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Vector-valued quasi-analytic functions and their applications to partial differential equations

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The unique-continuation property of solutions of partial differential equations is closely related with the analyticity of solutions. So in this paper we intend to study relations between the unique-continuation property of solutions in some variables and the generalized analyticity of solutions in these variables. First we introduce various notions of generalized analyticity of vector-valued functions, *relative analyticity, relative quasi-analyticity,* and those in weak sense. Then we study the generalized analyticity of solutions of partially elliptic or partially hypo-elliptic equations.

Only partial differential equations with constant coefficients are treated here. In special cases the analyticity of solutions has been discussed even for non-analytic coefficients. (For instance, see [5]). Generalization of our results to the case of variable coefficients will be interesting but it seems to be difficult.

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§ 1. Quasi-analyticity of vector-valued functions.

In this chapter we consider generalized analyticity and unique-continuation property of a family $\{f_\alpha(t)=f_\alpha(t_1, t_2, \cdots, t_n)\}$ of continuous functions defined on a real domain $\Omega^n \subset R^n$, whose range is in a locally convex linear space E. We say that a family $\{f_\alpha(t)\}$ has the *unique-continuation property* if any two elements $f_\alpha(\cdot)$ and $f_\beta(\cdot)$ which are equal on some open subset of Ω^n, are identically equal on the whole domain Ω^n, and say that it has the *strict unique-continuation property* if any two elements $f_\alpha(\cdot)$ and $f_\beta(\cdot)$ whose difference $f_\alpha(\cdot) - f_\beta(\cdot)$ has a zero point of infinite order, are identically equal on the whole domain Ω^n.

1. Relatively analytic functions. As is well known, an E-valued function $f(\cdot)$ defined on a complex domain $D^n \subset C^n$ or on a real domain $\Omega^n \subset R^n$ is called analytic if and only if $f(\cdot)$ has a power series expansion

$$f(t_1, \cdots, t_n) = \sum a_{p_1 \cdots p_n}(t_1 - t_1^0)^{p_1} \cdots (t_n - t_n^0)^{p_n},$$
$$a_{p_1 \cdots p_n} \in E.$$

in a neighbourhood of each point $(t_1^0, \cdots, t_n^0) \in D^n$ or Ω^n.

E' denotes the dual space of E. If E is sequentially complete and $f(\cdot)$ is scalarly analytic (i. e., $\langle f(\cdot), u\rangle$ is analytic for each $u \in E'$) on a complex domain D^n, then $f(\cdot)$ is analytic. However, if $f(\cdot)$ is scalarly analytic on a real domain Ω^n, then $f(\cdot)$ need not be analytic on Ω^n when E is infinite-dimensional, for the analyticity of each $\langle f(\cdot), u\rangle$ on some complex neighbourhood of Ω^n depending on u does not imply the analyticity of all $\langle f(\cdot), u\rangle$ on a fixed complex neighbourhood of Ω^n.

We consider the subspace of E',

$$\{u \in E' \mid \langle f(\cdot), u\rangle \text{ is analytic on a complex domain } D^n\}$$

which contains at least one element 0, and does not coincide with E' if $f(\cdot)$ is not analytic on D^n. We give a generalization of the analyticity as follows.

DEFINITION 1. An E-valued continuous function $f(\cdot)$ defined on a complex domain D^n (or on a real domain Ω^n) is called *relatively analytic* if the subspace $\{u \in E' : \langle f(\cdot), u\rangle$ is analytic on $D^n\}$ (or resp. the subspace $\{u \in E' \mid \langle f(\cdot), u\rangle$ is analytic on $D^n\}$ for some complex neighbourhood D^n of Ω^n) is total on E.

Relative analyticity is characterized as follows.

PROPOSITION 1. *An E-valued continuous function $f(\cdot)$ defined on a complex domain D^n is relatively analytic if and only if there exists some linear space F containing E, endowed with a locally convex separated topology weaker than that of E, such that $f(\cdot)$ is analytic on D^n as an F-valued function.*

PROOF. If $f(\cdot)$ is relatively analytic, then we put

$$G = \{u \in E' \mid \langle f(\cdot), u\rangle \text{ is analytic on } D^n\}$$

and

$$F = \text{the set of all linear functionals on } G \text{ with}$$
$$\text{the weak topology } \sigma(F, G).$$

Then F is complete and $\langle f(\cdot), u\rangle$ is analytic for any $u \in F' = G$, hence $f(\cdot)$ is analytic as an F-valued function. Moreover $F \supset E$, since G is total on E. Conversely, if such a space F exists, F' is total on E. Since F' is contained in $\{u \in E' ; \langle f(\cdot), u\rangle$ is analytic$\}$, the subspace $\{u \in E' \mid \langle f(\cdot), u\rangle$ is analytic$\}$ is total on E. q. e. d.

A family of E-valued functions $\{f_\alpha(t) \mid t \in D^n\}$ may be called *uniformly relatively analytic* if the set $\bigcap_\alpha \{u \in E' \mid \langle f_\alpha(\cdot), u\rangle$ is analytic on $D^n\}$ is total on E, and called merely *relatively analytic* if all of its finite subset are uniformly relatively analytic. It is easy to see that a relatively analytic family has the strict unique-continuation property. Note that even if each element of a family is relatively analytic, the family has not necessarily the unique-continuation property. (See Example 2.)

PROPOSITION 2. *For a continuous n-parameter group of bounded linear transformations* $\{U_t | t \in R^n\}$ *on a Banach space* E, *the family* $\{U_t f | f \in E\}$ *is uniformly relatively analytic in* t.

PROOF. For simplicity we shall prove our Proposition in case of a one-parameter group. Since we have

$$\lim_{k\to\infty} \langle f, \frac{k}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-k^2t^2} U_t^* u \, dt \rangle = \langle f, u \rangle \qquad \text{for } f \in E,\ u \in E',$$

where U_t^* means the transposed operator of U_t, the set

$$\left\{ \int_{-\infty}^{\infty} e^{-k^2t^2} U_t^* u \, dt \,|\, u \in E',\ k = 1, 2, \cdots \right\}$$

is total on E. Hence it suffices to show that $\langle U_s f, \int_{-\infty}^{\infty} e^{-k^2t^2} U_t^* u \, dt \rangle$ is analytic in s for any $f \in E$.

$$\begin{aligned} \langle U_s f, \int_{-\infty}^{\infty} e^{-k^2t^2} U_t^* u \, dt \rangle &= \int_{-\infty}^{\infty} \langle U_s f, e^{-k^2t^2} U_t^* u \rangle dt \\ &= \int_{-\infty}^{\infty} \langle U_{s+t} f, e^{-k^2t^2} u \rangle dt \\ &= \int_{-\infty}^{\infty} e^{-k^2(t-s)^2} \langle U_t f, u \rangle dt\,. \end{aligned}$$

Since $|\langle U_t f, u \rangle| < A e^{B|t|}$ for some constants A and B, the last integral is convergent uniformly in s when s is in a bounded complex domain. Hence the above function is analytic in s. q. e. d.

Note that for a group $\{U_t\}$ on a locally convex linear space or for a semi-group $\{U_t | 0 \leqq t < \infty\}$ on a Banach space, a function $U_t f$ is not necessarily relatively analytic. We shall give such an example.

EXAMPLE 1. Let E be a Banach space $C_0[0, \infty) = \{f | f(x) \text{ is continuous in } [0, \infty), \lim_{x\to\infty} f(x) = 0\}$ with the uniform norm $\|f\|_\infty = \sup |f(x)|$, or a locally convex linear space $C(-\infty, \infty)$ = the set of all continuous functions in $(-\infty, \infty)$ with the topology of uniform convergence on every compact set in $(-\infty, \infty)$. We consider the translation operator $U_t : f(x) \to f(x+t)$ on E. Let $f_0(x)$ be a non-zero continuous function in E with compact carrier. For any $g \in E'$ we have

$$\langle U_t f_0, g \rangle = 0 \qquad \text{for sufficiently large } t\,.$$

Hence if $\langle U_t f_0, g \rangle$ is analytic in t, it is identically zero. This means that the subspace $\{u \in E' | \langle U_t f_0, u \rangle \text{ is analytic}\} = \{0\}$ is not total on E.

EXAMPLE 2. Let E be a Banach space $C_0(-\infty, \infty)=$ the set of all continuous functions vanishing at $\pm\infty$ with the uniform norm. For bounded continuous non-zero functions $u(x)$ and $v(x)$, we consider groups of transformations $U_t: f(x)\to u(x)f(x+t)/u(x+t)$ and $V_t: f(x)\to v(x)f(x+t)/v(x+t)$. When $0<\varepsilon<u(x),\ v(x)<M$, they are continuous groups of transformations on E. We pick up a non-zero function $f_0(x)$ with a compact carrier in $[-1, 1]$. If $u(x)=v(x)$ for $x\in[-2, 2]$, then $U_t f_0 = V_t f_0$ for $t\in[-1, 1]$, and in general $U_t f_0 \neq V_t f_0$ for $t\notin[-1, 1]$. However, by virtue of Proposition 2, our functions $U_t f_0$ and $V_t f_0$ are relatively analytic E-valued functions.

In Example 1, an element U_t of the semi-group on the Banach space $C_0[0, \infty)$ is not a one-to-one operator, hence the family $\{U_t f \mid f\in C_0[0, \infty)\}$ naturally has not the unique-continuation property. However for any group $\{U_t\}$ of transformations on a locally convex linear space E the family $\{U_t f \mid f\in E\}$ has evidently the unique-continuation property. Later we shall introduce a weaker notion of analyticity applicable to groups of transformations. For that purpose we need the theory of quasi-analytic functions.

2. Scalar-valued quasi-analytic functions. For a multi-index $p=(p_1, p_2, \cdots, p_n)$, we denote $D^p=\left(\frac{1}{i}\frac{\partial}{\partial t_1}\right)^{p_1}\left(\frac{1}{i}\frac{\partial}{\partial t_2}\right)^{p_2}\cdots\left(\frac{1}{i}\frac{\partial}{\partial t_n}\right)^{p_n}$.

