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On 12-manifolds of a special kind

Dedicated to Professor Atuo Komatu on his 60th birthday

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§ 1. Preliminary.

Let K be a simply connected CW-complex whose cohomology groups are as follows

$$H^0(K)=H^4(K)=H^8(K)=H^{12}(K)\cong Z \quad \text{and} \quad H^i(K)=0 \quad \text{for other } i.$$

In this paper we shall consider conditions under which K has a homotopy type of a compact C^∞-manifold without boundary. By using Novikov-Browder theory[1] we can partially solve the above problem. By an orientation of K we mean a pair of generators of $H^8(K)$ and $H^{12}(K)$[2]. Since K is homotopy equivalent to a CW-complex $S^4\cup e^8\cup e^{12}$ we can associate with K elements $\alpha\in\pi_7(S^4)$ and $\beta\in\pi_{11}(S^4\cup e^8)$ which are ∂-images of the generators of $\pi_8(K, S^4)$ and $\pi_{12}(K, S^4\cup e^8)$ carried by the orientation of K respectively. Here ∂ denotes the boundary homomorphism: $\pi_8(K, S^4)\to\pi_7(S^4)$ and $\pi_{12}(K, S^4\cup e^8)\to\pi_{11}(S^4\cup e^8)$ respectively. Let $h: S^7\to S^4$ be the Hopf map and let τ be the element of $\pi_7(S^4)$ such that $2[h]+\tau=[\iota_4, \iota_4]$[3]. It is known that $\pi_7(S^4)$ is isomorphic to the direct sum of Z and Z_{12} which are generated by $[h]$ and τ respectively. Hence we can replace α by two integers a, b $(0\leqq b\leqq 11)$ such that $\alpha=a[h]+b\tau$. In this paper the case $b=0$ shall be treated in which case we can replace β by numerical invariants. Let K_a be the CW-complex which is obtained by attaching e^8 to S^4 by a representative of $a[h]$, and let $\varphi_a: K_a\to K_1$ be a map which is the identity on S^4 and of degree a on e^8. Obviously, $K_1=P_2(Q)$, the quaternion projective plane. Denote by ξ_1 $(S^{11}\to P_2(Q)=K_1)$ the canonical S^3-bundle. Then let ξ_a be the bundle induced by φ_a, and by the same symbol ξ_a we denote also the total space of this bundle. We consider the group $\pi_{11}(K_a)$ and the diagram

1) Concerning Browder's theorem and another application, see [**3**] and [**5**].
2) We suppose that an orientation of e^4 is fixed.
3) $[f]$ denotes the homotopy class of f, and $[\ ,\ ]$ Whitehead product.

$$\begin{tikzcd}
 & \pi_{11}(\xi_a) \arrow[d, "P_{a*}"] & & \\
\pi_{11}(S^4) \arrow[r, "i_a"'] & \pi_{11}(K_a) \arrow[r, "j_a"'] \arrow[d, "\partial_a"] & \pi_{11}(K_a, S^4) \arrow[r, "\partial_a"'] & \pi_{10}(S^4) \\
 & \pi_{10}(S^3) \arrow[ul, "E"] & &
\end{tikzcd}$$

