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Interpolation by the real method preserves compactness of operators

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In this paper we will prove the following

THEOREM. *Let* $[E_0, E_1]$ *and* $[F_0, F_1]$ *be arbitrary interpolation pairs, and let* T *be a continuous linear operator from the couple* $[E_0, E_1]$ *to the couple* $[F_0, F_1]$. *If the mappings* $T: E_0 \to F_0$ *and* $T: E_1 \to F_1$ *are compact, then for* $1 \leqq p < \infty$, $0 < \theta < 1$ $T: S(\theta, p; E_0, E_1) \to S(\theta, p; F_0, F_1)$ *is compact. Here* $S(\theta, p; E_0, E_1)$ *is the interpolation space by the real method of Lions and Peetre* [**1**].

When the couple $[F_0, F_1]$ satisfies a certain approximation hypothesis, A. Persson [**3**] proved that if $T: E_0 \to F_0$ is compact, then $T: E_\theta \to F_\theta$ is also compact, where E_θ and F_θ are the interpolation spaces by the real or the complex method.

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§ 1. Notations, definitions and fundamental facts.

For two linear topological spaces $\mathcal{E}$ and $\mathcal{F}$, we write $\mathcal{E} \subset \mathcal{F}$ if $\mathcal{E}$ is a linear subspace of $\mathcal{F}$ and the identity map is continuous.

A pair of Banach spaces $[E_0, E_1]$ is said to be an interpolation pair if there exists a Hausdorff linear topological space $\mathcal{E}$ such that $E_0 \subset \mathcal{E}$ and $E_1 \subset \mathcal{E}$. In this paper, when we write $[E_0, E_1]$ or $[F_0, F_1]$ we always assume that the pair is an interpolation pair.

For $[E_0, E_1]$ we can define Banach spaces $E_0 \cap E_1$ and $E_0 + E_1$ with norms

$$\|x\|_{E_0 \cap E_1} = \mathrm{Max}\,(\|x\|_{E_0},\ \|x\|_{E_1}),$$

and

$$\|x\|_{E_0 + E_1} = \mathrm{Inf}\,(\|x_0\|_{E_0} + \|x_1\|_{E_1};\ x = x_0 + x_1)$$

respectively.

Given a Banach space E and real numbers p and θ ($1 \leqq p \leqq \infty$), we consider E-valued sequences $\{a_m\}_{m=-\infty}^{\infty}$ such that $\{e^{m\theta}\|a_m\|_E\} \in l^p$. In the linear space of all those sequences, which is denoted by $l_\theta^p(E)$, we introduce the norm

$$\|\{a_m\}\|_{l_\theta^p(E)} = \|\{\|e^{m\theta}a_m\|_E\}\|_{l^p} = \{\sum_{m=-\infty}^{\infty}\|e^{m\theta}a_m\|_E^p\}^{\frac{1}{p}}.$$

In case $p=\infty$, we modify this norm in the usual manner.

DEFINITION 1.1. Given real numbers p_0, p_1 and θ (where $1\leqq p_0, p_1\leqq\infty$, $0<\theta<1$) we denote by $w(p_0, \theta, E_0; p_1, \theta-1, E_1)$ the Banach space $l_\theta^{p_0}(E_0) \cap l_{\theta-1}^{p_1}(E_1)$.

DEFINITION 1.2. Under the same condition, if $\{a_m\}\in w(p_0, \theta, E_0; p_1, \theta-1, E_1)$, then the sum $\sum_{m=-\infty}^{\infty} a_m$ converges in E_0+E_1. The set of all such elements $\sum a_m$ in E_0+E_1 forms a Banach space with the norm

$$\|a\|_S = \mathrm{Inf}\ \|\{a_m\}\|_{w(p_0,\theta,E_0;p_1,\theta-1,E_1)}; \qquad \sum a_m = a\}.$$

We denote this Banach space by $S(p_0, \theta, E_0; p_1, \theta-1, E_1)$ and the mapping $\{a_m\}\to\sum a_m$ by $\sum$.

PROPOSITION A (Lions-Peetre [1]). *If* $p_0\leqq q_0$ *and* $p_1\leqq q_1$, *then we have*

$$S(p_0, \theta, E_0; p_1, \theta-1, E_1)\subset S(q_0, \theta, E_0; q_1, \theta-1, E_1).$$

PROPOSITION B (Peetre [2]). *If* $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$, *then we have*

$$S(p_0, \theta, E_0; p_1, \theta-1, E_1)=S(p, \theta, E_0; p, \theta-1, E_1).$$

In this paper we denote $S(p, \theta, E_0; p, \theta-1, E_1)$ by $S(\theta, p; E_0, E_1)$ and $w(p, \theta, E_0; p, \theta-1, E_1)$ by $w(\theta, p; E_0, E_1)$ for short.