DEFINITION 2. Let $\{b_q \mid q=(q_1, q_2, \cdots, q_n)\}$ be a sequence of positive numbers with multi-indices. Then a family $C\{b_q\}$ of C^∞-functions on R^n is defined by

$$C\{b_q\}=\{f(t) \mid \sup_{t\in K}|D^q f(t)|\leqq B^{|q|}b_q \quad \text{for any compact } K\subset R^n \text{ and for some constant } B=B(f, K)\}.$$

$C\{b_q\}$ is the family of all analytic functions if b_q is $q!=q_1!q_2!\cdots q_n!$. The family $C\{b_q\}$ is called *quasi-analytic* if $C\{b_q\}$ has the unique-continuation property. It is easily seen that a quasi-analytic family $C\{b_q\}$ has the strict unique-continuation property. The following fundamental theorem (see [3]) is well known;

THEOREM. *Let the dimension $n=1$. A family $C\{b_q\}$ is quasi-analytic if and only if*

$$\int_1^\infty \frac{\log \Gamma(r)}{r^2}dr=\infty \qquad \textit{for } \Gamma(r)=\sup_q \frac{r^q}{b_q}.$$

As special cases of the above theorem, we have the following two corollaries.

COROLLARY 1. *A family $C\{b_q\}$ $(n=1)$ is quasi-analytic if*

$$\sum_q \frac{1}{\sqrt[q]{b_q}}=\infty.$$

COROLLARY 2. *A family $C\{b_q\}$ is quasi-analytic if*

(1) $$b_q = a_1 a_2 \cdots a_{|q|} \quad \textit{with} \quad \sum_{i=1}^{\infty} \frac{1}{a_i} = \infty, \quad 0 < a_1 \leqq a_2 \leqq \cdots$$

The following theorem concerning regularization by quasi-analytic mollifiers is the main purpose in this section.

THEOREM 1. *For an arbitrary positive continuous function* $H(x) \in C(R^n)$, *there exists a sequence* $\{b_q\}$ *satisfying* (1) *and exist* $\{f_k(\cdot) \mid k = 1, 2, \cdots\} \subset C\{b_q\}$ *such that*

(2) $$\|f_k(\cdot)\|_1 = 1, \qquad f_k(t) \geqq 0,$$

(3) $$h(t) * f_k(t) \in C\{b_q\} \quad \textit{for any } h \textit{ with } |h(t)| \leqq H(t),$$

(4) $$h(t) * f_k(t) \to h(t) \qquad \textit{uniformly on every compact set, as } k \to \infty.$$

For the proof, we need some lemmas.

LEMMA 1. *Let* $\{a_k\}$ *be an increasing sequence of positive numbers such that* $\sum \frac{1}{a_k^2} < \infty$ *with* $a_1 \geqq 1$, *and let* $\varphi_k(t)$ *be functions defined for* $t \in R^n$ *with the properties*

$$\|\varphi_k\|_1 = 1, \quad \varphi_k(t) = 0 \qquad \textit{for } |t| > \frac{1}{a_k}, \quad k = 1, 2, \cdots$$

and

$$\varphi_k(t) \geqq 0, \ \varphi_k(-t) = \varphi_k(t) \qquad \textit{for } k \geqq N \ (= \textit{a fixed positive integer} \geqq 3).$$

Put $f_k(t) = \varphi_1 * \varphi_2 * \cdots \varphi_k(t)$. *If* φ_1 *and* φ_2 *satisfy the Lipschitz condition* $|\varphi_i(t+h) - \varphi_i(t)| \leqq M_1 |h|$ *for all* t *and* $h \in R^n$, *then* $\{f_k(t)\}$ *converges uniformly to some function* $f(t)$ *satisfying*

(5) $$|f(t)| \leqq \sum_{i=k+1}^{N} \frac{M}{a_i} + \sum_{i=\max(k+1, N+1)}^{\infty} \frac{M}{a_i^2} \qquad \textit{for } |t| \geqq \sum_{i=1}^{k} \frac{1}{a_i}, \ k = 1, 2, \cdots$$

with some constant M *depending on* φ_1 *and* φ_2.

PROOF. For $k < N$ we have easily

(6) $$|f_{k+1}(t) - f_k(t)| = \left| \int_{R^n} (f_k(t-s) - f_k(t)) \varphi_{k+1}(s) ds \right| \leqq \frac{M_1}{a_{k+1}}.$$

Put $\phi_k = \varphi_3 * \varphi_4 * \cdots * \varphi_k$. Since $\|\varphi_k\|_1 = 1$, we have $\|\phi_k\|_1 \leqq 1$. Now $f_k = \varphi_1 * \varphi_2 * \phi_k$ and therefore for $k \geqq 3$

$$\begin{aligned} &|f_k(t+s) + f_k(t-s) - 2f_k(t)| \\ &= \left| \iint (\varphi_1(\sigma+s) - \varphi_1(\sigma))(\varphi_2(\tau) - \varphi_2(\tau - s)) \phi_k(t - \sigma - \tau) d\sigma d\tau \right| \\ &\leqq M_1^2 |s|^2 2 \sup_{t \in R^n} \iint_{|\sigma| \leqq \frac{1}{a_1}} |\phi_k(t - \sigma - \tau)| d\tau d\sigma \\ &\leqq C M_1^2 |s|^2, \end{aligned}$$

where C depends on a_1 only. Using this estimate we get for $k \geqq N$

$$|f_{k+1}(t)-f_k(t)| = \left|\int\{f_k(t-s)-f_k(t)\}\varphi_{k+1}(s)\,ds\right| \tag{7}$$

$$= \frac{1}{2}\left|\int\{f_k(t+s)+f_k(t-s)-2f_k(t)\}\varphi_{k+1}(s)ds\right|$$

$$\leqq \frac{1}{2}CM_1^2\int|s|^2|\varphi_{k+1}(s)|\,ds \leqq \frac{CM_1^2}{2a_{k+1}^2} = \frac{M}{a_{k+1}^2}\,;$$

Since $f_k(t)=0$ for $|t| \geqq \sum_{i=1}^{k}\frac{1}{a_i}$, we have

$$f_{k+m}(t) = \sum_{i=1}^{m}\{f_{k+i}(t)-f_{k+i-1}(t)\} \qquad \text{for } |t| \geqq \sum_{i=1}^{k}\frac{1}{a_i}\,.$$

The statement of the lemma now follows from (6) and (7). q. e. d.

We denote $\varphi_1 * \varphi_2 * \cdots * \varphi_k(t) = \mathop{*}\limits_{j=1}^{k}\varphi_j(t)$.

LEMMA 2. *For the function $\bar{\varphi}(s)$ of one-variable such that $\bar{\varphi}(s) = \max(0, 1-|s|)$, we put $\varphi(t) = \prod_{j=1}^{n}\bar{\varphi}(t_j)$ $(t=(t_1, t_2, \cdots, t_n))$ and $\varphi_k(t) = a_k\varphi(a_k t)$. Then each derivative of $f(t) = \mathop{*}\limits_{k=1}^{\infty}\varphi_k(t)$ satisfies*

$$|D^p f(t)| \leqq 2^{|p|}a_1a_2\cdots a_{|p|}M\Big(\sum_{i=\max(|p|+1,k+1)}^{\infty}\frac{1}{a_i^2}+\sum_{i=k+1}^{|p|}\frac{1}{a_i}\Big) \tag{8}$$

$$\textit{for } |t| \geqq \sum_{i=1}^{k}\frac{1}{a_i}\,,$$

where M is a constant independent of a_k for $k>4$.

PROOF. We assume $p_1 = \max(p_1, p_2, \cdots, p_n)$ without loss of generality. Then for $|p| \geqq 4n$, we have $p_1 \geqq 4$. For the one-variable function

$$\bar{\phi}_k(s) = \begin{cases} -a_k^2 \operatorname{sign} s & \text{for } |s| \leqq a_k^{-1} \\ 0 & \text{for } |s| > a_k^{-1}, \end{cases}$$

we put $\phi_{k,j}(t) = \phi_{k,j}(t_1, \cdots, t_n) = \bar{\phi}_k(t_j)\prod_{i \neq j}a_k\bar{\varphi}(a_k t_i)$. Then we have

$$D^p f(t) = \lim_{l\to\infty}(-i)^{|p|}\Big(\mathop{*}\limits_{j=1}^{p_1}\phi_{j,1}\Big) * \Big(\mathop{*}\limits_{j=p_1+1}^{p_1+p_2}\phi_{j,2}\Big) * \cdots$$

$$* \Big(\mathop{*}\limits_{j=p_1+\cdots+p_{n-1}+1}^{|p|}\phi_{j,n}\Big) * \varphi_{|p|+1} * \cdots * \varphi_l(t)\,.$$

Note that $\phi_{1,1} * \phi_{2,1}$ and $\phi_{3,1} * \phi_{4,1}$ satisfy the Lipschitz condition. Applying Lemma 1 to $\left(\frac{\phi_{1,1}}{2a_1}\right) * \left(\frac{\phi_{2,1}}{2a_2}\right) * \cdots * \left(\frac{\phi_{|p|,n}}{2a_{|p|}}\right) * \varphi_{|p|+1} * \cdots * \varphi_l(t)$, which coincides with $(2^{|p|}a_1a_2\cdots a_{|p|})^{-1}D^p\big(\mathop{*}\limits_{i=1}^{l}\varphi_i\big)$, we have for $|p| \geqq 4n$

$$|D^p f(t)| \leqq 2^{|p|} a_1 a_2 \cdots a_{|p|} M' \Big(\sum_{i=\max(|p|+1,k+1)}^{\infty} \frac{1}{a_i^2} + \sum_{i=k+1}^{|p|} \frac{1}{a_i} \Big)$$

$$\text{for } |t| \geqq \sum_{i=1}^{k} \frac{1}{a_i}.$$

M' depends only on $\phi_{1,1} * \phi_{2,1}$ and $\phi_{3,1} * \phi_{4,1}$. Therefore the inequality (8) holds good for any p also.