where the horizontal line is the homotopy exact sequence of the pair (K_a, S^4), the vertical line is that of the bundle ξ_a and E is the suspension homomorphism which is an isomorphism and satisfies $\partial_a \circ i_a \circ E =$ identity. We know that $\pi_{10}(S^3) \cong Z_{15}$, $\pi_{10}(S^4) \cong Z_{24} + Z_3$ generated by $[h] \circ E^3[h]$ and $\tau \circ E^3\tau$, and $\pi_{11}(K_a, S^4) \cong Z + Z_{24}$ generated by $[\chi_a, \iota_4]_r$ and $\chi_a \circ \tilde{h}$, where $\chi_a \in \pi_8(K_a, S^4)$ is the class of the characteristic map of the cell e^8, $\partial\tilde{h} = E^3[h]$ for $\partial: \pi_{11}(D^8, S^7) \cong \pi_{10}(S^7)$ and $[\ ,\ ]_r$ is the relative Whitehead product. Since the above E is an isomorphism $\pi_{11}(K_a)$ is isomorphic to the direct sum of $\pi_{10}(S^3)$ and $\mathrm{Im}\, j_a = \mathrm{Ker}\, \partial_a$. We have also $\partial_a[\chi_a, \iota_4]_r = -[\partial_a \chi_a, \iota_4] = -a[[h], \iota_4] = -2a([h] \circ E^3[h])$ by use of Lemma (4.6) of [2], and $\partial_a(\chi_a \circ \tilde{h}) = a([h] \circ E^3[h])$. Therefore the following lemma is obtained.

LEMMA 1.1. *The group $\pi_{11}(K_a)$ is isomorphic to $Z_{15} + Z + Z\rho_a$. The summands are generated by elements μ_a, λ_a and ν_a respectively which are characterized as follows:*

i) *μ_a is the i_a-image of a (fixed) generator of $\pi_{11}(S_4) \cong Z_{15}$*

ii) $\partial_a(\lambda_a) = \partial_a(\nu_a) = 0$

iii) $j_a(\lambda_a) = [\chi_a, \iota_4]_r + 2(\chi_a \circ \tilde{h})$ *and* $j_a(\nu_a) = (24/\rho_a)(\chi_a \circ \tilde{h})$, *where* $\rho_a = (24, a)$.

By virtue of this lemma we can associate numerical invariants (a, m, l, n), $(0 \leqq m < 15,\ 0 \leqq n < \rho_a)$, of integers with K where $\alpha = a[h]$ and $\beta = m \cdot \mu_a + l \cdot \lambda_a + n \cdot \nu_a$. We shall call K a complex of type (a, m, l, n). In [4] James has proved

LEMMA 1.2. *If K is of type (a, m, l, n) then*

$$e^4 \cup e^4 = ae^8, \quad e^4 \cup e^8 = le^{12} \quad \text{in } H^*(K)$$

where e^{4i} denotes the oriented generator carried by each cell.

For example, let $\mathcal{P}_3(Q)$ be the quaternion projective 3-space $P_3(Q)$ with the orientation $((e^4)^2, (e^4)^3)$. Then $\mathcal{P}_3(Q)$ is a complex of type $(1, 0, 1, 0)$, because $a = 1 = l$ by lemma 1.2 and $m = 0$. Here $m = 0$ follows from $\partial_a \circ \partial = 0: \pi_{12}(P_3(Q), P_2(Q)) \to \pi_{11}(P_2(Q)) \to \pi_{10}(S^3)$ and $\nu_1 = 0$.

It is important for our purpose that the Poincaré duality holds in $H^*(K)$, that is, $e^4 \cup e^8 = \pm e^{12}$. Thus the above lemma implies

PROPOSITION 1. *If K is of type (a, m, l, n), then Poincaré duality holds in $H^*(K)$ if and only if $l = \pm 1$.*

§ 2. Stable vector bundles over the complex K of type (a, m, l, n).

It is easily seen that a stable vector bundle over K is uniquely determined by it's Pontrjagin classes. In our case Pontrjagin classes may be considered as a triple (p_1, p_2, p_3) of integers. Then by an elementary argument of vector bundles (see [**5**]) we can show

LEMMA 2.1. *There exist a basis of $KO(K)$ $\xi_1(K)$, $\xi_2(K)$, $\xi_3(K)$ such that*

$$\xi_1(K)=(2, a, \alpha(K)), \quad \xi_2(K)=(0, 6, \beta(K)), \quad \xi_3(K)=(0, 0, 240),$$

where $\alpha(K)$ and $\beta(K)$ are integers which are determined mod 240, *up to homotopy type of K.*

By using the product formula for Pontrjagin classes and Lemma 2.1 we obtain

PROPOSITION 2. *A triple of integers (p_1, p_2, p_3) is Pontrjagin classes of a stable vector bundle over K if and only if*

$$p_1=2r, \qquad p_2=ar(2r-1)+6s$$

and

$$p_3=240t+s\beta(K)+r\alpha(K)+12rls+2alr(r-1)+alr(r-1)(r-2)4/3$$

for some integers r, s, t.

