.1. Phillips [**5**] and Hasegawa [**4**] gave characterizations of $G^+(1, 0)$ prior to [**6**]. But Hasegawa's dispersiveness is not convenient for perturbation questions because the functional τ' introduced by him does not possess the subadditivity property (1.4). Phillips used a special type of semi-inner-product dispersiveness, and all of our theorems remain true (this is easily checked) if we define α-dispersiveness in terms of his semi-inner-product instead of σ. But our definition has an advantage in applications since one can concretely express α-dispersiveness (s) and (w) in many Banach lattices (see the discussion and examples in [**6**]).

REMARK 1.2. The proofs in this paper could be essentially shortened if the following were true: a 0-dispersive (w) operator B is dissipative in at least one semi-inner-product. Phillips [**5**, p. 298] mentions a similar question, and these questions can be answered in the affirmative for many Banach lattices, or if B is bounded, or more generally if $B \in G(M, \alpha)$, etc. Nonetheless the general relationship among the different definitions of dispersiveness is not known.

Most of our results are analogous to the perturbation theorems for infinitesimal generators $G(1, 0)$ of strongly continuous contraction semigroups studied in [**1, 2, 3, 7**] and others, wherein the condition of dissipativeness plays a key role. However, due to the situation just mentioned, it was necessary to obtain our results independent of the discussion of $G(1, 0)$.

§ 2. Additive perturbation.

THEOREM 2.1. *Let $A \in G^+(1, \alpha)$ and let B be a linear operator with $\mathfrak{D}(B) \supset \mathfrak{D}(A)$ such that for some $a < 1$ and $b < +\infty$*

$$\|Bf\| \leqq a\|Af\| + b\|f\|, \quad \text{for all } f \in \mathfrak{D}(A). \tag{2.1}$$

If $A + B$ is $(\alpha + \beta)$-dispersive (w), then $A + B \in G^+(1, \alpha + \beta)$.

Gustafson [1, Theorem 2] proved a similar theorem for $G(1, 0)$, extending the previous limit from $a<\frac{1}{2}$ to $a<1$. On the other hand, for $\alpha=\beta=0$, Sato [6, Lemma 5.2] proved the above theorem under the assumption $a<\frac{1}{2}$. There 0-dispersiveness (w) of B was assumed, but the proof needs no change if $A+B$ is 0-dispersive (w). For the present case, we need a lemma.

LEMMA 2.1. *If A is α-dispersive (s) and $A+B$ is $(\alpha+\beta)$-dispersive (w), then $A+cB$ is $(\alpha+c\beta)$-dispersive (w) for $0\leqq c\leqq 1$.*

PROOF. By (1.2) and (1.4) we have

$$\begin{aligned}\sigma(f^+, -(A+cB)f) &\geqq \sigma(f^+, -c(A+B)f)-\sigma(f^+, (1-c)Af)\\ &= c\sigma(f^+, -(A+B)f)-(1-c)\sigma(f^+, Af)\\ &\geqq -c(\alpha+\beta)\|f^+\|-(1-c)\alpha\|f^+\|\\ &= -(\alpha+c\beta)\|f^+\| .\end{aligned}$$

PROOF OF THEOREM 2.1. Since (2.1) implies

$$\|(B-\beta I)f\|\leqq a\|(A-\alpha I)f\|+(a|\alpha|+|\beta|+b)\|f\| ,$$

the theorem reduces to the case $\alpha=\beta=0$ by Lemmas 1.1 and 1.2. Hence, assume that $\alpha=\beta=0$. It remains only to handle the case $\frac{1}{2}\leqq a<1$. We can find $c_j>0$, $(j=1, 2, \cdots, n)$, $a'<\frac{1}{2}$ and $b'<+\infty$ such that $\sum_{j=1}^{n} c_j=1$ and

$$\|c_k Bf\|\leqq a'\|(A+\sum_{j=1}^{k-1} c_j B)f\|+b'\|f\| , \qquad k=1, 2, \cdots, n ,$$

exactly in the same way as in [1]. Thus we have $A+\sum_{j=1}^{k} c_j B\in G^+(1, 0)$ for $k=1, 2, \cdots, n$, noting that Lemma 2.1 guarantees their 0-dispersiveness (w), and the theorem is proved.