DEFINITION 1.3. Let E and F be Banach spaces. We denote by $\mathcal{B}(E, F)$ the Banach space of all continuous linear operators on E into F. If $T\in\mathcal{B}(E, F)$ is compact from E to F, we write $T\in K(E, F)$. We denote by $\mathcal{B}([E_0, E_1], [F_0, F_1])$ the Banach space of all linear operators on E_0+E_1 to F_0+F_1 which transform continuously E_0 into F_0 and E_1 into F_1 respectively. If, in addition, $T; E_0\to F_0$ and $E_1\to F_1$ are compact, we write $T\in K([E_0, E_1], [F_0, F_1])$.

DEFINITION 1.4. A Banach space $X\supset E_0\cap E_1$ is said to be of class $\underline{\mathcal{K}}_\theta(E_0, E_1)$ if there exists a constant $C>0$ such that $\|a\|_X\leqq C\|a\|_{E_0}^{1-\theta}\|a\|_{E_1}^{\theta}$ for all $a\in E_0\cap E_1$. Also a Banach space $X\subset E_0+E_1$ is said to be of class $\overline{\mathcal{K}}_\theta(E_0, E_1)$ if there exists a constant $C>0$ such that for any $a\in X$ and for any $t>0$, we can choose $a_i(t)$ in E_i $(i=0, 1)$ with the property that $a_0(t)+a_1(t)=a$, $\|a_0(t)\|_{E_0}\leqq Ct^{-\theta}\|a\|_X$, and $\|a_1(t)\|_{E_1}\leqq Ct^{1-\theta}\|a\|_X$. A Banach space is said to be of class $\mathcal{K}_\theta(E_0, E_1)$ if it is of class $\underline{\mathcal{K}}_\theta(E_0, E_1)$ and of class $\overline{\mathcal{K}}_\theta(E_0, E_1)$.

Then the space $S(p_0, \theta, E_0; p_1, \theta-1, E_1)$ is of class $\mathcal{K}_\theta(E_0, E_1)$, (see [1]).

PROPOSITION C (Lions-Peetre [1]). *Let* p_0, p_1 *and* θ *be real numbers as in Definition* 1.1. *If* $T\in\mathcal{B}([E_0, E_1], [F_0, F_1])$, *then we have*

i) $T\in\mathcal{B}(S(p_0, \theta, E_0; p_1, \theta-1, E_1), S(p_0, \theta, F_0; p_1, \theta-1, F_1))$ *and*

ii) $\|T\|_\theta\leqq C\|T\|_0^{1-\theta}\|T\|_1^{\theta}$.

where $\|T\|_\theta$, $\|T\|_0$ *and* $\|T\|_1$ *are the norms in the spaces* $\mathscr{B}(S(p_0, \theta, E_0; p_1, \theta-1, E_1), S(p_0, \theta, F_0; p_1, \theta-1, F_1))$, $\mathscr{B}(E_0, F_0)$ *and* $\mathscr{B}(E_1, F_1)$ *respectively.*

PROPOSITION D (Lions-Peetre [1]). *Let* X_{θ_0} *and* X_{θ_1} *be Banach spaces of class* $\mathscr{K}_{\theta_0}(E_0, E_1)$ *and* $\mathscr{K}_{\theta_1}(E_0, E_1)$ *respectively (where* $0<\theta_0\neq\theta_1<1$*). Then we have*

$$S(q_0, \nu, X_{\theta_0}; q_1, \nu-1, X_{\theta_1}) = S(p_0, \theta_\nu, E_0; p_1, \theta_\nu-1, E_1),$$

where $\theta_\nu=(1-\nu)\theta_0+\nu\theta_1$ *and* $\frac{1}{q_i}=\frac{1-\theta_i}{p_0}+\frac{\theta_i}{p_1}$ $i=0, 1$.

PROPOSITION E (Peetre [2]). *We have*

$$S(\theta, r; E_0, E_1) = S\Big(\frac{1}{r}, r; S(\theta, \infty; E_0, E_1), S(\theta, 1; E_0, E_1)\Big).$$

PROPOSITION F (Lions-Peetre [1], Peetre [2]). *Let E be a Banach space. Then we have*

$$S(p_0, \nu, l_{\theta_0}^{p_0}(E); p_1, \nu-1, l_{\theta_1}^{p_1}(E)) = l_{\theta_\nu}^{p_\nu}(E),$$

where

$$\theta_\nu=(1-\nu)\theta_0+\nu\theta_1 \quad \text{and} \quad \frac{1}{p_\nu}=\frac{1-\nu}{p_0}+\frac{\nu}{p_1}. \tag{1.1}$$

§2. Interpolation space of the spaces $w(\theta, p; E_0, E_1)$.

PROPOSITION 2.1. *For* $0<\nu<1$, $1\leqq p_0, p_1\leqq\infty$ *and* $0<\theta_0, \theta_1<1$, *we have*

$$S(p_0, \nu, w(\theta_0, p_0; E_0, E_1); p_1, \nu-1, w(\theta_1, p_1; E_0, E_1))$$
$$= w(\theta_\nu, p_\nu; E_0, E_1),$$

where p_ν *and* θ_ν *are given by* (1.1).