LEMMA 3. *For a monotone-decreasing sequence $\{\varepsilon_m\}$ of positive numbers, there exists a C^∞-function $f(t)$ satisfying the following two conditions:*

(i) $f(t) \geqq 0, \quad \|f\|_1 = 1.$

(ii) *For any $p=(p_1, \cdots, p_n)$, we have*

$$\sup_{|t| \geqq m} |D^p f(t)| \leqq M \varepsilon_m a_1 a_2 \cdots a_{|p|} 2^{|p|}, \qquad m \geqq \sum_{i=1}^{|p|} \frac{1}{a_i}$$

for some constant M, where $\{a_k\}$ is a sequence such that

$$\sum_{k=1}^{\infty} \frac{1}{a_k} = \infty, \qquad 1 \leqq a_1 \leqq a_2 \leqq \cdots \tag{9}$$

PROOF. Since the function $f(t)$ in Lemma 2 satisfies conditions (i) and (8), it suffices to choose a sequence $\{a_k\}$ satisfying (9) such that

$$\mu_m \leqq \varepsilon_m, \quad \text{for } \varepsilon_m = \sum_{i=k_m+1}^{\infty} \frac{1}{a_i^2}, \quad k_m = \max \{k \mid m \geqq \sum_{i=1}^{k} a_i^{-1}\}. \tag{10}$$

Put $\lambda_m = \min\Big(\frac{\lambda_{m-1}}{2}, \frac{\varepsilon_m}{2}\Big)$, $\lambda_0 = \varepsilon_1$. Suppose that $\{a_i \mid i \leqq k_j\}$ is determined such that, for $m<j$,

$$\sum_{i=k_m+1}^{k_{m+1}} \frac{1}{a_i^2} < \lambda_m, \qquad \sum_{i=k_m+1}^{k_{m+1}} \frac{1}{a_i} = 1. \tag{11}$$

Then we define

$$k_{j+1} = k_j + [\lambda_j^{-1}+1], \quad a_i = [\lambda_j^{-1}+1] \qquad \text{for } k_j+1 \leqq i \leqq k_{j+1},$$

and so by induction we obtain a sequence $\{a_k\}$. We see easily that (11) is satisfied for $m=j$, hence (11) is satisfied for all m. Moreover we have

$$\sum_{i=k_j+1}^{\infty} \frac{1}{a_i^2} = \sum_{m=j}^{\infty} \sum_{i=k_m+1}^{k_{m+1}} \frac{1}{a_i^2} \leqq \sum_{m=j}^{\infty} \lambda_m \leqq 2\lambda_j \leqq \varepsilon_{j+1} \leqq \varepsilon_j,$$

and

$$\sum_{i=1}^{k_j} \frac{1}{a_i} = \sum_{m=1}^{j} \sum_{i=k_m+1}^{k_{m+1}} \frac{1}{a_i} = \sum_{m=1}^{j} 1 = j.$$

Hence (9) and (10) hold good.

LEMMA 4. *Under the same assumption as in Lemma 3 there exists a non-trivial function $f(t)$ such that*

$$\sup_{|t| \geqq m} |D^p f(t)| \leqq M \varepsilon_m 4^{|p|} a_1 \cdots a_{|p|}.$$

PROOF. By Lemma 2 and Lemma 3 there exists a function $f(t)$ such that

$$\sup_{|t|\geqq m}|D^p f(t)|\begin{cases}\leqq M\varepsilon_m 2^{|p|}a_1\cdots a_{|p|} & \text{for } m>\sum_{i=1}^{|p|}\frac{1}{a_i}\\ \leqq M2^{|p|}a_1\cdots a_{|p|}\Big(\sum_{i=|p|+1}^{\infty}\frac{1}{a_i^2}+\sum_{i=1}^{|p|}\frac{1}{a_i}\Big) & \text{in general.}\end{cases}$$

Hence, if $m\geqq \sum_{i=1}^{|p|}\frac{1}{a_i}$, our assertion is trivial. Let $k_m=\max\Big\{k\,|\,m>\sum_{j=1}^{i}\frac{1}{a_j}\Big\}$. Then without loss of generality we may assume that

$$\frac{\varepsilon_m 2^{k_m}}{i_m+1}\geqq 1 \quad\text{and}\quad \sum_{i=1}^{\infty}\frac{1}{a_i^2}\leqq 1\,. \tag{12}$$

In fact, the function $f(t)$ constructed in the proof of Lemma 3 satisfies (12), since $k_m\geqq[\lambda_{m-1}^{-1}+1]\geqq\frac{1}{\varepsilon_m}$.

For $|p|>i_m$ $\Big($i.e. $m\leqq\sum_{i=1}^{|p|}\frac{1}{a_i}\Big)$ we have

$$\sup_{|t|\geqq m}|D^p f(t)|\leqq M2^{|p|}a_1\cdots a_{|p|}(1+|p|)=M4^{|p|}a_1\cdots a_{|p|}\frac{1+|p|}{2^{|p|}}$$
$$\leqq M4^{|p|}a_1\cdots a_{|p|}\frac{k_m+1}{2^{k_m}}\leqq M\varepsilon_m 4^{|p|}a_1\cdots a_{|p|}\,.$$

PROOF OF THEOREM 1. For a double sequence

$$\lambda_k^{(m)}=\sup_{\substack{k\leqq|t|\leqq k+2\\ |s|=m}}H(t-s),\qquad k=0,1,2,\cdots \quad\text{and}\quad m=0,1,2,\cdots,$$

there exists a sequence $\{\lambda_k>0\}$ such that

$$\sup_k\frac{\lambda_k^{(m)}}{\lambda_k}<\infty,\qquad\text{for any } m\,.$$

We put $M_m=\sup_k\frac{\lambda_k^{(m)}}{\lambda_k}$. Apply Lemma 4 for $\varepsilon_k=\frac{1}{2^k\lambda_k(2k+4)^n}$. For a function $f(t)$ in Lemma 4 and for $m=[|s|+1]$, we have

$$\Big|D^p\int h(t-s)f(t)dt\Big|\leqq\int H(t-s)|D^pf(t)|\,dt$$
$$\leqq\sum_{k=0}^{\infty}(2k+4)^n\lambda_k^{(m)}M\varepsilon_k a_1a_2\cdots a_{|p|}4^{|p|}\leqq 4^{|p|}\sum_{k=0}^{\infty}\frac{MM_m}{2^k}a_1\cdots a_{|p|}$$
$$=4^{|p|+1}MM_ma_1a_2\cdots a_{|p|}\,.$$

For an arbitrary constant $K>0$, we put $B=4M\max_{|m|\leqq K}M_m$. Then we have

$$\sup_{|t|\leqq K}\Big|D^p\int h(s-t)f(s)ds\Big|\leqq B4^{|p|}a_1a_2\cdots a_{|p|}\,. \tag{13}$$

Moreover for a fixed k, we can choose $f(x)=f_k(x)$ such that

$$(14)\qquad \int_{|x|\geqq\frac{1}{k}} |H(y-x)f_k(x)|\,dx<\frac{1}{k}, \qquad \text{for } |y|\leqq k\,.$$

The sequence $\{f_k(x)\,|\,k=1, 2, \cdots\}$ satisfies our requirements (3) and (4), by virtue of (13) and (14). q. e. d.

For future use we shall prove the following

COROLLARY TO LEMMA 2. *For any $\varepsilon>0$ there exists a non-zero positive function $f(t)$ with a compact carrier such that*

$$(15)\qquad |D^p f(t)|\leqq M|p|^{|p|}(\log|p|)^{(1+\varepsilon)|p|}$$

PROOF. Put $a_k=ck(\log(k+1))^{1+\varepsilon}$. Then $\sum_{k=1}^{\infty}\frac{1}{a_k}=\frac{K}{c}$, where $K=\frac{1}{(\log 2)^{1+\varepsilon}}+\int_1^{\infty}\frac{dx}{x(\log(1+x))^{1+\varepsilon}}<\infty$. Hence the function $f(t)$ in Lemma 2 satisfies

$$f(t)=0 \qquad \text{for } |t|\geqq\frac{K}{c},$$

and

$$(16)\qquad |D^p f(t)|\leqq 2^{|p|}MK' a_1 a_2\cdots a_{|p|}, \qquad \text{for } K'=\sum_{i=1}^{\infty}\frac{1}{a_i^2}+\frac{K}{c}.$$

The above inequality (16) implies evidently (15).

3. Vector-valued quasi-analytic functions. Now we return to the case of E-valued functions for some locally convex linear topological space E. Let $f(t)$ be an E-valued continuous function defined on R^n. Then for any natural number m the set $\{f(t):\,|t|\leqq m\}$ is compact in E, hence bounded in E. Therefore we can choose a sequence $\{B_i:\,i=1, 2, \cdots\}$ of convex circular bounded sets in E, such that $f(R^n)=\{f(t):\,t\in R^n\}\subset\bigcup_i B_i$. Similarly, for any finite number of E-valued continuous functions $\{f_k(t):\,k=1, 2, \cdots, m\}$, we can choose a sequence $\{B_i\}$ of convex circular bounded sets in E such that $\bigcup_{k=1}^{m} f_k(R^n)\subset\bigcup_i B_i$.

We consider a family $\{f_\alpha(t):\,\alpha\in A\}$ of E-valued continuous functions defined on R^n. If any finite subset $\{f_{\alpha_i}(t):\,i=1, 2, \cdots, m\}$ of $\{f_\alpha(t):\,\alpha\in A\}$ has the unique-continuation property, then the family $\{f_\alpha(t):\,\alpha\in A\}$ itself has the unique-continuation property. Hence it is sufficient to consider the unique-continuation property on each subfamily $\{f_\alpha\,|\,f_\alpha(R^n)\subset\bigcup_i E_{B_i}\}$, where E_{B_i} is the normed space generated by B_i, for each sequence $\{B_i\}$ of convex circular bounded sets in E. We give the limit inductive topology on $\bigcup_i E_{B_i}$, and so the dual $(\bigcup_i E_{B_i})'$ is the set of all linear functionals bounded on each B_i. Thus we are led to the following definition, giving a weaker notion of quasi-analyticity.