REMARK 2.1. In Theorem 2.1 we can replace the assumption of $(\alpha+\beta)$-dispersiveness (w) of $A+B$ by β-dispersiveness (w) of B. The new assumption is stronger since we have

LEMMA 2.2. *If A is α-dispersive (s) and B is β-dispersive (w), then $A+B$ is $(\alpha+\beta)$-dispersive (w).*

PROOF. From (1.4) we have

$$\sigma(f^+, -(A+B)f)\geqq \sigma(f^+, -Bf)-\sigma(f^+, Af)\geqq -(\alpha+\beta)\|f^+\| .$$

REMARK 2.2. Theorem 2.1 cannot be extended to $a\leqq 1$. For example, let $\mathfrak{B}$ be the Banach lattice of continuous functions on the real line which vanish at infinity with norm $\|f\|=\max|f(x)|$. Let φ be the continuous function defined by $\varphi(x)=1$ for $x\leqq 0$, $\varphi(x)=1+\sqrt{x}$ for $0<x\leqq 1$ and $\varphi(x)=2$ for $x>1$

and let $A=\varphi(x)D$, $B=-D$, where $D=\frac{d}{dx}$, the domain of D being the set of functions f such that $f\in\mathfrak{B}$ and $f'\in\mathfrak{B}$. Then $A, B\in G^+(1, 0)$, hence $A+B$ is 0-dispersive (w), and (2.1) is satisfied for $a=1$ and $b=0$. But, $A+B\notin G^+(1, 0)$. In fact, no extension of $A+B$ belongs to $G^+(1, 0)$, as shown by Trotter [**7**, Example 2].

§ 3. Multiplicative perturbation.

The following result incorporates into one statement dispersive analogues of the left and right bounded multiplicative perturbation of $G(1, 0)$ results of [**2**] and [**3**]; we also state (Theorem 3.3 below) an unbounded version. A more detailed investigation of the individual cases for unbounded multipliers may be found in [**3**] (in a dissipative context, for $G(1, 0)$); that paper should also be seen for examples of specific applications of multiplicative perturbation. For the reasons mentioned in Remark 1.2, our proofs are somewhat different from those employed in [**2, 3**].

THEOREM 3.1. *Let $A\in G^+(1, \alpha)$ and let B be a bounded linear operator such that $\mathfrak{D}(B)=\mathfrak{B}$ and $-B$ is $(-\beta)$-dispersive (w) for some $\beta>0$. Let C denote BA, AB, $B^{-1}AB$, or BAB^{-1}. Then $C\in G^+(1, \gamma)$, provided that C is γ-dispersive (w).*

Note that, by the following lemma, B has an everywhere defined inverse B^{-1} under the conditions in Theorem 3.1.

LEMMA 3.1. *Let B be a bounded linear operator with $\mathfrak{D}(B)=\mathfrak{B}$ and suppose that $-B$ is $(-\beta)$-dispersive (w) for some $\beta>0$. Then $\mathfrak{R}(B)=\mathfrak{B}$, B^{-1} exists and is bounded, and $-B$ is $(-\beta)$-dispersive (s).*

PROOF. Everything follows from the fact that $-B\in G^+(1, -\beta)$.

Also by this lemma, Theorem 3.1 consists of special cases of the next more general result.

THEOREM 3.2. *Let $A\in G^+(1, \alpha)$. Let B_1 and B_2 be bounded linear operators such that $\mathfrak{D}(B_j)=\mathfrak{R}(B_j)=\mathfrak{B}$, $j=1, 2$, B_1 has a bounded inverse, and $-B_2B_1$ is $(-\gamma)$-dispersive (w) for some $\gamma>0$. If B_2AB_1 is α'-dispersive (w), it belongs to $G^+(1, \alpha')$.*