PROOF. We shall denote by S the left hand side of our identity. By Proposition F and Definition 1.1, we have

$$S = S(p_0, \nu, l_{\theta_0}^{p_0}(E_0)\cap l_{\theta_0-1}^{p_0}(E_1); p_1, \nu-1, l_{\theta_1}^{p_1}(E_0)\cap l_{\theta_1-1}^{p_1}(E_1))$$
$$\subset S(p_0, \nu, l_{\theta_0}^{p_0}(E_0); p_1, \nu-1, l_{\theta_1}^{p_1}(E_0)) = l_{\theta_\nu}^{p_\nu}(E_0).$$

Similarly we have $S\subset l_{\theta_\nu-1}^{p_\nu}(E_1)$. So we get $S\subset l_{\theta_\nu}^{p_\nu}(E_0)\cap l_{\theta_\nu-1}^{p_\nu}(E_1) = w(\theta_\nu, p_\nu; E_0, E_1)$.

Let λ be a real number satisfying $p_0(1-\lambda\nu)=p_\nu$. For $\{a_m\}\in w(\theta_\nu, p_\nu, E_0, E_1)$, we set

$$u_{m,n}=\begin{cases} a_m & \text{if } n=0 \\ 0 & \text{if } n\neq 0 \end{cases}$$

and

$$v_{m,n}=u_{m,n+[\alpha m]},$$

where

$$\alpha_m = \text{Max}\Big\{m\frac{p_\nu}{p_0}(\theta_0-\theta_1)+\lambda(m\theta_0+\log\|a_m\|_{E_0}),$$

$$m\frac{p_\nu}{p_0}(\theta_0-\theta_1)+\lambda(m(\theta_0-1)+\log\|a_m\|_{E_1})\Big\}.$$

Then we can verify the following lemma.

CALCULUS LEMMA.

i) $$e^{m\theta_0 p_0}\sum_{n=-\infty}^{\infty}\|e^{n\nu}v_{m,n}\|_{E_0}^{p_0}\leqq e^{\nu p_0}(e^{m\theta_\nu}\|a_m\|_{E_0})^{p_\nu}$$

ii) $$e^{m(\theta_0-1)p_0}\sum_{n=-\infty}^{\infty}\|e^{n\nu}v_{m,n}\|_{E_1}^{p_0}\leqq e^{\nu p_0}(e^{m(\theta_\nu-1)}\|a_m\|_{E_1})^{p_\nu}$$

iii) $$e^{m\theta_1 p_1}\sum_{n=-\infty}^{\infty}\|e^{n(\nu-1)}v_{m,n}\|_{E_0}^{p_1}\leqq (e^{m\theta_\nu}\|a_m\|_{E_0})^{p_\nu}+(e^{m(\theta_\nu-1)}\|a_m\|_{E_1})^{p_\nu}$$

iv) $$e^{m(\theta_1-1)p_1}\sum_{n=-\infty}^{\infty}\|e^{n(\nu-1)}v_{m,n}\|_{E_1}^{p_1}\leqq (e^{m\theta_\nu}\|a_m\|_{E_0})^{p_\nu}+(e^{m(\theta_\nu-1)}\|a_m\|_{E_1})^{p_\nu}.$$

PROOF OF THIS LEMMA. We will prove iii) and others can be proved similarly. By the definition of $v_{m,n}$, we have

$$e^{m\theta_1 p_1}\sum_{n=-\infty}^{\infty}\|e^{n(\nu-1)}v_{m,n}\|_{E_0}^{p_1}=e^{m\theta_1 p_1}e^{(1-\nu)p_1[\alpha_m]}\|a_m\|_{E_0}^{p_1}.$$

We set $\log\|a_m\|_{E_0}=c_m$ and $\log\|a_m\|_{E_1}=d_m$, then it is sufficient for us to prove

v) $$m\theta_1 p_1+(1-\nu)p_1[\alpha_m]+p_1 c_m\leqq \text{Max}\{m\theta_\nu p_\nu+p_\nu c_m,\ m(\theta_\nu-1)p_\nu+p_\nu d_m\}.$$

But this is evident since, by the definition of λ and θ_ν, p_ν we have

$$\frac{1}{1-\nu}\Big(\frac{\theta_\nu}{p_1}-\frac{\theta_1}{p_\nu}\Big)=\frac{\theta_0}{p_1}-\frac{\theta_1}{p_0},\quad \frac{1}{1-\nu}\Big(\frac{\theta_\nu-1}{p_1}-\frac{\theta_1-1}{p_\nu}\Big)=\frac{\theta_0-1}{p_1}-\frac{\theta_1-1}{p_0}$$

and

$$(m\theta_\nu p_\nu+p_\nu c_m-m\theta_1 p_1-p_1 c_m)/(1-\nu)p_1$$
$$=mp_\nu\Big(\frac{\theta_0}{p_1}-\frac{\theta_1}{p_0}\Big)+\lambda c_m,$$

$$\{m(\theta_\nu-1)p_\nu+p_\nu d_m-m\theta_1 p_1-p_1 c_m\}/(1-\nu)p_1$$
$$=mp_\nu\Big(\frac{\theta_0-1}{p_1}-\frac{\theta_1-1}{p_0}\Big)+\lambda d_m+\frac{1}{1-\nu}(d_m-c_m-m)$$

and by the definition of α_m, we have v). Hence we have iii).