Now we shall determine $\alpha(K)$ and $\beta(K)$. Since, for $k=1, 2, 3$, $\pi_{N+4k-1}(S^N)$[4] is isomorphic to the cyclic group $Z\rho_k$ where $\rho_k=24, 240, 504$ for $k=1, 2, 3$ respectively we can put $[f]=c[h_k]$[5] for a map $f\colon S^{N+4k-1}\to S^N$. Let Y_c be the CW-complex $S^N\cup e^{N+4k}$ which is obtained from attaching e^{N+4k} to S^N by f. Concerning $KO(Y_c)$[6] we have

LEMMA 2.2. *There exist elements η_1^c, η_2^c of $KO(Y_c)$ whose Pontrjagin classes are*

$$P_{N/4}(\eta_1^c)=(N/2-1)!\,e^N, \quad P_{N/4+k}(\eta_1^c)=\lambda_c e^{N+4k}$$

$$P_{N/4}(\eta_2^c)=0, \quad P_{N/4+k}(\eta_2^c)=(N/2+2k-1)!\,a_k e^{N+4k}$$

where $a_k=1$ for even k, $a_k=2$ for odd k and $\lambda_c=(-1)^k cB_k(N/2+2k-1)a_k$[7]. *And any η of $KO(Y_c)$ is represented by a linear combination of η_1^c and η_2^c.*

PROOF. Since the J-homomorphism $J\colon KO(S^{4k})\to\pi_{N+4k-1}(S^N)$ is onto $(k=1, 2, 3)$ we may consider Y_c as the Thom complex of a N-vector bundle ξ over S^{4k} whose k-th Pontrjagin class is $a_k\cdot c\cdot(2k-1)!$. Then the proof is established by using the formula [**1**, Corollary 5.4].

By using Lemma 2.2 we can easily obtain

LEMMA 2.3. *Let $\alpha\in\pi_{N+4k-1}(S^N)$ $(k=1, 2, 3)$. Then α is zero if and only if*

4) N is sufficiently large and we suppose $N\equiv 0 \bmod 8$.
5) h_k denotes a generator of $\pi_{N+4k-1}(S^N)$ and c is an integer.
6) $KO(Y_c)$ is the group consisted of stable vector bundles over Y_c.
7) B_k is the k-th Bernouille number.

$P_{N/4+k}(\eta)\equiv 0 \bmod (2k-1+N/2)!\,a_k$ *for any* η *of* $KO(Y_\alpha)$, *where* Y_α *means the complex* $S^N\overset{\alpha}{\cup} e^{N+4k}$.

Now let Y be the CW-complex $S^8\overset{f}{\cup} e^{12}$ ($[f]=(2l+(n24/(a, 24)h_1)$. Since K is of type (a, m, l, n) there exists a map $P: K\to Y$ such that the restriction $P|S^4$ is constant and P is of degree one on each cells e^8, e^{12}. Therefore it is clear that $P_1(\xi_2(K))$, $P_2(\xi_2(K))$ coincide with those of the bundle induced by P from η_1^c in the case of Lemma 2.2, $c=(2l+(n24/(a, 24)))$, $k=1$ and $N=8$. Hence we have

LEMMA 2.4. $\beta(K)=-10(2l+n24/(a, 24)) \bmod 240$.