PROOF. Let $A'=B_2AB_1$ and suppose that A' is α'-dispersive (w). Choose a positive number λ so large that $\lambda>\alpha$ and $\alpha'-\lambda\alpha<0$, and let $E=A'-\lambda B_2B_1$. Since $-B_2B_1$ is $(-\gamma)$-dispersive (s) by Lemma 3.1, $-\lambda B_2B_1$ is $(-\lambda\gamma)$-dispersive (s), and hence E is $(\alpha'-\lambda\gamma)$-dispersive (w) by Lemma 2.2. $\mathfrak{D}(E)$ is dense because $\mathfrak{D}(E)=\mathfrak{D}(A')=\mathfrak{D}(AB_1)$ and $\mathfrak{D}(AB_1)$ is dense by the bounded invertibility of B_1 and denseness of $\mathfrak{D}(A)$. Furthermore, since $E=B_2(A-\lambda I)B_1$, $A\in G^+(1, \alpha)$ and $\lambda>\alpha$, we have $\mathfrak{R}(E)=\mathfrak{B}$. Therefore $E\in G^+(1, \alpha'-\lambda\gamma)$ by Theorem 1.1, noting that $\alpha'-\lambda\gamma<0$. A' being a bounded perturbation of E, we obtain

$A' \in G^{+}(1, \alpha')$ by Theorem 2.1.

THEOREM 3.3. *Let* $A \in G^{+}(1, \alpha)$, *let* $\mathfrak{D}(B_2) = \mathfrak{R}(B_2) = \mathfrak{B}$, *let* $\mathfrak{D}(B_1)$ *be dense and either:* (i) B_1^{-1} *bounded and* $\mathfrak{R}(B_1) \supset \mathfrak{D}(A)$; *or* (ii) B_1 *closed and* $\mathfrak{R}(B_1) = \mathfrak{B}$. *Suppose that* $-B_2B_1$ *is* $(-\gamma)$*-dispersive* (*w*) *for some* $\gamma > 0$, *and bounded. Then if* B_2AB_1 *is* α'*-dispersive* (*w*), *it belongs to* $G^{+}(1, \alpha')$.

PROOF. It is easy to check (e. g., it follows directly from [**6**, Theorem 4] and Lemma 1.2) that $-B_2B_1$ is $(-\gamma)$-dispersive (s). Choosing λ as in the proof of Theorem 3.2, let $A' = B_2AB_1$, $E = A' - \lambda B_2B_1$, and $E' = B_2(A - \lambda I)B_1$. Then $\mathfrak{R}(E') = \mathfrak{B}$, because $\mathfrak{R}(B_2) = \mathfrak{B}$, $A \in G^{+}(1, \alpha)$, and $\mathfrak{R}(B_1) \supset \mathfrak{D}(A)$. Since $\mathfrak{D}(B_2) = \mathfrak{B}$, it follows that $\mathfrak{D}(E') = \mathfrak{D}(E) = \mathfrak{D}(A') = \mathfrak{D}(AB_1)$; in particular $E' = E$. We can conclude that $A' \in G^{+}(1, \alpha')$ as before if $\mathfrak{D}(AB_1)$ is dense. The latter is assured in case (i) as in Theorem 3.2 by B_1^{-1} bounded and $\mathfrak{D}(B_1)$ dense; in case (ii), $\mathfrak{D}(AB_1) = \mathfrak{D}((A - \lambda I)B_1)$ is dense by the well-known Fredholm theory, since $A - \lambda I$ is Fredholm and B_1 is closed with finite (zero) deficiency index.

REMARK 3.1. Suppose $\beta = 0$ in the assumption of Theorem 3.1. Then BA can fail to be in $G^{+}(1, \gamma)$ even if it is γ-dispersive (w). For example, let $\mathfrak{B}$, $\varphi(x)$, and D be the same as in Remark 2.2, let $A = D$, and let B be multiplicative by the function $\varphi(x) - 1$. Then $A \in G^{+}(1, 0)$, $-B$ is bounded 0-dispersive (s), and BA is 0-dispersive (s), but any extension of BA does not belong to $G^{+}(1, 0)$ as mentioned before. To check dispersiveness, note that the 0-dispersiveness (s) of an operator C in this space is equivalent to the following maximum principle, as is shown in [**6**, Example 6.2]: if $f \in \mathfrak{D}(C)$ attains its positive maximum at x_0, then $Cf(x_0) \leqq 0$.

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