Now, using i), ii) and Definition 1.1, we have

$$2\|\{a_m\}\|_{w(\theta_\nu,p_\nu;E_0,E_1)}$$
$$\geqq\Big(\sum_m\|e^{m\theta_\nu}a_m\|_{E_0}^{p_\nu}\Big)^{\frac{1}{p_\nu}}+\Big(\sum_m\|e^{m(\theta_\nu-1)}a_m\|_{E_1}^{p_\nu}\Big)^{\frac{1}{p_\nu}}$$

$$\geqq C(\sum_n e^{n\nu p_0}\sum_m \|e^{m\theta_0}v_{m,n}\|_{E_0}^{p_0})^{\frac{1}{p_\nu}}+C(\sum_n e^{n\nu p_0}\sum_m \|e^{m(\theta_0-1)}v_{m,n}\|_{E_1}^{p_0})^{\frac{1}{p_\nu}}$$

$$\geqq C_{p_0}(\sum_n e^{n\nu p_0}\{[\sum_m \|e^{m\theta_0}v_{m,n}\|_{E_0}^{p_0}]^{\frac{1}{p_0}}+[\sum_m \|e^{m(\theta_0-1)}v_{m,n}\|_{E_1}^{p_0}]^{\frac{1}{p_0}}\}^{p_0})^{\frac{1}{p_\nu}}.$$

So, for any fixed n, $\{v_{m,n}\}_{m=-\infty}^{\infty}\in {}_{(m)}l_{\theta_0}^{p_0}(E)\cap {}_{(m)}l_{\theta_0-1}^{p_0}(E_1)=w_{(m)}(\theta_0, p_0; E_0, E_1)$[1] and

$$\{\|\{v_{m,n}\}_{m=-\infty}^{\infty}\|_{w(m)(\theta_0,p_0;E_0,E_1)}\}_{n=-\infty}^{\infty}\in {}_{(n)}l_\nu^{p_0}.$$

Similarly, using iii), iv), we have

$$\{\|\{v_{m,n}\}_{m=-\infty}^{\infty}\|_{w(m)(\theta_1,p_1;E_0,E_1)}\}_{n=-\infty}^{\infty}\in {}_{(n)}l_{\nu-1}^{p_1}.$$

So, we have

$$\{\{v_{m,n}\}_{m=-\infty}^{\infty}\}_{n=-\infty}^{\infty}\in w_{(n)}(p_0, \nu, w_{(m)}(\theta_0, p_0; E_0, E_1); p_1, \nu-1, w_{(m)}(\theta_1, p_1; E_0, E_1)).$$

On the other hand, from the definition of $\{v_{m,n}\}$, we have

$$\sum_{n=-\infty}^{\infty} v_{m,n}=a_m \qquad \text{for any } m.$$

Then we have $\{a_m\}\in S$. Hence $S\supset w(\theta_\nu, p_\nu; E_0, E_1)$.

PROPOSITION 2.2. *Under the same assumption as in Proposition 2.1, we have*

$$S(p_0, \nu, S(\theta_0, p_0; E_0, E_1); p_1, \nu-1, S(\theta_1, p_1; E_0, E_1))=S(\theta_\nu, p_\nu; E_0, E_1).$$

This can be proved from Propositions D and E.

§ 3. Spaces $S(0, 1; E_0, E_1)$ and $S(1, 1; E_0, E_1)$.

Though for $p>1$ we cannot define the spaces $S(\theta, p; E_0, E_1)$ in the case when $\theta=0$ or 1, we can define them for $p=1$ as in § 1.

DEFINITION 3.1. We set

$$w(0, 1; E_0, E_1)=l_0^1(E_0)\cap l_{-1}^1(E_1),\quad S(0, 1; E_0, E_1)=\sum w(0, 1; E_0, E_1),$$

$$w(1, 1; E_0, E_1)=l_1^1(E_0)\cap l_0^1(E_1),\quad S(1, 1; E_0, E_1)=\sum w(1, 1; E_0, E_1).$$

Then we have following lemmas.

LEMMA 3.2. $E_0\cap E_1\subset S(0, 1; E_0, E_1)$ *and* $\|a\|_{S(0,1;E_0,E_1)}\leqq \|a\|_{E_0\cap E_1}$.

LEMMA 3.3. $E_0\cap E_1$ *is dense in* $S(0, 1; E_0, E_1)$.

LEMMA 3.4. $S(0, 1; E_0, E_1)\subset E_0$ *and* $\|a\|_{E_0}\leqq \|a\|_{S(0,1;E_0,E_1)}$ *for all* $a\in S(0, 1; E_0, E_1)$.

LEMMA 3.5. *For any* $a\in E_0\cap E_1$, *we have* $\|a\|_{S(0,1;E_0,E_1)}=\|a\|_{E_0}$.