DEFINITION 2. A family $\{f_\alpha(t): \alpha \in A\}$ of E-valued continuous functions defined on R^n is called *relatively quasi-analytic in weak sense* if for every sequence $\{B_i\}$ of convex circular bounded sets in E there exists a total subset $F \subset (\bigcup_i E_{B_i})'$ such that for any $u \in F$

(17) $$\{\langle f_\alpha(t), u\rangle : f_\alpha(R^n) \subset \bigcup_i E_{B_i}\} \subset C\{b_q\},$$

where $C\{b_q\}$ is a quasi-analytic family (depending on u).

Evidently a relatively quasi-analytic family in weak sense has the strict unique-continuation property. We consider a case in which this notion is more simply defined.

THE FIRST COUNTABILITY CONDITION OF MACKEY: For any sequence of bounded sets $\{B_i\}$ in E, there exists a sequence $\{\varepsilon_i\}$ of positive numbers such that the union $\bigcup_i \varepsilon_i B_i$ is bounded in E.

This condition is satisfied for instance by (F)-spaces. When E satisfies the first countability condition of Mackey, a family $\{f_\alpha(t): \alpha \in A\}$ of E-valued continuous functions is relatively quasi-analytic in weak sense if and only if the condition in Definition 2 is satisfied by E_B for every convex circular bounded set B in E, instead of $\bigcup_i E_{B_i}$.

§ 2. Unique-continuation of solutions of partial differential equations.

For a linear partial differential operator $P(D)$ with constant coefficients, as is well known, the following three conditions are equivalent to each other:

(i) *$P(D)$ is elliptic.*

(ii) *The family of solutions of the equation $P(D)u=0$ has the* (*strict*) *unique-continuation property.*

(iii) *All solutions of the equation $P(D)u=0$ are analytic.*

Our main purpose in this section is to generalize the above theorem for partial ellipticity.

4. Relative quasi-analyticity of solutions of partially conditionally elliptic equations. We begin with a brief explanation of the concept of partially conditionally elliptic operator as defined in [1]. Let $P=P(D_x, D_y)$ be a linear partial differential operator with constant coefficients on $x=(x_1, x_2, \cdots, x_m) \in R^m$, $y=(y_1, y_2, \cdots, y_n) \in R^n$. P is called *partially conditionally elliptic in x* if any solution $u(x, y)$ of the equation $Pu=0$ analytic in y is analytic in x also. This notion is characterized as follows.

THEOREM. *P is partially conditionally elliptic in x if and only if the following two equivalent conditions are satisfied:*

(18) $$|\xi'| \leqq c(1+|\eta|+|\xi''|) \qquad \text{for } P(\xi, \eta)=0,$$

where $\xi'=\mathrm{Re}(\xi)$ and $\xi''=\mathrm{Im}(\xi)$.

(19) $$P(D_x, D_y)=P_0(D_x)+\sum_{i>0} P_i(D_x)Q_i(D_y),\ P_0 \text{ is elliptic}$$

$$\deg P_i<\deg P_0 \quad\text{and}\quad \deg P_i+\deg Q_i \leqq \deg P_0 .$$

Now we state one of our main results.

THEOREM 3. *The following three conditions are equivalent to each other:*

(i′) *P is partially conditionally elliptic in x.*

(ii′) *The family of solutions in $\Omega^m\times R^n$ of the equation $Pu=0$ has the unique-continuation property in x.*

(iii′) *The family $\{u: Pu=0\}$ is relatively quasi-analytic in weak sense in x, where $\{u(x)\}$ are $C(R^n)$-valued functions.*

We use similar notations to those in [1]:

$$|D^\alpha g|^2=\sum_{|p|=\alpha}|D^p g|^2 \qquad \alpha=\text{an integer},\ p=(p_1,\cdots,p_m)$$

and for a sphere K in R^m with radius r

$$|g, K|^2=\int_K |g(x)|^2dx,$$

$$|D^\alpha g, K|_\sigma=\sum_{0\leqq|\nu|+|\mu|\leqq q_0+\alpha}|D_x^\nu D_y^\mu g, K|\sigma^{|\nu|+|\mu|}$$

where $q_0=\max\left(\deg P, \left[\frac{n}{2}\right]+1\right)$.

LEMMA 5. *For a C^∞-solution v of $Pv=0$, we have*

$$|D_x v, K|_\sigma \leqq C(\sigma^{-1}|v, L|_\sigma+|D_y v, L|_\sigma),$$

where L is the sphere with radius $r+\sigma$ having the common center with K, and C is a suitable positive constant.

PROOF. We cite the following inequality ([2, Lemma 7.5.1]).

$$\sigma^{p_0}|D_x^\alpha v, K|\leqq C'(\sigma^{p_0}|P_0 v, L|+\sum_{|p|<p_0}\sigma^{|p|}|D_x^p v, L|)$$

$$\text{for } \alpha\leqq p_0,\quad \sigma\leqq 1,\quad \text{where}\quad p_0=\deg P.$$

Hence we have

$$\begin{aligned}|D_x v, K|_\sigma &\leqq |D_x^{q_0+1}v, K|\sigma^{q_0}+\sum_{\substack{|\nu|+|\mu|=q_0+1\\|\mu|>0}}|D_x^\nu D_y^\mu v, K|\sigma^{q_0}\\&\quad+\sum_{|\nu|+|\mu|\leqq q_0}|D_x^\nu D_y^\mu v, K|\sigma^{|\nu|+|\mu|-1}\\&\leqq C'(\sigma^{q_0}|D_x^{q_0-p_0+1}P_0 v, L|+\sum_{|p|<q_0}\sigma^{|p|}|D_x^{p+1}v, L|)\\&\quad+\sum_{|\nu|+|\mu|=q_0}|D_y D_x^\nu D_y^\mu v, L|\sigma^{q_0}+\sum_{|\nu|+|\mu|\leqq q_0}|D_x^\nu D_y^\mu v, L|\sigma^{|\nu|+|\mu|-1}\end{aligned}$$

$$\leqq C'(\sigma^{q_0}|D_x^{q_0-p_0+1}\sum_{j>0}P_j(D_x)Q_j(D_y)v, L|+\sum_{|p|<q_0}\sigma^{|p|}|D_x^{p+1}v, L|)$$

$$+|D_y v, L|_\sigma+\sigma^{-1}|v, L|_\sigma$$

$$\leqq C'(\sigma^{q_0}\sum_{|\nu|\leqq q_0}|c_\nu D_x^\nu v, L|+\sigma^{q_0}\sum_{|\nu|+|\mu|\leqq q_0}|c_{\nu\mu}D_y D_x^\nu D_y^\mu v, L|)$$

$$+C'\sigma^{-1}|v, L|_\sigma+|D_y v, L|_\sigma+\sigma^{-1}|v, L|_\sigma,$$

where c_ν, $c_{\nu\mu}$ depend only on P. Hence our inequality is proved for a new constant C.

PROOF OF THEOREM 3. (iii′)→(ii′). This implication is evident from the definition of relative quasi-analyticity in weak sense.

(ii′)→(i′). Assume that P is not partially conditionally elliptic in x. We put

$$P(\xi, \eta)=P_0(\xi)+\sum_{j>0}P_j(\xi)\eta^{\beta(j)} \qquad |\beta(j)|\geqq 1.$$

Then P_0 is not elliptic in x, or $\deg P_0<\deg P$, by virtue of (19). If P_0 is not elliptic in x, then there exists a null solution $u_0(x)\neq 0$ of the equation $P_0(D_x)u(x)=0$, such that $u_0(x)=0$ for $\langle x, N\rangle>0$, with respect to a characteristic plane $\{x: \langle x, N\rangle=0\}$ of P_0 (see [2]). Since $|\beta(j)|\geqq 1$, the null solution $u_0(x)$ satisfies the equation $Pu=0$. Hence the family $\{u: Pu=0\}$ has not the unique-continuation property. If $\deg P_0<\deg P=p$, the principal part of P is

$$\sum_{|\alpha|+|\beta|=p}C_{\alpha_1\cdots\alpha_m\beta_1\cdots\beta_n}\xi_1^{\alpha_1}\cdots\xi_m^{\alpha_m}\eta_1^{\beta_1}\cdots\eta_n^{\beta_n} \qquad |\beta|>0.$$

Hence the hyperplane $x_1=0$ is a characteristic plane. Since a null solution with respect to the characteristic plane $x_1=0$ exists, the family $\{u(x)=u(x, y): Pu=0\}$ has not the unique-continuation property.

(i′)→(iii′). The space $E=C(R^n)$, which is an (F)-space, satisfies the first countability condition of Mackey. Hence it suffices to show that the family $\{u: Pu=0 \text{ and } u(x)\in E_B \text{ for any } x\in R^m\}$ is relatively quasi-analytic for any convex circular bounded set B in E of the form

$$B=\{h(y)\in C(R^n)\,|\,|h(y)|\leqq H_0(y)\}$$

where $H_0(y)$ is a continuous positive function.