Next we shall determine $\alpha(K)$. Let $\mathcal{P}_n(Q)$ be the quaternion projective n-space with the orientation $e^4, (e^4)^2, \cdots, (e^4)^n$. Since the tangent bundle of $\mathcal{P}_8(Q)$ has the cohomology class $1+14e^4+97e^8+428e^{12}+\cdots$ as it's total Pontrjagin class the restriction of it on $\mathcal{P}_3(Q)$ is represented by our notation such as

$$\tau(\mathcal{P}_8(Q))\,|\,\mathcal{P}_3(Q)=7\xi_1(\mathcal{P}_3(Q))+\xi_2(\mathcal{P}_3(Q)) \qquad \bmod \xi_3(\mathcal{P}_3(Q))\,.$$

Then we have

LEMMA 2.5. *If* K *is of type* $(a, 0, l, n)$ $\alpha(K)\equiv 0 \bmod 240$.

PROOF. If K is of type $(1, 0, 1, 0)$ we may consider K as $P_3(Q)$. Therefore $\alpha(K)=0$ by the above. Now if K is of type $(a, 0, l, n)$ there exists a map; $F: K\to P_3(Q)$ such that F is of degree 1 on e^4, a on e^8 and al on e^{12} respectively, because by easy computations it is seen that the composition map $i\cdot\varphi_a$ (defined in § 1) is extendable over K as above. Hence by comparing the Pontrjagin classes of $F^{-1}(\xi_1(P_3(Q)))$ with that of $\xi_1(K)$ we have $\alpha(K)=0$.

Let φ be a correspondence: $\pi_{11}(K_a)\to Z_{240}$ defined by $\varphi(g)=\alpha(K)$, where K denotes the CW-complex which is obtained from attaching e^{12} to K_a by a map g of $\pi_{11}(K_a)$. It can be easily seen that φ is a homomorphism. By Lemma 2.5 we have that $\varphi(\lambda_a)=\varphi(\nu_a)=0$. Hence the problem is to determine $\varphi(\mu_a)$. Consider the iterated suspension $E^{12}: \pi_{11}(S^4)\to\pi_{23}(S^{16})=Z_{240}[\sigma]$. We suppose $E^{12}\alpha=m[\sigma]$ for $\alpha\in\pi_{11}(S^4)$ and let Y_α be the CW-complex $S^4\overset{\alpha}{\cup} e^{12}$. Our problem is to find the third Pontrjagin class of an element ξ of $KO(Y_\alpha)$ whose first Pontrjagin class is $2e^4$. By using the formula $chE^{12}=E^{12}ch$ and Lemma 2.2 (take $Y_m=E^{12}Y_\alpha$) we obtain $P_3(\xi)=me^{12} \bmod 120e^{12}$. On the other hand, since $15\alpha=0$, we have $15p_3(\xi)\equiv 0$, i. e. $15m\equiv 0 \bmod 240$. Hence we have $p_3(\xi)\equiv me^{12} \bmod 240$. By choosing the generator of $\pi_{11}(S^4)$ whose E^{12}-image is $16[\sigma]$ we obtain

LEMMA 2.6. $\varphi[\mu_a]=16$ *and* $\alpha(K)\equiv 16m \bmod 240$ *if* K *is of type* (a, m, l, n).

§ 3. Reducibility of Thom complexes.

Let K be a CW-complex $S^4\cup e^8\cup e^{12}$ and let K_i be the $4i$-skelton of K. Let ξ be a stable vector bundle over K and let T_ξ be the Thom complex of

ξ. Now we may consider that T_ξ, $T_{\xi|k_2}$, $T_{\xi|k_1}$ have a CW-decomposition as follows

$$T_\xi = T_{\xi|k_2} \overset{\varphi}{\cup} e^{12+N},\ T_{\xi|k_2} = T_{\xi|k_1} \cup e^{N+8},\ T_\xi = T_{\xi|k_1} \cup e^{N+8} \cup e^{N+12}.$$