LEMMA 3.6. *For any* $a\in S(0, 1; E_0, E_1)$, *we have* $\|a\|_{S(0,1;E_0,E_1)}=\|a\|_{E_0}$.

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1) $\{v_{m,n}\}\in {}_{(m)}l_{\theta_0}^{p_0}(E_0)$ (or $w_{(m)}(\theta_0, p_0; E_0, E_1)$) means that $\{v_{m,n}\}$ is an element of $l_{\theta_0}^{p_0}(E_0)$ (or $w(\theta_0, p_0; E_0, E_1)$ resp.) considered as a sequence in m for the fixed n.

Lemma 3.2. and Lemma 3.3 are proved in [1] for $0<\theta<1$. Our proof is the same.

PROOF OF LEMMA 3.4. For any $a\in S(0,1;E_0,E_1)$ and for any $\varepsilon>0$, there exists $\{a_m\}\in w(0,1;E_0,E_1)$ such that $\sum\{a_m\}=a$ and $\|\{a_m\}\|_w\leqq(1+\varepsilon)\|a\|_S$. From the definition we have $\{a_m\}\in l_0^1(E_0)$, hence we have $\sum_m a_m\in E_0$. Then $a\in E_0$, and

$$\|\{a_m\}\|_w\geqq\|\{a_m\}\|_{l_0^1(E_0)}=\sum_m\|a_m\|_{E_0}\geqq\|\sum_m a_m\|_{E_0}=\|a\|_{E_0}.$$

Then $\|a\|_{E_0}\leqq(1+\varepsilon)\|a\|_S$ for any $\varepsilon>0$. Hence we have $\|a\|_{E_0}\leqq\|a\|_S$.

PROOF OF LEMMA 3.5. We may assume that $a\neq 0$. Then we set

$$N=\left[\log\frac{\|a\|_{E_1}}{\|a\|_{E_0}}\right]+1,\quad\text{and}\quad a_m=\begin{cases}a & \text{if } m=N\\ 0 & \text{if } m\neq N.\end{cases}$$

Then we have $\|\{a_m\}\|_w=\|a\|_{E_0}$. Hence by the definition we obtain $\|a\|_S\leqq\|a\|_{E_0}$.

Lemma 3.6 is proved immediately from Lemmas 3.3 and 3.5.

From the above lemmas we have

PROPOSITION 3.7. *$S(0,1;E_0,E_1)=\overline{E_0\cap E_1}^{E_0}=$ the closure of $E_0\cap E_1$ in E_0. Similarly we have $S(1,1;E_0,E_1)=\overline{E_0\cap E_1}^{E_1}$.*

REMARK 3.8. Propositions 2.1 and 2.2 are valid in the case when $\theta_0=0$, $\theta_1=1$ and $p_0=p_1=1$.

§ 4. Proof of Theorem.

We shall state our theorem rigorously with our notations.

THEOREM 4.1. *Let $[E_0,E_1]$, $[F_0,F_1]$ be interpolation pairs, and $1\leqq p<\infty$, $0<\theta<1$. If $T\in K([E_0,E_1],[F_0,F_1])$, then $T: S(\theta,p;E_0,E_1)\to S(\theta,p;F_0,F_1)$ is compact.*

For any θ, p fixed as before and for any $T\in\mathscr{B}([E_0,E_1],[F_0,F_1])$ we can define an operator $\tilde{T}$ on $w(\theta,p;E_0,E_1)$ into $w(\theta,p;F_0,F_1)$ induced by T. That is

$$\tilde{T}\{a_m\}=\{Ta_m\}\in w(\theta,p;F_0,F_1)\qquad\text{for any }\{a_m\}\in w(\theta,p;E_0,E_1).$$

Now we remark the following fact:

REMARK 4.2. For $T\in\mathscr{B}([E_0,E_1],[F_0,F_1])$, T is compact from $S(\theta,p;E_0,E_1)$ into $S(\theta,p;F_0,F_1)$ if and only if $\sum\circ\tilde{T}$ is compact from $w(\theta,p;E_0,E_1)$ into $S(\theta,p;F_0,F_1)$. Here $\sum\circ\tilde{T}$ is the composition of operators $\tilde{T}$ and $\sum$.

For the proof of Theorem 4.1, we will prepare two propositions.

DEFINITION 4.3. Let E be a linear space. For any element x in the linear space $E^{\pm\infty}$ of all E-valued sequences;

$$x=(\cdots,x^{(-k-1)},x^{(-k)},x^{(-k+1)},\cdots,x^{(0)},\cdots,x^{(k-1)},x^{(k)},\cdots)$$

we define the projections P_k, P_+ and P_- from $E^{\pm\infty}$ to $E^{\pm\infty}$ by

$$P_k x = (\cdots, 0, 0, x^{(-k+1)}, \cdots, x^{(0)}, \cdots, x^{(k-1)}, 0, 0, \cdots)$$

$$P_+ x = (\cdots, 0, 0, 0, \cdots, 0, x^{(0)}, \cdots, x^{(k-1)}, x^{(k)}, \cdots)$$

$$P_- x = x - P_+ x .$$

PROPOSITION 4.4. *If* $\hat{T} \in K([w(0, 1; E_0, E_1), w(1, 1; E_0, E_1)], [F_0, F_1])$, *then for any* θ $(0<\theta<1)$ $\hat{T} \in K(w(\theta, 1; E_0, E_1), S(\theta, 1; F_0, F_1))$.