For a fixed function $w(y)\in(C_0^\infty)$ and a fixed solution u of $Pu=0$, $u(x, y)*w(y)$ is infinitely differentiable in x, since P is partially hypoelliptic in x. Let K_1 be a compact set in R^n. Then for any pair of indices μ, ν there exists a continuous function $H_{\nu\mu}(y)$ such that

$$\sup_{x\in K_1}|D_x^\mu D_y^\nu u*w|\leqq H_{\nu\mu}(y) \qquad \text{for all } y\in R^n.$$

It is easy to see that in the above inequality we can replace all $H_{\nu\mu}$ with one continuous positive function H if we take suitable constants $C_{\nu\mu}>0$, that is,

$$\sup_{x\in K_1} |D_x^\mu D_y^\nu u * w| \leqq C_{\nu\mu} H(y) \qquad \text{for all } y\in R^n .$$

For a solution $u(x, y)$ of the equation $Pu=0$ in E_B, we put

$$v(x, y)=u(x, y)*w(y)*f_k(y),$$

where $f_k(y)$ is a function associated with $H(y)$ in Theorem 1. Then we have

$$\sup_{x\in K_1, |y|\leqq R} |(D_x^\nu D_y^\mu u * w) * D_y^q f_k| \leqq C_{\nu,\mu,k} b_q A^{|q|}$$

for some constants $C_{\nu,\mu,k}$ and A. Note that v is also a solution of $Pv=0$. Let K be a sphere in R^m with radius r and L be the sphere with radius $r+\sigma$ $(0<\sigma\leqq 1)$ having the common center with K. Then we have by Lemma 5 for $|y|\leqq R$,

$$\begin{aligned}
|D_x^\alpha v, K|_{\sigma/\alpha} &\leqq C^\alpha \sum_{|q|\leqq\alpha} \binom{\alpha}{|q|}\Big(\frac{\alpha}{\sigma}\Big)^{\alpha-|q|} |D_y^q v, L|_{\sigma/\alpha} \\
&\leqq C^\alpha \sum_{|q|\leqq\alpha} 2^\alpha\Big(\frac{\alpha}{\sigma}\Big)^{\alpha-|q|} \sum_{|\nu|+|\mu|\leqq q_0} |D_x^\nu D_y^{q+\mu} v, L|\Big(\frac{\sigma}{\alpha}\Big)^{|\nu|+|\mu|} \\
&\leqq \Big(\frac{2C}{\sigma}\Big)^\alpha \sum_{|q|\leqq\alpha} \alpha^{\alpha-|q|} \sum_{|\nu|+|\mu|\leqq q_0} C' \sup_{x\in L} |D_y^q(D_x^\nu D_y^\mu u * w)*f_k| \\
&\leqq C'\Big(\frac{2C}{\sigma}\Big)^\alpha \sum_{q\leqq\alpha} \alpha^{\alpha-|q|} \sum_{|\nu|+|\mu|\leqq q_0} \sup_{x\in L} |(D_x^\nu D_y^\mu u*w)*(D_y^q f_k)| \\
&\leqq C'\Big(\frac{2C}{\sigma}\Big)^\alpha \alpha^\alpha \sum_{|q|\leqq\alpha} \frac{b_q}{\alpha^{|q|}} A^{|q|} \max_{|\nu|+|\mu|\leqq q_0} C_{\mu,\nu,k} .
\end{aligned}$$

Hence we have for a new constant C

$$|D_x^\alpha v, K|_\sigma \leqq C^\alpha \alpha^\alpha \sum_{|q|\leqq\alpha} \frac{b_q}{\alpha^{|q|}} \leqq C^\alpha \alpha^\alpha \alpha^n \max_{|q|\leqq\alpha} \frac{b_q}{\alpha^{|q|}} .$$

By virtue of Condition (1) in § 1, we have for any q' with $|q'|=|q|-1$

$$\frac{b_q}{\alpha^{|q|}} \geqq \frac{b_{q'}}{\alpha^{|q|-1}} \qquad \text{for } a_q\geqq\alpha ,$$

$$\frac{b_q}{\alpha^{|p|}} \leqq \frac{b_{q'}}{\alpha^{|q|-1}} \qquad \text{for } a_q<\alpha .$$

Hence $\max_{|q|\leqq\alpha} \frac{b_q}{\alpha^{|q|}} = \max_{|p|=\alpha} \Big(b_0, \frac{b_p}{\alpha^\alpha}\Big)$.

For $|p|=\alpha$ we have thus by Sobolev's inequality

$$\sup_{|x|\leqq r-\varepsilon} |D_x^p v| \leqq C|D_x^\alpha v, K|_\sigma \sigma^{-q_0} \leqq B^\alpha(b_0\alpha^\alpha+b_p) ,$$

where B is a constant depending on K and on σ. Thus $v(x, 0)\in C\{b_q+q!\}$. By Proposition below the family $C\{b_q+q!\}$ is quasi-analytic. Since $v(x, 0)$

$=\langle u(x), w*f_k\rangle$, and since $\{w*f_k: w\in(C_0^\infty), k=1, 2, \cdots\}$ is total, $u(x)$ is relatively quasi-analytic in E_B.

PROPOSITION (T. Yamanaka). *For a sequence* $\{a_i\}$ *such that*

$$0<a_1\leqq a_2\leqq a_3\cdots,$$

we put

$$b_q=a_1a_2\cdots a_q, \qquad c_q=b_q+q! \qquad (q=1, 2, \cdots).$$

If $\sum\frac{1}{a_i}=\infty$ *(i.e. the family* $C\{b_q\}$ *is quasi-analytic), then*

$$\sum_q\frac{1}{\sqrt[q]{c_q}}=\infty,$$

(i.e. the family $C\{c_q\}$ *is quasi-analytic).*

PROOF. At first we shall verify that

$$\sum_{i=1}^{\infty}\frac{1}{a_i+i}=\infty.$$

Put $S_1=\{i\,|\,a_i\leqq i\}$, $S_2=\{i\,|\,a_i>i\}$. If $\sum_{i\in S_2}\frac{1}{a_i}=\infty$, we have

$$\sum_{i=1}^{\infty}\frac{1}{a_i+i}\geqq\sum_{i\in S_2}\frac{1}{a_i+a_i}=\infty.$$

Hence it is done. Thus we may assume that $\sum_{i\in S_2}\frac{1}{a_i}<\infty$. Then $\sum_{i\in S_1}\frac{1}{a_i}=\infty$, and so S_1 is an infinite set. Let i_0 be an arbitrary index. Then there exists an index i_1 such that

$$i_1\geqq 2i_0, \qquad i_1\in S_1.$$

We have

$$\sum_{i=i_0}^{i_1}\frac{1}{a_i+i}\geqq\sum_{i=i_0}^{i_1}\frac{1}{a_{i_1}+i_1}\geqq\sum_{i=i_0}^{i_1}\frac{1}{2i_1}\geqq\frac{1}{4}.$$

This implies the divergence of $\sum_{i=1}^{\infty}\frac{1}{a_i+i}$, since i_0 is arbitrary.

Put $d_q=(a_1+1)(a_2+2)\cdots(a_q+q)$, for $q=1, 2, 3, \cdots$. By the equality $\sum_{i=1}^{\infty}\frac{1}{a_i+i}=\infty$, we have

$$\sum_q\frac{1}{\sqrt[q]{d_q}}=\infty.$$

Since $c_q\leqq d_q$ for $q=1, 2, 3, \cdots$, we have

$$\sum_q\frac{1}{\sqrt[q]{c_q}}=\infty.$$

It is to be noted that, there exist two sequences $\{a_i\}$ and $\{a_i'\}$ satisfying

$$0<a_1\leqq a_2\leqq\cdots, \quad 0<a_1\leqq a_2\leqq\cdots, \quad \sum\frac{1}{a_i}=\infty, \quad \sum\frac{1}{a_i}=\infty,$$

nevertheless

$$\sum \frac{1}{a_i+a'_i} < \infty .$$

Such an example was given also by T. Yamanaka.

COROLLARY TO THEOREM 3. *If the solutions $u(x)=u(x, y)$ of an equation $Pu=0$ has the unique-continuation property in x, then it has the strict unique-continuation property.*

EXAMPLE 3. The wave equation $\left(\frac{\partial}{\partial t^2}-\frac{\partial^2}{\partial x_1^2}-\cdots-\frac{\partial^2}{\partial x_m^2}\right)u=0$ is partially conditionally elliptic in x. If every point of t-axis is an infinite order zero point of a solution u, then u is identically zero in the whole space. In particular, if a solution u is zero in the double characteristic cone $x_1^2+x_2^2+\cdots+x_m^2<t^2$, and if u is infinitely differentiable at the origin, then u is zero in the whole space. The fact, that a solution u which is zero in the cylinder $x_1^2+\cdots+x_m^2<r^2$ is zero in the whole space, is a direct consequence of Holmgren's theorem. Recently Lax-Morawetz-Phillips proved ([2, Theorem IV]) that if a weak C^1-solution u with finite energy is zero in the double characteristic cone $x_1^2+x_2^2+\cdots+x_m^2<t^2$, then u is zero in the whole space. It happens that there exists a non-zero distribution solution u (with infinite energy) which is zero in the double characteristic cone. In case of C^∞-coefficients, Kumano-go [3] showed that an equation of the form $\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial x_2^2}+f\frac{\partial}{\partial t}+g\right)u=0$ has a non-zero C^∞-solution which is zero in the cylinder $x_1^2+x_2^2<1$.