Now we seek the conditions for reducibility of T_ξ, i.e. $\varphi=0$. Let j_2 be the inclusion homomorphism: $\pi_{N+11}(T_{\xi|k_2}) \to \pi_{N+11}(T_{\xi|k_2}, T_{\xi|k_1}) = \pi_{N+11}(S^{N+8})$. By Lemma 2.3 $j_2(\varphi)=0$ is equivalent to that $p_{N/4+3}(\eta)=0 \bmod (N/2+5)!\,2$ for any η of $KO(T_\xi)$ such that $\eta | T_{\xi|k_1}=0$. Under this condition there exists an element φ' of $\pi_{N+11}(T_{\xi|k_1})$ such that $i(\varphi')=\varphi$ where i is the inclusion homomorphism: $\pi_{N+11}(T_{\xi|k_1}) \to \pi_{N+11}(T_{\xi|k_2})$. Since we have that $\pi_{N+12}(T_{\xi|k_2}, T_{\xi|k_1}) = \pi_{N+12}(S^{N+8})=0$ i is a monomorphism. Thus $\varphi=0$ is equivalent to $\varphi'=0$. Let j_1 be the inclusion homomorphism: $\pi_{N+11}(T_{\xi|k_1}) \to \pi_{N+11}(T_{\xi|k_1}, S^N) = \pi_{N+11}(S^{N+8})$. Again by Lemma 2.3 $j_1(\varphi')=0$ is equivalent to $p_{N/4+3}(\eta)=0 \bmod (N/2+5)!\,2$ for any η of $KO(T_\xi)$ such that the restriction $\eta | S^N$ is trivial. Under these conditions ($j_2(\varphi)=j_1(\varphi')=0$) consider the part of the homotopy exact sequence of the pair $(T_{\xi|k_1}, S^N)$,

$$\pi_{N+12}(T_{\xi|k_1}, S^N) \xrightarrow[\partial]{} \pi_{N+11}(S^N) \xrightarrow[i]{} \pi_{N+11}(T_{\xi|k_1}).$$

Let φ'' be an element of $\pi_{N+12}(S^N)$ such that $i(\varphi'')=\varphi'$. Since ∂ is trivial, $\varphi'=0$ is equivalent to $\varphi''=0$. And moreover $\varphi''=0$ is true if and olny if $p_{N/4+3}(\eta)=0 \bmod (N/2+5)!\,2$ for any η of $KO(T_\xi)$ by Lemma 2.3. Thus we have

PROPOSITION 3. *T_ξ is reducible if and only if $ch_{N/2+6}(\eta)$ is integral for all η of $KO(T_\xi)$.*

Now by the formula $ch\Phi_{KO}(\bar\eta) = \Phi_\xi(ch(\bar\eta)\cdot \hat{A}^{-1}(\xi))$ for $\bar\eta \in KO(K)$[8] we have $ch_{N/2+6}(\eta) = A_3 m + A_2 ch_2(\bar\eta) + A_1 ch_4(\bar\eta) + ch_6(\bar\eta)$ for some integer m and $\bar\eta \in KO(K)$ where $A_1 = -\hat{A}_1(\xi)$, $A_2 = \hat{A}_1(\xi)^2 - \hat{A}_2(\xi)$ and $A_3 = 2\hat{A}_1(\xi)\hat{A}_2(\xi) - \hat{A}_1(\xi)^3 - \hat{A}_3(\xi)$[9]. Since $KO(T_\xi)$ is one to one corresponded to $KO(K)$ by Φ_{KO} the integrality of $ch_{N/2+6}(\eta)$ for all $\eta \in KO(T_\xi)$ is equivalent to that of $A_3 m + A_2 ch_2(\bar\eta) + A_1 ch_4(\bar\eta)$ $ch_6(\bar\eta)$ for all $\bar\eta \in KO(K)$. Any $\bar\eta$ of $KO(K)$ is represented by a linear combination of $\xi_i(K)$ $(i=1, 2, 3)$ so that we can replace $\bar\eta$ by $\xi_i(K)$ $(i=1, 2, 3)$. On the ohter hand, from easy computation we have