PROOF. First we notice that $\hat{T}P_k \in K(w(\theta, 1; E_0, E_1), S(\theta, 1; F_0, F_1))$ for any k, where P_k acts on the space of $E_0 \cap E_1$-valued sequences. So it is enough for the proof to show that the sequence of operators $\hat{T}P_k$ approximates $\hat{T}$ uniformly in $\mathscr{B}(w(\theta, 1; E_0, E_1), S(\theta, 1; F_0, F_1))$. We shall denote the norm of this space of operators by $\|\cdot\|_\theta$ for short. Then

$$\|\hat{T}(I-P_k)\|_\theta \leqq \|\hat{T}(I-P_k)P_+\|_\theta + \|\hat{T}(I-P_k)P_-\|_\theta .$$

From Propositions 2.1 and 2.2, and Proposition C in § 1, we have

$$\|\hat{T}(I-P_k)P_\pm\|_\theta \leqq C\|\hat{T}(I-P_k)P_\pm\|_0^{1-\theta}\|\hat{T}(I-P_k)P_\pm\|_1^\theta .$$

We shall prove that $\|\hat{T}(I-P_k)P_-\|_\theta \to 0$ as $k\to\infty$. Since $\|\hat{T}(I-P_k)P_-\|_1 \leqq M$ for any k, it is enough to prove that

$$\|\hat{T}(I-P_k)P_-\|_0 \to 0 .$$

This is proved immediately from the next lemma.

LEMMA 4.5. *For any* $\varepsilon>0$ *there exists* N_ε *such that*

$$\|\hat{T}(I-P_N)P_-x\|_{F_0} \leqq \varepsilon\|(I-P_N)P_-x\|_{w(0,1;E_0,E_1)}$$

for all $x \in w(0, 1; E_0, E_1)$ *and for all* $N \geqq N_\varepsilon$.

PROOF OF LEMMA 4.5. If Lemma be not true, we can choose a sequence $\{v_{n_j}\}_j$ in $w(0, 1; E_0, E_1)$ and $\varepsilon_0>0$ satisfying

(1) $$n_j \uparrow \infty ,$$

(2) $$\|v_{n_j}\|_w = 1 \qquad \text{for all } j ,$$

(3) $$(I-P_{n_j})P_- v_{n_j} = v_{n_j} ,$$

(4) $$\|\hat{T}v_{n_j}\|_{F_0} \geqq \varepsilon_0 \|v_{n_j}\|_w .$$

On the other hand by Definition 4.2 we have

$$v_{n_j} = \sum_{k \geqq n_j} (P_{k+1}-P_k)P_- v_{n_j} ,$$

and

$$\|(P_{k+1}-P_k)P_- v_{n_j}\|_w \leqq \|v_{n_j}\|_w .$$

Now for $k \geqq n_j$ we have

$$\|e^k(P_{k+1}-P_k)P_-v_{n_j}\|_{w(1,1;E_0,E_1)}$$
$$=\|(P_{k+1}-P_k)P_-v_{n_j}\|_{w(0,1;E_0,E_1)}\leqq\|v_{n_j}\|_{w(0,1;E_0,E_1)}=1\,.$$

Since $\hat{T}: w(1,1;E_0,E_1)\to F_1$ is bounded, we have,

$$(5)\qquad e^k\|\hat{T}(P_{k+1}-P_k)P_-v_{n_j}\|_{F_1}\leqq M\qquad\text{for all } n_j \text{ and } k\geqq n_j\,,$$

from which we have

$$\|\hat{T}v_{n_j}\|_{F_0+F_1}\leqq\|\hat{T}v_{n_j}\|_{F_1}=\|\hat{T}\sum_{k\geqq n_j}(P_{k+1}-P_k)P_-v_{n_j}\|_{F_1}\leqq\sum_{k\geqq n_j}e^{-k}M\,.$$

By (1), we have $\|\hat{T}v_{n_j}\|_{F_0+F_1}\to 0$ as $j\uparrow\infty$. On the other hand, from (2), $\{\hat{T}v_{n_j}\}$ is a totally bounded set in F_0. Hence we have $\hat{T}v_{n_j}\to 0$ in F_0. This contradicts (4). The lemma is proved.

Similarly we have $\|\hat{T}(I-P_k)P_+\|_1\to 0$ and $\|\hat{T}(I-P_k)P_+\|_\theta\to 0$ as $k\to\infty$.

Then $\|\hat{T}(I-P_k)\|_\theta\to 0$ as $k\to\infty$. The proposition is proved.