5. Relative analyticity of solutions in a bounded domain. When we consider the unique-continuation property in $R^m\times\Omega^n$ for a bounded domain $\Omega^n\subset R^n$, the situation is a little different from Theorem 3. In fact, the condition i) does not imply the unique-continuation property. We shall only prove the following

THEOREM 4. *Let Ω^n be a bounded domain in R^n. Then concerning following three conditions, the implication i″)→iii″)→ii″) holds good.*

(i″) $P=P_0(D_x)+\sum_{j>0} P_j(D_x)Q_j(D_y)$, *$P_0$ is elliptic,*

$$\deg P_j+\deg Q_j<\deg P_0 \qquad \text{for } j>0 .$$

(ii″) *The family $\{u(x, y)\in C(R^m\times\Omega^n)\,|\,Pu=0\}$ has the unique-continuation property in x.*

(iii″) *The family $\{u(x)=u(x, y)\in C(R^m\times\Omega^n)\,|\,Pu=0\}$ is relatively analytic in x.*

Notice that (ii″) does not imply (i″). The implication (iii″)→(ii″) is evident from the definition of relative analyticity. We shall show that (i″) implies (iii″). Let $f(t)$ be a function in Corollary to Lemma 2 for $\varepsilon=\frac{1}{2(p-1)}$, ($p_0=\deg P_0$). For a solution u of $Pu=0$ in $R^m\times\Omega^n$, we put $v(x)=\langle u, f\rangle_y$

$=\int_{\Omega^n} u(x, y)f(y)dy$. Then in a similar way to the proof of Theorem 3, we obtain the analyticity of $v(x)$. In fact, since

$$\begin{aligned}|D_x^p v, K|_\sigma &\leqq C(\sigma^{-p_0}|D_x^{p-p_0}v, L|_\sigma + |D_x^{p-p_0}\langle P_0 u, f\rangle_y, L|_\sigma)\\ &\leqq C(\sigma^{-p}|D_x^{p-p_0}v, L|_\sigma + \sum_{j>0}|D_x^{p-p_0}\langle P_j Q_j u, f\rangle_y, L|_\sigma)\\ &\leqq C'(\sigma^{-p}|D_x^{p-p_0}v, L|_\sigma \sum_{0\leqq\mu\leqq p_0-1}|\langle D_x^{\mu+p-p_0}u, D_y^{p_0-\mu-1}f\rangle_y, L|_\sigma),\end{aligned}$$

we have for $kp_0/\sigma > 1$

$$\begin{aligned}|D_x^{kp_0}v, K|_{\sigma/kp_0} &\leqq A^{kp_0}\sum_{|q|=0}^{k(p_0-1)}\left(\frac{kp_0}{\sigma}\right)^{kp_0-\left[\frac{|q|p_0}{p_0-1}\right]}(p_0+1)^{kp_0}|\langle u, D_y^q f\rangle_y, L|_{\sigma/kp_0}\\ &\leqq B^{kp_0}\sum_q k^{kp-(1+\varepsilon)|q|}|q|^{(1+\varepsilon)|q|}|v, L|_{\sigma/kp_0}\\ &\leqq C^k k^{kp_0}|v, L|_{\sigma/kp}.\end{aligned}$$

§ 3. Unique-continuation of solutions with some growth conditions.

As is well known, the Cauchy problem of heat equation $(\partial/\partial t-\Delta)u=0$ is solved uniquely when solutions of exponential order at ∞ are considered. (On the uniqueness of solutions of Kowalevskaja system, see Yamanaka [8].) Our purpose in this section is to consider a generalization of the above fact, the unique-continuation property of solutions of partially hypoelliptic equations under some growth conditions.

6. Relative analyticity of solutions with some growth conditions of partially hypoelliptic equations. A linear partial differential operator $P(D_x, D_y)$ with constant coefficients is called *partially hypoelliptic in* x if any distribution solution $u(x, y)$ of the equation $Pu=0$ is infinitely differentiable in x as a (D_y')-valued function. This notion is characterized as follows. (See [1].)

THEOREM. *P is partially hypoelliptic in x if and only if the following two equivalent conditions are satisfied:*

$$P(\xi, \eta)=0,\ \xi'' \text{ and } \eta \text{ bounded} \Rightarrow \xi' \text{ bounded}. \tag{20}$$

$$(\xi'=\mathrm{Re}(\xi),\quad \xi''=\mathrm{Im}(\xi).)$$

$$P(D_x, D_y)=P_0(D_x)+\sum_{j>0}P_j(D_x)Q_j(D_x), \tag{21}$$

where P_0 is hypoelliptic and $P_j \ll P_0$. ($P_j \ll P_0$ means $P_j(\xi')/P_0(\xi')\to 0$ as $\xi'\to\infty$.)

Let $E_y=\{v(y)\in C(R^n)\,|\,v(y)$ is of exponential order at $\infty\}$, that is, $E_y=\bigcup_{k=1}^{\infty}\{v(y)\in C(R^n)\,|\,v(y)=0(\exp(k|y|))\}$. Each subset $\{v(y)\in C(R^n)\,|\,v(y)=0(\exp(k|y|))\}$ for any fixed k is a Banach space with respect to the norm $\|v\|_k=\sup_y|v(y)e^{-k|y|}|$. We consider E_y the limit inductive space of the sequence of Banach spaces above.

THEOREM 5. *A linear partial differential operator $P(D_x, D_y)$ with constant coefficients is partially hypoelliptic in x if and only if the family $\{u(x, y) \in C(R^m \times R^n) | Pu=0$ and $u(x, y)=0(\exp C(|x|+|y|))\}$ is uniformly relatively analytic in x as E_y-valued functions.*

COROLLARY. *$P=P(D_x)$ is hypoelliptic if and only if all solutions $u(x)$ of $Pu=0$ satisfying $u(x)=0(\exp C|x|)$ are analytic functions.*

A better result than this corollary was already given in [**6**].

For the proof of Theorem 5, we need some lemmas.

LEMMA 6. *For any fixed hypoelliptic operator P_0, there exist some integer ν and an operator $S: L^2 \to L^2$ such that (Δ is Laplacian)*

$$S^\nu = -\Delta+1, \quad \textit{and}$$

$$P_0 \gg Q \quad \textit{implies} \quad \|S^h Q v\| < \|P_0 S^{h-1} v\| + C^h \|v\|$$

$$\textit{for} \quad v \in (C_0^\infty), \quad h=1, 2, \cdots,$$

where C depends only on Q.

PROOF. Since the space $\{Q | Q \ll P_0\}$ is finite-dimensional, there exist positive constants ε and k_Q such that

$$|Q(\xi')(1+|\xi'|^2)^\varepsilon| \leqq |P_0(\xi')| \qquad \text{for any } Q \ll P_0, \ |\xi'| \geqq k_Q .$$

We pick up an integer $\nu > 1/\varepsilon$, and define $S=\mathcal{F}^{-1}(1+|\xi|^2)^{1/\nu}\mathcal{F}$ ($\mathcal{F}$ means Fourier transformation, and $\mathcal{F}^{-1}$ the inverse), $C=\sup_{|\xi'| \leqq k_Q} \{(1+|\xi'|^2)^{1/\nu}(1+|Q(\xi')|)\}$. Since

$$(1+|\xi'|^2)^{h/\nu}|Q(\xi')| \leqq \max\{(1+|\xi'|^2)^{\frac{h-1}{\nu}}|P_0(\xi')|, C^h\},$$

we obtain the required inequality by the Fourier transformation. q. e. d.

We put for an integer $k > \nu$ ($S^\nu = -\Delta+1$)

$$\begin{cases} f(x) = \mathcal{F}^{-1}(\exp(-\xi_1^{2k} - \cdots - \xi_m^{2k})) \\ g(y) = \mathcal{F}^{-1}(\exp(-\eta_1^{2k} - \cdots - \eta_m^{2k})) . \end{cases} \tag{22}$$

LEMMA 7. *The following inequalities hold good.*

$$\begin{aligned} &\| e^{ax} D_x^p f \| \leqq C^{|p|+1} N!, \\ &\| e^{ay} D_y^q g \|_1 \leqq C'^{|q|} N'!, \end{aligned} \tag{23}$$

where $N=(N_1, \cdots, N_m)$, $N'=(N_1', \cdots, N_n')$, $N_i = \left[\frac{2p_i+1}{2k}\right]$, $N_j' = \left[\frac{2q_j+1}{2k}\right]$, $N! = \prod_{i=1}^m N_i!$, $N'! = \prod_{j=1}^n N_j'!$ and C, C' are constants not depending on p, q.

PROOF. We show only the inequality concerning f. The another is similarly obtained. We have formally for $a=(a, a, \cdots, a)$

$$\mathcal{F}(e^{ax}D_x^p f)=\frac{1}{(2\pi)^{\frac{m}{2}}}\int_{R^m}e^{(a-i\xi)x}D_x^p f(x)\,dx$$
$$=(\xi+ia)^p(\mathcal{F}f)(\xi+ia)=(\xi+ia)^p e^{-\Sigma(\xi_j+ia)^{2k}}\,.$$

Since $(\xi+ia)^p e^{-\Sigma(\xi_j+ia)^{2k}}\in L^2$, $\mathcal{F}(e^{ax}D_x^p f)$ exists and satisfies the above equality. Let us estimate the norm of $\mathcal{F}(e^{ax}D_x^p f)$.

$$\|e^{ax}D_x^p f\|^2=\|\mathcal{F}(e^{ax}D_x^p f)\|^2 \tag{24}$$
$$=\int_{R^m}|(\xi+ia)^p e^{-\Sigma(\xi_j+ia)^{2k}}|^2\,d\xi$$
$$=\prod_{j=1}^m\int_{-\infty}^{\infty}|(\xi_j+ia)^{2p_j}e^{-2(\xi_j+ia)^{2k}}|\,d\xi\,.$$

Put $0<\alpha_0<\tan\frac{\pi}{4k}$. Then $\mathrm{Re}\,(\xi_j+ia)^{2k}>0$ for $\left|\frac{a}{\xi_j}\right|\leqq\alpha_0$ $(j=1,2,\cdots,m)$. Since for $\beta=\mathrm{Re}\,(1+i\alpha_0)^{2k}$ we have

$$\mathrm{Re}\,(\xi_j+ia)^{2k}=\mathrm{Re}\left(\xi_j^{2k}\left(1+\frac{ia}{\xi_j}\right)^{2k}\right)$$
$$\geqq\xi_j^{2k}\,\mathrm{Re}\,(1+i\alpha_0)^{2k}=\beta\xi_j^{2k},\qquad\text{for }\left|\frac{a}{\xi_j}\right|\leqq\alpha_0\,,$$

we have

$$|(\xi_j+ia)^{2p_j}e^{-2(\xi_j+ia)^{2k}}|\leqq|\xi_j(1+\alpha_0)|^{2p_j}e^{-\beta\xi_j^{2k}}\,,$$
$$\text{for }\left|\frac{a}{\xi_j}\right|\leqq\alpha_0\,.$$