$$\begin{array}{lll} ch_2(\xi_1(K)) = 2e^4, & ch_2(\xi_2(K)) = 0, & ch_2(\xi_3(K)) = 0, \\ ch_4(\xi_1(K)) = (a/6)e^8, & ch_4(\xi_2(K)) = -e^8, & ch_4(\xi_3(K)) = 0, \\ ch_6(\xi_1(K)) = ((4al+6\alpha(K))/6!)e^{12}, & ch_6(\xi_2(K)) = (\beta(K)/5!)e^{12}, & ch_6(\xi_3(K)) = 2e^{12}. \end{array}$$

Thus Proposition 3 is transformed to

PROPOSITION 3'. *$T(\xi)$ is redusible if and only if the following three classes are all integer classes which are divisible by 2.*

8) R-R theorem.

9) $\hat{A}_i(\xi)$ denotes i-th $\hat{A}$-genus.

$$\hat{A}_3(\xi)+\hat{A}_1(\xi)^3-2\hat{A}_1(\xi)\hat{A}_2(\xi)\,, \qquad (\beta(K)/5\,!)e^{12}+\hat{A}_1(\xi)e^8\,,$$

and

$$((4al+6\alpha(K))/6\,!)e^{12}-(a/6)\hat{A}_1(\xi)e^8+2(\hat{A}_1(\xi)^2-\hat{A}_2(\xi))e^4\,.$$

Now let us put $p_1(\xi)=xe^4$, $p_2(\xi)=ye^8$, $p_3(\xi)=ze^{12}$. Then we have

$$\hat{A}_1(\xi)=(-x/24)e^4, \quad \hat{A}_2(\xi)=((7ax-4y)/2^7\cdot 45)e^8$$

and

$$\hat{A}_3(\xi)=((44xyl-16z-31alx)/2^{10}\cdot 3^3\cdot 5\cdot 7)e^{12}\,.$$

In our case, by Proposition 2, Proposition 3 is transformed to

PROPOSITION 4. *Let K be type of (a, m, l, n) and let ξ be $r\xi_k^1+s\xi_k^2+t\xi_k^3$.* Then T_ξ is reducible if and only if the following relations are satisfied.

$$\beta(K)=10rl \qquad \mathrm{mod}\ 240$$

$$5ar^2(2-r)+9alr+6\alpha(K)+4al+6sl=0 \qquad \mathrm{mod}\ 1440$$

$$35alr^3-21alr^2+(14al+126sl+6\alpha(K))_r+6s\beta(K)+2^5\cdot 45t\equiv 0 \quad \mathrm{mod}\ 2^8\cdot 3^4\cdot 5\cdot 7.$$

At last for applications of Novikov-Browder theory we must find conditions under which index formula holds for $\tilde{\xi}$ dual of ξ of $KO(K)$, i.e.

$$I(K)=L_3(\tilde{p}_1,\ \tilde{p}_2\ \tilde{p}_3)=(62\tilde{p}_3-13\tilde{p}_1\tilde{p}_2+2\tilde{p}_1^3)/3^5\cdot 5\cdot 7\,.$$

Since $I(K)=0$ is trivial in our case, from Proposition 2 and $\tilde{\xi}=-r\xi_1(K)-s\xi_2(K)-t\xi_3(K)$ for $\xi=r\xi_1(K)+s\xi_2(K)+t\xi_3(K)$ we can obtain

PROPOSITION 5. *Index formula for $\tilde{\xi}$ is satisfied if and only if*

$$140alr^3-294alr^2+(124al+180\alpha(K)+6^2\cdot 7^2\cdot sl)_r+186s\beta(K)+2^5\cdot 3^2\cdot 5\cdot 31t=0\,.$$

Thus by combining Propositions 1, 2, 4 and 5 we can obtain an application of Browder theory to K.

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