PROPOSITION 4.6. *Let D_0 and D_1 be Banach spaces with $D_0\subset D_1$. If T_1 in $\mathscr{B}([w(\theta,1;E_0,E_1), w(\theta,\infty;E_0,E_1)], [D_0,D_1])$ has the property that $T_1: w(\theta,1;E_0,E_1)\to D_0$ is compact, then the mapping $T_1: W(\theta,p_\nu;E_0,E_1)\to S(\nu,p_\nu;D_0,D_1)$ is compact, where $0<\nu<1$ and $\dfrac{1}{p_\nu}=1-\nu$.*

PROOF. As in the proof of Proposition 4.4, it is enough for us to prove that T_1P_k approximates T_1 uniformly in $\mathscr{B}(w(\theta,p_\nu;E_0,E_1), S(\nu,p_\nu;D_0,D_1))$. To show this fact we prepare

LEMMA 4.7. *For any $\varepsilon>0$, there exists an integer $N_\varepsilon>0$ such that*

$$\|T_1(I-P_N)x\|_{D_0}\leqq\varepsilon\|(I-P_N)x\|_{w(\theta,1;E_0,E_1)}$$

for any $x\in w(\theta,1;E_0,E_1)$ and for any $N\geqq N_\varepsilon$.

PROOF OF LEMMA 4.7. If Lemma does not hold true, there exists $\varepsilon_0>0$ such that for any k we can find $N_k\geqq k$ and $x_{N_k}\in w(\theta,1;E_0,E_1)$ with the property that

$$\|T_1(I-P_{N_k})x_{N_k}\|_{D_0}>\varepsilon_0\|(I-P_{N_k})x_{N_k}\|_{w(\theta,1;E_0,E_1)}\,.$$

Now we set

$$y_{N_k}=(I-P_{N_k})x_{N_k},\quad z_{N_k}=y_{N_k}/\|y_{N_k}\|_{w(\theta,1;E_0,E_1)}$$

z_{N_k} has the following properties.

$$(1)\qquad \|T_1z_{N_k}\|_{D_0}>\varepsilon_0\qquad\text{for any } k\,.$$

$$(2)\qquad \|z_{N_k}\|_{w(\theta,1;E_0,E_1)}=1\qquad\text{for any } k\,.$$

Then from the assumption that $T_1: w(\theta,1;E_0,E_1)\to D_0$ is compact, we can choose a subsequence $\{z_{N'_k}\}$ of $\{z_{N_k}\}$ and $v\in D_0$ such that $T_1z_{N'_k}\to v$ in D_0. In view of (1) we have $\|v\|_{D_0}>0$. Then $v\neq 0$, and $\|v\|_{D_1}=\delta>0$. We set $\varepsilon_1=\delta/3\|T_1\|_1$.

Again we choose a subsequence $\{z_{n_j}\}$ of $\{z_{N'_k}\}$ in the following manner.

We set $n_1=N_1$. From (2) we have $\|z_{n_1}\|_{w(\theta,1;E_0,E_1)}=1$, then there exists $m_1\geqq n_1$ such that $\|(I-P_{m_1})z_{n_1}\|_{w(\theta,1;E_0,E_1)}\leqq\varepsilon_1$. From the definition we have $N'_k\uparrow\infty$, then there exists N'_k satisfying $N'_k>m_1$. Now we define n_2 by this N'_k. So we have $\|(I-P_{n_2})z_{n_1}\|_w\leqq\varepsilon_1$. We define the subsequence $\{z_{n_j}\}$ inductively in this manner so that it satisfies the following conditions

$$n_j\uparrow\infty, \tag{0}$$

$$\|T_1z_{n_j}\|_{D_0}\geqq\varepsilon_0, \tag{1}$$

$$\|z_{n_j}\|_{w(\theta,1;E_0,E_1)}=1 \quad \text{for all } j, \tag{2}$$

$$T_1z_{n_j}\to v \quad \text{in } D_0, \tag{3}$$

$$(I-P_{n_j})z_{n_j}=z_{n_j} \quad \text{for all } j, \tag{4}$$

$$\|(I-P_{n_{j+1}})z_{n_j}\|_{w(\theta,1;E_0,E_1)}\leqq\varepsilon_1 \quad \text{for all } j, \tag{5}$$

where

$$\varepsilon_1=\frac{\|v\|_{D_1}}{3\|T_1\|_1}; \tag{6}$$

From (3) and the assumption $D_0\subset D_1$, we have

$$T_1z_{n_j}\to v \quad \text{in } D_1.$$

On the other hand, from the fact that $w(\theta,1;E_0,E_1)\subset w(\theta,\infty;E_0,E_1)$ and from (5), we have

$$\begin{aligned}&\left\|\frac{1}{m}\sum_{j=1}^m z_{n_j}\right\|_{w(\theta,\infty;E_0,E_1)}\\ &\leqq\left\|\frac{1}{m}\sum_{j=1}^m P_{n_{j+1}}z_{n_j}\right\|_{w(\theta,\infty;E_0,E_1)}+\frac{1}{m}\sum_{j=1}^m\|(I-P_{n_{j+1}})z_{n_j}\|_{w(\theta,\infty;E_0,E_1)}\\ &\leqq\left\|\frac{1}{m}\sum_{j=1}^m P_{n_{j+1}}z_{n_j}\right\|_{w(\theta,\infty;E_0,E_1)}+\varepsilon_1.\end{aligned}$$