Hence it holds that

$$\int_{\left|\frac{a}{\alpha_0}\right|}^{\infty}|(\xi_j+ia)^{2p_j}e^{-2(\xi_j+ia)^{2k}}|\,d\xi_j \tag{25}$$
$$\leqq(1+\alpha_0)^{2p_j}\int_0^{\infty}\xi_j^{2p_j}e^{-2\beta\xi_j^{2k}}\,d\xi_j\,.$$

Set $s=\xi_j^{\mu}$ for $\mu=\frac{2p_j+1}{N_j+1}$. Since $\frac{2k}{\mu}\geqq1$, we have

$$\int_0^{\infty}\xi_j^{2p_j}e^{-2\beta\xi_j^{2k}}\,d\xi_j=\frac{1}{\mu}\int_0^{\infty}s^{\frac{2p_j+1}{\mu}-1}e^{-2\beta s^{\frac{2k}{\mu}}}\,ds \tag{26}$$
$$\leqq1+\int_0^{\infty}s^{N_j}e^{-2\beta s}ds=1+\frac{N_j\,!}{(2\beta)^{N_j+1}}\,.$$

On the other hand, it is easy to see that

$$|(\xi_j+ia)^{2p_j}e^{-2(\xi_j+ia)^{2k}}|\leqq\left|a\left(1+\frac{1}{\alpha_0}\right)\right|^{2p_j}e^{2\left|a\left(1+\frac{1}{\alpha_0}\right)\right|^{2k}}\,,$$
$$\text{for }\left|\frac{a}{\xi_j}\right|\geqq\alpha_0\,.$$

Hence

$$\int_0^{\left|\frac{a}{\alpha_0}\right|} |(\xi_j+ia)^{2p_j}e^{-2(\xi_j+ia)^{2k}}|\,d\xi_j \leqq \left|a\left(1+\frac{1}{\alpha_0}\right)\right|^{2p_j} e^{2\left|a\left(1+\frac{1}{\alpha_0}\right)\right|^{2k}}\left|\frac{a}{\alpha_0}\right|. \tag{27}$$

From (24), (25), (26) and (27) it follows that

$$\begin{aligned}\|e^{ax}D_x^p f\|^2 &\leqq \prod_{j=1}^m \Big\{\frac{|a|}{\alpha_0}^{2p_j+1}\Big(1+\frac{1}{\alpha_0}\Big)^{2p_j} e^{2\left|a\left(1+\frac{1}{\alpha_0}\right)\right|^{2k}} \\ &\qquad + (1+\alpha_0)^{2p_j}\Big(1+\frac{N_j!}{(2\beta)^{N_j+1}}\Big)\Big\} \\ &\leqq \prod_{j=1}^m N_j!\Big\{A\,B^{2p_j}+(1+\alpha_0)^{2p_j}\Big(\max\Big(1,\frac{1}{2\beta}\Big)\Big)^{2p_j+1}\Big\} \\ &\leqq C^{2|p|}N! \leqq C^{2|p|}(N!)^2. \qquad \text{q. e. d.}\end{aligned}$$

We use the following notations for functions $u(x, y)$ and $v(x, y)$:

$$\langle u, v\rangle_x = \int u\cdot v\,dx, \quad \|u\|_x^2 = \int |u|^2 dx$$

$$\langle u, v\rangle_y = \int u\cdot v\,dy, \quad \|u\|_y^2 = \int |u|^2 dy.$$

LEMMA 8. *Let u be a solution of $Pu=0$. Using the notation ${}^tQ_0=1$,*

$$\|\langle S^h P_0(fu), g\rangle_y\|_x < C\sum_{j\geqq 0}\sum_{|\alpha|\leqq \deg P_0}(\|\langle S^{h-1}P_0(D^\alpha f\cdot u), {}^tQ_j g\rangle_y\|_x + C^{h-1}\|\langle D^\alpha f\cdot u, {}^tQ_j g\rangle_y\|_x), \tag{28}$$

where C is a positive constant independent of h.

PROOF. Since

$$\begin{aligned}\langle S^h P_0(fu), g\rangle_y &= \sum_{|\alpha|\geqq 0}\frac{1}{\alpha!}\langle S^h(D^\alpha f\cdot P_0^{(\alpha)}u, g\rangle_y \\ &= \langle S^h(f\cdot P_0 u), g\rangle_y + \sum_{|\alpha|>0}\frac{1}{\alpha!}\langle S^h(D^\alpha f\cdot P_0^{(\alpha)}u), g\rangle_y \\ &= \sum_j \langle S^h(f\cdot P_j u), {}^tQ_j g\rangle_y + \sum_{|\alpha|>0}\frac{1}{\alpha!}\langle S^h(D^\alpha f\cdot P_0^{(\alpha)}u), g\rangle_y,\end{aligned}$$

we have

$$\|\langle S^h P_0(fu), g\rangle_y\|_x \leqq \sum_j \|\langle S^h(f\cdot P_j u), {}^tQ_j g\rangle_y\|_x + \sum_{|\alpha|>0}\frac{1}{\alpha!}\|\langle S^h(D^\alpha f\cdot P_0^{(\alpha)}u), g\rangle_y\|_x. \tag{29}$$

Since

$$D^\alpha f\cdot P_0^{(\alpha)}u = P_0^{(\alpha)}(D^\alpha f\cdot u) - \sum_{|\beta|>0}\frac{1}{\beta!}D^{\alpha+\beta}f\cdot P_0^{(\alpha+\beta)}u,$$

it holds that

(30) $$\sum_{|\alpha|>0}\frac{1}{\alpha!}\|\langle S^h(D^\alpha f\cdot P_0^{(\alpha)}u),\ g\rangle_y\|_x\leqq C'\sum_{|\alpha|>0}\|\langle S^hP_0^{(\alpha)}(D^\alpha f\cdot u),\ g\rangle_y\|_x\,.$$

Similarly we have

(31) $$\|\langle S^h(f\cdot P_ju),\ {}^tQ_jg\rangle_y\|_x\leqq C''\sum_{|\alpha|>0}\|\langle S^hP_j^{(\alpha)}(D^\alpha f\cdot u),\ {}^tQ_jg\rangle_y\|_x\,,$$

where C' and C'' depend only on P.

By Lemma 6 we have

(32) $$\|\langle S^hP_0^{(\alpha)}(D^\alpha f\cdot u),\ g\rangle\|\leqq\|\langle S^{h-1}P_0(D^\alpha f\cdot u),\ g\rangle\|+C_1^h\|\langle D^\alpha f\cdot u,\ g\rangle\|\,,$$

and

(33) $$\|\langle S^hP_j^{(\alpha)}(D^\alpha f\cdot u),\ {}^tQ_jg\rangle\|\leqq\|\langle S^{h-1}P_0(D^\alpha f\cdot u),\ {}^tQ_jg\rangle\|+C_2^h\|\langle D^\alpha f\cdot u,\ {}^tQ_jg\rangle\|\,.$$

We calculate (30) using (32), and (31) using (33). Then we can estimate (29) as follows.

$$\|\langle S^hP_0(fu),\ g\rangle_y\|_x\leqq C'\sum_j\sum_{|\alpha|>0}(\|\langle S^{h-1}P_0(D^\alpha f\cdot u),\ {}^tQ_jg\rangle\|+C_2^h\|\langle D^\alpha f\cdot u,\ {}^tQ_jg\rangle\|)$$
$$+C''\sum_{|\alpha|>0}(\|\langle S^{h-1}P_0(D^\alpha f\cdot u),\ g\rangle\|+C_1^h\|\langle D^\alpha f\cdot u,\ g\rangle\|)\,.$$

LEMMA 9. *Every solution u of $Pu=0$ with $|u(x,y)|<Ke^{C(|x|+|y|)}$ satisfies*

$$\|\langle(D_x^\alpha f)u,\ D_y^\beta g\rangle_y\|_x<KAB^{|\alpha|+|\beta|}\lambda^{m\lambda}\mu^{n\mu}\,,$$

where $\lambda=\left(\frac{|\alpha|+1}{k}\right)$, $\mu=\left(\frac{|\beta|+1}{k}\right)$.

PROOF. We have by Lemma 7 for N, N' with $N_j=\left[\frac{2\alpha_j+1}{2k}\right]$, $N'_j=\left[\frac{2\beta_j+1}{2k}\right]$,

$$\|\prod_{j=1}^n e^{a|y_j|}D_y^\beta g\|_1\leqq\|\prod_{j=1}^n(e^{ay_j}+e^{-ay_j})D_y^\beta g\|_1\leqq 2^nB'^{|\beta|+1}N'!\,,$$

and similarly

$$\|\prod_{j=1}^m e^{a|x_j|}D_x^\alpha f\|\leqq 2^mB'^{|\alpha|+1}N!\,.$$

Combining above two inequalities we have

$$\begin{aligned}\|\langle(D_x^\alpha f)u,\ D_y^\beta g\rangle_y\|_x^2&\leqq\int\Big\{\int|uD_y^\beta g|\,dy\Big\}^2|D_x^\alpha f|^2dx\\&\leqq K^2\Big\{\int|e^{B|y|}D_y^\beta g|\,dy\Big\}^2\int|e^{B|x|}D_x^\alpha f|^2dx\\&=K^2\|e^{B|y|}D_y^\beta g\|_1^2\|e^{B|x|}D_x^\alpha f\|^2\\&\leqq K^22^{2n}B'^{2|\alpha|+2}(N'!)^22^{2m}B''^{2|\beta|+2}(N'!)^2\\&\leqq(KAB^{|\alpha|+|\beta|}N!N'!)^2\,.\end{aligned}$$

Since $\lambda^{m\lambda}\geqq N!$, $\mu^{n\mu}\geqq N'!$, our assertion is proved.

LEMMA 10. *Let $u(x)$ be a continuous function. If there exists an integer k such that $u(x)*\varphi(x)$ is analytic for any $\varphi(x)\in(D^k)$ $((D^k)=\{f\in(C^k)|$ carrier of f is compact$\})$, then $u(x)$ itself is analytic.*

PROOF. Let T be the operator: $(D^k)\ni\varphi\to u*\varphi\in A(R^m)$, where $A(R^m)=$ the limit inductive space of $\{A(U)=$ the space of all analytic functions on U with the topology of uniform convergence on every compact subset of $U\}$ and U runs over all complex neighbourhood of R^m. Then the transposed operator ${}^tT: A'(R^m)\ni\phi\to u*\phi\in(D^k)'$ is defined on $A'(R^m)$. Since every element φ of $A(R^m)$ is infinitely differentiable, the element ϕ of $A'(R^m)$ is also differentiable:

$$D^p\phi\in A'(R^m)\,,\qquad \langle D^p\phi,\varphi\rangle=(-1)^{|p|}\langle\phi,D^p\varphi\rangle\,.$$

Hence we have $u*D^p\phi=D^p(u*\phi)\in(D^k)'$, which implies $u*\phi\in(C)$ since p is arbitrary. This means ${}^tT$ is an operator: $A'(R^m)\to(C)$. By the closed graph theorem ${}^tT$ is continuous from $A'(R^m)$ to (C). The scalar product $\langle u,\phi\rangle$ is defined by $u*\phi(0)$. Then $u\in A(R^m)''=A(R^m)$.

PROOF OF THEOREM 5. At first we shall prove the sufficiency. By virtue of Lemma 8, for a solution u of $Pu=0$ such that $\max_\alpha|P_0^{(\alpha)}u|\leqq Ke^{a(|x|+|y|)}$, we have ($\beta_j$ is a multi-index with $|\beta_j|\leqq\deg P_0$)

$$\|\langle S^hP_0(fu),g\rangle_y\|_x\leqq C^h\sum_{k_1\cdots k_h}\sum_{\beta_1\cdots\beta_h}\|\langle P_0(D^{\beta_1+\cdots+\beta_h}f\cdot u),{}^tQ_{k_1}\cdots{}^tQ_{k_h}g\rangle_y\|_x$$
$$+C^h\sum_{i=1}^h\sum_{k_1\cdots k_i}\sum_{\beta_1\cdots\beta_i}\|\langle D^{\beta_1+\cdots+\beta_i}f\cdot u,{}^tQ_{k_1}\cdots{}^tQ_{k_i}g\rangle_y\|_x\,.\tag{34}$$

We denote $Q_k(D_y)=\sum_\alpha L_k^{(\alpha)}D_y^\alpha$. Let $L=\max|L_k^{(\alpha)}|$, $\lambda=\dfrac{(h+1)\deg P_0+1}{k}$ and $\mu=\max_j\dfrac{(h+1)\deg Q_j+1}{k}$. Since $P_0(D^{\beta_1+\cdots\beta_h}f\cdot u)=\sum_\beta\dfrac{1}{\beta!}D^{\beta_1+\cdots+\beta_h+\beta}f\cdot P_0^{(\beta)}u$, we have by Lemma 9 for $q=\max_j\deg Q_j$

$$\|\langle S^hP_0(fu),g\rangle_y\|_x\leqq C^h\sum_{k_1\cdots k_h}\sum_{\beta_1\cdots\beta_h,\beta}\|\langle D^{\beta_1+\cdots+\beta_h+\beta}f\cdot P_0^{(\beta)}u,{}^tQ_{k_1}\cdots{}^tQ_{k_h}g\rangle_y\|_x$$
$$+C^h\sum_{j=1}^h\sum_{k_1\cdots k_j}\sum_{\beta_1\cdots\beta_j}\|\langle D^{\beta_1+\cdots+\beta_j}f\cdot u,{}^tQ_{k_1}\cdots{}^tQ_{k_j}g\rangle_y\|_x$$
$$\leqq C^h\sum_{k_1\cdots k_h}\sum_{\beta_1\cdots\beta_h,\beta}K.A.B^{|\beta_1|+\cdots+|\beta_h|+|\beta|+hq}L^h\lambda^{m\lambda}\mu^{n\mu}$$
$$+C^h\sum_{i=1}^h\sum_{k_1\cdots k_i}\sum_{\beta_1\cdots\beta_i}K.A.B^{|\beta_1|+\cdots+|\beta_i|+|\beta|+jq}L^j\lambda^{m\lambda}\mu^{n\mu}\,.$$

Let $k_0=$ the number of Q_k, $l=$ the number of $P_0^{(\beta)}$ with $P_0^{(\beta)}\neq0$. Then $\sum_{k_1\cdots k_h}1=k_0^h$, $\sum_{\beta_1\cdots\beta_h,\beta}1=l^{h+1}$. So we have for $p=\deg P_0$, $\bar B=\max(B,1)$ and $\bar L=\max(L,1)$,

$$\|S^h\langle P_0(fu),g\rangle_y\|_x\leqq k_0^hl^{h+1}C^h(h+1)KA\bar B^{(h+1)p+hq}\bar L^h\lambda^{m\lambda}\mu^{n\mu}\,.$$

This implies that $\langle P_0(fu), g\rangle_y = P_0(f\langle u\cdot g\rangle_y)$ is an entire function, for $\nu \geqq 2$ and for $k > 2\nu$ $(m \deg P_0 + \max_j n \deg Q_j)$, since $S^\nu = -\Delta + 1$. Then $f\langle u, g\rangle_y$ is an entire function, since $\lim_{|\xi'|\to\infty} P_0(\xi') > 0$. Hence $\langle u, g\rangle_y$ is analytic except zero point of f, especially in a neighbourhood of the origin. This implies the analyticity of $\langle u, g\rangle_y$: $u(x+x_0, y)$ is also a solution of $Pu=0$ for any $x_0 \in R^m$ and hence $u(x+x_0, y)$ is analytic in a neighbourhood of the origin.

If u is a solution satisfying $|u(x, y)| < Ke^{a(|x|+|y|)}$, then for any $\varphi \in (D^k)$ $(k = \text{degree } P_0)$ $u * \varphi$ is a solution satisfying $|u*\varphi| \leqq K_\varphi e^{(|x|+|y|)}$ hence $u*\varphi$ is analytic. By Lemma 10 u itself is analytic.

Next we shall prove the necessity. Let $E_{x,y}$ = the space $\{v(x, y) \in C(R^m \times R^n)$ $|v(x, y)$ is of exponential order at $\infty\}$ with the limit inductive topology concerning the sequence of norms $\|v\|_k = \sup|v(x, y)e^{-k(|x|+|y|)}|$, and $A(R^m)$ = the limit inductive space of $\{A(U)$ = the space of all analytic functions with the topology of uniform convergence on every compact subset of $U\}$, where U runs over all complex neighbourhood of R^m. Let u be a solution of $Pu=0$. By the assumption there exists a total subspace F of E'_y such that

$$v(x) = \int u(x, y)\varphi(y)dy \text{ is analytic in } x \text{ for any } \varphi \in F.$$

Since the linear mapping associated with fixed $\varphi: u \to v = \int u\varphi dy$ is a closed operator, it is continuous from $E_{x,y}$ to $A(R^m)$. This continuity means that for any compact set $K \subset R^m$ there exists a function $\Phi(x, y)$ with $|\Phi(x, y)e^{j(|x|+|y|)}|$ $\to 0$ as $|x|+|y| \to \infty$, $j = 1, 2, \cdots$ such that

$$\sup_{x\in K}\left|\frac{\partial v}{\partial x_j}\right| \leqq C_j(\varphi)\sup_{x,y}|\Phi(x, y)u(x, y)|.$$

Put $u(x, y) = e^{(i\xi x + i\eta y)}$ for $P(\xi, \eta) = 0$. Then for $\xi'' = \mathrm{Im}(\xi)$, $\eta'' = \mathrm{Im}(\eta)$,

$$\sup_K |e^{-\xi'' x}\mathcal{F}\varphi(\eta)| \sum_i |\xi_i| \leqq C(\varphi)\sup_{x,y}|\Phi(x, y)e^{-x\xi''-y\eta''}|.$$

We fix a compact set $K' \subset R^n$. For any point $p \in K'$ there exists an element $\varphi_p \in F$ such that $\mathcal{F}\varphi_p(p) \neq 0$ since F is total on E_y, and so we can choose $p_1, \cdots, p_s$ such that

$$\sum_{j=1}^s |\mathcal{F}\varphi_{p_j}(\eta)| > 0, \qquad \text{for any } \eta \in K'.$$

Hence we have

$$\sum_{j=1}^m |\xi_j| \sup_K |e^{-\xi'' x}| \sum_{j=1}^s |\mathcal{F}\varphi_{p_j}(\eta)| \leqq \sum_{j=1}^s C(\varphi_{p_j}) \sup_{x,y}|\Phi(x, y)e^{-x\xi''-y\eta''}|,$$

$$\text{for } \eta \in K'.$$

This means that the boundedness of η and ξ'' implies the boundedness of ξ.

q. e. d.

In the same way as above, we can prove the following theorem.

THEOREM 6. *Let Ω^m be an arbitrary open domain in R^m. The family $\{u(x, y) \in C(\Omega^m \times R^n) \mid Pu = 0,\ |u(x, y)| \leqq C(x)e^{B|y|}$ for some constant B and $C(x) \in C(\Omega^m)\}$ is relatively analytic in x if and only if P can be expressed in the form*

$$P(D_x, D_y) = P_0(D_x) + \sum_j P_j(D_x)Q_j(D_y),$$

where P_0 is elliptic and $\deg P_0 > \deg P_j$.

EXAMPLE. Let $P = \frac{\partial}{\partial t} - A$, where A is a differential operator with constant coefficients in n-dimensional x-space. Then a solution $u(t, x)$ of the equation $Pu = 0$ in $(-\lambda, \lambda) \times R^n$, such that $|u(t, x)| \leqq C(t)e^{B|x|}$ and $u(t, x) = 0$ for $t < 0$, is identically zero in the domain.

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