By the definitions of $w(\theta,\infty;E_0,E_1)$ and P_k, we have

$$\begin{aligned}\|z\|_{w(\theta,\infty;E_0,E_1)}=\mathrm{Max}\,\{&\|P_+P_kz\|_{w(\theta,\infty;E_0,E_1)},\\ &\|P_-P_kz\|_{w(\theta,\infty;E_0,E_1)};\ k=0,1,2,\cdots\},\end{aligned}$$

for any $z\in w(\theta,\infty;E_0,E_1)$. So, in view of (0), (2) and (4), we have

$$\begin{aligned}&\left\|\sum_{j=1}^m P_{n_{j+1}}z_{n_j}\right\|_{w(\theta,\infty;E_0,E_1)}\\ &=\mathrm{Max}\,\Big\{\Big\|P_+P_k\sum_{j=1}^m P_{n_{j+1}}(I-P_{n_j})z_{n_j}\Big\|_{w(\theta,\infty;E_0,E_1)},\\ &\qquad\Big\|P_-P_k\sum_{j=1}^m P_{n_{j+1}}(I-P_{n_j})z_{n_j}\Big\|_{w(\theta,\infty;E_0,E_1)};\ k=0,1,2,\cdots\Big\}\end{aligned}$$

$$= \mathrm{Max}\,\{\|P_+\sum_{j=1}^{m} P_k(P_{n_{j+1}}-P_{n_j})z_{n_j}\|_w\,,$$

$$\|P_-\sum_{j=1}^{m} P_k(P_{n_{j+1}}-P_{n_j})z_{n_j}\|_w\,;\ k=0,1,2,\cdots\}$$

$$\leqq \mathrm{Max}\,\{\|P_+P_kz_{n_j}\|_w\,,\ \|P_-P_kz_{n_j}\|_w\,;\ k=0,1,2,\cdots$$

$$j=1,2,\cdots,m\}$$

$$\leqq \mathrm{Max}\,\{\|z_{n_j}\|_{w(\theta,\infty;E_0,E_1)}\,;\ j=1,2,\cdots,m\}$$

$$\leqq \mathrm{Max}\,\{\|z_{n_j}\|_{w(\theta,1;E_0,E_1)}\,;\ j=1,2,\cdots,m\}$$

$$=1\,.$$

Then we have

$$\left\|\frac{1}{m}\sum_{j=1}^{m} z_{n_j}\right\|_{w(\theta,\infty;E_0,E_1)} \leqq \frac{1}{m}+\varepsilon_1$$

and

$$\left\|\frac{1}{m}\sum_{j=1}^{m} T_1z_{n_j}\right\|_{D_1} \leqq \frac{\|T_1\|_1}{m}+\frac{1}{3}\|v\|_{D_1}\,.$$

But, since $T_1z_{n_j}\to v$ in D_1 we have $\frac{1}{m}\sum_{j=1}^{m} T_1z_{n_j}\to v$ in D_1. Hence the left hand side of the above inequality goes to $\|v\|_{D_1}$ as $m\to\infty$. That is a contradiction. The lemma is proved.

From this lemma we can prove Proposition 4.5, since we have

$$\|T_1(I-P_k)\|_\nu\to 0 \quad \text{in} \quad \mathscr{B}(w(\theta,p_\nu;E_0,E_1),\ S(\nu,p_\nu;D_0,D_1))$$

by Proposition C and Propositions 2.1 and 2.2.

PROOF OF THEOREM 4.1. From the assumption $T\in K([E_0,E_1],[F_0,F_1])$ and Proposition 3.7 we have

$$T\in K([S(0,1;E_0,E_1),\ S(1,1;E_0,E_1)],\ [F_0,F_1])\,.$$

In view of Remark 4.2, we have

$$\Sigma\circ\tilde{T}\in K([w(0,1;E_0,E_1),\ w(1,1;E_0,E_1)],\ [F_0,F_1])\,.$$

By Proposition 4.4, we have

$$\Sigma\circ\tilde{T}\in K(w(\theta,1;E_0,E_1),\ S(\theta,1;F_0,F_1))\,.$$

From Proposition C in § 1,

$$\Sigma\circ\tilde{T}\in \mathscr{B}(w(\theta,\infty;E_0,E_1),\ S(\theta,\infty;F_0,F_1))\,.$$

From Proposition A, the assumption in Proposition 4.6 is satisfied by the couple $[S(\theta,1;F_0,F_1),\ S(\theta,\infty;F_0,F_1)]$. Hence we have

$$\Sigma\circ\tilde{T}\in K(w(\theta,p;E_0,E_1),\ S(\theta,p;F_0,F_1)) \qquad \text{for all } p\ (1\leqq p<\infty)$$

Again from Remark 4.2, we obtain

$$T \in K(S(\theta, p; E_0, E_1), S(\theta, p; F_0, F_1)).$$

This proves our theorem.

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